DOCUMENT RESUME

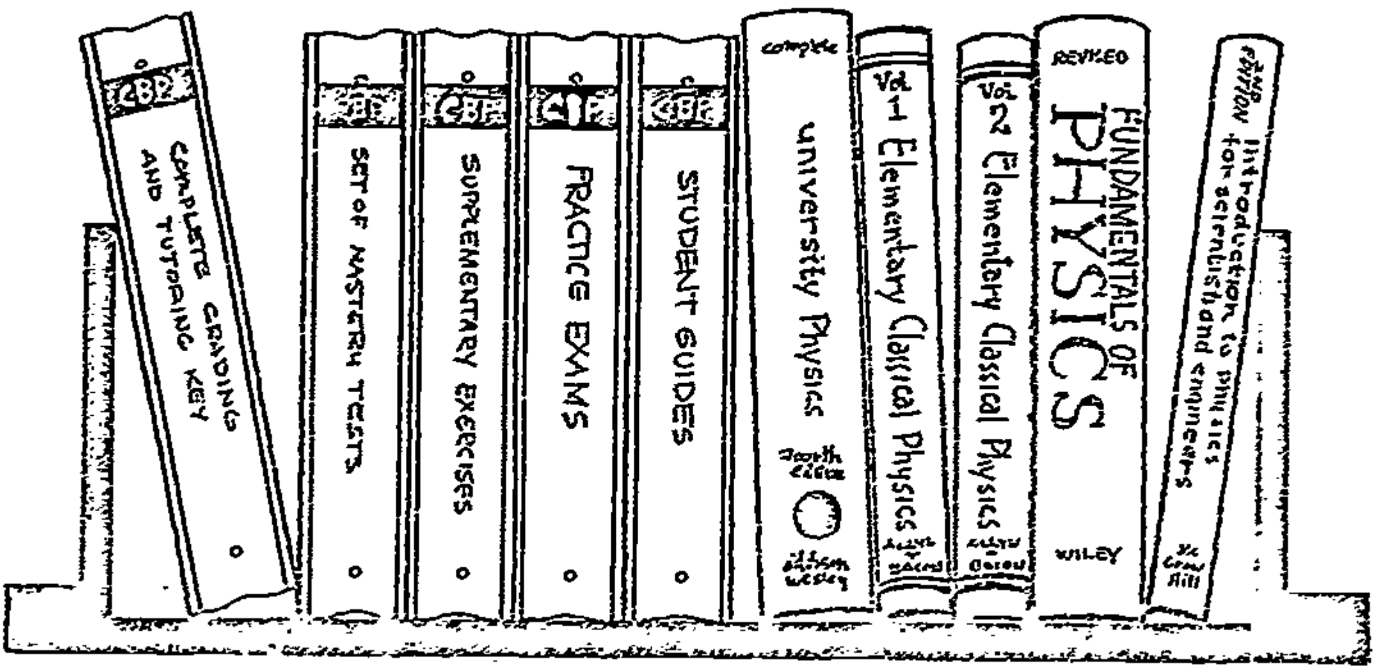
ED 121 583 SE 020 453

AUTHOR	Fuller, Robert G., Ed.; And Others
TITLE	Study Modules for Calculus-Based General Physics. [Includes Modules 15-17: Gravitation; Simple Harmonic Motion; and Traveling Waves; plus a Partial Derivatives Review].
INSTITUTION	Nebraska Univ., Lincoln.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	75
NOTE	85p.; Prepared at a College Faculty Workshop (University of Colorado, Boulder, June 23-July 11, 1975); directions for the CBP modules are found in SE 020 448; Related documents are SE 020 449-461
AVAILABLE FROM	University of Nebraska, CBP Workshop, Behlen Laboratory of Physics, Lincoln, Nebraska 68588 (entire set of 42 modules, $15.00)
EDRS PRICE	MF-$0.83 HC-$4.67 Plus Postage
DESCRIPTORS	*Calculus; College Science; *Course Descriptions; Higher Education; Individualized Curriculum; Individualized Instruction; *Instructional Materials; *Physics; *Programed Instruction; Science Education; Undergraduate Study
IDENTIFIERS	Gravity; Keller Plan; *Personalized System of Instruction; PSI; Simple Harmonic Motion

ABSTRACT

This is part of a series of 42 Calculus Based Physics (CBP) modules totaling about 1,000 pages. The modules include study guides, practice tests, and mastery tests for a full-year individualized course in calculus-based physics based on the Personalized System of Instruction (PSI). The units are not intended to be used without outside materials; references to specific sections in four elementary physics textbooks appear in the modules. Specific modules included in this document are: Module 15--Gravitation, Module 16--Simple Harmonic Motion, Module 17--Traveling Waves, plus a Partial Derivatives Review. (CP)

STUDY MODULES FOR CALCULUS-BASED GENERAL PHYSICS*



CBP Workshop
Behlen Laboratory of Physics
University of Nebraska
Lincoln, NE 68508

*Supported by The National Science Foundation

Comments

These modules were prepared by fifteen college physics professors for use in self-paced, mastery-oriented, student-tutored, calculus-based general physics courses. This style of teaching offers students a personalized system of instruction (PSI), in which they increase their knowledge of physics and experience a positive learning environment. We hope our efforts in preparing these modules will enable you to try and enjoy teaching physics using PSI.

Robert G. Fuller
Director
College Faculty Workshop

MODULE AUTHORS

OWEN ANDERSON	Bucknell University
STEPHEN BAKER	Rice University
VAN BLEUMEL	Worcester Polytechnic Institute
FERNAND BRUNSCHWIG	Empire State College
DAVID JOSEPH	University of Nebraska - Lincoln
ROBERT KARPLUS	University of California - Berkeley
MICHAEL MOLONEY	Rose Hulman Institute of Technology
JACK MUNSEE	California State University - Long Beach
GARY NEWBY	Boise State University
IVOR NEWSHAM	Olivet Nazarene College
WILLIAM SNOW	University of Missouri - Rolla
WILLARD SPERRY	Central Washington State College
ROBERT SWANSON	University of California - San Diego
JAMES TANNER	Georgia Institute of Technology
DAVID WINCH	Kalamazoo College

These modules were prepared by the module authors at a College Faculty Workshop held at the University of Colorado - Boulder, from June 23 to July 11, 1975.

Workshop Staff

Albert A. Bartlett	University of Colorado
Thomas C. Campbell	Illinois Central College
Harold Q Fuller	University of Missouri - Rolla

Calculus-Based Physics (CBP) Modules Production Staff

Robert G. Fuller	Editor
Thomas C. Campbell	Assistant Editor
William D. Snow	Illustrator
Catherine A. Caffrey	Production Editor

COMMENT TO USERS

In the upper right-hand corner of each Mastery Test you will find the "pass" and "recycle" terms and a row of numbers "1 2 3 ..." to facilitate the grading of the tests. We intend that you indicate the weakness of a student who is asked to recycle on the test by putting a circle around the number of the learning objective that the student did not satisfy. This procedure will enable you easily to identify the learning objectives that are causing your students difficulty.

ERRATA

Traveling Waves: p, 10, K(4), line 5, "...0.00300 m in amplitude." P.11. line 3 in F(1)(b) "...4) (2..." The Practice Test Answers on p. 13 are:

1. (a) $y_1 = A \sin(kx - \omega t)$. ... $y_2 = A \sin(kx + \omega t)$...$A = 0.00100$ m, $k = 5.24\ \mathrm{m}^{-1}$, and $\omega = 157\ \mathrm{s}^{-1}$. . . .(b) $v = \omega/k = \sqrt{T/\mu}$, $T = \omega^2\mu/k^2 = (157)^2(2 \times 10^{-3})/(5.24)^2 = 1.80$ N.

2. (b)...+x direction....(c) ...by $\sqrt{2}$."

We shall correct these and any other errors brought to our attention when the CBP Modules are reprinted. We would be happy to receive your suggestions or any corrections that you discover necessary in using the modules.

COMMENT TO USERS

It is conventional practice to provide several review modules per semester or quarter, as confidence builders, learning opportunities, and to consolidate what has been learned. You the instructor should write these modules yourself, in terms of the particular weaknesses and needs of your students. Thus, we have not supplied review modules as such with the CBP Modules. However, fifteen sample review tests were written during the Workshop and are available for your use as guides. Please send $1.00 to CBP Modules, Behlen Lab of Physics, University of Nebraska - Lincoln, Nebraska 68588.

FINIS

This printing has completed the initial CBP project. We hope that you are finding the materials helpful in your teaching. Revision of the modules is being planned for the Summer of 1976. We therefore solicit your comments, suggestions, and/or corrections for the revised edition. Please write or call

CBP WORKSHOP
Behlen Laboratory of Physics
University of Nebraska
Lincoln, NE 68588

Phone (402) 472-2790
(402) 472-2742

STUDY GUIDE

GRAVITATION

INTRODUCTION

The members of the solar system - the Sun, the Moon, and the planets - have held a strong fascination for mankind since prehistoric times. The motions of these heavenly bodies were thought to have important specific influences on persons' lives - a belief that is reflected even today in horoscopes and astrological publications. A revolution in man's thinking that occurred about four hundred years ago established the concept of a solar system with planets orbiting about the Sun and moons orbiting about some of the planets. Copernicus, Kepler, Galileo, and Newton were the four scientific leaders chiefly responsible for establishing this new viewpoint. One of its very practical aspects, yet difficult for us earth-bound creatures to grasp, is that the force of gravity gradually diminishes as one recedes from the Earth, in a way beautifully stated by Newton in his universal law of gravitation.

Gravity is a universal force: It acts on every material thing from the smallest nuclear particle to the largest galaxy. It even acts on objects that have zero rest mass, such as photons - the fantastically minute "chunks" in which light comes. One of the most exciting areas of astronomical research today is the "black hole," where the gravitational field may be so immense that not even light can escape!

Newton's law of gravitation is important not only in itself, but also because it serves as a model for the interaction of electric charges, which you will study later. Not only are the force law and the potential-energy function nearly the same, but the concept of a field carries over and becomes even more useful in the calculation of forces between electrically charged particles.

PREREQUISITES

Before you begin this module, you should be able to:	Location of Prerequisite Content
*Relate the resultant force acting on a particle to the particle's mass and acceleration (needed for Objectives 1 and 2 of this module)	Newton's Laws Module
*Relate the acceleration of a particle moving in a circular path to its speed and the radius of the path (needed for Objective 2 of this module)	Planar Motion Module
*Use the principle of conservation of total mechanical energy to solve problems of particle motion (needed for Objective 3 of this module)	Conservation of Energy Module

LEARNING OBJECTIVES

After you have mastered the content of this module, you will be able to:

1. Law of gravitation - Use Newton's law of universal gravitation to determine (a) the (vector) gravitational force exerted by one object on another - or the distance or a mass when the force is known; and (b) the gravitational field of an object.

2. Circular orbits - Use the gravitational force law, together with the expression for centripetal acceleration, to find the speed, period, orbital radius, and/or masses of objects moving in circular orbits as a result of gravitational forces.

3. Energy conservation - Determine the potential energy of one object in the gravitational field of another; and use energy conservation to relate changes in this potential energy to changes in kinetic energy and speed of the first object.

TEXT: Frederick J. Bueche, Introduction to Physics for Scientists and Engineers (McGraw-Hill, New York, 1975), second edition

SUGGESTED STUDY PROCEDURE

Read the General Comments on the following pages of this study guide, along with Sections 10.8, 10.9, and 10.11 in Chapter 10 of Bueche. Recommended: Read Chapter 14, Sections 14-4 and 14-8 thru 14-10 of Halliday and Resnick (HR)* or Sections 13-5, 13-6, and 13-8 in Chapter 13 of Weidner and Sells (WS).* Optional: Read Sections 10.10, 10.12, and 10.13 of Bueche.

If possible, you should read through a derivation of the result for large, spherically symmetric objects mentioned in General Comment 2 below; and you may wish further discussion of gravitational potential energy, beyond that given in General Comment 4. Both these topics are covered in the Recommended Readings above.

A correction to Figure 10.13: The quartz fiber in a Cavendish balance is actually very fine, and not twisted! In use, the mirror rotates through only a small angle.

Work the problems for Objective 2 starting from the fundamental gravitational and centripetal force expressions. Do not try to remember the equations in Illustrations 10.5 and 10.7.

BUECHE

Objective Number	Readings	Problems with Solutions: Study Guide	Problems with Solutions: Text[b]	Assigned Problems: Study Guide	Additional Problems (Chap. 10)
1	General Comments 1, 2; Secs. 10.8, 10.9	A	Illus. 10.6	D, E(b), F(a)	J, 12-14, 24
2	General Comment 3; Sec. 10.11	B	Illus. 10.5, 10.7	E(a), F(b), G(c), I(a, c)	K, L, 23, 25, Quest.[c] 1, 11, 13
3	General Comment 4; HR*: Secs. 14-8 thru 14-10; or WS*: Secs. 13-5, 13-6	C a		G(a, b), H, I(b)	M, N, 17

[a]See Example in General Comment 4. [b]Illus. = Illustration(s). [c]Quest = Question(s).

*HR = David Halliday and Robert Resnick, Fundamentals of Physics (Wiley, New York, 1970; revised printing, 1974).

WS = Richard T. Weidner and Robert L. Sells, Elementary Classical Physics (Allyn and Bacon, Boston, 1973), second edition, Vol. 1.

TEXT: David Halliday and Robert Resnick, Fundamentals of Physics (Wiley, New York, 1970; revised printing, 1974)

SUGGESTED STUDY PROCEDURE

Read the General Comments on the following pages of this study guide, along with Chapter 14 in your text; but Sections 14-3 and 14-5 are optional. Also, you will not be expected to reproduce the derivation in Section 14-4; however, you should read it through because the result, Eq. (14-8), is very important.

At this point, it's quite likely that you haven't yet studied oscillations (as in the module Simple Harmonic Motion). But you need not be dismayed at the two references to simple harmonic motion - they're not critical to your understanding of this module, and you can simply take the stated results about oscillations at face value.

On p. 266 in Section 14-8, the quantity "F(r)" would more appropriately be called "$F_r(r)$." It is really the component of $\vec{F}(r)$ along the outward radial direction (which can be negative and is, in this instance), whereas the notation used makes it look like the magnitude of a force (which cannot be negative).

Work the problems for Objective 2 starting from the fundamental gravitational and centripetal force expressions. Do not try to remember Eq. (14-13) and the subsequent equation.

HALLIDAY AND RESNICK

Objective Number	Readings	Problems with Solutions: Study Guide	Problems with Solutions: Text	Assigned Problems: Study Guide	Additional Problems (Chap. 14)
1	General Comments 1, 2; Secs. 14-1, 14-2, 14-6	A		D, E(b), F(a)	J, 3, Quest. 1, 26
2	General Comment 3	B	Example*	E(a), F(b), G(c), I(a, c)	K, L, 20, 22-27, Quest.† 5, 8, 12, 21
3	General Comment 4; Secs. 14-8 thru 14-10	C	General Comment 4 Example; Ex.† 4, 5	G(a, b), H, I(b)	M, N, 30-34, 43-45

*Study derivation of Eqs. (14-12) and (14-13) in Section 14-7 (pp. 262, 263).

†Ex. = Example(s). Quest. = Question(s).

TEXT: Francis Weston Sears and Mark W. Zemansky, University Physics (Addison-Wesley, Reading, Mass., 1970), fourth edition

SUGGESTED STUDY PROCEDURE

Read the General Comments on the following pages of this study guide, along with Chapter 5, Sections 5-4 and 5-5, Chapter 6, Section 6-9, and Chapter 7, Section 7-4. Recommended: Read Sections 14-4 and 14-8 through 14-10 in Chapter 14 of Halliday and Resnick (HR),* or Sections 13-5, 13-6, and 13-8 in Chapter 13 of Weidner and Sells (WS).* Optional: Read Section 6-10 of the text.

If possible, you should read through a derivation of the result for large, spherically symmetric objects mentioned in General Comment 2 below; and you may wish further discussion of gravitational potential energy, beyond that given in General Comment 4 and Section 7-4 in the text. Both these topics are covered in the Recommended Readings above.

Work the problems for Objective 2 starting from the fundamental gravitational and centripetal force expressions. Do not try to remember the equations in Section 6-9.

SEARS AND ZEMANSKY

Objective Number	Readings	Problems with Solutions: Study Guide	Problems with Solutions: Text	Assigned Problems: Study Guide	Additional Problems
1	General Comments 1, 2; Secs. 5-4, 5-5	A	Sec. 5-4, Ex. 1, 2; Sec. 5-6, Ex. 8	D, E(b), F(a)	J, 5-2, 5-4, 5-8 to 5-12
2	General Comment 3; Sec. 6-9	B	Sec. 6-9, Ex.	E(a), F(b), G(c), I(a, c)	K, L, 6-43 to 6-47, 7-36(a), 7-50(e, f)
3	General Comment 4; Sec. 7-4 after Ex.[a] 5; HR*: Secs. 14-8 thru 14-10; or WS*: Secs. 13-5, 13-6	C b		G(a, b) H, I(b)	M, N, 7-36(b, c)

[a]Ex. = Examples(s). [b]See Example in General Comment 4.

*HR = David Halliday and Robert Resnick, Fundamentals of Physics (Wiley, New York, 1970; revised printing, 1974).

WS = Richard T. Weidner and Robert L. Sells, Elementary Classical Physics (Allyn and Bacon, Boston, 1973), second edition, Vol. 1.

TEXT: Richard T. Weidner and Robert L. Sells, Elementary Classical Physics (Allyn and Bacon, Boston, 1973), second edition, Vol. 1

SUGGESTED STUDY PROCEDURE

Read the General Comments on the following pages of this study guide, along with Chapter 13 in your text; but Sections 13-4 and 13-7 are optional. Also, you will not be expected to reproduce the derivation in Section 13-8; however, you should read it through because the result, Eq. (13-18) and the implied equation for the force due to a large spherical object, is very important.

Work the problems for Objective 2 starting from the fundamental gravitational and centripetal force expressions. Do not try to remember the equations derived for planetary and satellite motion.

WEIDNER AND SELLS

Objective Number	Readings	Problems with Solutions: Study Guide	Problems with Solutions: Text	Assigned Problems: Study Guide	Additional Problems
1	General Comments 1, 2; Secs. 13-1 thru 13-3	A	Ex.* 13-2	D, E(b), F(a)	J, 13-2, 13-10
2	General Comment 3	B	Ex. 13-1	E(a), F(b), G(c), I(a, c)	K, L, 13-5, 13-8, 13-9, 13-20
3	General Comment 4; Secs. 13-5, 13-6	C †	Ex. 13-4	G(a, b), H, I(b)	M, N, 13-12(b), 13-14 to 13-16, 13-21, 13-26, 13-27

*Ex. = Example(s).

†See Example in General Comment 4.

GENERAL COMMENTS

1. The Gravitational Force Law and the Gravitational Field $\vec{g}$

The LAW OF UNIVERSAL GRAVITATION is easily expressed:

> Every particle (with mass M_2, say) in the universe is attracted toward every other particle (with mass M_1, say) by a force with magnitude
>
> $$F = G(M_1 M_2/r^2),$$
>
> where r is the distance between the two particles, and
>
> $$G = 6.67 \times 10^{-11}\ \mathrm{N\ m^2/kg^2}$$
>
> is an experimentally measured universal constant (see Fig. 1).

(In the figure, M_1 has been placed at the origin for later convenience.) Note that the direction of $\vec{F}$ is exactly along the line joining the two masses. This law can also be expressed vectorially:

> The gravitational force experienced by M_2 due to the presence of M_1 is
>
> $$\vec{F}_{21} = -G(M_1 M_2/r^2)\hat{r},$$
>
> where r is again the distance of separation, and $\hat{r}$ is a unit vector pointing in the direction from M_1 toward M_2 (see Fig. 1).

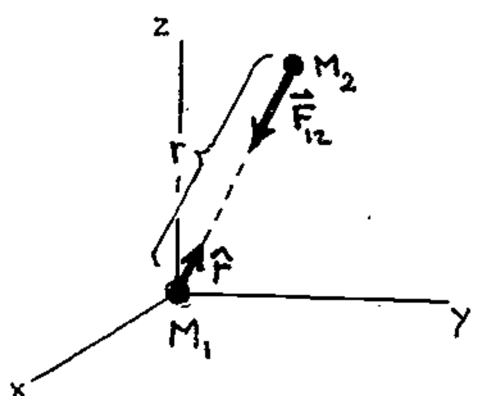


Figure 1

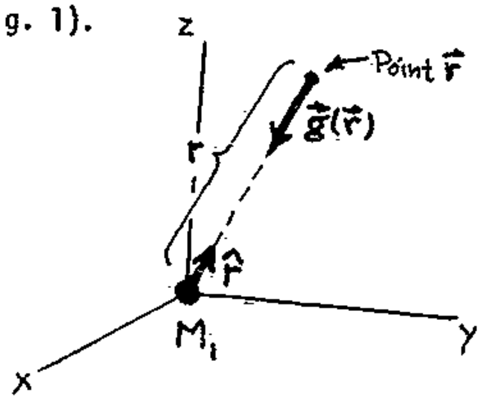


Figure 2

Two masses are required in order to talk about the gravitational force $\vec{F}_{21}$. But notice that if we divide through by M_2, we obtain a quantity

$$\vec{g}(\vec{r}) \equiv \frac{\vec{F}_{21}}{M_2} = -G\left(\frac{M_1}{r^2}\right)\hat{r}$$

that depends only upon the magnitude of M_1 and the point where M_2 is located, relative to M_1. [Since $\vec{r} \equiv r\hat{r}$ is simply the position vector of that point relative to M_1, we have indicated this latter dependence by writing $\vec{g}(\vec{r})$, instead of simply $\vec{g}$.] See Figure 2.

In fact, we don't need M_2 at all in thinking about this quantity $\vec{g}(\vec{r})$, which is called the gravitational field intensity or, more simply, just the gravitational field due to M_1.

This abstraction $\vec{g}$ associated with a single mass M_1 occupies all space surrounding M_1 whether other masses are present or not; and at each point, it can be represented by a vector pointing toward M_1, with the magnitude GM/r^2. Notice that, physically, the gravitational field intensity at a given point is simply the acceleration a very small object would experience if it were placed at that point.

The use of the concept of a force field to describe an interaction at a distance is an exceedingly important technique, and will be developed further in later modules on electric and magnetic interactions. The gravitational field is a central conservative field; central because it acts along the line joining the interacting particles, and conservative (for energy) because it is possible to define a potential energy function of distance. If you have already studied torque and angular momentum, you will recognize that a particle subject only to the centrally directed gravitational force of another body experiences no torque; thus its angular momentum is constant, or "conserved." (For circular orbits, this reduces to the simple result that the speed is constant.) Furthermore, as the particle changes its position (in whatever kind of an orbit), all decreases in kinetic energy are accompanied by equal increases in gravitational potential energy, and vice versa, so that the total energy remains constant. In this module, you will make extensive use of the conservation of energy. Note that because the moving particle is subject to a force, its linear momentum is not constant.

2. "Large" Spherically Symmetric Objects

One of the interesting and useful consequences of the functional form of the law of gravitation (namely, the dependence $1/r^2$) is that the gravitational field of an extended spherically symmetric object of mass M and radius R (see Fig. 3) is exactly the same as the field of a "point" (i.e., very small) object of mass M located at the same place as the center of the sphere. That is, the gravitational field due to 3(a) is exactly the same as the field due to 3(b) for every point $r > R$. (For $r < R$, the fields are very different for these two situations.) Therefore, whenever you encounter a spherically symmetric object, you can simplify the situation by replacing the object with a point object of equal mass - assuming you are interested only in points outside the spherical object, and not, say, in a tunnel through its middle.

Caution: When using the gravitational law in such situations, be sure to use the distance from the center of the gravitational attraction, and not the height above the surface of the Earth or the other body!

Figure 3

(a)

(b)

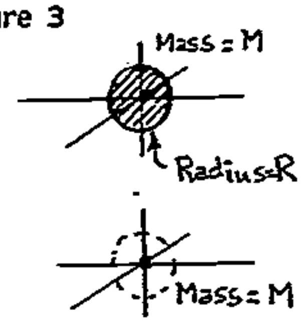
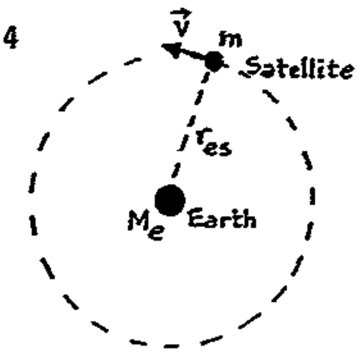


Figure 4



3. Circular Planetary and Satellite Orbits

In reality, the orbits of planets and satellites are never exactly circles, but, rather, more general ellipses. However, the orbits of most planets and of many satellites are near enough to circular that only a very small error results from treating them as circular. This simplifies the calculations greatly, since then you can use what you learned about circular motion in the module Planar Motion. In fact, you found in Problem G of Planar Motion that a circular trajectory such as

$$\vec{r}(t) = 2[\cos(\pi t/4)\hat{i} + \sin(\pi t/4)\hat{j}]\ \mathrm{m}$$

has an acceleration

$$\vec{a}(t) = -\omega^2\vec{r}(t),$$

where $\omega = v/r$. [See, in particular, parts (c) and (d) of that problem.] That is, a particle moving in a circular path of radius r at the constant speed $v = \omega r$ has a centripetal acceleration with magnitude $a_c = \omega^2 r$.

Furthermore, you found in the module Newton's Laws that it takes a force $\vec{F} = m\vec{a}$ to give a particle with mass m the acceleration a. From these last three equations, it follows that the centripetal force

$$F_c = m\omega^2 r = mv^2/r$$

is required to hold a particle in a circular path.

In the case of a satellite moving around Earth as in Figure 4, this centripetal force is provided by the gravitational force of attraction between Earth and the satellite. Since $M_e \gg m$, we can ignore the motion of Earth; it acts just like a fixed force center. As you learned in your studies for Objective 1 of this module, the gravitational force acting on the satellite has the magnitude

$$F_g = GM_e m/r_{es}^2.$$

Equating F_c to F_g yields

$$\frac{mv^2}{r_{es}} = G\frac{M_e m}{r_{es}^2} \quad \text{or} \quad v^2 = \frac{GM_e}{r_{es}}.$$

This relation allows us to calculate, say, v in terms of M_e and r_{es}. Once we have found v from a relation such as the above, then it is, of course, easy to find the period T of the circular motion, since vT is just the circumference $2\pi r$ of the circular orbit.

The motion of planets around the Sun is very similar; since the Sun has a mass much greater than that of any planet, it can be treated as a fixed force center, just as the Earth was above.

4. Gravitational Potential Energy

In an earlier unit, you learned that an object of mass m that is raised a distance h in the vicinity of Earth's surface gains potential energy in the amount mgh. Let's show this directly from the law of gravitation, $F = GM_e m/r^2$. The work done lifting m is the integral of the force we must exert through the given distance:

$$W = \int_R^{R+h} \left(\frac{GM_e m}{r^2}\right) dr = GM_e m \int_R^{R+h} \frac{dr}{r^2} = -GM_e m\left[\frac{1}{r}\right]_R^{R+h}$$

$$= -GM_e m\left[\frac{1}{R+h} - \frac{1}{R}\right] = GM_e m \frac{h}{R(R+h)}.$$

When the height h is much less than the radius R of the Earth, this yields the approximate value $W \simeq GM_e mh/R^2 = mgh$.

Note that this is valid only when $h \ll R$.

What if we go all the way from the surface of Earth to infinity? This time we get

$$W(R \text{ to } \infty) = -GM_e m\left[\frac{1}{r}\right]_R^{\infty} = -GM_e m\left[0 - \frac{1}{R}\right] = +\frac{GM_e m}{R}.$$

This is the amount of energy that must be expended to carry a mass m from the surface of Earth to a point infinitely far away. (We are neglecting the presence of the other planets and the Sun.) Since it is convenient and customary to say that an object has zero potential energy at infinity, we see that the gravitational potential energy of an object is a negative quantity (zero only at infinity) that becomes more negative as the object approaches any other massive object. For instance, as an object approaches Earth, it loses more and more potential energy (its potential energy becomes more and more negative), and its kinetic energy becomes correspondingly greater. Recall that space capsules returning from the Moon attain extremely high velocities just before reaching Earth's atmosphere. This is exactly analogous to the example of a car gaining speed as it coasts down a steep hill - potential energy is being transformed into kinetic energy.

If we use the customary symbol $U(r)$ to denote gravitational potential energy, then our result above is just

$$U(r) = -G(M_e m/r),$$

for a point or spherically symmetric mass. A particle in such a gravitational field (with no other forces present) always moves in such a way that the sum of its kinetic and gravitational potential energies is constant:

$$E_i = \text{Total initial (mechanical) energy} = K_i + U_i$$

$$= K_f + U_f = \text{Total final (mechanical) energy} = E_f.$$

This energy-conservation equation is very useful in solving many problems.

Example

Space scientists wish to launch a 100-kg probe to infinity (i.e., far from Earth). How much energy does this require? What initial speed is needed? Ignore the presence of the Sun for this example. ("Initial" means at the time of burnout, which for this problem is only a negligible distance above the surface of Earth.)

Solution

The minimum required energy K_i is that which gets the space probe to "infinity" with zero kinetic energy. That is,

$$K_i + U_i = K_f + U_f = 0 + 0,$$

or

$$K_i = -U_i = \frac{GM_e m}{R_e} = \frac{(6.67 \times 10^{-11})(6.0 \times 10^{24})(100)}{6.4 \times 10^6}\ \text{J} = 6.3 \times 10^9\ \text{J}.$$

The needed initial speed follows from the relation

$$K_i = (1/2)mv_i^2 = GM_e m/R_e,$$

which yields

$$v_i = \sqrt{2GM_e/R} = 1.12 \times 10^4\ \text{m/s}.$$

ADDITIONAL LEARNING MATERIALS

Auxilliary Reading

Stanley Williams, Kenneth Brownstein, and Robert Gray, Student Study Guide with Programmed Problems to Accompany Fundamentals of Physics and Physics, Parts I and II, by David Halliday and Robert Resnick (Wiley, New York, 1970).

Objective 1: Sections 14-1 and 14-2;

Objective 2: Section 14-8, especially Problems 1 to 7, and 15 to 17;

Objective 3: Sections 14-5 through 14-7, and 14-8, Problems 19 to 28.

Film Loop

Ealing #80-212: Measurement of "G"/The Cavendish Experiment.

Various Texts

Frederick J. Bueche, Introduction to Physics for Scientists and Engineers (McGraw-Hill, New York, 1975), second edition: Sections 10.8 through 10.13.

David Halliday and Robert Resnick, Fundamentals of Physics (Wiley, New York, 1970; revised printing 1974): Chapter 14.

Francis Weston Sears and Mark W. Zemansky, University Physics (Addison-Wesley, Reading, Mass., 1970), fourth edition: Sections 5-4, 5-5, 6-9, 6-10, 7-4.

Richard T. Weidner and Robert L. Sells, Elementary Classical Physics (Allyn and Bacon, Boston, 1973), second edition, Vol. 1: Chapter 13.

PROBLEM SET WITH SOLUTIONS

Since some of the problems for this module are numerically arduous, you may use the simplified numerical values below (accurate to within a few percent) when working these problems.

$G = (2/3) \times 10^{-10}$ N m^2/kg^2	$\pi^2 = 10$
Mass of Sun = 2.0×10^{30} kg	Radius of Sun = 7.0×10^{8} m
Mass of Earth = 6.0×10^{24} kg	Radius of Earth = 6.4×10^{6} m
Mass of Moon = $(3/4) \times 10^{23}$ kg	Radius of Moon = $(1/6) \times 10^{7}$ m
Mass of Mars = 6.3×10^{23} kg	Radius of Mars = $(1/3) \times 10^{7}$ m
Earth to Moon = $(3/8) \times 10^{9}$ m	Earth to Sun = 1.5×10^{11} m
g on Earth = 10 m/s^2	Saturn to Sun = 1.5×10^{12} m

A(1). A space probe determines that the magnitude of the gravitational field $\vec{g}$ is 1.1 times as large at the surface of Uranus as it is at the surface of Earth. Use the radius of Uranus, $R_u = 2.4 \times 10^7$ m, to determine its mass.

Solution

Since $g = GM/r^2$, we have

$$M_u = gR_u^2/G = \frac{[(1.1)(2.4 \times 10^7)^2]}{[(2/3) \times 10^{-10}]} \text{ kg} = 9.5 \times 10^{25} \text{ kg}.$$

B(2). Communications satellites, such as Telestar, are placed in synchronous orbits around Earth. (A synchronous orbit is an orbit in which the satellite is constantly above the same spot on the surface of Earth.) How far above the surface of Earth must such a satellite be? Be sure to start from the fundamental gravitational and centripetal force equations.

Solution

For a synchronous orbit, the angular velocity of the satellite must be the same as that of Earth, namely, $\omega = 2\pi \times$ (number of revolutions per second) $= (2\pi/24 \times 60 \times 60)$ rad/s.

Since the Earth is much heavier, it is nearly stationary. Therefore, the radius of the satellite's orbit is virtually the same as the distance between the center of the Earth and the satellite; call this distance r. Then the equation $F_{centrip} = F_{grav}$ becomes just

$$mr\omega^2 = GM_e m/r^2$$

or

$$r^3 = GM_e/\omega^2 = [(2/3) \times 10^{-10}](6.0 \times 10^{24})[12 \times 3600)^2/\pi^2] \text{ m}^3 = 74 \times 10^{21} \text{ m}^3.$$

And thus $r = 4.2 \times 10^7$ m. But the problem asked for the height above the surface of Earth:

$$h = r - R_e = 4.2 \times 10^7 \text{ m} - 0.60 \times 10^7 \text{ m} = 3.6 \times 10^7 \text{ m}.$$

C(3). If an object is fired from the surface of Earth with a great enough speed v_0, it will escape from the gravitational field of Earth and will not return. What initial speed is needed for an object fired vertically to rise to a maximum height $R_e/3$ above the surface of Earth, before it returns? Express your answer in terms of v_0; R_e is the radius of Earth. (You do not need to use any numerical values!)

Solution

At its maximum height the object will not be moving, so that $K_f = 0$. Therefore, $K_i + U_i = K_f + U_f$ becomes

$$\frac{1}{2}(mv_i^2) - \frac{GM_e m}{R_e} = 0 - \frac{GM_e m}{(4R_e/3)}, \qquad v_i^2 = \frac{1}{2}\left(\frac{GM_e}{R_e}\right).$$

Thus

$$v_i = \left(\frac{GM_e}{2R_e}\right)^{\frac{1}{2}} = \left(\frac{[(2/3) \times 10^{-10}](6.0 \times 10^{24})}{2(6.4 \times 10^6)}\right)^{\frac{1}{2}} = 3.8 \times 10^3 \text{ m/s}.$$

Problems

D(1). (a) At what height above the surface of Earth is the gravitational field equal to 5.0 m/s^2? Express your answer in terms of the radius of Earth R_e.

(b) At what point between the Earth and the Sun does an object feel no gravitational force? Express your answer in terms of the masses M_e and M_s, and the Earth-to-Sun distance r_{es}.

E(1,2). Jupiter has a moon with an approximately circular orbit of radius 4.2×10^8 m and a period of 42 h.

(a) What is the magnitude of the gravitational field $\vec{g}$ due to Jupiter at the orbit of this moon?

(b) From (a) and the value of G, find the mass of Jupiter.

F(1,2). Answer the questions below, using only Newton's law of universal gravitation, the centripetal force law, and the following data:

At the surface of the Earth, $g = 9.8 \text{ m/s}^2$.

The radius of Earth is 6400 km.

The Moon completes one orbit around the Earth every 27.3 d $= 2.40 \times 10^6$ s.

From the Cavendish experiment, $G = 6.7 \times 10^{-11} \text{ N m}^2/\text{kg}^2$.

(a) What is the mass of Earth?

(b) What is the radius of the Moon's orbit?

G(2,3). Typical satellite orbits back around 1960 were 1.6×10^5 m above the Earth's surface.

(a) What is the potential energy, relative to infinity, of a 1000-kg satellite in such an orbit?

(b) What is its potential energy relative to the Earth's surface?

(c) Find the time that such a satellite requires to complete one orbit. Be sure to start from the fundamental gravitational and centripetal force laws!

H(3). A space traveler in interstellar space is working near her craft when her safety line breaks. At that moment she is 3.00 m away from the center of mass of the craft and drifting away from it at the speed of 1.00 mm/s. If the mass of the craft is 10 000 kg, will she reach a maximum distance and be drawn back, or will she drift away indefinitely?

I(2,3). A 1.00×10^6 kg spaceship making observations in interplanetary space is in a circular orbit about the Sun at a radius of 1.50×10^{11} m (approximately the orbit of Earth).

(a) What is its kinetic energy while in this orbit? [You must start from the gravitational and centripetal force (or acceleration) laws.]

(b) Having completed their observations here, the crew next depart on a voyage to the vicinity of Jupiter's orbit, five times as distant from the Sun (7.5×10^{11} m). There, however, it is not necessary to establish a circular orbit; the ship can arrive there with essentially zero kinetic energy. Purely on the basis of energy conservation, what is the minimum energy that the engines must provide for this voyage?

(c) While in the orbit of part (a), the ship made two complete trips around the Sun. How long, in seconds, was it there?

J(1). Astronauts on the Moon can jump considerably higher than they can on Earth; that is, the acceleration due to gravity is much less. In fact, $g_m = 0.17\ g_e$. The moon is also much smaller: $R_m = 0.27\ R_e$.

(a) Use these data to find the ratio of the masses of the Moon and the Earth, M_m/M_e.

(b) On the straight line between the Earth and the Moon there is a point where a spaceship experiences no gravitational field, because the fields of the Earth and the Moon cancel. How far is this point from the Moon? The Moon is 3.5×10^5 km from the Earth.

K(2). Certain neutron stars are believed to be rotating at about one revolution per second. If such a star has a radius of 30 km, what must be its mass in order that objects on its surface will not be thrown off by the rapid rotation?

L(2). An asteroid revolves about the Sun in a circular orbit once every eight years. Approximately how far is it from the Sun in astronomical units (1 AU is the mean distance from the Sun to Earth)? You must start from the fundamental gravitational and centripetal force (or acceleration) equations.

M(3). What speed is necessary for a 1000-kg spaceship at a distance from the Sun equal to the radius of Saturn's orbit to escape from the Sun's gravitational field?

N(3). A star of mass 2.0×10^{30} kg and another star of mass 4.0×10^{34} kg are initially at rest infinitely far away. They then move directly toward one another under the influence of the gravitational force. Calculate the speed of their impact, which occurs when their centers are separated by 3.0×10^{10} m. The radii of the stars are 1.0×10^{10} m and 2.0×10^{10} m, respectively. Neglect the motion of the more massive star.

Solutions

D(1). (a) $h = r - R_e = 0.41R_e$; (b) $r_s = \sqrt{M_s/M_e}\, r_e$, and thus

$$r_e = \frac{r_{es}}{1 + \sqrt{M_s/M_e}}.$$

E(1,2). (a) $0.73\ \mathrm{m/s^2}$; (b) 1.9×10^{27} kg.

F(1,2). (a) 5.9×10^{24} kg; (b) $gR_e^2/r_m^2 = g'$ (at the Moon's orbit)

$$g' = \frac{v^2}{r_m} = \frac{(2\pi r_m/T)^2}{r_m} = \frac{(2\pi)^2 r_m}{T^2}, \qquad r_m = [gR_e^2T^2/(2\pi)^2]^{1/3} = 3.9 \times 10^8\ \mathrm{m}.$$

G(2,3). (a) -6.1×10^{10} J; (b) 1.6×10^9 J; (c) 88 min.

H(3). Let's hope she has a wrench in her hand that she can throw, since her total energy is

$$E = K + U = (1/2)mv^2 - GMm/r = m(1/2 - 2/9) \times 10^6\ \mathrm{J} > 0!$$

I(2,3). (a) 4.4×10^{14} J; (b) 2.7×10^{14} J; (c) $2\pi \times 10^7$ s.

J(1). (a) $m/M = 0.012$;
(b) 0.35×10^5 km.

K(2). 1.6×10^{25} kg.

L(2). 4.0 AU.

M(3). 1.3×10^4 m/s.

N(3). 1.3×10^5 m/s.

PRACTICE TEST

Use the simplified numerical values below (accurate to within a few percent) in the Practice Test.

$G = (2/3) \times 10^{-10}$ N m^2/kg^2	$\pi^2 = 10$
Mass of Sun = 2.0×10^{30} kg	Radius of Sun = 7.0×10^{8} m
Mass of Earth = 6.0×10^{24} kg	Radius of Earth = 6.4×10^{6} m
Mass of Moon = $(3/4) \times 10^{23}$ kg	Radius of Moon = $(1/6) \times 10^{7}$ m
Mass of Mars = 6.3×10^{23} kg	Radius of Mars = $(1/3) \times 10^{7}$ m
Earth to Moon = $(3/8) \times 10^{9}$ m	Earth to Sun = 1.5×10^{11} m
g on Earth = 10 m/s^2	Saturn to Sun = 1.5×10^{12} m

1. The radius of the planet Jupiter is 11 times that of Earth, and its mass is 310 times as large as that of Earth. Using only these data, find out how the acceleration due to gravity on the surface of Jupiter compares with that on Earth.

2. A 1.0×10^{6} kg spaceship making observations in interplanetary space is in a circular orbit about the Sun at a radius of 1.5×10^{11} m (approximately the orbit of Earth).

   (a) What is its kinetic energy while in this orbit? [You must start from the gravitational and cetripetal force (or acceleration) laws.]

   (b) Having completed their observations here, the crew next depart on a voyage to the vicinity of Jupiter's orbit, five times as distant from the Sun (7.5×10^{11} m). There, however, it is not necessary to establish a circular orbit; the ship can arrive there with essentially zero kinetic energy. Purely on the basis of energy conservation, what is the minimum energy that the engines must provide for this voyage?

   (c) While in the orbit of part (a), the ship made two complete trips around the Sun. How long, in seconds, was it there?

Practice Test Answers

1. $g_J = 2.6\ g_e$.

2. (a) 4.4×10^{14} J; (b) 2.7×10^{14} J; (c) $2\pi \times 10^{7}$ s.

GRAVITATION

Mastery Test Form A

Date________________

pass recycle
1 2 3

Name__________________________

Tutor________________

Use the simplified numerical values below (accurate to within a few percent) in this Mastery Test.

$G = (2/3) \times 10^{-10}$ N m^2/kg^2	$\pi^2 = 10$
Mass of Sun = 2.0×10^{30} kg	Radius of Sun = 7.0×10^{8} m
Mass of Earth = 6.0×10^{24} kg	Radius of Earth = 6.4×10^{6} m
Mass of Moon = $(3/4) \times 10^{23}$ kg	Radius of Moon = $(1/6) \times 10^{7}$ m
Mass of Mars = 6.3×10^{23} kg	Radius of Mars = $(1/3) \times 10^{7}$ m
Earth to Moon = $(3/8) \times 10^{9}$ m	Earth to Sun = 1.5×10^{11} m
g on Earth = 10 m/s^2	Saturn to Sun = 1.5×10^{12} m

1. On certain Saturdays in the autumn, large numbers of people experience a strong attraction for the football stadium. Do you suppose that this attraction could be gravitational inorigin? That is, estimate the gravitational force of attraction exerted by the stadium on an 80-kg football fan one block (133 m) away. Assume a total mass of 1.0×10^7 kg for the stadium (including the fans already assembled there).

2. In 1944, when the first group of astronauts landed on Mars, they discovered the third moon of Mars, which was in a circular orbit 1.0 km above the surface of the planet.

   (a) What was the speed of this moon in its orbit? Remember that you are to start from the fundamental centripetal and gravitational force equations!
   Since a moon this close to the surface of Mars interfered with their explorations, the astronauts decided to move it into an orbit at a higher altitude. Fortuitously, the angular speed for this new orbit was

   $$\omega = \sqrt{42} \times 10^{-6} \text{ rad/s.}$$

   (This is especially fortuitous if you use a calculator that does not take cube roots!)
   The mass of the moon was 2.0×10^3 kg.

   (b) What was the radius of the new orbit?

   (c) How much energy did it take to move the moon to its new orbit?

GRAVITATION

Mastery Test Form 8

Date____________________

pass recycle

1 2 3

Name______________________________

Tutor____________________

Use the simplified numerical values below (accurate to within a few percent) in this Mastery Test.

$G = (2/3) \times 10^{-10}$ N m^2/kg^2	$\pi^2 = 10$
Mass of Sun = 2.0×10^{30} kg	Radius of Sun = 7.0×10^{8} m
Mass of Earth = 6.0×10^{24} kg	Radius of Earth = 6.4×10^{6} m
Mass of Moon = $(3/4) \times 10^{23}$ kg	Radius of Moon = $(1/6) \times 10^{7}$ m
Mass of Mars = 6.3×10^{23} kg	Radius of Mars = $(1/3) \times 10^{7}$ m
Earth to Moon = $(3/8) \times 10^{9}$ m	Earth to Sun = 1.5×10^{11} m
g on earth = 10 m/s^2	Saturn to Sun = 1.5×10^{12} m

1. A meteorite originally at rest in interstellar space falls to the surface of Earth; find the speed with which it hits. For this problem, make the simplifying assumptions (not really justified) that the effect of the Sun, the motion of the Earth, and the retarding force of the atmosphere can be neglected.

2. Because of an accident on a space flight, a 70-kg man is left in deep space, 1.0×10^4 m from the spherical asteroid Juno of mass 1.5×10^{14} kg.

   (a) How fast must he move, propelled by his rocket pack, to achieve a circular orbit around the asteroid at this distance, rather than crashing to its surface? [You must start from the fundamental gravitational and centripetal force (or acceleration) equations.]

   (b) It takes 8 h and 45 min for the rescue ship to arrive. Where should they look for him, relative to the place of the accident?

GRAVITATION

Mastery Test Form C

Date ______________

pass recycle
1 2 3

Name ______________________

Tutor ______________

Use the simplified numerical values below (accurate to within a few percent) in this Mastery Test.

$G = (2/3) \times 10^{-10}$ N m^2/kg^2	$\pi^2 = 10$
Mass of Sun = 2.0×10^{30} kg	Radius of Sun = 7.0×10^{8} m
Mass of Earth = 6.0×10^{24} kg	Radius of Earth = 6.4×10^{6} m
Mass of Moon = $(3/4) \times 10^{23}$ kg	Radius of Moon = $(1/6) \times 10^{7}$ m
Mass of Mars = 6.3×10^{23} kg	Radius of Mars = $(1/3) \times 10^{7}$ m
Earth to Moon = $(3/8) \times 10^{9}$ m	Earth to Sun = 1.5×10^{11} m
g on earth = 10 m/s^2	Saturn to Sun = 1.5×10^{12} m

1. When Deathwish Hershey reported for his Space Corps preinduction physical, he brought with him a letter from his psychiatrist to certify that he showed suicidal tendencies in low-g environments. In spite of this, he was assigned to a tour of duty on Mudberry, a perfectly spherical airless asteroid of radius 200 km, where $g = 0.20$ m/s^2. After a month of solitary duty, Deathwish could stand it no longer. Having completed a quick calculation, he fired a bullet parallel to the asteroid's surface at speed v and stood at attention, waiting for it to hit him on the back of the head. Little did he know that his appeal had been successful and that a ship was due to arrive in one and one-half hours to return him to civilian life.

   (a) At what speed v did Deathwish fire the bullet? [You must start from the fundamental gravitational and centripetal force (or acceleration) equations.]

   (b) Did the ship or the fatal bullet arrive first?

2. If Deathwish had fired the bullet vertically instead of horizontally, but with the same speed v as in the preceding problem, how high would it have gone? Or would it have escaped from the asteroid entirely?

MASTERY TEST GRAOING KEY - Form A

1. Solution: Hardly, since

$$F_g = \frac{GMm}{r^2} = \frac{[(2/3) \times 10^{-10}](1.0 \times 10^7)(80)}{(133)^2} N = 3.0 \times 10^{-6} N.$$

2. What To Look For: (a) Check that the student really started by equating the centripetal force to the gravitational force. (Equating accelerations is essentially equivalent.) If some other (correct) equation was used, make her/him derive it!

   (c) Make sure that both kinetic and potential energies have been included for both orbits. If the student used one of the relations $E = -K = (1/2)U$ (valid only for such orbits), ask her/him how the problem could be done without assuming this relation. (But don't require the student to go through the arithmetic again.)

   Solutions: (a) $F_{centrip} = F_{grav}$ or $mv^2/R_M = GM_m m/R_M^2$. ($a_{centrip} = a_{grav}$ is also acceptable.) Thus

$$v = \left(\frac{GM_m}{R_m}\right)^{1/2} = \left(\frac{[(2/3) \times 10^{-10}](6.3 \times 10^{23})}{(1/3) \times 10^7}\right)^{1/2} \text{ms} = 3.5 \times 10^3 \text{ m/s}.$$

   (b) As above,

$$mr\omega^2 = \frac{GM_m}{r^2} \text{ or } r = \left(\frac{GM_m}{\omega^2}\right)^{1/2} = \left(\frac{[(2/3) \times 10^{-10}](6.3 \times 10^{23})}{42 \times 10^{-12}}\right)^{1/2} \text{m}$$

$$r = 1.0 \times 10^8 \text{ m}.$$

   (c) $r_i = (1/3) \times 10^7$ m, $r_f = 1.0 \times 10^8$ m, $v_i = 3.5 \times 10^3$ m/s,

   $\omega_f = \sqrt{42} \times 10^{-6}$ rad/s, and $v_f = r_f\omega_f = \sqrt{42} \times 10^2$ m/s;

   thus the energy needed is $\Delta E = K_f + U_f - K_i - U_i = m[\frac{1}{2}v_f^2 - \frac{1}{2}v_i^2 + GM_m(\frac{1}{r_i} - \frac{1}{r_f})]$

$$= (2.0 \times 10^3) \times \{\tfrac{1}{2}(\sqrt{42} \times 10^2)^2 - \tfrac{1}{2}(3.5 \times 10^3)^2$$

$$+ [(2/3) \times 10^{-10}](6.3 \times 10^{23})$$

$$\times (3.0 \times 10^{-7} - 1.0 \times 10^{-8})\}$$

$$= (2.0 \times 10^3)[0.21 - 6.1 + (4.2)(2.9)] \times 10^6$$

   or $\Delta E = 1.3 \times 10^9$ J.

MASTERY TEST GRADING KEY - Form B

1. Solution: $E_f = E_i$ or $K_f + U_f = K_i + U_i = 0 + 0$; thus

   $(1/2)mv_f^2 = K_f = -U_f = +G(M_e/R_e)$,

   $$v_f = \left(\frac{2GM_e}{R_e}\right)^{1/2} = \left(\frac{2(2/3 \times 10^{-10})(6.0 \times 10^{24})}{(6.4 \times 10^6)}\right)^{1/2} = 1.1 \times 10^4 \text{ m/s}.$$

2. What To Look For: (a) Check that the student really started by equating the centripetal force to the gravitational force. (Equating accelerations is essentially equivalent.) If some other (correct) equation was used, make her/him derive it.

   Solutions: (a) $F_{centrip} = F_{grav}$ or $mv^2/r = GMm/r^2$. Thus,

   $$v = \left(\frac{GM}{r}\right)^{1/2} = \left(\frac{[(2/3) \times 10^{-10}](1.5 \times 10^{14})}{1.0 \times 10^4}\right)^{1/2} \text{ m/s} = 1.0 \text{ m/s},$$

   the speed he needs to orbit.

   (b) $T = \frac{2\pi r}{v} = \frac{2\pi(1.0 \times 10^4)}{(1.0)} = 6.3 \times 10^4$ s.

   $8 \text{ h} + 45 \text{ min} = 8(3600) + 45(60) = 3.15 \times 10^4$ s.

   They should, therefore, look for him almost exactly on the other side of Juno.

MASTERY TEST GRADING KEY - Form C

1. What To Look For: (a) Check that the student really started by equating the centripetal force to the gravitational force. (Equating accelerations is essentially equivalent.) If some other (correct) equation was used, make her/him derive it!

Solutions: (a) $F_c = F_g$ (Centripetal force = gravitational force) or

$$\frac{mv^2}{R} = G\frac{Mm}{R^2} = gm,$$

$$v = (gR)^{1/2} = [(0.20)(2.0 \times 10^5)]^{1/2}\ \mathrm{m/s} = 200\ \mathrm{m/s}.$$

(b) $T = 2\pi R/v = [2\pi(2.0 \times 10^5)/200]\ \mathrm{s} = 2\pi \times 10^3\ \mathrm{s} = 6280\ \mathrm{s}.$

By comparison, $1\frac{1}{2}$ hour = (1.5)(3600 s) = 5400 s. Cheers!

2. What To Look For: Make sure that both kinetic and potential energies have been included. If the student used one of the relations $E = -K = (1/2)U$ (valid only for such orbits), ask her/him how the problem could be done without assuming this relation. (But don't require the student to go through the arithmetic again.)

Solution: Use energy conservation: $E_f = E_i$, or $K_f + U_f = K_i + U_i$.
Let h be the maximum height of the bullet, the point at which $v_f = 0$; then

$$0 - \frac{GMm}{R + h} = \frac{1}{2}mv_i^2 - \frac{GMm}{R}.$$

Since $GM = gR^2$ [see Problem 1(a)], and $v_i = \sqrt{gR}$, this becomes

$$\frac{mgR^2}{R + h} = -\frac{1}{2}mgR + mgR = \frac{1}{2}mgR.$$

Thus, $R = (1/2)(R + h)$, and $h = R = 2.0 \times 10^5$ m.

(If the energy had been so high that the bullet escaped, there would have been no value of h that satisfied this equation.)

Module____ 1

STUDY GUIDE

SIMPLE HARMONIC MOTION

INTRODUCTION

Have you ever felt you were the slave of a clock? Clocks are mechanisims that include a pendulum or balance wheel whose repeated patterns of movement define equal time intervals, one after another. Such repeated movements are called periodic motion. Periodic motion may occur when a particle or body 's confined to a limited region of space by the forces acting on it and does not have sufficient energy to escape.

In this module you will study the special kind of periodic motion that results when the net force acting on a particle, often called the restoring force, is directly proportional to the particle's displacement from its equilibrium position; this is known as simple harmonic motion. Actually, simple harmonic motion is an idealization that applies only when friction, finite size, and other small effects in real physical systems are neglected. But it is a good enough approximation that it ranks in importance with other special kinds of motion (free fall, circular and rotational motion) that you have already studied. Examples of simple harmonic motion include cars without shock absorbers, a child's swing, violin strings, and, more importantly, certain electrical circuits and vibrations of a tuning fork that you may study in later modules.

PREREQUISITES

Before you begin this module, you should be able to:	Location of Prerequisite Content
*Define kinetic energy (needed for Objective 3 of this module)	Work and Energy Module
*Define potentail energy, and use the conservation of energy to solve simple problems (needed for Objectives 2 and 4 of this module)	Rotational Motion Module
*Define angular velocity, acceleration, displacement, and torque (needed for Objectives 2 and 4 of this module)	Rotational Motion Module
*Apply Newton's second law for rotation to solve simple problems (needed for Objectives 2 and 4 of this module)	Rotational Dynamics Module

LEARNING OBJECTIVES

After you have mastered the content of this module, you will be able to:

1. Definitions - Define the following terms or relate them to the solution of Newton's second law for simple harmonic motion, $x = A \cos(\omega t + \theta)$ (instead of θ you may see ϕ or δ):

simple harmonic motion,	amplitude,
frequency,	phase constant (or phase angle),
angular frequency,	period,
spring constant,	restoring force.

2. Identify simple harmonic motion - Analyze the motion of a particle to determine whether simple harmonic motion occurs, and if so, determine its angular frequency.

3. Linear simple harmonic motion - Organize the necessary data about a particle undergoing linear simple harmonic motion to find any or all of the following quantities: the particle's position as a function of time, angular frequency, period, amplitude, phase, frequency, velocity, acceleration, mass, and the restoring force, kinetic energy, or potential energy of the system.

4. Rotational or approximate simple harmonic motion - Apply Newton's second law or conservation of energy to simple physical systems carrying out rotational or approximately linear simple harmonic motion to determine any or all of the quantities listed in Objective 3.

TEXT: Frederick J. Bueche, Introduction to Physics for Scientists and Engineers (McGraw-Hill, New York, 1975), second edition

SUGGESTED STUDY PROCEDURE

Read Chapter 13, Sections 13.1 through 13.5, 13.7, 13.8, and work at least Problems A through H, and 1, 4, 16, 9, 20 in Chapter 13 before attempting the Practice Test.

The general solution of a differential equation is discussed in General Comment 1. Study that carefully. Read the discussion on the small-angle approximation, $\sin\theta = \theta$, in General Comment 3.

Conservation of Energy

If you forgot how to obtain the potential energy of a simple harmonic oscillator, read Section 9.6 in Chapter 9. Since the forces within a spring that make it resist compression and extension are conservative, the sum of kinetic and potential energy in any harmonic oscillator is a constant. This observation can often be used to solve for the amplitude of vibration. For instance, if we know the velocity of the oscillator as it passes equilibrium (when the potential energy is zero), we can find its maximum displacement (when kinetic energy is zero) from

$$0 + (1/2)mv_0^2 = (1/2)kx_0^2 + 0, \qquad x_0 = v_0\sqrt{m/k}.$$

BUECHE

Objective Number	Readings	Problems with Solutions	Assigned Problems		Additional Problems (Chap. 13)
		Study Guide	Study Guide	Text (Chap. 13)	
1	Secs. 13.1 to 13.4, General Comment 1	A	E	1	2
2	Sec. 13.3, General Comments 2, 3	B	F		
3	Secs. 13.5, 13.7	C	G	4, 16	11
4	Secs. 13.7, 13.8, General Comment 3	D	H	9, 20	6, 15, 18

Similarly, if we know x_0, we can solve for the maximum velocity of the oscillator, $v_0 = x_0(k/m)^{1/2}$ without even thinking about derivatives [Note, however, that this is $v(t)_{max}$.]:

$$v(t)_{max} = (\frac{dx}{dt})_{max} = [\frac{d[x_0 \cos(\omega t + \theta)]}{dt}]_{max} = [-x_0\omega \sin(\omega t + \theta)]_{max}.$$

The maximum value of $[-\sin(\omega t + \theta)]$ is +1. Thus

$$v(t)_{max} = x_0\omega = x_0\sqrt{k/m}.$$

TEXT: David Halliday and Robert Resnick, Fundamentals of Physics (Wiley, New York, 1970; revised printing, 1974)

SUGGESTED STUDY PROCEDURE

Read Chapter 13, Sections 13-1 through 13-6, and work at least Problems A through H, and 1, 9, 17, 27, 37 in Chapter 13 before attempting the Practice Test.

The general solution of a differential equation is discussed in General Comment 1. Study that carefully. Read the discussion on the small-angle approximation, $\sin\theta \approx \theta$, in General Comment 3. There are no problems on Section 13-6, but reading it should help to clarify angular frequency.

Example: Physical Pendulum

A rather simple example of simple harmonic motion is the physical pendulum or compound pendulum as shown in Figure 1. A rigid body of mass m is suspended from an axis O. The center of mass is a distance h from the axis. The torque about the axis O on the body is equal to the moment of inertia about the axis, I, times the angular acceleration:

$$\tau = I\alpha = I(d^2\theta/dt^2).$$

The restoring torque is provided by the weight $m\vec{g}$:

$$\tau = -mgh \sin\theta.$$

HALLIDAY AND RESNICK

Objective Number	Readings	Problems with Solutions: Study Guide	Assigned Problems: Study Guide	Assigned Problems: Text (Chap. 13)	Additional Problems (Chap. 13)
1	Secs. 13-1 to 13-3, General Comment 1	A	E	1, 17	16, 18
2	Sec. 13-3, General Comments 2, 3	B	F		
3	Secs. 13-3 to 13-4	C	G	9, 27	5, 14, 32
4	Sec. 13-5, General Comment 3	D	H	37	35, 40, 42

Therefore,

$$-mgh \sin\theta = I(d^2\theta/dt^2).$$

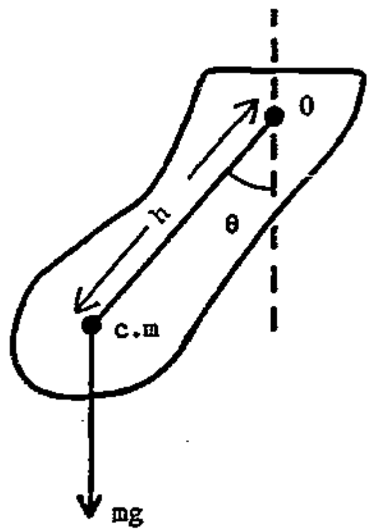
If θ is small, $\sin\theta = \theta$, thus

$$-mgh\theta = I(d^2\theta/dt^2).$$

This is the equation for simple harmonic motion. Compare this with Eq. (13-21) in the text, setting k equivalent to mgh, and thus

$$\omega^2 = mgh/I.$$

Figure 1



TEXT: Francis Weston Sears and Mark W. Zemansky, University Physics (Addison-Wesley, Reading, Mass., 1970), fourth edition

SUGGESTED STUDY PROCEDURE

Read Chapter 11, Sections 11-1 through 11-6, 11-8 and 11-9, and work at least Problems A through H, and 11-1, 11-5, 11-13, and 11-29 before attempting the Practice Test.

In Section 11-4, note the integral on the left-hand side of Eq. (11-7). You will not be expected to do integrals like this by yourself; instead, you can look them up in a book of tables. The general solution of a differential equation is discussed in General Comment 1. Study that carefully. Read the discussion on the small-angle approximation, $\sin\theta \approx \theta$, in General Comment 3.

Reference Circle

First, let us define "simple harmonic motion." If a particle, displaced a distance x from a position of rest and released, experiences a force toward that position of rest of magnitude proportional to the magnitude of its displacement, the particle will move with simple harmonic motion. Expressed algebraically, this is $F = -kx$, where x is the displacement from the position of rest, F the restoring force, and k a positive constant.

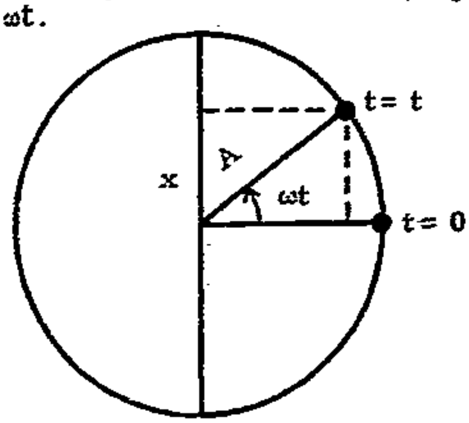
Now, leave this for a moment, and consider the projection of the motion of a particle, moving at constant angular speed ω in a circular path of radius A,

SEARS AND ZEMANSKY

Objective Number	Readings	Problems with Solutions	Assigned Problems		Additional Problems
		Study Guide	Study Guide	Text	
1	Secs. 11-1 to 11-4, General Comment 1	A	E	11-1	
2	Sec. 11-2, General Comments 2, 3	B	F		
3	Secs. 11-4 to 11-5	C	G	11-5, 11-13	11-2, 11-3, 11-4, 11-18
4	Secs. 11-6, 11-8, 11-9, General Comment 3	D	H	11-29	11-22, 11-26, 11-30

upon a diameter of the circle, as in Figure 2. Arbitrarily, let the projection be at the center of the diameter at $t = 0$, and call x the displacement of the projection from the center. Then, at some later time t, the particle will have turned through an angle θ, equal to ωt, and the projection will then have moved a distance $x = A \sin \omega t$.

Figure 2



Now, differentiate this expression twice with respect to time, obtaining

$dx/dt = \omega A \cos \omega t$ and $d^2x/dt^2 = -\omega^2 A \sin \omega t$.

By definition, d^2x/dt^2 is acceleration, ω^2 is a positive constant, and $A \sin \omega t$ is x, the displacement of the projection from the center of the circle. From Newton's second law:

$F = md^2x/dt^2$, $F = -m\omega^2 A \sin \omega t$.

Therefore our final equation is $F = -m\omega^2 x = -kx$, and the projection moves with simple harmonic motion, with the center of the diameter as the position of rest. The circle used here is referred to as the circle of reference.

The amplitude of the simple harmonic motion is defined in Section 11-3 as the maximum value of x. Since $\sin \omega t$ cannot be greater than one, the maximum value of x is A in our equation $x = A \sin \omega t$, and A, the coefficient of the trigonometric term, is the amplitude of the simple harmonic motion. The period of the motion is the time required for one complete vibration. In this time, then, the projection must move from the center of the diameter, up to a maximum positive displacement, down to a maximum negative displacement, and back to the central point. In the same time, then, the particle moving with angular speed ω in a circular path will go just once around the circle. The angle turned through by this particle is 2π rad, and we see, from the definition of angular velocity $\omega = \theta/t$, that $\omega = 2\pi/T$, where T is the period of the simple harmonic motion. Also, since the frequency $f = 1/T$, $\omega = 2\pi f$. Thus, in our equation $x = A \sin \omega t$, the coefficient of t is $2\pi f$ or $2\pi/T$.

By definition, ds/dt is the velocity, and d^2s/dt^2 is the acceleration. By the use of just our equation $x = A \sin \omega t$ and its derivatives, we may now find the velocity and acceleration in our particular simple harmonic motion for any position x or time t.

The constant θ_0 in Eq. (11-11) is of use only when the position of the projection is specified when $t = 0$ at some point other than the position of rest, and in the absence of such specification may arbitrarily be set equal to zero. Suppose, though, that x has some value x_0 at $t = 0$. Then Eq. (11-11) becomes

$$x_0 = A \sin \theta_0,$$

and θ_0 may be evaluated.

Another use of the circle of reference is to simplify the kind of problem we encounter in part (c) of Problems 11-3 and 11-4, where we are asked for the minimum time necessary to move from point x_1 to point x_2 in the simple harmonic motion. See Figure 3. We can determine the angles ϕ_1 and ϕ_2, since $\sin \phi_1 = x_1/A$ and $\sin \phi_2 = x_2/A$. Their sum is the angle θ turned through by the particle in the circle of reference while the projection moves from x_1 to x_2. As the angular velocity ω is a constant, we may say that $\omega = \theta/t = 2\pi/T$ or $t = (\theta/2\pi)T$, and, knowing θ and the period T, the time is immediately determined.

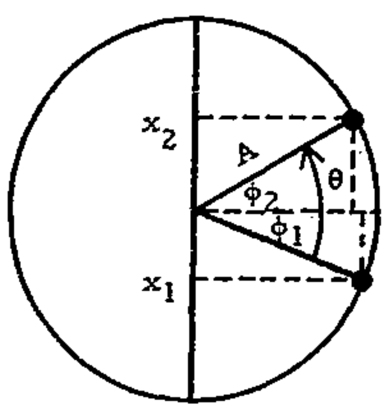


Figure 3

TEXT: Richard T. Weidner and Robert L. Sells, Elementary Classical Physics (Allyn and Bacon, Boston, 1973), second edition, Vol. 1

SUGGESTED STUDY PROCEDURE

Read Chapter 14, Sections 14-1 to 14-4, and work at least Problems A through H, and 14-1, 14-5, 14-12, 14-15, and 14-29 before attempting the Practice Test.

The general solution of a differential equation is discussed in General Comment 1. Study that carefully. Example 14-3 should be studied before working Problems B and F for Objective 2. Read the discussion on the small-angle approximation, $\sin \theta \approx \theta$, in the General Comment 3.

WEIDNER AND SELLS

Objective Number	Readings	Problems with Solutions		Assigned Problems		Additional Problems
		Study Guide	Text	Study Guide	Text	
1	Sec. 14-1, General Comment 1	A		E	14-1	14-4
2	Sec. 14-3, General Comments 2, 3	B	Ex.* 14-3	F	14-12	
3	Secs. 14-1, 14-2	C		G	14-5, 14-15	14-9, 14-10, 14-11
4	Sec. 14-4, General Comment 3	D		H	14-29	14-22, 14-25, 14-28

*Ex. = Example(s).

GENERAL COMMENTS

1. Differential Equations

Your present calculus course may not have acquainted you with differential equations. Hence we shall discuss them briefly without getting too fancy or formal. The equation

$$d^2x/dt^2 = -\omega^2 x \tag{1}$$

is called a second-order differential equation because it contains a second derivative. It is not like an algebraic equation for which certain constant values of x satisfy the equality. As is shown in your text the solution of Eq. (1) is a function of the time. Although the function

$$x = A\cos(\omega t + \theta) = A\cos\theta\cos\omega t - A\sin\theta\sin\omega t \tag{2}$$

can be thought of simply as being arrived at by a very clever guess, it can be shown (by advanced mathematical techniques) to be the most general possible solution of Eq. (1).

Equation (2) can also be written in terms of two new constants B and C as

$$x(t) = B\cos\omega t + C\sin\omega t. \tag{3}$$

(What are the relations among B, C, A, and θ?) The velocity is

$$v(t) = dx(t)/dt = -\omega B\sin\omega t + C\cos\omega t. \tag{4}$$

These last two equations are especially helpful. For instance, if you are told that the particle begins its simple harmonic motion from rest at the point x_0, you know that $x(0) = x_0$ and $v(0) = 0$, hence since $\cos(0) = 1$ and $\sin(0) = 0$ you immediately have $B = x_0$ and $C = 0$. If the particle starts at the origin with velocity v_0, then you can conclude that $B = 0$ and $C = v_0$. Look at the equations and check these results for yourself. If you have a more complicated case in which the particle starts at x_0 with velocity v_0, then you can find B and C yourself, using the same method. Try it. Once you have found B and C, you can find then A and θ.

Note in the above discussion that a change in phase of $\pi/2$ does not change the solution. That is, let $\theta = \theta' + \pi/2$:

$$x = A\cos(\omega t + \theta), \quad x = A\cos(\omega t + \theta' + \pi/2), \quad x = A\sin(\omega t + \theta').$$

This last equation is just as valid a solution of the differential equation as the cosine function. Try substituting it in Eq. (1) and see for yourself.

2. Outline of Method for Investigating a System for Simple Harmonic Motion

$$F(x) = -kx, \tag{5}$$

$$m(d^2x/dt^2) = -kx. \tag{6}$$

I. Determine the net force acting on the particle.
(a) Identify forces acting on the particle by drawing a free-body diagram. Choose a convenient coordinate system.
(b) Find the net force acting on the particle as a function of its position in the chosen coordinate system.

II. Describe the particle's displacement from the equilibrium position.
(a) Find the position where the net force is equal to zero. That is the equilibrium position of the particle.
(b) If necessary, introduce a new coordinate system with origin at the equilibrium position.

III. Use the coordinate system introduced in II(b) to state Eqs. (5) and (6).
(a) Express the net force as a function of the new coordinates. Compare this with Eq. (5).
(b) Express the acceleration in terms of the second time derivative of the new coordinates.
(c) Use the expressions derived in steps (a) and (b) to state Newton's second law $\vec{F} = m\vec{a}$ in terms of the new coordinates. Compare its form with Eq. (6).

3. Approximation: $\sin\theta = \theta$ for Small Angles

The first thing to note is that this is true only if θ is in radians. Obviously $\sin 1.00° \neq 1$. But $1° = 0.0174$ rad, and $\sin(0.0174 \text{ rad}) = 0.0174$. This approximation is good up to about 15.0° or 0.262 rad. $\sin(0.262 \text{ rad}) = 0.258$. This is only an error of about 1% so the approximation is pretty good. The error in the period of a pendulum when the amplitude = 15.0° is only 0.50%. Thus, even though a system actually does not execute simple harmonic motion, if the angular displacement is kept small enough its motion will be essentially simple harmonic.

A few of the many examples of simple harmonic oscillators are listed below, along with the expressions for F_x or τ. You should verify these expressions for yourself. The determination of ω, f, and T for each is left as an exercise.

Examples of Simple Harmonic Motion

a. Object on a spring (Fig. 4). Equilibrium occurs at the height for which the spring force equals $-m\vec{g}$. When the object is displaced, the spring force changes, but mg remains the same. Restoring force is $F_x = -kx$.

b. Object fastened to two stretched springs (Fig. 5). When the object is displaced, one spring pulls more, and the other pulls less. Restoring force is $F_x = -(2k)x$.

c. Object fastened to two stretched springs, but displaced sideways (Fig. 6). If the displacement is small, the forces exerted by the springs change in direction, but hardly at all in magnitude. Restoring force is $F_x = -2F_0(x/\ell)$.

Figure 4

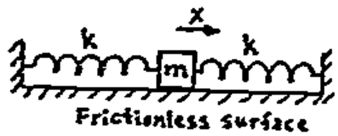


Figure 5

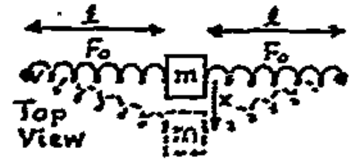


Figure 6

d. <u>Object fastened to two stretched elastic strings</u> (Fig. 7). Essentially the same as above. For small displacements, the tension F_0 in the strings does not change appreciably. Restoring force is $F_x = -2F_0(x/\ell)$.

e. <u>Massive object on a "massless flagpole"</u> (Fig. 8). For small displacements, the motion is almost linear. Restoring force is $F_x = -Kx$.

f. <u>Object on a string</u> (pendulum, Fig. 9). The restoring force is the component of $m\vec{g}$ perpendicular to the string, $-mg \sin\theta$. For small displacements, the motion is almost linear, and $\sin\theta \approx \theta$. Restoring force is $F_x = -mg\theta = -mg(x/\ell)$.

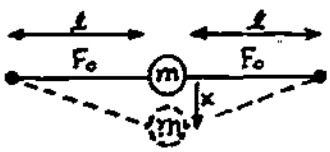


Figure 7

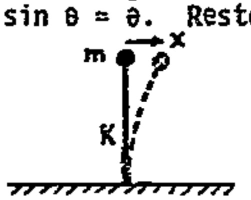


Figure 8

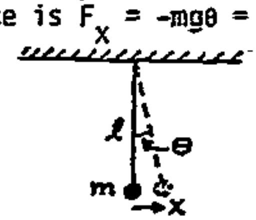


Figure 9

g. <u>Small object sliding in a frictionless spherical bowl</u> (Fig. 10). Same as above. Restoring force is $F_x = -mg\theta$. Or, use the restoring <u>torque</u> $\tau = -mg\ell\theta$ with $I = \ell^2 m$ and $\tau = I\alpha = I\, d^2\theta/dt^2$.

h. <u>Pivoted plank on spring</u> (Fig. 11). Same as a car with good shocks in front, very bad shocks in back. As the object bounces up and down, the force of gravity is constant, but the spring force changes. Restoring torque is $\tau = -\ell k(\ell\theta) = -k\ell^2\theta$.

i. <u>Object hung from wire and rotating about a vertical axis</u> (torsion pendulum, Fig. 12). Some mantlepiece clocks use a pendulum of this kind. Generally, the wire provides a restoring torque $\tau = -\kappa\theta$.

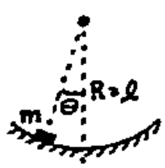


Figure 10

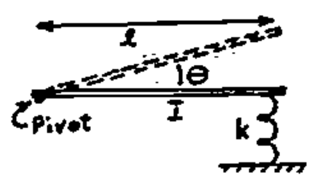


Figure 11

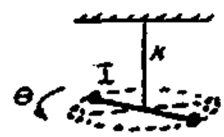


Figure 12

ADDITIONAL LEARNING MATERIALS

Film loops: "Simple Harmonic Motion"

"Velocity and Acceleration in Simple Harmonic Motion"

Available from Ealing Corporation.

PROBLEM SET WITH SOLUTIONS*

A(1). Define the following terms: amplitude, angular frequency, phase constant, period.

Solution

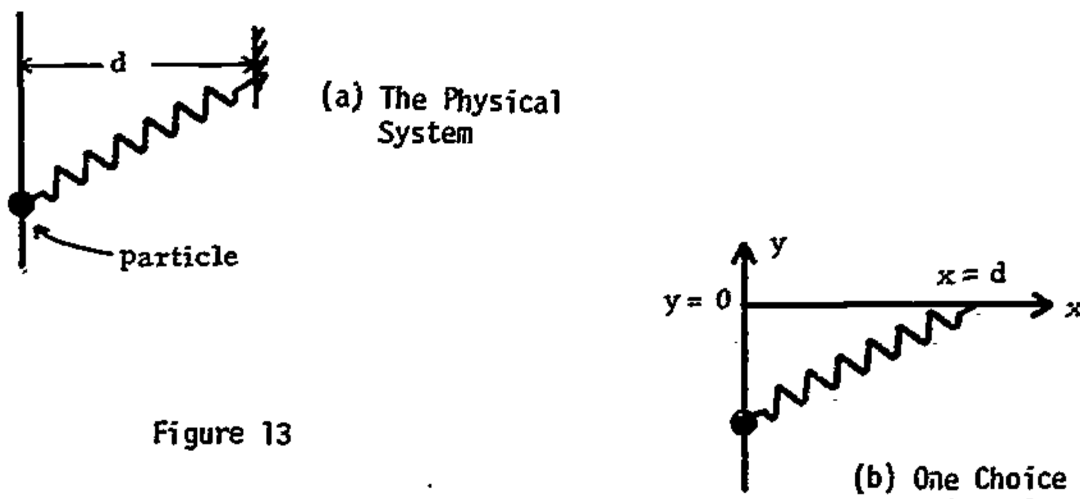
Amplitude: the maximum displacement from equilibrium of an oscillating particle.

Angular frequency: 2π divided by the time required to complete one cycle of the motion (or $2\pi f$). Another definition is the number of radians completed per second; knowing that 2π rad equal one cycle.

Phase constant: if $x = A\cos(\omega t + \theta)$ then $(\omega t + \theta)$ is called the phase of the motion. θ is called the phase constant (phase angle). The amplitude and the phase of the motion determine the initial velocity and position of the particle (or vice versa if you like). For example, if $\phi = \pi/2$ at $t = 0$, then $x = A\cos(\pi/2) = 0$ and the particle starts at $x = 0$. The unit of the phase will be radians.

Period: the time required to complete one cycle.

B(2). A particle of mass m is restricted to move on a vertical frictionless track. It is attached to one end of the massless spring with spring constant k and unextended length $\ell_0 = 0$ m (small compared to other lengths in the problem). The other end of the spring is hooked to a peg at the distance d from the track (see Fig. 13). (Use $g = 9.8\ m/s^2$.)
(a) Show that the particle carries out simple harmonic motion when displaced from its equilibrium position.
(b) Find the period of oscillation of the particle.



(a) The Physical System

(b) One Choice of a Coordinate System

Figure 13

*The key to Problems B and F is to find the net force acting on the particle, find the particle's equilibrium position, and then express the force as a function of the displacement from equilibrium. If and only if the force can be written as $F(x) = -kx$, then the motion is simple harmonic. The three-step procedure is described in General Comment 2.

Solution

(a) I. In Figure 14, $\vec{S}$ is the force of the spring, $\vec{N}$ is the constraining force of the track, $\vec{W}$ is the gravitational force, y is the vertical position of the particle,

$$\vec{W} = -mg\hat{j}, \quad \vec{N} = -N\hat{i}, \quad \vec{S} = -(\text{spring constant}) \times (\text{extended length}) = -k(-d\hat{i} + y\hat{j}).$$

The total force is then $\vec{F}(y) = \vec{S} + \vec{W} + \vec{N} = (kd - N)\hat{i} + (ky - mg)\hat{j}$.

II. Where is the net force = 0? Set $\vec{F} = 0$ and solve for y.

$$\vec{F} = 0 = (kd - N)\hat{i} + (ky - mg)\hat{j}.$$

Thus the net force is zero when $N = dk$ and $y = -mg/k$. Introduce a new coordinate system at the equilibrium position:

$$x' = x, \quad y' = y + mg/k.$$

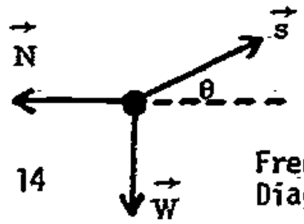


Figure 14 Free-Body Diagram

III. In this new system, the total force is

$$\vec{F}(y') = -ky'\hat{j}.$$

This is the same form of the force that gives rise to simple harmonic motion, $F(x) = -kx$, thus the motion of the mass on the vertical track is simple harmonic.

(b) Newton's second law is $\vec{F} = m\vec{a}$, so

$$-ky'\hat{j} = m(d^2y'/dt^2)\hat{j}, \quad -ky' = m(d^2y'/dt^2), \quad \omega^2 = k/m, \quad T = 2\pi/\omega = 2\pi\sqrt{m/k}.$$

C(3). One day you visit a friend who has a chair suspended on springs. When you sit down on the chair, it oscillates vertically at 0.50 Hz. After the oscillations have died down, you stand up slowly, and the chair rises 0.50 m. Next, your friend sits in the chair, and you find that the oscillations have a period of 2.10 s. Assume that your mass is 60 kg:
(a) What is the spring constant for the two springs together?
(b) What is the mass of the chair?
(c) What is the mass of your friend?
(d) While you are sitting in the chair, at a certain instant (t = 0) the chair is 0.300 m above its equilibrium position, and momentarily at rest. Find the expression for y(t), its displacement from equilibrium as a function of time.
(e) Under these conditions, what is the maximum kinetic energy of you and the chair? What is your maximum speed?

Solution

(a) $F = -kx, \quad -mg = -kx, \quad k = mg/x = 60(9.8)/0.50 = 1180 \approx 1200$ N/m.

(b) $\omega^2 = k/m, \quad m = m_{chair} + m_{you} = k/\omega^2 = k/(2\pi f)^2,$

$$m_{chair} = k/(2\pi f)^2 - m_{you} = \frac{1200\ N/m}{[(2\pi)(0.50)]^2(1/s^2)} - 60\ kg$$

$$= (120 - 60)\ kg = 60\ kg.$$

(c) $m_{friend} = k/(2\pi f)^2 - m_{chair} = 72$ kg.

(d) $A = 0.300$ m $= 2\pi f = 2\pi(0.50) = \pi$ rad/s. $y = A\cos(\omega t + \theta)$.

At $t = 0$, $y = A$, $A = A\cos(0 + \theta)$. Therefore $\cos\theta = 1$, $\theta = 0$.

$y(t) = 0.300\cos(\pi t)$ m.

(e) Maximum kinetic energy = maximum potential energy

$$= (1/2)kx^2 = (1/2)(1200)(0.300)^2 = 54\ J.$$

Maximum speed $= (dy/dt)_{max} = (-A\omega\sin\omega t)_{max} = A\omega = 0.94$ m/s.

Figure 15

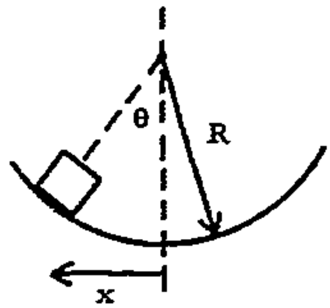


Figure 16

D(4). "A challenging problem." A particle with mass M slides freely in a hemispherical bowl of radius R, as shown in Figure 16.
(a) Find the potential energy $U(x)$, making the approximations

$$1 - \cos\theta = 1 - [1 - (1/2)\theta^2 + \cdots] = (1/2)\theta^2 \quad \text{and} \quad \theta = x/R.$$

(b) Suppose the particle starts from rest with a displacement x_0; what is its kinetic energy $K(x)$?
(c) What is its velocity v_x as a function of x?
(d) What is its acceleration a_x as a function of x? (Hint: Differentiate!)
(e) Does this particle undergo simple harmonic motion? If not, explain why not; if so, find the angular frequency ω for this motion.

Solution

(a) As the particle slides up the slope to an angle θ it has increased in height a distance $R - R\cos\theta = R(1 - \cos\theta)$. If θ is small then $(1 - \cos\theta) \approx (1/2)\theta^2$ and $\theta \approx x/R$, so that

$$R(1 - \cos\theta) \approx R[(1/2)\theta^2] = R[(1/2)(x^2/R^2)] = (1/2)(x^2/R).$$

Potential energy $= mgh = (mg/2R)x^2$.

(b) Kinetic energy = loss in potential energy

$$= (mg/2R)(x_0^2) - (mg/2R)(x^2) = (mg/2R)(x_0^2 - x^2).$$

(c) Total kinetic energy is $(1/2)mv^2$. Gain in kinetic energy = loss in potential energy:

$$(1/2)mv^2 = (mg/2R)(x_0^2 - x^2), \qquad v_x = \pm[(g/R)(x_0^2 - x^2)]^{1/2}.$$

Is this reasonable? First check the units. The resulting units should be the same on both the left- and right-hand sides of the equation:

$$[\frac{(g)}{(R)}(x)^2]^{1/2} = ? \qquad [\frac{(g)}{(R)}(x^2)]^{1/2} = [\frac{(m/s^2)m^2}{m}]^{1/2} = (m^2/s^2)^{1/2} = m/s = \text{unit of velocity.}$$

Thus the units are correct. Now what would happen if $x = x_0$? $(x_0 - x) = 0$ and $v = 0$. As we expect, the velocity is zero at the starting point. Now let's take x approaching zero. As $x \to 0$, $(x_0^2 - x^2) \to x_0^2$, a maximum value. Thus the velocity becomes maximum at $x = 0$, which is where we have the maximum kinetic energy.

(d) Before differentiating by brute force, remember that $d(y^2) = 2y\,dy$, therefore differentiate

$$v_x^2 = (g/R)(x_0^2 - x^2).$$

Thus $\quad 2v_x\,dv_x = (g/R)(-2dx\,dx).$

Divide by dt: $\quad v_x(dv_x/dt) = -(g/R)(dx/dt).$

Now since $dx/dt = v_x, \quad dv_x/dt = a_x,$

divide each side by v_x to get

$$a_x = (-g/R)x.$$

(e) From Newton's second law $F = ma = m(dx^2/dt^2)$, thus

$$m(d^2x/dt^2) = -(g/R)x \quad \text{or} \quad d^2x/dt^2 \quad (g/Rm)z = 0.$$

This is a form of the equation for simple harmonic motion:

$$d^2x/dt^2 + (k/m)x = 0 \quad \text{or} \quad F = -kx.$$

So yes, simple harmonic motion does occur. The angular frequency $\omega = \sqrt{k/m} = \sqrt{g/mR}$. Part (e) could also be answered by finding the restoring force.

Problems

E(1). The displacement of an object undergoing simple harmonic motion is given by the equation

$$x(t) = \underset{\text{m}}{3.00}\sin(\underset{\frac{\text{rad}}{\text{s}}}{8\pi}\,\underset{\text{s}}{t} + \underset{\text{rad}}{\pi/4}).$$

Note that the mks units of each term are shown underneath.

(a) What is the amplitude of motion?

(b) What is the frequency of the motion?

(c) Sketch the position of the particle as a function of time, starting at $t = 0$.

F(2). In the book Tik-Tok of Oz, Queen Anne, Hank the mule, the Rose Princess, Betsy, Tik-Tok, Polychrome, the Shaggy Man, and the entire Army of Oogaboo all fall through the straight Hollow Tube to the opposite side of the earth. The retarding force of the air is evidently negligible during this trip, since they all pop out neatly at the other end. For an object at a distance r from the center of such a spherical mass distribution the gravitational force has the magnitude (you will not have to derive something like this):

$$F_g(r) = (r/R_e)^3 GM_e(m/r^2) = mgr/R_e,$$

and is directed toward the center of the earth. Use $R_e = 6.4 \times 10^6$ m for the radius of the earth:

(a) Do they undergo simple harmonic motion? How do you know?

(b) How long does their trip last?

G(3). An automobile with very bad shock absorbers behaves as though it were simply mounted on a spring, as far as vertical oscillations are concerned. When empty, the car's mass is 1000 kg, and the frequency of oscillation is 2.00 Hz.

(a) What is the spring constant?

(b) How much energy does it take to set this car into oscillation with an amplitude of 5.0 cm (assuming all damping can be neglected)?

(c) What is the maximum speed of the vertical motion in (b)?

(d) Suppose that four passengers with an average weight of 60 kg now enter the car. What is the new frequency of oscillation?

H(4). The rotor of the electric generator in Figure 17 is to be driven by a long shaft. Since any rotational oscillations about axis AA′ of the rotor would cause fluctuations in the electrical output, an engineer decides to investigate this possibility, starting with the case of completely undamped motion (i.e., no friction).

(a) It takes a torque $\tau = \kappa\theta$ to twist the shaft an angle θ. When one end is clamped as in the figure, will the rotor undergo simple harmonic motion? How do you know?

(b) With the parameters indicated in the figure, what will she find for the frequency?
(c) What are the maximum potential and kinetic energies of the oscillation if the amplitude is 0.00100 rad? What is the total mechanical energy?
(d) What change in the stiffness of the shaft (κ) would be necessary to double the frequency of oscillation?

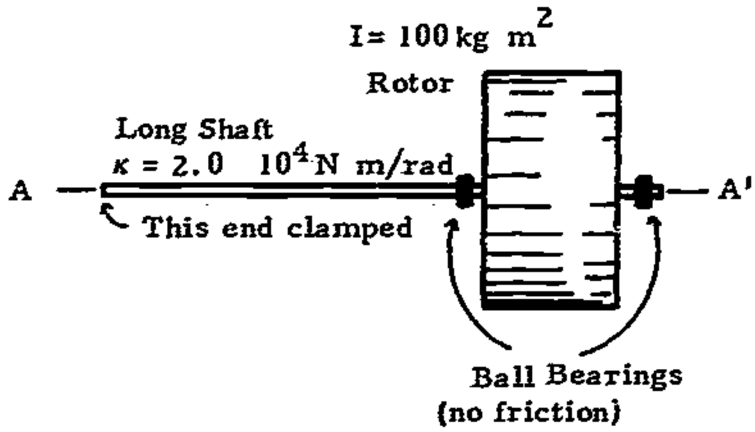


Figure 17

Solutions

E(1). (a) 3.00 m.

(b) $\omega = 8\pi$ rad/s,

if $x = 3.00 \sin(8\pi t + \theta)$ m.

$f = \omega/2\pi = 4.0$ Hz.

(c)

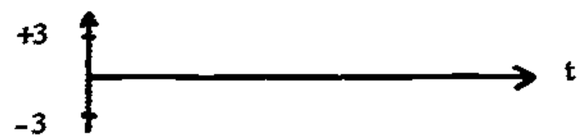


Figure 18

F(2). (a) Yes; the gravitational force they experience has a magnitude proportional to the distance from the center of the earth, and is directed toward the center.
(b) If you take the x axis to lie along the tube, then $m(d^2x/dt^2) = F_x = -mgx/R_e$, or $d^2x/dt^2 = -(g/R_e)x$. But from (a) above we know that their motion is given by an expression of the form $x = A\cos(\omega t + \theta)$, for which $d^2x/dt^2 = -\omega^2 x$. Therefore, $\omega = \sqrt{g/R_e}$; and the duration of their trip is $(1/2)T = \pi\sqrt{R_e/g} = 800\pi$ s $= 0.70$ h.

G(3). (a) $k = m\omega^2 = m(2\pi f)^2 = 1.60 \times 10^5$ N/m. Does this seem reasonable?
(b) Total energy = maximum potential energy = 2.00×10^2 J.
(c) From conservation of energy,

$$v_{max} = \sqrt{k/m}A = 0.63 \text{ m/s}.$$

(d) $f = (1/2\pi)\sqrt{k/m} = 1.80$ Hz. (Which mass should you use?)

H(4). (a) Yes, τ is proportional to θ.
(b) $I\alpha = \tau = -\kappa\theta$, or $d^2\theta/dt^2 = -(\kappa/I)\theta$. But $\theta = \theta_0 \cos(\omega t + \theta)$, from part (a), so that $d^2/dt^2 = -\omega^2\theta$. Therefore, $\omega = \sqrt{\kappa/I}$, and $f = \omega/2\pi = 2.25$ Hz.
(c) 0.0100 J each.
(d) Since κ varies as the square of ω, if ω is doubled, κ becomes four times as stiff.

PRACTICE TEST

1. Seesaws at parks often go unused because two small children seldom decide to play on them simultaneously. As Technical Consultant to the Park Board, your first assignment is to provide specifications for a One-Tot-Teeter. The design is partly determined by the existing equipment. (See Figure 19.) A child of mass m is to receive a ride with a period of T seconds. Without child or counterweight, the teeter-totter has a rotational inertia I_1; the child and counterweight, of course, have I_2. Note: the spring is attached to the teeter. Start with Newton's second law for rotational motion to find the answer.
   (a) Does simple harmonic motion occur? Why?
   (b) What spring constant is needed?

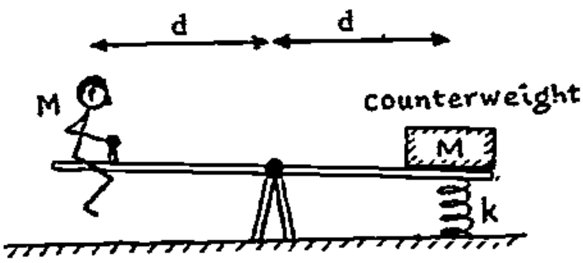


Figure 19

Figure 20

2. A child is bouncing a 50-g rubber ball on the end of a rubber string, in such a way as to give the ball and string a total energy of 0.050 J (counting the potential energy as zero at the equilibrium position). If the ball were just hanging at rest, it would stretch the string 20.0 cm. For the motion of the bouncing ball, find
   (a) the angular frequency ω,
   (b) the amplitude,
   (c) the maximum kinetic energy,
   (d) the maximum speed of the ball, and
   (e) the expression for its acceleration as a function of time, if the position = +0.200 m at t = 0 s.

Practice Test Answers

1. (a) Total moment of inertia equals $I_1 + I_2 = I_{total} = I$.

Total torque = $\tau_{child} + \tau_{counterweight} + \tau_{spring}$, $\tau_{child} = -\tau_{counterweight}$.

Total torque = τ_{spring} = (force) × (distance) = $-kyd$,

$y = (\sin\theta)d = \theta d$ approximately.

Thus $\tau = -k\theta d^2$, which is a restoring torque proportional to displacement. Therefore motion will be simple harmonic.

(b) $\tau = I\alpha = I(d^2\theta/dt^2) = k\ d^2\theta$, $d^2\theta/dt^2 = (kd^2/I)\theta = \omega^2\theta$,

$\omega^2 = kd^2/I$, $\omega^2 = (2\pi f)^2 = (2\pi/T)^2$,

thus

$kd^2/I = (2\pi/T)^2$, $k = I(2\pi/dT)^2$.

2. (a) $F = -kx$, $-mg = -kx$, $k = mg/x$,

$\omega = \sqrt{k/m} = \sqrt{mg/xm} = \sqrt{g/x} = \sqrt{9.8/0.200} = 7.0$ rad/s.

(b) $A = ?$ $k = mg/x = (0.50\text{ kg})(9.8\text{ m/s}^2/0.200\text{ m}) = 2.45$ N/m.

$U = (1/2)kx^2$, $U_{max} = (1/2)kA^2$, $A = \sqrt{2U_{max}/k} = \sqrt{2(0.050)/2.45}$

$= \sqrt{0.0408} = 0.200$ m.

(c) $K_{max} = U_{max} = E_{total} = 0.050$ J.

(d) $K_{max} = (1/2)mv_{max}^2$, $v_{max} = \sqrt{2K_{max}/m} = \sqrt{20.05/0.050} = 1.40$ m/s.

(e) $a = A\omega^2\cos(\omega t + \theta) = (0.200)(7^2)\cos(7t + \theta) = 9.8\cos(7t + \theta)$ m/s^2.

At $t = 0$, $x = A\cos\theta = 0.200\cos\theta = 0.200$. Thus $\cos\theta = 1$, $\theta = 0$ rad.

SIMPLE HARMONIC MOTION

Mastery Test Form A

Name______________________

Date__________________

pass recycle

1 2 3 4

Tutor__________________

1. A block sliding on a frictionless surface is held in place by two springs attached to opposite sides and <u>stretched</u> to clamps at the edge of the surface as in Figure 1. The springs have an unextended length = ℓ and a force constant k.

   (a) Show that the block will carry out simple harmonic motion on the surface if it is pushed closer to one of the clamps and then released.

   (b) Find the angular frequency ω for this motion.

2. You have been retained as a consultant to a traveling circus to advise on problems of a trapeze act. See Figure 2. The artists each have a mass of 80 kg and will use a trapeze hung from ropes 30.0 m long to travel between platforms 15.0 m apart. The approximation, $\sin\theta \approx \theta$ is valid here.

   (a) The musical director wants to know how long it will take them to swing back and forth with one of the two persons on the trapeze.

   (b) What is the magnitude of the maximum velocity?

   (c) The property manager wants to know what the maximum tension on the ropes will be with <u>two</u> people swinging.

   (d) Write an expression for the displacement as a function of time, assuming the artists start at the right-hand side at $t = 0$.

Furthermore, the circus owner insists that you <u>start from Newton's second law</u> to find the answer to (a). We await your answers.

Figure 1

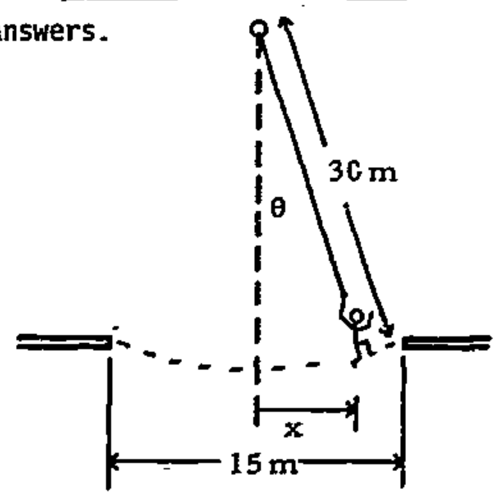


Figure 2

SIMPLE HARMONIC MOTION

Mastery Test Form B

Name________________________________

Date________________________

pass recycle

1 2 3 4

Tutor________________________

1. A car with good shocks in front and very bad shocks in the back can be modeled by a pivoted plank on a spring as in Figure 1. The spring constant is k, the length is ℓ, and the moment of inertia I about the pivot is $m\ell^2/3$.

   (a) Show that the plank will carry out simple harmonic motion if it is pushed up at a small angle θ and then released.

   (b) Find the angular frequency for this motion.

2. You have just been hired by Tinseltown Movie Studios to design a "jungle elevator" for Tarzan. We want 90-kg Tarzan to grab the end of a hanging elastic vine, step off his tree branch, and be brought to rest at the ground 15.0 m below.

   (a) Where should the equilibrium point be for the system of Tarzan-plus-vine?

   (b) What must be the force constant of the vine?

   (c) How long does the trip take?

   (d) What is the maximum pull on Tarzan's hands?

   (e) What is Tarzan's maximum velocity.

   (f) Write Tarzan's velocity as a function of time if he starts from the limb at $t = 0$.

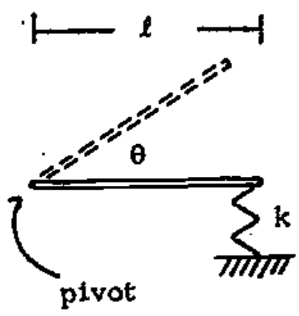


Figure 1

Figure 2

SIMPLE HARMONIC MOTION

Date____________________

Mastery Test Form C

pass recycle

1 2 3 4

Name____________________

Tutor____________________

1. A long thin rod of length ℓ and mass m is pivoted about a point on the rod that is a distance h above the center of the rod. The moment of inertia about the pivot point is $m(\ell^2/12 + h^2)$. See Figure 1.

   (a) Show that the rod will carry out simple harmonic motion if it is pushed to one side and then released.

   (b) Find the angular frequency for this motion.

2. A body of mass 100 g hangs on a long spiral spring. When pulled down 10.0 cm below its equilibrium position and released, it vibrates with a period of 2.00 s.

   (a) What is its velocity as it first passes through the equilibrium position?

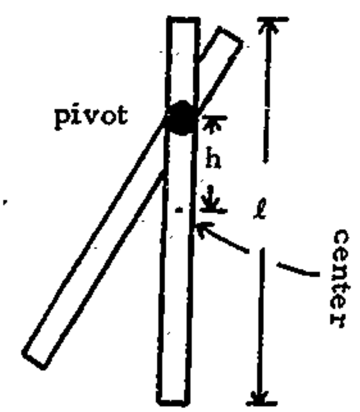
   (b) Write the position of the body as a function of time assuming that $x = -10.0$ cm at $t = 0$..

   (c) What is its acceleration when it is 5.0 cm above the equilibrium position?

   (d) When it is moving upward, how long a time is required for it to move from a point 10.0 cm below its equilibrium position to its equilibrium position.

   (e) How much will the spring shorten if the body is removed?

Figure 1



MASTERY TEST GRADING KEY - Form A

1. What To Look For: If student shows that $F \propto x$, that's enough for part (a).

   Solution: (a) See Figure 21. Let origin be at midpoint. Since at that point the springs are not extended, the horizontal forces are equal and opposite. Thus, that is the equilibrium position as well. If it is displaced a distance x from equilibrium then

   $\vec{F}_1 = -kx\hat{i}, \quad \vec{F}_2 = -kx\hat{i}, \quad \vec{F}_{total} = -2kx\hat{i} = m(d^2x/dt^2)\hat{i}.$

   Thus $m(d^2x/dt^2) = -2kx$: same form for simple harmonic motion. Thus simple harmonic motion does occur.

   (b) $\dfrac{d^2x}{dt^2} + \dfrac{2k}{m}x = 0, \ \omega = \sqrt{2k/m}.$

2. What To Look For: (a) See if student uses Newton's second law to get equation of motion. (c) Could also use

   $F = -mg \sin\theta \simeq -mg\,\theta \simeq -mg(x/L) = (d^2x/dt^2),$

   so that $d^2x/dt^2 + (g/L)x = 0.$

   (d) Why is it not $7.5\cos(0.57t + \theta)$, where θ is some angle other than zero?

   Solution: (a) $\tau = -mgL \sin\theta$. For small θ, $\sin\theta \approx \theta$,

   $\tau = mgL\theta = I(d^2\theta/dt^2), \qquad I = mL^2$ for a particle around an axis.

   $-mgL\theta = mL^2(d^2\theta/dt), \qquad d^2\theta/dt^2 + (g/L)\theta = 0. \qquad \omega = \sqrt{g/L}.$

   $T = 1/f = 2\pi/\omega = 2\pi\sqrt{L/g} = 2\pi\sqrt{30.0/9.8} = 11$ s, the time for one cycle, back and forth. T is independent of mass, thus time is same for one or two persons.

   (b) From conservation of energy:

   $(1/2)mv^2_{max} = mgh,$

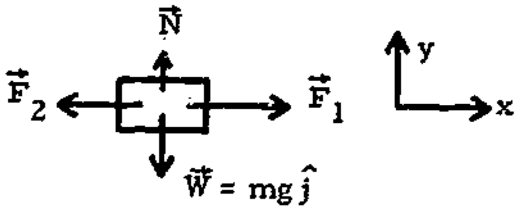


Figure 21

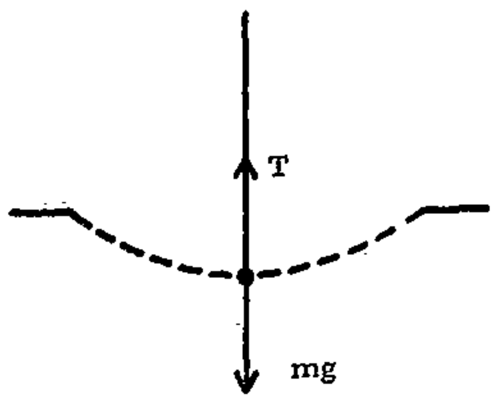


Figure 22

where h is the distance from ledge to bottom of swing: $h = L(1 - \cos\theta)$ $= L\{1 - \cos[\sin^{-1}(7.5/30)]\}$, $v_{max} = \sqrt{2gh} = 4.32$ m/s.

(c) See Figure 22. For circular motion: Net force = centripetal force,

$T - mg = mv^2/L$, $T = mg + mv^2/L$.

T_{max} occurs at v_{max} which occurs at the bottom of the swing. From Part (b) $v^2 = 2gL(1 - \cos\theta)$.

Tension $= 2mgh/L + mg = 2mg\{1 - \cos[\sin^{-1}(7.5/30)]\} + mg = 1.66 \times 10^3$ N.

(d) $x = 7.5 \cos(0.57t)$.

MASTERY TEST GRADING KEY - Form B

1. What To Look For: (a) If they show that $\tau \propto \theta$, that is enough for part (a). Note: $y'/\ell = \sin\theta \approx \theta$.

Solution: (a) See Figure 23.

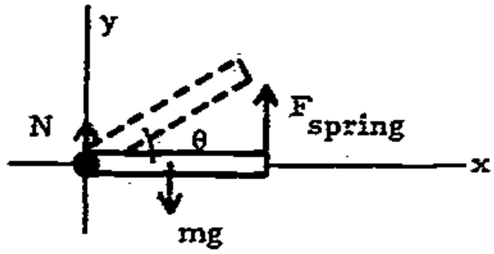


Figure 23

Considering rotational motion only, the plank is in equilibrium when $\text{torque}_{mg} = \text{torque}_{spring}$, $-mg\ell/2 = ky\ell$, $y = -mg/2k$ = equilibrium position. If it is displaced a distance y' from equilibrium then the net restoring torque is given by

$$\tau_{restoring} = -ky'\ell = I(d^2\theta/dt^2) = (m\ell^2/3)(d^2\theta/dt^2).$$

Thus

$$d^2\theta/dt^2 + \frac{3k}{m}\frac{y'}{\ell} = 0, \quad \frac{y'}{\ell} \approx \theta, \quad \frac{d^2\theta}{dt^2} + \frac{3k}{m}\theta = 0.$$

Thus it is simple harmonic motion: $\omega = \sqrt{3k/m}$.

2. What To Look For: (c) How long would it take to go from the tree limb to 7.5 m above ground? (d) If Tarzan were oscillating up and down, is there a point where the pull would be zero? If so, where? (f) Check for minus sign if they used the sine function.

Solution: (a) Since the points of zero kinetic energy are the tree limb and ground, the point of maximum kinetic energy or equilibrium is halfway or 7.5 m.

(b) If Tarzan were to oscillate slowly and finally come out to rest, he would be at equilibrium and

$$F_{vine} = -kx_{eq} = -mg,$$

Thus

$$k = mg/x_{eq} = 90(9.8)/7.5 = 118 \text{ N/m}.$$

(c) Trip would take one-half cycle or (1/2)T.

$T = 1/f = 2\pi/\omega = 2\pi\sqrt{m/k}$,

$(1/2)T = \pi\sqrt{m/k} = \pi\sqrt{\frac{m}{mg/x_{eq}}} = \pi\sqrt{x_{eq}/y} = \pi\sqrt{7.5/9.8} = 2.75$ s.

(d) Maximum pull = maximum restoring force of vine

$= F_{vine\ (max)} = -kx_{max} = (mg/x_{eq})x_{max} = 90(9.8)(15/7.5) = 1760$ N.

(e) $(1/2)kx^2 = (1/2)mv^2$ from conservation of energy.

$v_{max} = \sqrt{k/m}\ x_{max} = \sqrt{118/90}\ 7.5 = 8.58$ m/s.

(f) $v = v_{max} \cos(\omega t + \theta)$.

At $t = 0$, $v = 0$, thus $\theta = \pi/2$.

$v(t) = 8.58 \cos(1.14t + \pi/2) = -8.58 \sin(1.14t)$ m/s.

MASTERY TEST GRADING KEY - Form C

1. What To Look For: (a) For how large an angle does $\sin\theta \approx \theta$ hold?

Solution: This is the same thing as a physical pendulum. See Figure 24.

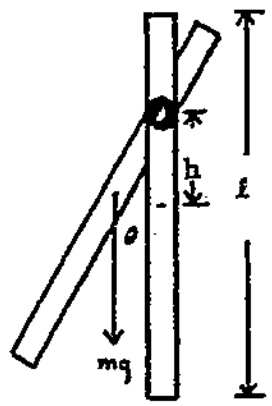


Figure 24

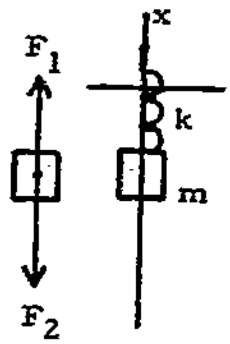


Figure 25

restoring torque = $-mgh\sin\theta$, torque = $I\alpha = I(d^2\theta/dt^2)$,

$-mgh\sin\theta = I(d^2\theta/dt^2)$.

$\sin\theta \approx \theta$ for small θ, $-mgh\theta = I(d^2\theta/dt^2)$, $d^2\theta/dt^2 + mgh\theta/I = 0$.

This is the same as for simple harmonic motion. Thus simple harmonic motion occurs.

$$\text{(b) } \omega = \sqrt{\frac{mgh}{I}} = \sqrt{\frac{mgh}{m(\ell^2/12 + h^2)}} = \sqrt{\frac{gh}{(\ell^2/12 + h^2)}} .$$

2. What To Look For: (b) It is more confusing if we say $A = +10.0$ cm, $\theta = \pi$, although it can be done.

(c) Alternate solution for (c):

$F = -kx$, $-kx = -ma$, $a = (-k/m)x$, $a = -\omega^2 x = -0.49\ m/s^2$.

(d) How long would it take to go from $x = -10.0$ cm to $x = 10.0$ cm?

Solution: (a) From conservation of energy:

$(1/2)mv_{max}^2 = (1/2)kx_{max}^2$, $\omega^2 = k/m$ or $k = m\omega^2$, $(1/2)mv^2 = (1/2)m\omega^2x^2$.

$v_{max}^2 = \omega^2 x_{max}^2$, $v_{max} = \omega x_{max} = 2\pi(1/T)x_{max} = 2\pi(1/2)(0.100) = 0.310$ m/s.

($x_{max} = A$, thus $v_{max} = A\omega$.)

(b) $x = A\cos(\omega t + \theta)$, $\omega = 2\pi/T = \pi$.

At $t = 0$, $x = -10.0$ cm, thus $\theta = 0$, $A = -10.0$.

$x = -10.0\cos(\omega t)$.

(c) at $x = +5.0$ cm, $t = t_1$,

5.0 cm $= -10.0\cos(\omega t_1)$ cm or $0.50 = \cos(\omega t_1)$ at $x = +5.0$.

Acc. (at $x = +5.0$ cm) $= \left.\frac{d^2x}{dt^2}\right|_{(x = +5.0)} = A\omega^2 \cos(\omega t_1)\Big|_{(x = +5.0)} = A\omega^2(0.50)$

$= 0.100[2\pi(1/2)]^2 0.50\ \text{m/s}^2 = 0.49\ \text{m/s}^2$.

(d) As in part (b), $A = -10.0$ and $\theta = 0$. Let time $= t_2$:

$x = A\cos(\omega t_2)$, $0 = -10.0\cos(\omega t_2)$, $\cos(\omega t_2) = 0$,

$\omega t_2 = \pi/2$, $t_2 = 0.50$ s.

(e) See Figure 25.

$k = m\omega^2 = m[2\pi(1/T)]^2$, $F_1 = -kx\tau$, $F_2 = -mg\tau$,

$-kx = -mg$, $m[2\pi(1/T)]^2 x = -mg$, $x = gT^2/(2\pi)^2 = -(9.8)(2)^2/(2\pi)^2 = -1.00$ m.

Thus, spring will shorten by one meter.

PARTIAL DERIVATIVES

Can you evaluate partial derivatives of functions of more than one variable? Try the following self-check test. If you get all the answers correct you should be able to handle the material in this course involving partial derivatives. If not, read the material that follows the test.

SELF-CHECK TEST

1. Suppose y is a function of the independent variables x and t:

$$y(x, t) = x^2 + t + axt^2,$$

where a is a constant. Determine the expression for

(a) $\partial y/\partial x$;

(b) $\partial y/\partial t$;

(c) $\partial^2 y/\partial t^2$.

(d) If $a = 3$, evaluate y at $(x, t) = (2, 5)$.

(e) If $a = 3$, evaluate $\partial y/\partial t$ at $(x, t) = (4, 2)$.

2. Try another function y of the independent variables x and t:

$$y = A \sin(\omega t - kx).$$

If the constants are $A = 3$, $\omega = 2$, $k = \pi$, evaluate:

(a) y at $(x, t) = (3/2, 3\pi/2)$;

(b) $\partial y/\partial t$ at $(x, t) = (-5/2, 3\pi/4)$;

(c) $\partial^2 y/\partial t^2$ at $(x, t) = (1/2, -\pi/8)$.

Answers are at the bottom of this page. If you did not get them right, continue reading in this review module.

At the end of this review we shall provide a more rigorous definition of a partial derivative, but go ahead and read straight through to get a feeling for what you need to do from an operational standpoint.

Self-Check Test Answers

1. (a) $2x + at^2$; (b) $1 + 2axt$; (c) $2ax$. (d) 159; (e) 49.

2. (a) -3; (b) 6; (c) -62.

If we have a function of more than one independent variable, than we can define a <u>partial derivative</u> with respect to one of the variables, which is simply the derivative of the function with all the other variables fixed. The notation using ∂, which we will use below, tells you it is a partial derivative. For example, suppose we have the function y that depends on the independent variables x and t:

$$y = A \sin(kx - \omega t),$$

where A, k, and ω are constants. Then, "the partial derivative of y with respect to x" is denoted by $\partial y/\partial x$ and is found by setting t constant and differentiating with respect to x:

$$\partial y/\partial x = kA \cos(kx - \omega t).$$

Similarly, "the partial derivative of y with respect to t" is denoted by $\partial y/\partial t$ and is found by setting x constant and differentiating with respect to t:

$$\partial y/\partial t = -\omega A \cos(kx - \omega t).$$

Note that the value of either of the partial derivatives depends on both independent variables x and t as well as the constants A, k, and ω.

Here are some Exercises to practice on (answers at bottom of page):

<u>Exercises</u>

1. If $y(x, t) = x^2 + 4x^3t + 5t^4$, determine the expression for the following partial derivatives and their values at $(x, t) = (4, 3)$:

   (a) $\partial y/\partial x$;

   (b) $\partial y/\partial t$;

   (c) $\partial^2 y/\partial t^2 \equiv (\partial/\partial t)(\partial y/\partial t)$ (i.e., perform the partial derivative with respect to t <u>two</u> times in succession).

2. If $y(x, t) = A \cos[k(x - ct) - \phi]$, determine the expression for the following partial derivatives.

   (a) $\partial y/\partial x$;

   (b) $\partial y/\partial t$;

   (c) $\partial^2 y/\partial x\, \partial t \equiv (\partial/\partial x)(\partial y/\partial t)$ (i.e., perform the partial derivative with respect to t followed by the partial derivative with respect to x).

<u>Exercise Answers</u>

1. (a) $2x + 12x^2t$. This is the result of setting t constant and performing the derivation with x as the variable. We evaluate this expression at $(x, t) = (4, 3)$ to get $(2 \times 4) + (12 \times 16 \times 3) = 584$.

   (b) $4x^2 + 20t^3$; 604. (c) $60t^2$; 540.

2. (a) $-kA \sin[k(x - ct) - \phi]$. (b) $kcA \sin[k(x - ct) - \phi]$.

   (c) $k^2cA \cos[k(x - ct) - \phi]$.

As promised, here is a more formal definition of a partial derivative of a function of more than one independent variable:

Suppose that we have a function y that depends on the independent variables x_1, x_2, x_3, etc. We can write it as $y(x_1, x_2, x_3, \ldots)$. The "partial derivative of y with respect to x_1" is then denoted by $\partial y/\partial x_1$ and is defined by the expression

$$\frac{\partial y}{\partial x_1} \equiv \lim_{\Delta x \to 0} \frac{y(x_1 + \Delta x, x_2, x_3, \ldots) - y(x_1, x_2, x_3, \ldots)}{\Delta x}.$$

Similarly,

$$\frac{\partial y}{\partial x_2} \equiv \lim_{\Delta x \to 0} \frac{y(x_1, x_2 + \Delta x, x_3, \ldots) - y(x_1, x_2, x_3, \ldots)}{\Delta x}.$$

As you can see, this is the same sort of limit used to define the derivative of a function of only one variable, the difference being that the function and its partial derivatives are functions of more than one variable. You should consult a calculus textbook for more details.

Module ____ 1

STUDY GUIDE

TRAVELING WAVES

INTRODUCTION

"For many people - perhaps for most - the word 'wave' conjures up a picture of an ocean, with the rollers sweeping onto the beach from the open sea. If you have stood and watched this phenomenon, you may have felt that for all its grandeur it contains an element of anticlimax. You see the crests racing in, you get a sense of the massive assault by the water on the land - and indeed the waves can do great damage, which means that they are carriers of energy - but yet when it is all over, when the wave has reared and broken, the water is scarcely any further up the beach than it was before. That onward rush was not to any significant extent a bodily motion of the water. The long waves of the open sea (known as the swell) travel fast and far. Waves reaching the California coast have been traced to origins in South Pacific storms more than 7000 miles away, and have traversed this distance at a speed of 40 mph or more. Clearly the sea itself has not traveled in this spectacular way; it has simply played the role of the agent by which a certain effect is transmitted. And here we see the essential feature of what is called wave motion. A condition of some kind is transmitted from one place to another by means of a medium, but the medium itself is not transported. A local effect can be linked to a distant cause, and there is a time lag between cause and effect that depends on the properties of the medium and finds its expression in the velocity of the wave. All material media - solids, liquids, and gases - can carry energy and information by means of waves....

"Although waves on water are the most familiar type of wave, they are also among the most complicated to analyze in terms of underlying physical processes. We shall, therefore, not have very much to say about them. Instead, we shall turn to our old standby - the stretched string - about which we have learned a good deal that can now be applied to the present discussion."*

PREREQUISITES

Before you begin this module, you should be able to:	Location of Prerequisite Content
*Provide a mathematical and pictorial description of a particle undergoing sinusoidal motion (needed for Objectives 1 through 4 of this module)	Simple Harmonic Motion Module
*Find the partial derivative of a simple function of two variables (needed for Objectives 1 through 4 of this module)	Partial Derivatives Review

*A. P. French, Vibrations and Waves (Norton, New York, 1971), pp. 201, 202.

LEARNING OBJECTIVES

After you have mastered the content of this module, you will be able to:

1. Description of wave - Describe and interpret descriptions of traveling transverse waves on a string, using both pictorial and mathematical formulations.
2. Wave velocity - Relate the wave speed to the physical properties of string.
3. Superposition - Apply the superposition principle to (a) reflections at a boundary, (b) waves moving in the same direction, and (c) waves moving in opposite directions (standing waves, resonance).
4. Power - Discuss the dependence of transmission of power in a wave in a string on the physical variables.

TEXT: Frederick J. Bueche, Introduction to Physics for Scientists and Engineers (McGraw-Hill, New York, 1975), second edition

SUGGESTED STUDY PROCEDURE

This module is limited to transverse waves on a string. Your text's treatment of waves on a string is combined with other wave phenomena, thus to satisfy the stated objectives in this module you will have to skip around a bit, reading some material that does not apply specifically to waves on a string. That should not hurt, and you may pick up some related ideas that are interesting.

Read the following material in Bueche: Appendix 9, Section 29.4 to 29.6, 31.1 to 31.3, and 34.1 to 34.3. Work at least Problems A through K before attempting the Practice Test.

BUECHE

Objective Number	Readings	Problems with Solutions		Assigned Problems	Additional Problems
		Study Guide	Text	Study Guide	
1	Appendix 9, Sec. 29.4	A		F, G	Chap. 29, Probs. 1, 3
2	Sec. 29.5	B	Illus.[a] 29.2	H	
3	Secs. 31.1 to 31.3, 34.1 to 34.3	C, D	Illus. 34.1	I, J	Chap. 31, Quest. 2, Probs. 7, 8; Chap. 34, Probs. 1, 3
4	Sec. 29.6	E	Illus. 29.3	K	Chap. 29, Prob. 4

[a]Illus. = Illustration(s).

TEXT: David Halliday and Robert Resnick, Fundamentals of Physics (Wiley, New York, 1970; revised printing, 1974)

SUGGESTED STUDY PROCEDURE

Read Chapter 16 and work at least Problems A through K before attempting the Practice Test.

HALLIDAY AND RESNICK

Objective Number	Readings	Problems with Solutions		Assigned Problems	Additional Problems[b]
		Study Guide	Text	Study Guide	(Chap. 16)
1	Secs. 16-1 to 16-3	A		F, G	3, 5
2	Sec. 16-4	B	Ex.[a] 1, 2	H	11, 13, 15
3	Secs. 16-6 to 16-8	C, D		I, J	25, 28, 32, 33, 39
4	Sec. 16-5	E		K	22

[a]Ex. = Example(s).

[b]Problem 28 should be worked with a wave velocity of 20.0 m/s. Problem 33 should be worked with a wave velocity of 15.0 m/s.

TEXT: Francis Weston Sears and Mark W. Zemansky, University Physics (Addison-Wesley, Reading, Mass., 1970), fourth edition

SUGGESTED STUDY PROCEDURE

Read Chapters 21 and 22. Since this module is limited to waves on a string, you can skip Sections 21-4, 21-5, 21-6, 22-6, 22-7, 22-8, and 22-9 and still achieve the objectives of this module. Work at least Problems A through K before attempting the Practice Test.

For Objective 4, look at the situation pictured in Figure 21-5. Not only is the transverse force F providing a transverse impulse, but it is also doing work on the string. This work goes into increasing the kinetic energy of the string, since more and more of the string is moving. We also see that energy is being transmitted along the string since to the left of the point P the string has kinetic energy and to the right of that point it has none, and point P is moving along the string with the wave speed c.

We can compute the instantaneous power furnished by F in the following way. The instantaneous power is $\vec{F} \cdot \vec{v}$. We see from Figure 21-5 that the negative of the slope of the string is given by F/S, so we have a value for F in terms of the slope of the string and of S.

Power = S × (negative of the slope of the string) × v.

Suppose we have a force that is producing a sinusoidal wave of the form in Eq. (21-3), the force being applied at x = 0. Now,

$$v = \partial y/\partial t = \omega Y \cos(\omega t - kx),$$

SEARS AND ZEMANSKY

Objective Number	Readings	Problems with Solutions: Study Guide	Problems with Solutions: Text	Assigned Problems: Study Guide	Additional Problems
1	Secs. 21-1, 21-2	A	Ex.[a] 2 (Sec. 21-3)	F, G	21-3, 21-7
2	Sec. 21-3	B	Ex. 1 (Sec. 21-3)	H	21-1
3	Secs. 22-1, 22-2, 22-3, 22-5	C, D		I, J	22-3
4	This module	E		K	

[a]Ex. = Example(s).

and the slope

$$\partial y/\partial x = -kY \cos(\omega t - kx),$$

so that the equation for the power gives, at $x = 0$,

$$\text{Power} = k\omega Y^2 S \cos^2 \omega t.$$

Using the facts that $c^2 = S/\mu$, $c = \omega/k$, and the fact that the time-average value of $\cos^2 \omega t = 1/2$ (which you can see by noting that $\cos^2 \omega t$ spends equal amounts of time at equal distances above and below 1/2), we find

$$\text{Power}_{av} = (1/2)\omega^2 Y^2 \rho c.$$

This is the time-average power that is put into the wave and the power that the wave carries off to the right. Note that the power is proportional to the <u>square</u> of the angular frequency ω, the <u>square</u> of the amplitude Y (reminiscent of the energy in a simple harmonic oscillator), and is directly proportional to the velocity of the wave and to the mass of the string.

TEXT: Richard T. Weidner and Robert L. Sells, Elementary Classical Physics (Allyn and Bacon, Boston, 1973), second edition, Vol. 1

SUGGESTED STUDY PROCEDURE

Read Chapter 16 through Section 16-18. Note the following points: (a) In Eq. (16-2a), note that F(x - ct) stands for "function of the quantity x - ct." Don't confuse this F with the F_t that represents the tension in the string. (b) In Figure 16-4, and in most of the figures following, the displacement of the string away from its equilibrium line has been greatly exaggerated. The superposition principle for waves on a string will only hold for waveforms with very small slopes. Work at least Problems A through K before attempting the Practice Test.

WEIDNER AND SELLS

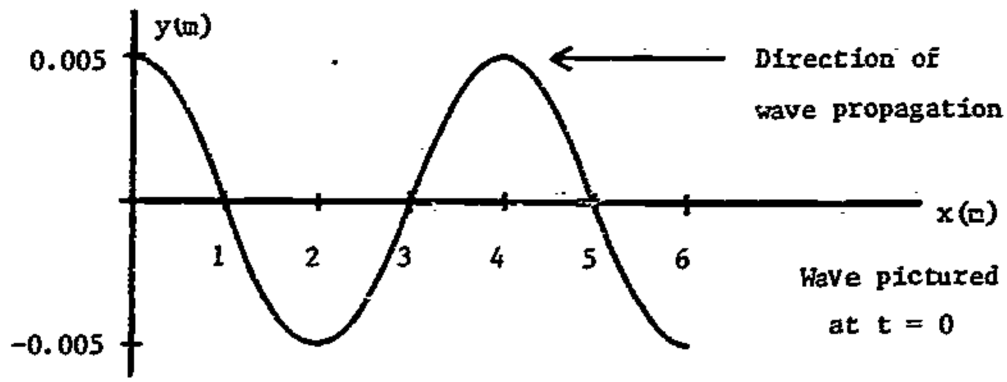
Objective Number	Readings	Problems with Solutions		Assigned Problems	Additional Problems
		Study Guide	Text (Ex.[a])	Study Guide	
1	Secs. 16-1, 16-4	A	16-2	F, G	16-3, 16-9, 16-10
2	Sec. 16-1	B	16-1, 16-2	H	16-1, 16-8
3	Secs. 16-3, 16-6, 16-7, 16-8	C, D	16-3	I, J	16-22
4	Sec. 16-5	E		K	16-11, 16-22, 16-14

[a]Ex. = Example(s).

PROBLEM SET WITH SOLUTIONS

A(1). A "snapshot" of a traveling sinusoidal wave at $t = 0$ traveling to the left is shown in Figure 1 along with some other information. (Note that the vertical scale is exaggerated.) At the position $x = 0$, the wave reaches a positive maximum 30 times each second.
(a) Determine the frequency, wavelength, wave speed, angular frequency, period, propagation constant (or wave number) and amplitude of this traveling wave.
(b) Sketch the "snapshot" of the wave that you would obtain at $t = 1/120$ s.
(c) Write the mathematical expression that represents the transverse displacement as a function of x and t.
(d) Determine an expression for the displacement of the point P that moves with the string at $x = 2.50$ m, and compute the velocity of P at $t = 1/60$ s.

Figure 1



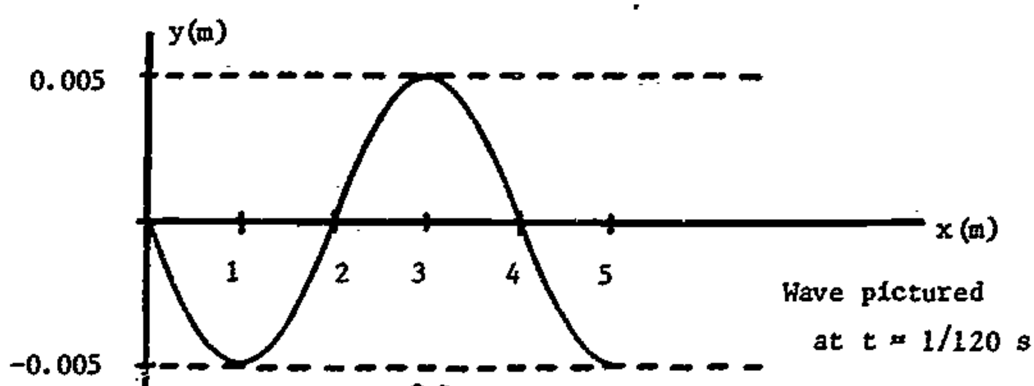
Solution

(a) From the text of the problem, the frequency = 30.0 Hz. Thus angular frequency = $2\pi f = 188\ s^{-1}$ and the period = $1/f = 1/30.0$ s. From the figure we see that the wavelength $\lambda = 4.0$ m. Thus the wave speed = frequency × wavelength = 120 m/s. We also know that

wave speed = (propagation constant)/(angular frequency) = ω/k, $k = 2\pi/\lambda = 1.57\ m^{-1}$.

From the figure we see that the amplitude of the wave is 0.0050 m.
(b) We could work part (c) first and then plug in $t = (1/120)$ s to see what we get, but let's try a graphical approach. We know that since the wave is moving to the left at 120 m/s, in 1/120 s it has moved 1.00 m to the left. We can immediately sketch the wave shifted 1.00 m to the left as in Figure 2.

Figure 2



(c) We recall that a sinusoidal wave traveling in the negative x direction is written in the following form (or one equivalent to it):

$$y = A \sin(kx + \omega t + \phi).$$

How do we recall this without resorting to rote memory? Here's how: The amplitude factor A is the number (whose dimension is a length for this kind of wave) that multiplies a sinusoidal function (which oscillates between +1 and -1). What's left is the argument of the sine function - the phase. It has to be dimensionless (radians) and has to depend independently on x and t. Therefore we multiply x by k and t by ω. To get a wave moving in the negative x direction we add: $kx + \omega t$. This guarantees that the phase (corresponding to, say, the maximum of the sine curve) remains constant as x decreases while t increases. (Here you should check to see that you can argue that $kx - \omega t$ is what you want for a wave traveling in the +x direction.) Finally, we need a phase constant ϕ to fix up the sine curve to agree with the picture at $t = 0$. If we write $y = A \sin(kx + \omega t + \phi)$, how do we determine ϕ? Well, if $\phi = 0$ we don't get the right picture at $t = 0$, thus we have to give ϕ a nonzero value. If we let ϕ go negative, this has the effect of shifting the sine curve to the right at $t = 0$. Since we want to shift the sine curve a quarter wavelength to the left, we need to make $\phi = +\pi/2$. Now we have it:

$$y = A \sin(kx + \omega t + \pi/2),$$

where A, k, and ω have already been determined in part (a). We could write it as $y = A \cos(kx + \omega t)$, since $\sin(\theta + \pi/2) = \cos\theta$.

(d) Before putting any numbers into this part, let's get symbolic expressions first. Use x_0 for the x coordinate of P. Then the transverse displacement of P is given by the expression for the wave with x_0 plugged in for x:

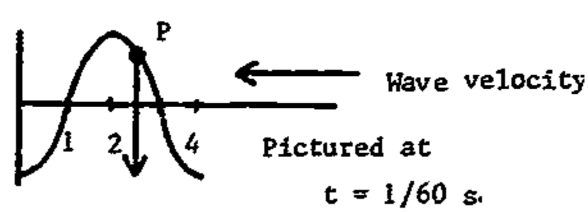
$$y = A \sin(kx_0 + \omega t + \pi/2).$$

This is the expression for the displacement of P. Note that the only variable is the time and that P is executing simple harmonic motion. The velocity of P is given by

$$\partial y_P/\partial t = \omega A \cos(kx_0 + \omega t + \pi/2).$$

Plugging in all the other numbers and $t = (1/60)$ s we get -0.666 m/s. (Take care that you look up the cosine for an argument in radians.) Note that we can check the sign by looking at a sketch of the wave for which the wave has been shifted another quarter wavelength to the left beyond the situation in part (b): Figure 3. In this figure we see that P is moving in the negative y direction, in agreement with our calculation.

Figure 3



B(2). A string with linear density 15.0 g/m propagates waves at a speed of 20.0 m/s. The string is driven transversely by an oscillating arm with an angular frequency 20.0 s^{-1} and an amplitude 0.0300 m.
(a) Determine the tension in the string.
(b) Determine the maximum velocity of a point on the string 2.00 m from the driver.

Solution

(a) We know the wave speed from the derivation in the text:

$$v = \left(\frac{\text{tension in string}}{\text{mass/unit length of string}}\right)^{1/2} = \left(\frac{F}{\rho}\right)^{1/2}$$

Before proceeding, we can check this formula to see it it is dimensionally correct (If it isn't, we have the wrong formula!):

$$[v] = [L/T]$$

$$\left(\frac{F}{\rho}\right)^{1/2} = \left[\left(\frac{ML}{T^2}\frac{L}{M}\right)^{1/2}\right] = \frac{L}{T}.$$

The formula is dimensionally correct. Thus we find $F = \rho v^2 = 6.0$ N.

(b) Since the maximum velocity of any point on the string is the same as any other (although the maximum velocity occurs at different times at different points) the 2.00 m is not relevant. We can, for example, consider the point at which the driver is attached to the string. You should get $(\partial y/\partial t)_{max} = 0.60$ m/s.

C(3). A pulse moves along a string with wave speed v as shown in Figure 4. The right end of the string is fixed to a wall. The situation is shown at t = 0. Sketch the vertical displacement of the point P as a function of time.

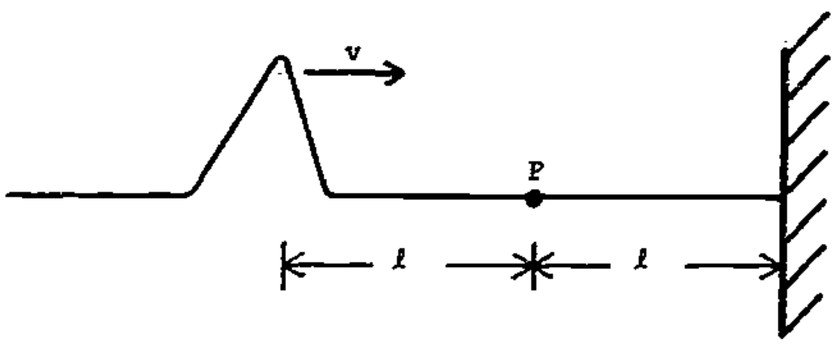


Figure 4

Solution

The boundary condition at the wall is that the string has no transverse displacement (i.e., it is fixed). We can guarantee this boundary condition by imagining that the string does not stop at the wall but extends off to the right where a symmetrically located wave is propagating to the left. See Figure 5. This new wave is shaped just like the other wave except that it is inverted and is reflected front to back. Now, the principle of superposition states that the displacement of the string is just the algebraic sum of the displacements in the two waves. The new wave has been set up so that, at the former position of the wall, the displacement of the string is always zero, and as far as the left half of the string is concerned, it is just as if there were a wall at that position still. The motion of point P thus is first affected by the passage of the wave pulse going to the right and, later, by the reflected wave going to the left.

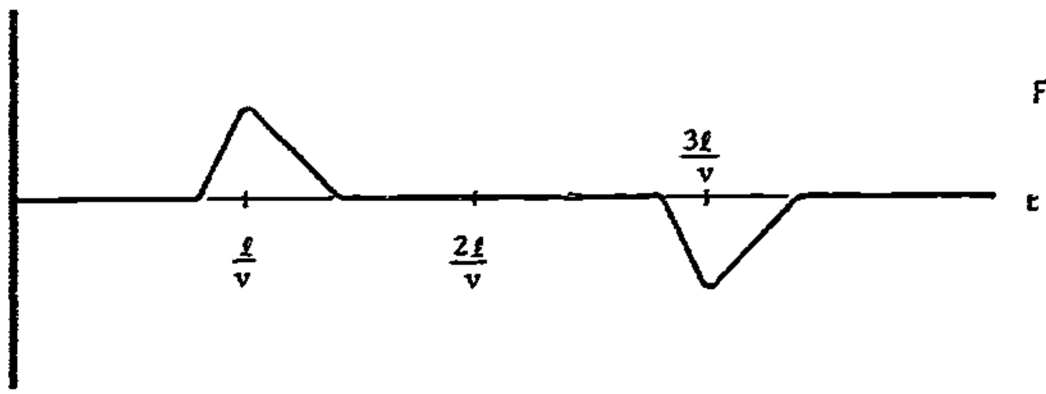


Figure 5

D(3). A string vibrates according to the equation

$$y = (0.0040\ \text{m}) \sin[(25.0\ \text{m}^{-1})x] \cos[(400\ \text{s}^{-1})t].$$

(a) What are the amplitude and velocity of the component waves whose superposition gives rise to this vibration?
(b) Determine the distance between nodes.
(c) Sketch the shape of the string at several different times to provide a "motion picture" of the motion of the string.

Solution

We hope that you recognized this to be the form of a standing wave, produced by two traveling waves of equal amplitude and frequency traveling in opposite directions. You can consult your textbook for a mathematical plug for the answer to part (a), but it is important that you also be able to reason out the answer pictorially. Let us answer part (c) first to get Figure 6. Before doing arithmetic, write the formula you were given in algebraic quantities:

$$y = A \sin kx \cos \omega t.$$

By inspection of the formula and Figure 6 we see that both the standing and the traveling waves take a distance $2\pi/k$ to go through one cycle along x, thus $\lambda = 2\pi/k$. It is also easy to see that the period of the standing wave is the same as that of the traveling waves, thus $f = \omega/2\pi$. The wave speeds are thus $v = \lambda f = \omega/k$, and for this example

$$v = (400\ s^{-1})/(25.0\ m^{-1}) = 16.0\ m/s.$$

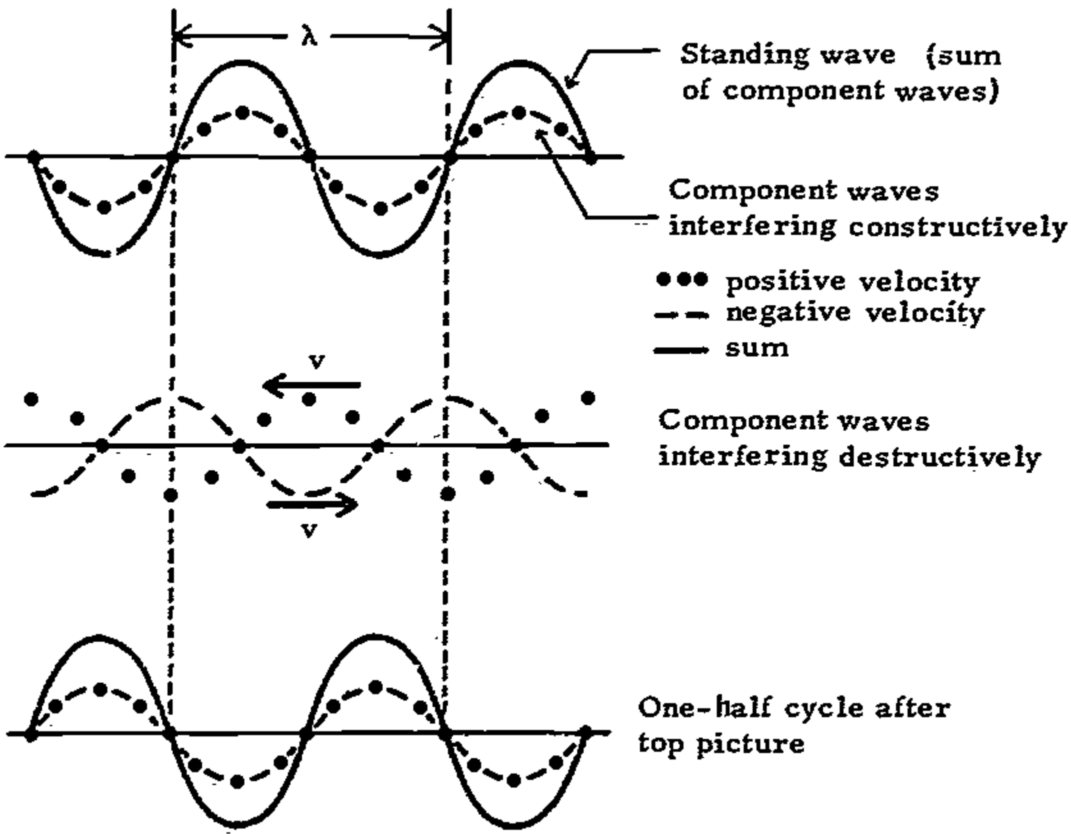
The amplitude of both traveling waves is one-half of the maximum amplitude of the standing wave, as we see from the figure, therefore the

$$\text{amplitude of traveling waves} = A/2 = 0.0020\ m.$$

Again, as we can see from the figure, the distance between nodes equals

$$\frac{\lambda}{2} = \frac{1}{2}\frac{2\pi}{k} = \frac{\pi}{k} = \frac{\pi}{25.0\ m^{-1}} = 0.126\ m.$$

Figure 6



E(4). Give a physical argument (give an analogy to some other, possibly simpler, physical situation) to justify that the power carried in a sinusoidal traveling wave depends on the square of the frequency. Does your analogy also provide an explanation for why the energy transmission also depends on the other variables the way it does, or do you have to dredge up something else? (After answering this question, you should not have to claim that you have to "memorize" the formula for power transmitted by a wave on a string.)

Solution

Suppose you start to wiggle, with a transverse motion, the end of a string that is initially at rest. What does the string look like at successive instants? Consider a small segment of the string. It is moving up and down with simple harmonic motion, pulling the next segment up or down and thus doing work on it. Power is work over time, and thus we find that the power for a sinusoidal wave is similar to that for simple harmonic motion, with the square of the frequency (which arose from the derivative) taking the place of angular velocity. How does the kinetic energy of the piece of string depend on the amplitude and frequency? How does it depend on the mass per unit length of the string?

Problems

F(1). A sinusoidal traveling wave has the form

$$y = \underset{\mathrm{m}}{0.0300}[\sin(\underset{\mathrm{m}^{-1}}{0.50}\underset{\mathrm{m}}{x} - \underset{\mathrm{s}^{-1}}{20.0}\underset{\mathrm{s}}{t} - \underset{\mathrm{rad}}{\pi/4})],$$

where the dimensions of each quantity are shown below the equation.
(a) Determine the frequency, wavelength, wave speed, angular frequency, period, amplitude, and direction of travel of this wave.
(b) Sketch "snapshots" of this wave at $t = 0$ and at $t = 0.0250$ s.

G(1). A pulse travels along a string with a speed of 5.0 m/s. Its shape and direction of motion at $t = 0$ are shown in Figure 7.

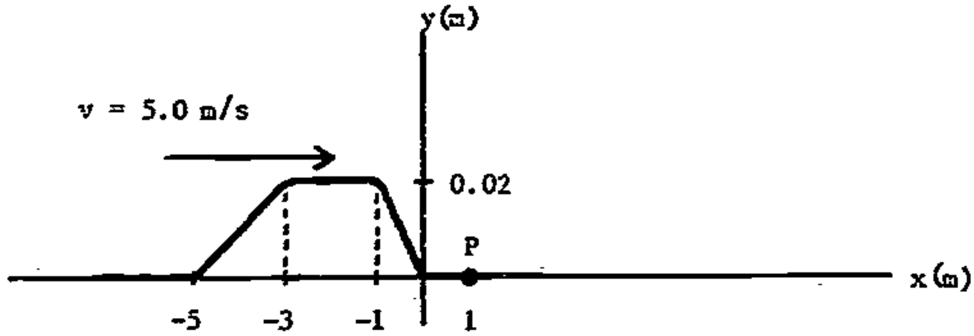


Figure 7

(a) Sketch the pulse at t = 0.80 s.
(b) The point P is fixed to the string at x = +1.00 m. Sketch the displacement and velocity of P versus time. Label the axes of your sketch carefully, showing the scales on each of the axes.

H(2). A rope under 80 N tension carries a sinusoidal wave of wave constant (wave number) $k = 4.0\ m^{-1}$ at a wave speed of 25.0 m/s. Determine the frequency of the waves and the mass per unit length of the rope.

I(3). Show that the superposition of two sinusoidal waves of equal amplitude, frequency, and wave speed that are traveling in the same direction gives a sinusoidal traveling wave regardless of the phase difference between them. Determine the amplitude of this traveling wave in terms of the amplitude of the two waves and the phase difference between them.

J(3). A string 2.00 m long is attached to the prong of an electrically driven tuning fork that vibrates perpendicularly to the length of the string at a frequency of 80 Hz. The mass of the string is 12.0 g. Determine the tension that must be applied to the string to make it resonate in three loops.

K(4). Two long wires of linear densities 2.00×10^{-2} kg/m and 5.0×10^{-2} kg/m, respectively, are joined at one end. The two free ends are pulled apart with a tension of 10.0 N. At the point at which the two wires are joined, an oscillating arm starts to shake the wires transversely with a sinusoidal oscillation of 25.0 Hz, 0.00300 in amplitude. Oetermine the energy supplied to the wires after 3.00 s.

Solutions

F(1). (a) The equation is of the form of a wave moving in the +x direction:

$$y = A \sin(kx - \omega t - \phi),$$

from which we immediately note the substitutions

$k = 0.50\ \mathrm{m}^{-1}$, angular frequency $\omega = 20\pi\ \mathrm{s}^{-1} = 62.8\ \mathrm{s}^{-1}$, amplitude $A = 0.0300$ m,

from which we can derive the other quantities:

wave speed $= \omega/k = 126$ m/s, frequency $= \omega/2\pi = 10.0$ Hz,

$$\text{wavelength} = \frac{\text{wave speed}}{\text{frequency}} = = \frac{\omega}{k}\frac{2\pi}{\omega} = \frac{2\pi}{k} = 12.6\ \text{m}, \qquad \text{period} = \frac{1}{f} = 0.100\ \text{s}.$$

(b) We note that this is a wave of amplitude 0.0300 m and wavelength 12.6 m. It is shifted at $t = 0$ from sin kx by the phase shift ϕ. For $\phi = \pi/4$, the wave slope is shifted $(\pi/4)/(2\pi)^{-1}$ wavelengths to the right, and thus the snapshot looks like Figure 8. At $t = 0.0250$ s we have

$$y = 0.030 \sin(0.50x - \pi/2 - \pi/4),$$

which is a sine function shifted $(3\pi/4)\ (2\pi)^{-1}$ wavelengths to the right, as in Figure 9.

G(1). (a) See Figure 10.
(b) See Figure 11.

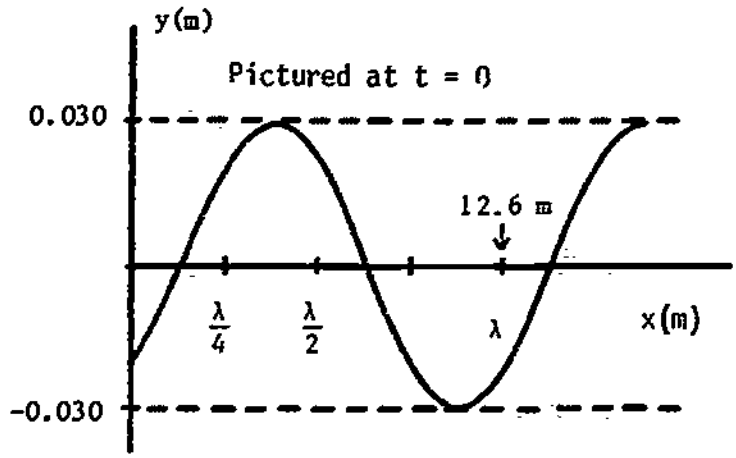
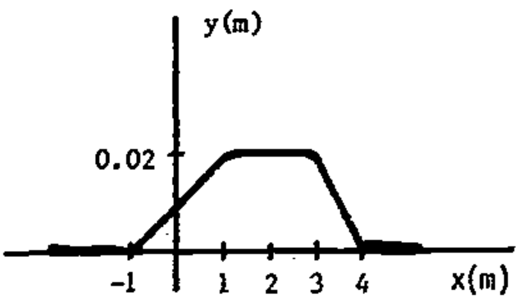


Figure 8

Figure 10



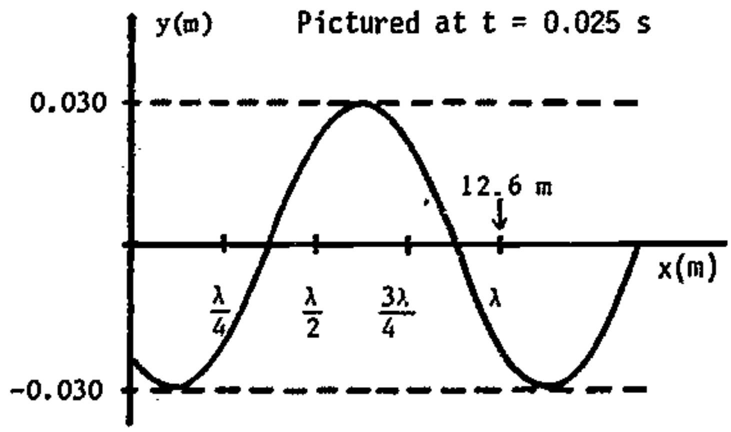


Figure 9

H(2). The frequency is 15.9 Hz, the density 0.128 kg/m.

I(3). This problem gives a little practice in applied trigonometry. Let us write expressions for the two waves:

$$y_1 = A\sin(kx - \omega t), \quad y_2 = A\sin(kx - \omega t - \phi),$$

that have the same amplitude, frequency, wave speed, and direction of propagation. The wave y_2 leads y_1 by a distance ϕ/k, which you should verify for yourself. We wish to calculate $y = y_1 + y_2$ but first we can reexpress y_1 and y_2, using the identity (which you should know)

$$\sin(\alpha + \beta) = (\sin\alpha)(\cos\beta) + (\cos\alpha)(\sin\beta)$$

to write

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta.$$

We can put $y_1 + y_2$ in this form by letting

$$\alpha + \beta = kx - \omega t, \quad \alpha - \beta = kx - \omega t - \phi,$$

which gives

$$\alpha = kx - \omega t - \phi/2, \quad \beta = \phi/2,$$

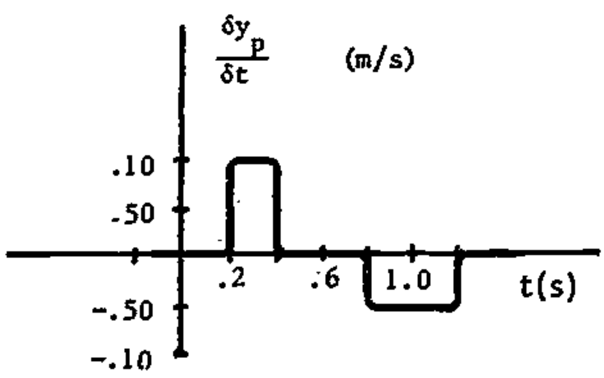
and

$$y = 2A\sin(kx - \omega t - \phi/2)\cos(\phi/2).$$

This is a traveling sinusoidal wave, regardless of the value of ϕ. The amplitude of the resultant wave is $2A\cos(\phi/2)$. If $\phi = 0, 2\pi, 4\pi$, etc., the waves are "in phase," and the amplitude is 2A - an example of constructive interference between the two waves. If $\phi = \pi, 3\pi, 5\pi$, etc., the amplitude is <u>zero</u> - destructive interference.

J(3). 68 N.

K(4). 0.39 J.



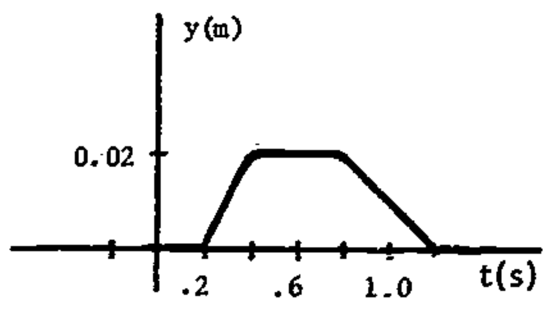
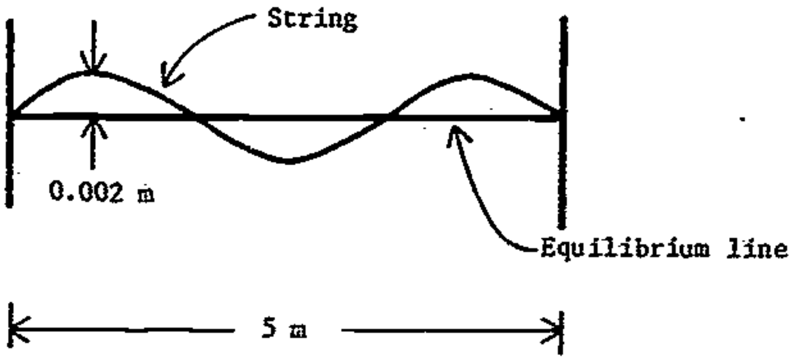


Figure 11

PRACTICE TEST

1. A standing wave on a string is shown at $t = 0$ in Figure 12. The frequency of its motion is 25.0 Hz, and the displacement of the string is a maximum.
(a) Write down symbolic expressions for the traveling waves whose superposition produces this standing wave, and give the numerical values for the symbols in your expressions.
(b) Determine the tension in the string if its mass per unit length is 2.00 g/m.

Figure 12



2. The motion of a string is given by

$y = A \sin(\omega t - kx)$,

where $A = 0.00200$ m, $\omega = 65\ s^{-1}$, and $k = 2.00\ m^{-1}$
(a) Determine the maximum speed of a point on the string at $x = 4.0$ m.
(b) In what direction is power being transmitted along the wave?
(c) If the amplitude and frequency are kept the same, but the tension in the string is increased by a factor of 2, how is the power transmitted by the string affected?

Practice Test Answers

1. (a) $y_1 = A \sin(\lambda x - vt)$ wave traveling to right, $y_2 = A \sin(\lambda x + vt)$ wave traveling to left, where $A = 0.00200$ m, $\lambda = 3.30$ m, and $v = f\lambda = (25.0)(3.30) = 83$ m/s.
(b) $\mu = 2.00$ g/m, $f = 25.0$ Hz, $\lambda = 3.30$ m.
$f = (1/\lambda)\sqrt{T/\mu}$, $f^2 = (1/\lambda^2)(T/\mu)$, $T = f^2\lambda^2\mu = (25.0)^2(3.30)^2(2.00) = 13.6$ N.

2. (a) $y = 0.00200 \sin(65t - 2.00x)$, $v = dy/dt = (0.00200)(65) \cos(65t - 2x)$,
$v_{x=4} = (0.00200)(65) = 0.130$ m/s.
(b) In the -x direction. (c) Power increases by $\sqrt{2}$.

TRAVELING WAVES

Date ____________________

Mastery Test   Form A

pass   recycle

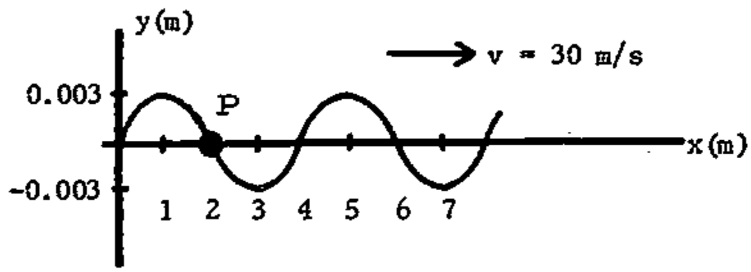
1   2   3   4

Name ______________________________   Tutor ______________________________

1. A standing wave of 5 loops (or antinodes) is set up on a flexible wire. The wire is 1.20 m long, and its total mass is 10.0 g. The maximum amplitude of oscillation of the wire is 0.50 mm, and its frequency of oscillation is 400 Hz.
(a) Sketch the shape of the wire at several instants, each separated by a quarter cycle from the previous instant.
(b) Determine the tension in the wire.
(d) Let one end of the wire be at x = 0 and the other end at x = 1.20 m. At t = 0 the wire is straight, but its tranverse velocity at x = 0.050 m is in the +y direction. Write the mathematical expression for the displacement y of the wire as a function of the variables x and t.

2. A sinusoidal traveling wave on a string moving to the right with a velocity 30.0 m/s is shown in Figure 1.
(a) Sketch the vertical velocity of the point P, which is at x = 2.00 m, as a function of time, labeling the axes of your sketch carefully.
(b) If the wavelength of this wave is increased by a factor of 3, keeping the amplitude the same (and presuming the tension and mass of the string are unchanged), determine the change in the rate of energy propagation on the string.

Figure 1



TRAVELING WAVES Date ______________

Mastery Test Form B

pass recycle

1 2 3 4

Name ______________________ Tutor ______________________

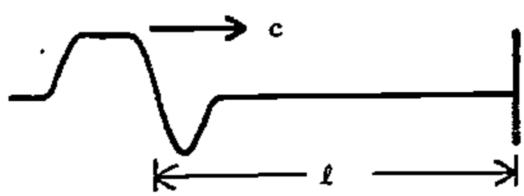
1. A wave propagates along a string whose mass per unit length is 0.0040 m kg/m. The transverse displacement of the wave is given by

   $y = A \sin(kx - \omega t - \phi)$,

   where $A = 0.00300$ m, $k = 6.0\ m^{-1}$, $\omega = 80\ s^{-1}$, and $\phi = \pi/2$.
   (a) Determine the tension in the string.
   (b) Sketch a "snapshot" of the string at $t = \pi/320$ s, showing the horizontal and vertical scales.
   (c) Determine (i) the direction and (ii) the magnitude of the wave velocity.
   (d) If the amplitude of the wave is halved, what effect does this have on the power transmitted by the wave? (Give your answer in a complete sentence.)

2. A pulse propagates with speed c down a long string that is fixed to a wall at one end. Figure 1 shows the string at t = 0. Sketch the shape of the string at $t = 3\ell/2c$ showing its location with respect to the wall.

Figure 1



TRAVELING WAVES

Date ____________________

Mastery Test Form C

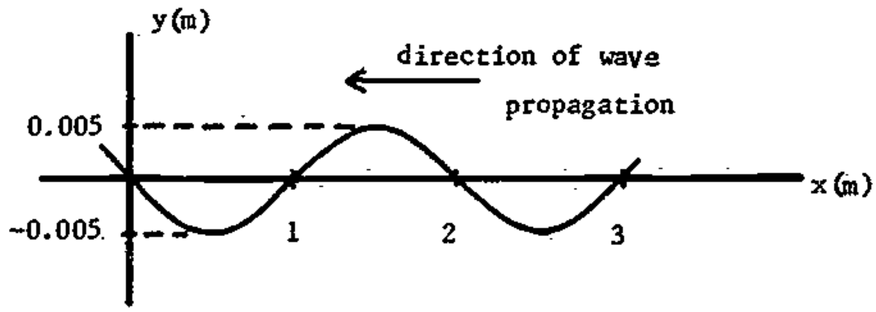
pass recycle

1 2 3 4

Name ______________________________ Tutor ______________________________

1. A wave propagating to the left is shown at $t = 0$ in Figure 1. The frequency of the wave is 60 Hz.
   (a) Sketch the wave at $t = 1/30$ s.
   (b) Sketch as a function of time the vertical position of the point on the string at $x = 0$. Show the scale clearly on the axes of your sketch.
   (c) Determine the mass per unit length of the string if its tension is 50 N.
   (d) If the appropriate traveling wave is superposed on the wave in the figure to produce a standing wave, determine the amplitude of the standing wave and determine the distance between nodes.

Figure 1



2. A wave given by

   $y_1 = A \sin(\omega t - kx)$

   transmits power P. Oetermine the power transmitted by a wave that is the superposition of y_1 with the wave

   $y_2 = A \cos(\omega t - kx)$.

MASTERY TEST GRADING KEY - Form A

1. What To Look For: (a) Clear picture. (b) Encourage algebraic manipulation before plugging in numbers. (c) Expression for standing wave.

   Solution: (a) See Figure 16, in which $\ell = 1.20$ m, $f = 400$ Hz, $m = 0.0100$ kg, and the amplitude is $5.0\ \ 10^{-4}$ m.

   (b) Since $\ell = 5(\lambda/2)$, $\lambda = (2/5)\ell$. $v = (F/\rho)^{1/2}$,

   $$F = \rho v^2 = \frac{m}{\ell}(f\lambda)^2 = \frac{m}{\ell}f^2\left(\frac{4}{25}\right)\ell^2 = \frac{4mf^2\ell}{25} = 307\ \text{N}.$$

   (c) See Figure 17.

   $y = A \sin \omega t \sin kx$,

   where $A = 5.0 \times 10^{-4}$ m from the problem. $\omega = 2\pi f = 2.51 \times 10^3\ s^{-1}$,

   $k = 2\pi/\lambda = 2\pi(2/5)\ell = 3.02\ m^{-1}$. Note that this form provides that $y = 0$ everywhere at $t = 0$ and $\partial y/\partial t > 0$ in the first loop.

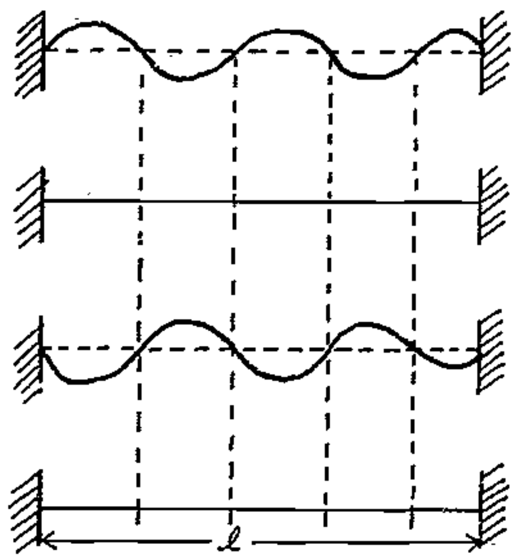
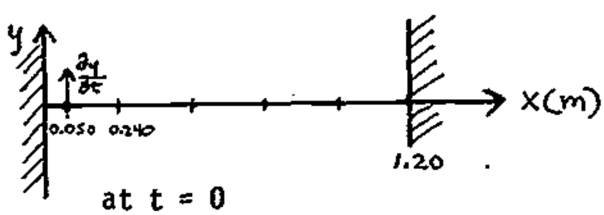


Figure 16



at $t = 0$

Figure 17

2. What To Look For: (a) Clear picture, appropriate mathematical justification. (b) Clear statement.

   Solution: (a) See Figure 18. Wave is written as $y = A \sin(kx - \omega t)$, where $A = 0.00300$ m, $k = 2\pi/\lambda = (2\pi/4)\ m^{-1} = (\pi/2)\ m^{-1}$,

   $\omega = 2\pi f = 2\pi(v/\lambda) = 2\pi(30/4) = 15\pi\ s^{-1}$.

   $(\partial y/\partial t)|_{x = 2} = -\omega A \cos(-\omega t + \pi) = \omega A \cos \omega t$, where $\omega A = 0.141$ m/s. By inspection of the figure we see that at $t = 0$, P has its maximum positive-y velocity.

(b) Power transmission is proportional to $(\omega A)^2 \rho v$. We keep A, ρ, and v the same but change λ:

$\lambda = v/f = v2\pi/\omega$, thus $P \propto 1/\lambda^2$

if A, ρ, and v remain constant. Therefore power, or rate of energy propagation on string, is <u>reduced</u> by a factor of 9 (is multiplied by 1/9) if λ is <u>increased</u> by a factor of 3 (is multiplied by 3).

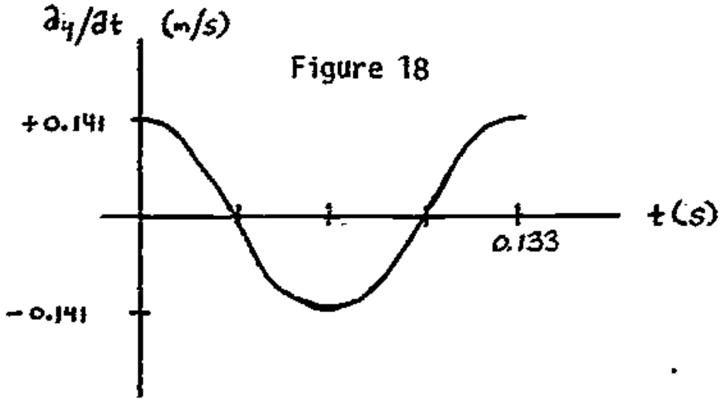


Figure 18

MASTERY TEST GRADING KEY - Form B

1. What To Look For: Correct scales on x and y axes; unambiguous statements.

   Solution: (a) $y = A \sin(kx - \omega t - \phi)$, where $A = 0.00300$ m, $k = 6.0\ m^{-1}$, $\rho = 0.0040$ kg/m, $\lambda = 1.050$ m, $\omega = 80\ s^{-1}$, period $= 2\pi/\omega = \pi/40$ s, $\phi = \pi/2$, wave speed $c = \omega/k = (F/\rho)^{1/2}$, and $F = \rho\omega^2/k^2 = 0.71$ N.

   (b) See Figure 19. At $t = \pi/320$ s, we write the equation of the displacement of the string as

   $y = A \sin(kx - \pi/4 - \pi/2)$, a sine curve shifted $3\pi/4k$ toward positive x.

   (c) (i) The direction is along +x. (ii) The magnitude of the wave velocity is $c = \omega/k = 80/6.0 = 13.3$ m/s.

   (d) $P \propto A^2$. If the amplitude of the wave is halved, the power transmitted by the wave is decreased by a factor of 4 (i.e., the power transmitted is multiplied by 1/4 if the amplitude is multiplied by 1/2).

2. Solution: See Figure 20. The string travels a distance $ct = c(3\ /2c) = (3/2)$ and is reflected at its fixed end.

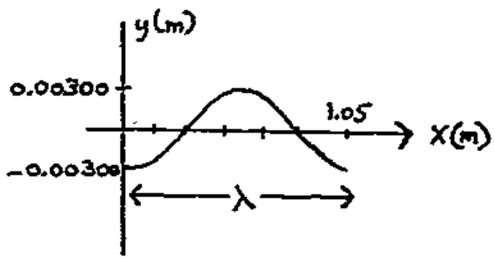


Figure 19

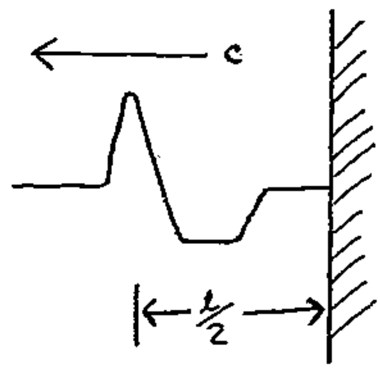


Figure 20

MASTERY TEST GRADING KEY - Form C

1. Solution: (a) See Figure 21. $t = 1/30$ s is two cycles later than $t = 0$. The figures look exactly the same.
   (b) See Figure 22.
   (c) $c = (F/\rho)^{1/2}$, thus $\rho = F/c^2 = F/(\lambda f)^2 = 50\ N/(2.00\ m)^2(60\ Hz)^2 = 3.47\ g/m$.
   (d) To get a standing wave, we must superpose a traveling wave of the same amplitude, traveling in the opposite direction. Thus the amplitude is $2.00 \times 0.0050\ m = 0.0100\ m$, and the distance between nodes is $\lambda/2 = 1.00\ m$.

2. Solution: $y_1 = A \sin(\omega t - kx)$, given $P_1 \propto A^2$.
   $y_2 = A \cos(\omega t - kx) = A \sin(\omega t - kx + \pi/2)$. If we do the trigonometry correctly, using $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$, we get
   $y_1 + y_2 = 2A \cos(\pi/4) \sin(\omega t - kx + \pi/4)$,* where $2A \cos(\pi/4)$ is the new amplitude. The new power is thus
   $P' \propto [2A \cos(\pi/4)]^2 = 4(1/\sqrt{2})^2 A^2$ and $P' = 2P$.
   (*This can be checked for reasonableness for the cases when y_1 and y_2 are exactly in phase and out of phase.)

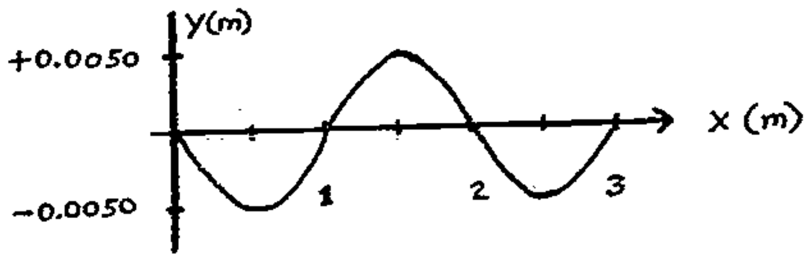


Figure 21

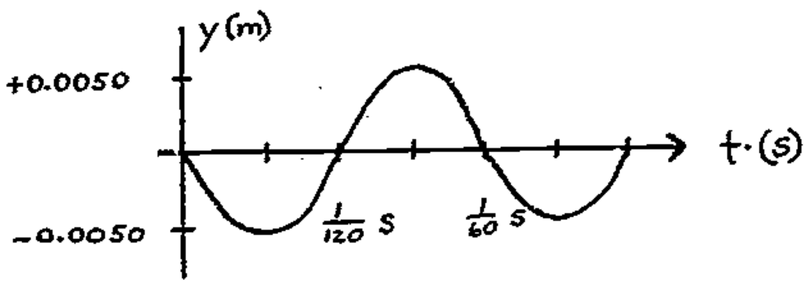


Figure 22