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IDENTIFIERS	Keller Plan; *Personalized System of Instruction; PSI; Vectors

ABSTRACT

This is part of a series of 42 Calculus Based Physics (CBP) modules totaling about 1,000 pages. The modules include study guides, practice tests, and mastery tests for a full-year individualized course in calculus-based physics based on the Personalized System of Instruction (PSI). The units are not intended to be used without outside materials; references to specific sections in four elementary physics textbooks appear in the modules. Specific modules included in this document are: Module 3--Planar Motion, Module 4--Newton's Laws, and Module 5--Vector Multiplication. (CP)

STUDY MODULES FOR CALCULUS-BASED GENERAL PHYSICS*

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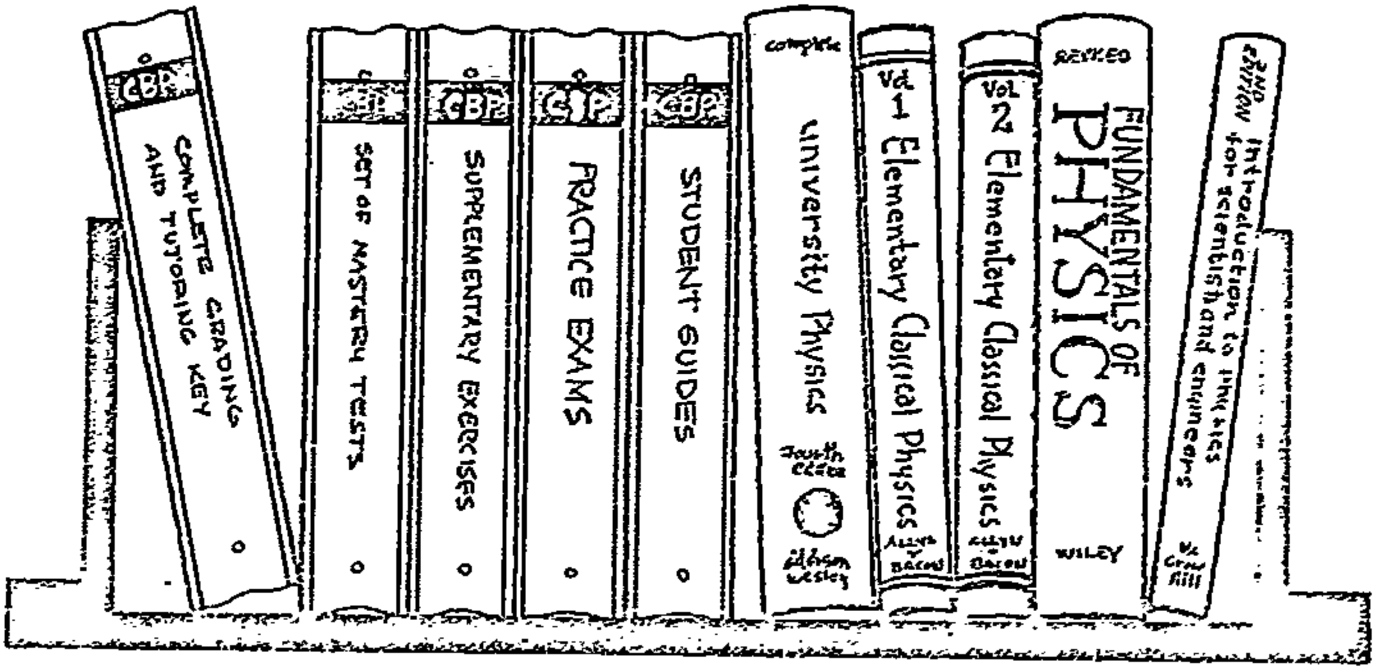

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Comments

These modules were prepared by fifteen college physics professors for use in self-paced, mastery-oriented, student-tutored, calculus-based general physics courses. This style of teaching offers students a personalized system of instruction (PSI), in which they increase their knowledge of physics and experience a positive learning environment. We hope our efforts in preparing these modules will enable you to try and enjoy teaching physics using PSI.

Robert G. Fuller
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These modules were prepared by the module authors at a College Faculty Workshop held at the University of Colorado - Boulder, from June 23 to July 11, 1975.

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COMMENT TO USERS

In the upper right-hand corner of each Mastery Test you will find the "pass" and "recycle" terms and a row of numbers "1 2 3 ..." to facilitate the grading of the tests. We intend that you indicate the weakness of a student who is asked to recycle on the test by putting a circle around the number of the learning objective that the student did not satisfy. This procedure will enable you easily to identify the learning objectives that are causing your students difficulty.

ERRATA

Planar Motion: p. 8, D(3)(c) "not changing (since v is perpendicular to a)." In Newton's Laws, on p. 6, Problem C, the heading "Solution" was left out before "From Newton's second law... ." In the Grading Key for Mastery Test C, Eq. (3) in Problem 3, the subscript on f and μ should be k, not R.

In Vector Multiplication, on p. 2(8ueche 1), the Reading for Objective 1 in the Table should be "...(last paragraph of Sec. 8.1)." On p. 9, the Solution to Problem A, part (a) should read

$$\vec{D} \cdot \vec{E} = D_x E_x + D_y E_y = (1)(4)\ m^2 + (3)(-5)\ m^2 + (-2)(1)\ m^2$$

$$= (4 - 15 - 2)\ m^2 = -13\ m^2.$$

In part (b), cos $\theta = -0.54$. The minus was left out. In the Practice Test on p. 11, the answer to Problem 2 should have absolute value signs around the $\vec{A} \times \vec{B}$, and the answer is 245°. In the Grading Key to Mastery Test B, the Solution to Problem 2 should be 37 cm^2.

We shall correct these and any other errors brought to our attention when the CBP Modules are reprinted. We would be happy to receive your suggestions or any corrections that you discover necessary in using the modules.

COMMENT TO USERS

It is conventional practice to provide several review modules per semester or quarter, as confidence builders, learning opportunities, and to consolidate what has been learned. You the instructor should write these modules yourself, in terms of the particular weaknesses and needs of your students. Thus, we have not supplied review modules as such with the CBP Modules. However, fifteen sample review tests were written during the Workshop and are available for your use as guides. Please send $1.00 to CBP Modules, Behlen Lab of Physics, University of Nebraska - Lincoln, Nebraska 68588.

FINIS

This printing has completed the initial CBP project. We hope that you are finding the materials helpful in your teaching. Revision of the modules is being planned for the Summer of 1976. We therefore solicit your comments, suggestions, and/or corrections for the revised edition. Please write or call

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PLANAR MOTION

INTRODUCTION

Enough of this physics where things move along straight lines only! We know that most interesting real-life motions involve curves of many and varied shapes. This module extends your understanding of kinematics from one dimension to two dimensions. To accomplish this, you will combine your knowledge of calculus and vectors with concepts like position, displacement, velocity, speed, and acceleration.

Two important applications that will be utilized many times in later modules are covered here. First is the motion of a particle experiencing constant acceleration, e.g., a baseball in flight. Second is the motion of a particle in a circular path with a constant speed, e.g., an earth satellite in circular orbit.

PREREQUISITES

Before you begin this module, you should be able to:	Location of Prerequisite Content
*Use vector algebra in the following operations (needed for Objectives 1 through 5 of this module): Addition; Multiplication by a scalar; Unit vectors; Magnitude of a vector; Scalar product	Dimensions and Vector Addition Module
*Differentiate polynomial, sine, and cosine functions (needed for Objectives 2, 4, and 5 of this module)	Calculus Review
*Use the chain rule for derivatives (needed for Objectives 2, 4, and 5 of this module)	Calculus Review
*Solve kinematics problems in one dimension (needed for Objectives 1 through 5 of this module)	Rectilinear Motion Module
*Compute angles in radians (needed for Objectives 1, 2, 4, and 5 of this module)	Trigonometry Review

LEARNING OBJECTIVES

After you have mastered the content of this module, you will be able to:

1. Graphing the trajectory - Given a particle's time-dependent position vector $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$, draw its path in the plane.

2. Velocity, speed, and acceleration - Given $\vec{r}(t)$, calculate velocity $\vec{v}(t)$, speed $v(t)$, and acceleration $\vec{a}(t)$.

3. Interpreting velocity and acceleration - Given a particle's position $\vec{r}$, velocity $\vec{v}$, and acceleration $\vec{a}$ at a specified time, determine whether at this instant:
   (a) its distance from the origin is increasing, decreasing, or not changing;
   (b) $\vec{r}$ is turning clockwise, counterclockwise, or not turning;
   (c) its speed is increasing, decreasing, or not changing;
   (d) $\vec{v}$ is turning clockwise, counterclockwise, or not turning.

4. Projectiles - Given that a particle moves with constant acceleration in two dimensions, solve problems involving position, velocity, acceleration, and time.

5. Uniform circular motion - Given that a particle moves in a circular path at a constant speed, solve problems involving position, velocity, acceleration, and time.

GENERAL COMMENTS

This study guide may be different from others in that a large portion of your studying will be done in the Problem Set. Each of the objectives is discussed in some detail in the 14 problems. Seven of these problems develop the basic ideas and present detailed solutions to typical problems. The remaining seven represent challenges for your attention.

Your text will be used to provide supplementary readings and problems. If you have a calculus text, you will also find it helpful.

TEXT: Frederick J. Bueche, Introduction to Physics for Scientists and Engineers (McGraw-Hill, New York, 1975), second edition

SUGGESTED STUDY PROCEDURE

Your primary reading for this module will be Sections 4.1, 4.2, and 4.8 of Chapter 4 and Section 10.6 of Chapter 10. For Objective 1, first study Problem A and the material preceding it, then work Problem H. Next review the definitions of velocity and acceleration in Sections 4.1 and 4.2 of the text (for Objective 2: calculating velocity, speed, and acceleration in two dimensions). Study Problems B and C, with their explanatory material, before working Problems I and J. Then study the section Interpreting Velocity and Acceleration and Problem D. Read Section 4.8 of the text before studying Problems E and F and working Problem K. Next study Problem G, with its preliminary material, before working Problems L, M, and N.

Study the text, Chapter 10, Section 10.6 and the first two paragraphs of Section 10.7. Don't let angular speed ω scare you. It's just the rate (in radians per second) at which the angle θ of Figure 10.7 is changing. You have already seen ω in Problem G. Do the Practice Test before attempting the Mastery Test.

BUECHE

Objective Number	Readings	Problems with Solutions: Study Guide	Assigned Problems: Study Guide	Assigned Problems: Text	Additional Problems
1		A	H		
2	Sec. 4.1 Sec. 4.2	B, C	I, J		
3		D	O		
4	Sec. 4.8	E, F	K	Chap. 4: Problem 17	Chap. 4: Problems 19, 21, 25
5	Sec. 10.6	G	L, M, N		

TEXT: David Halliday and Robert Resnick, Fundamentals of Physics (Wiley, New York, 1970; revised printing, 1974)

SUGGESTED STUDY PROCEDURES

Study Trajectories and Problem A before working Problem H. Then study the text, Chapter 3, Sections 3-3, 3-4, 3-6 and Chapter 4, Section 4-1, along with the sections Velocity and Speed and Acceleration of this module. Study Problems B and C and work Problems I and J for Objective 2. Read Interpreting Velocity and Acceleration before working through Problem O.

For Objectives 4 and 5, study Sections 4-2 to 4-4 of the text, and Problems E, F, and G of the module along with their preliminary material. Then work Problems K through N along with Problems 37 and 39 in Chapter 4 of the text. Do the Practice Test before attempting the Mastery Test.

HALLIDAY AND RESNICK

		Problems with Solutions	Assigned Problems		
Objective Number	Readings	Study Guide	Study Guide	Text	Additional Problems
1		A	H		
2	Secs. 3-3, 3-4, 3-6, 4-1	B, C	I, J		Chap. 4: Problems 3, 5
3		D	D		
4	Secs. 4-2, 4-3	E, F	K		Chap. 4: Problems 9, 11, 13
5	Sec. 4-4	G	L, M, N	Chap. 4: Problems 37, 39	

TEXT: Francis Weston Sears and Mark W. Zemansky, University Physics (Addison-Wesley, Reading, Massachusetts, 1970), fourth edition

SUGGESTED STUDY PROCEDURE

Study Trajectories and Problem A and work Problem H. Then study the text, Chapter 6, Sections 6-1 through 6-3, and Problems B and C along with their accompanying material, before working Problems I and J. Study Problem D. Next study the text, Section 6-5, and Problems E and F (Objective 4) before working Problems 6-1 and 6-3 of the text, Problem K of this module. Read Section 6-6 and study Uniform Circular Motion and Problem G before working Problems 6-27 and L to N.

Try the Practice Test before doing the Mastery Test.

SEARS AND ZEMANSKY

Objective Number	Readings	Problems with Solutions: Study Guide	Assigned Problems: Study Guide	Assigned Problems: Text
1		A	H	
2	Secs. 6-1, 6-2, 6-3	B, C	I, J	
3		D	D	
4	Sec. 6-5	E, F	K	6-1, 6-3
5	Sec. 6-6	G	L, M, N	6-27

TEXT: Richard T. Weidner and Robert L. Sells, Elementary Classical Physics (Allyn and Bacon, Boston, 1973), second edition, Vol 1

SUGGESTED STUDY PROCEDURE

Study Trajectories and Problem A and work Problem H. Then read the text, Chapter 4, Section 4-1, and study Problems B and C before working Problems I and J. Next study Interpreting Velocity and Acceleration and work through Problem D. Then read Sections 4-2 and 4-3 and study Motion with Constant Acceleration: Projectiles with Problems E and F before working Problems 4-7, 4-9 (text) and K on your own. Read Section 4-4 (text) and study Problem G with its discussion before working Problems L, M, N, and 4-23 of the text.

Try the Practice Test before working the Mastery Test.

WEIDNER AND SELLS

		Problems with Solutions	Assigned Problems	
Objective Number	Readings	Study Guide	Study Guide	Text
1		A	H	
2	Sec. 4-1	B, C	I, J	
3		D	D	
4	Secs. 4-2, 4-3	E, F	K	4-7, 4-9
5	Sec. 4-4	G	L, M, N	4-23

PROBLEM SET WITH SOLUTIONS

Trajectories

As a particle moves in a plane it traces out a curve, or path. A knowledge of this path and of the time at which the particle passed through each point constitutes a knowledge of the particle's trajectory.

At each instant the particle's position is specified by its position vector:

$$\vec{r} = x\hat{i} + y\hat{j}. \tag{1}$$

As the particle moves x, y, and $\vec{r}$ change. This time dependence is emphasized by writing Eq. (1) as

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}. \tag{2}$$

A vector equation such as (2) is a compact method for representing a trajectory in two dimensions.

A(1). A particle's position is given by

$$\vec{r}(t) = [(6-t^2)\hat{i} + (2t)\hat{j}]\ \text{m},$$

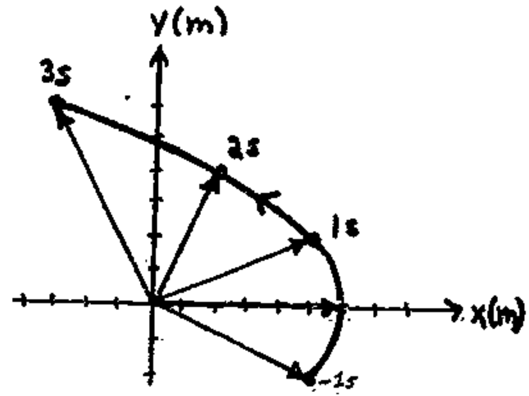
where t is measured in seconds.

(a) Draw the particle's trajectory for $-1 \leq t \leq 3$ s.
(b) Calculate the particle's distance from the origin at t = 2 s.
(c) Determine the particle's distance from the origin as a function of time.

Solution

(a) First evaluate $\vec{r}$ for

$t = -1$ s, $\vec{r} = (5\hat{i}-2\hat{j})$ m,
$t = 0$, $\vec{r} = (6\hat{i})$ m,
$t = 1$ s, $\vec{r} = (5\hat{i}+2\hat{j})$ m,
$t = 2$ s, $\vec{r} = (2\hat{i}+4\hat{j})$ m,
$t = 3$ s, $\vec{r} = (-3\hat{i}+6\hat{j})$ m;



Plot these vectors and connect their tips by a smooth curve. The arrow along the path between t = 1 s and t = 2 s is used to indicate the direction of motion along the path.

(b) The distance from the origin to the particle at t = 2 s is just the magnitude of $\vec{r}$ at this time. Thus,

$$r = \sqrt{2^2 + 4^2} = 2\sqrt{5}\ \text{m}.$$

(c) At any time t the distance from the origin to the particle is

$$r(t) = \sqrt{(6-t^2)^2 + (2t)^2} = \sqrt{36-8t^2+t^4}\ \text{m}.$$

Velocity and Speed

The <u>velocity</u> $\vec{v}$ for a particle is defined as the time derivative of the particle's position vector, that is,

$$\vec{v} = d\vec{r}/dt. \tag{3}$$

If $\vec{r}$ is written in component form, i.e.,

$$\vec{r} = x\hat{i} + y\hat{j},$$

then

$$\vec{v} = (dx/dt)\hat{i} + (dy/dt)\hat{j}, \tag{4}$$

so that

$$v_x = dx/dt, \qquad v_y = dy/dt. \tag{5}$$

The particle's speed v at any instant is defined as the magnitude of its velocity at that time,

$$v = \sqrt{v_x^2 + v_y^2}\ . \tag{6}$$

Speed, a scalar quantity, measures the time rate of change of distance traveled by the particle.

At each instant the velocity vector (if drawn with its initial point on the particle) is tangential to the particle's path. Thus, the magnitude of the velocity indicates "how fast" the particle is moving, whereas its direction points in the direction of motion of the particle.

B(2). For

$$\vec{r}(t) = [(6-t^2)\hat{i} + (2t)\hat{j}]\ \text{m},$$

as in Problem A,
(a) determine the velocity and speed as functions of time;
(b) draw the trajectory for $-1 \leq t \leq 3$ s and show $\vec{v}$ at t = 0 and t = 2 s.

Solution

(a) From the definition of velocity,

$$\vec{v}(t) = d\vec{r}/dt = (-2t\hat{i} + 2\hat{j})\ \text{m/s};$$

and thus the speed is

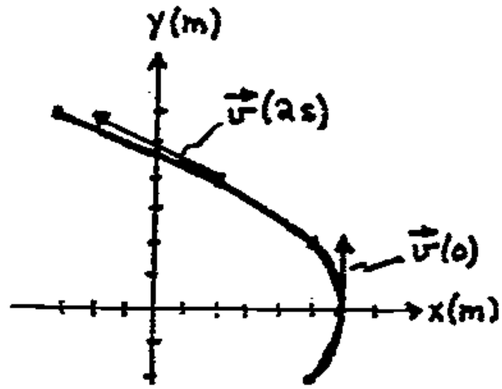
$$v(t) = \sqrt{(-2t)^2 + 2^2}\ \text{m/s} = 2\sqrt{t^2 + 1}\ \text{m/s}.$$

(b) The trajectory was drawn in Problem A. Now

$$\vec{v}(0) = 2\hat{j}\ \text{m/s},$$

$$\vec{v}(2\ \text{s}) = (-4\hat{i} + 2\hat{j})\ \text{m/s}.$$

(Notice that the velocity vectors are tangential to the path.)



Acceleration

The acceleration $\vec{a}$ for a particle is defined as the time derivative of the particle's velocity, i.e.,

$$\vec{a} = d\vec{v}/dt. \tag{7}$$

If $\vec{v}$ is written in component form:

$$\vec{v} = v_x\hat{i} + v_y\hat{j},$$

then

$$\vec{a} = (dv_x/dt)\hat{i} + (dv_y/dt)\hat{j}, \tag{8}$$

so that

$$a_x = dv_x/dt, \qquad a_y = dv_y/dt. \tag{9}$$

C(2). For

$$\vec{r}(t) = [(6-t^2)\hat{i} + (2t)\hat{j}]\ \text{m},$$

as in Problems A and B, determine $\vec{a}(t)$, $a_x(t)$, $a_y(t)$.

Solution

From Problem B,

$$\vec{v}(t) = (-2t\hat{i} + 2\hat{j})\ \text{m/s},$$

so that

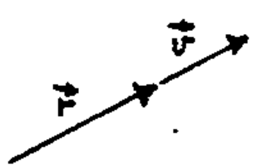
$$\vec{a}(t) = d\vec{v}/dt = (-2\hat{i})\ \text{m/s}^2.$$

Hence

$$a_x(t) = -2\ \text{m/s}^2 \qquad \text{and} \qquad a_y(t) = 0.$$

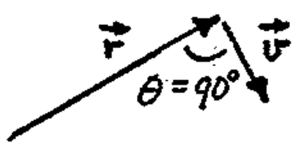
Interpreting Velocity and Acceleration

Remember that the velocity is defined as the time derivative of position. Since position is a vector, it may change by changing magnitude only, by changing direction only, or by changing both magnitude and direction. If the position vector changes its magnitude only, then the velocity vector is parallel to $\vec{r}$. If $\vec{r}$ changes direction only, then $\vec{v}$ is perpendicular to $\vec{r}$. Examples are shown below.

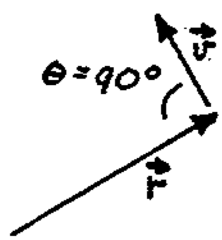


$\vec{r}$ increasing in length but not changing direction.

$\vec{r}$ decreasing in length but not changing direction.

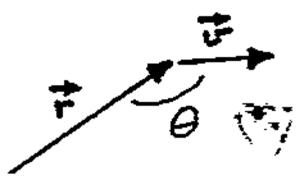


$\vec{r}$ turning clockwise but not changing length.

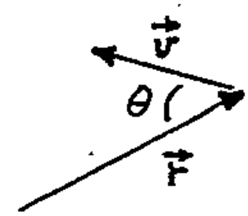


$\vec{r}$ turning counterclockwise but not changing length.

If $\vec{r}$ is both changing in magnitude and turning, then $\vec{v}$ will have components parallel to $\vec{r}$ and perpendicular to $\vec{r}$. Examples are shown below.

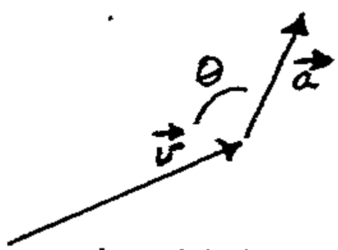


$\vec{r}$ increasing in length and turning clockwise.

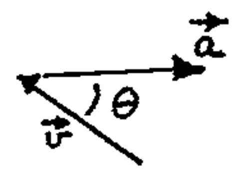


$\vec{r}$ decreasing in length and turning counterclockwise.

Since acceleration is obtained from velocity in precisely the same manner that velocity is determined from position, similar conclusions can be deduced about $\vec{v}$ from a knowledge of $\vec{a}$. Examples are shown below.



Speed increasing, $\vec{v}$ turning counterclockwise.



Speed decreasing, $\vec{v}$ turning clockwise.

D(3). At a certain instant a particle's position, velocity, and acceleration are

$$\vec{r} = 4\hat{i}\ \text{m}, \qquad \vec{v} = (-2\hat{i} + 3\hat{j})\ \text{m/s}, \qquad \vec{a} = (6\hat{i} + 4\hat{j})\ \text{m/s}^2.$$

(a) Is the distance from the origin to the particle increasing, decreasing, or not changing at this time?
(b) How is $\vec{r}$ turning at this instant?
(c) Is the particle's speed increasing, decreasing, or not changing at this instant?
(d) How is $\vec{v}$ turning at this instant?

<u>Solution</u>

(a) Decreasing;
(b) counterclockwise;
(c) not changing (since $\vec{v}$ is perpendicular to $\vec{r}$);
(d) clockwise.

Motion with Constant Acceleration: Projectiles

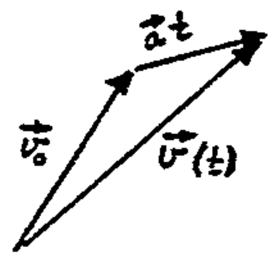
Suppose an object is known to move with a constant acceleration $\vec{a}$. Then

$$d\vec{v}/dt = \vec{a} = \text{const}; \tag{10}$$

and, just as with rectilinear motion,

$$\vec{v}(t) = \vec{v}_0 + \vec{a}t \qquad (\vec{a} = \text{const}), \tag{11}$$

where $\vec{v}$ is the velocity at time t and $\vec{v}_0$ is the velocity at $t = 0$. This equation for velocity is given graphical display below.



As t increases the $\vec{a}t$ term increases proportionately in magnitude without changing direction.

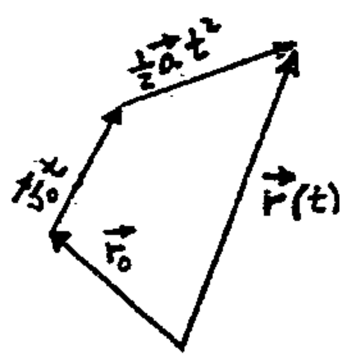
The position vector is obtained from

$$d\vec{r}/dt = \vec{v} = \vec{v}_0 + \vec{a}t. \tag{12}$$

Again, just as in rectilinear motion,

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t + (1/2)\vec{a}t^2 \qquad (\vec{a} = \text{const}), \tag{13}$$

where $\vec{r}$ is the position vector at time t and $\vec{r}_0$ is the position at $t = 0$. This equation is also graphically displayed below.



A word of caution is in order here. Equations (11) and (13) for $\vec{v}(t)$ and $\vec{r}(t)$ were obtained from an assumption of <u>constant acceleration</u> and <u>should be used only in such a circumstance.</u>

Perhaps the most familiar constant-acceleration phenomenon is that of a body in free fall near the earth's surface. If air resistance and other small effects are neglected, such an object moves with the downward acceleration

$$\vec{a} = -g\,\hat{j},$$

where $g = 9.8\ m/s^2$ and $\hat{j}$ is a unit vector pointing vertically upward.

E(4). A ball is projected from ground level above flat land with an initial velocity $\vec{v}_0 = v_{0x}\hat{i} + v_{0y}\hat{j}$ where $v_{0x} = 30.0$ m/s and $v_{0y} = 50$ m/s. Assuming that $\vec{r} = 0$ at $t = 0$, determine

(a) its velocity $\vec{v}(t)$,

(b) its position $\vec{r}(t)$.

Solution

(a) Since $\vec{a}$ ($a_x = 0$, $a_y = -g$) is constant,

$$\vec{v} = \vec{v}_0 + \vec{a}t = v_{0x}\hat{i} + v_{0y}\hat{j} + (-g\hat{j})t$$
$$= v_{0x}\hat{i} + (v_{0y} - gt)\hat{j}.$$

Substituting numerical values:

$$\vec{v} = [30.0\hat{i} + (50 - 9.8t)\hat{j}]\ m/s.$$

(b) Again, since $\vec{a}$ is constant,

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + (1/2)\,\vec{a}t^2$$
$$= (x_0\hat{i} + y_0\hat{j}) + (v_{0x}\hat{i} + v_{0y}\hat{j})t + (1/2)(-g\hat{j})t^2$$
$$= (x_0 + v_{0x}t)\hat{i} + [y_0 + v_{0y}t - (1/2)gt^2]\hat{j}.$$

Substituting numerical values, we find

$$\vec{r} = [(30.0t)\hat{i} + (50t - 4.9t^2)\hat{j}]\ m.$$

Comment. From the equations for $\vec{v}$ and $\vec{r}$,

$$v_x = v_{0x} = 30.0\ m/s, \qquad x = x_0 + v_{0x} = (30.0t)\ m.$$

Since the acceleration has a zero x component, the x coordinate is identical to that of a particle moving along the x axis with a constant speed of 30 m/s. Now consider the y components of $\vec{v}$ and $\vec{r}$:

$$v_y = v_{0y} - gt = (50 - 9.8t)\ \text{m/s},$$

$$y = y_0 + v_{0y}t - (1/2)gt^2 = (50t - 4.9t^2)\ \text{m}.$$

Not surprisingly the y coordinate is identical to that of a particle projected vertically with an initial speed of 50 m/s.

F(4). For the ball of Problem E,
(a) draw its flight path;
(b) determine its maximum height;
(c) determine its horizontal range (the distance between the origin and the point where the ball returns to its original height).

Solution

(a) From the previous problem,

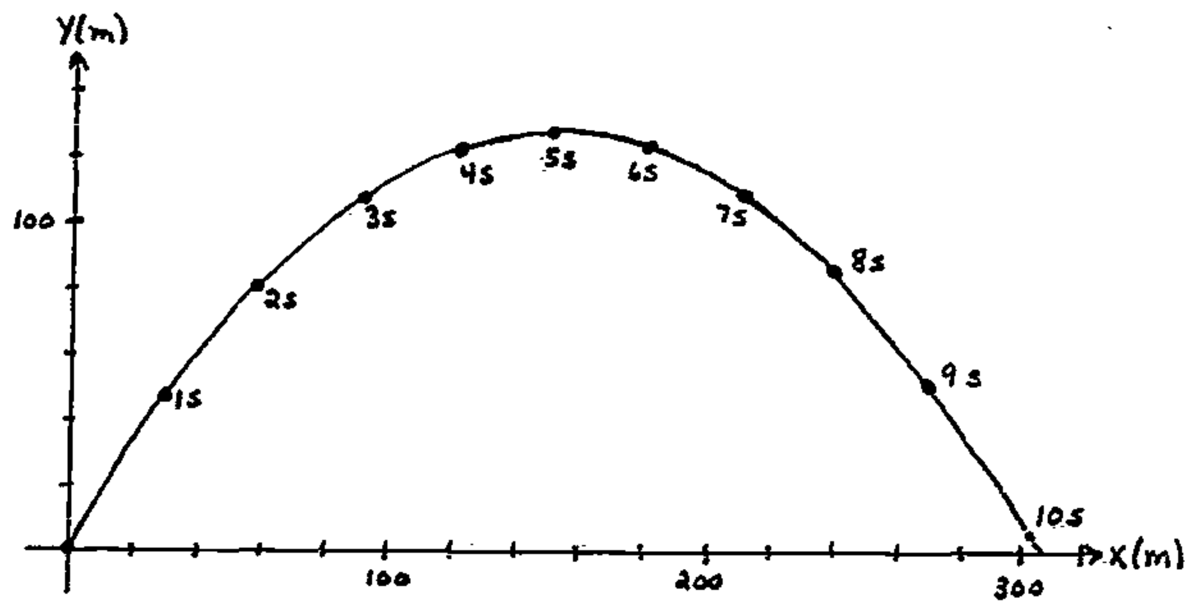
$$\vec{r}(t) = [(30.0t)\hat{i} + (50t - 4.9t^2)\hat{j}]\ \text{m}.$$

Evaluate $\vec{r}$ for integral values of $t \geq 0$, e.g.,

$$\vec{r}(1.00\ \text{s}) = 30.0\hat{i} + (50 - 49)\hat{j} = (30.0\hat{i} + 45\hat{j})\ \text{m},$$

$$\vec{r}(10.0\ \text{s}) = 300\hat{i} + (500 - 490)\hat{j} = (300\hat{i} + 10\hat{j})\ \text{m}.$$

Graphing these gives



(b) From the graph the maximum height is apparently near 130 m. To determine this value accurately, use the fact that at the point of maximum height, the y component of the velocity is necessarily zero:

$$v_y = v_{0y} - gt = 0.$$

Thus the time t_{max} for this maximum height is

$$t_{max} = \frac{v_{0y}}{g} = \frac{50\ m/s}{9.8\ m/s^2} = 5.1\ s.$$

The maximum height is obtained by substituting this value for t into the expression for y, i.e.,

$$y_{max} = v_{0y}t_{max} - \frac{1}{2}gt_{max}^2 = \frac{v_{0y}^2}{g} - \frac{1}{2}g\left(\frac{v_{0y}}{g}\right)^2 = \frac{v_{0y}^2}{2g}.$$

Substituting numerical values gives

$$y_{max} = \frac{50^2\ m^2/s^2}{2(9.8)\ m/s^2} = 128\ m.$$

(c) The horizontal range is just the x coordinate when the ball returns to its initial height. From the graph this distance is apparently slightly greater than 300 m. To determine this value more accurately, first determine the time t_R when the ball strikes the ground:

$$y = 0 = v_{0y}t_R - (1/2)gt_R^2 = t_R[v_{0y} - (1/2)gt_R].$$

Of the two solutions to this quadratic equation, the solution $t_R = 0$ is discarded (why?), and thus

$$t_R = 2v_{0y}/g.$$

(Notice that this time of flight is exactly twice the time to achieve the maximum height.) The horizontal range R is obtained by substituting t_R into the x equation:

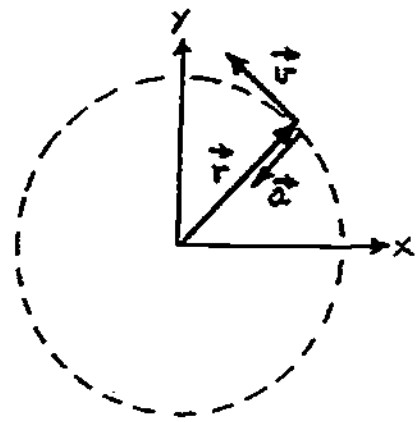
$$R = v_{0x}t_R = \frac{2v_{0x}v_{0y}}{g} = \frac{2(30.0\ m/s)(50\ m/s)}{9.8\ m/s^2} = 310\ m.$$

Uniform Circular Motion

First look back over your solutions of Problems H, I, and J. In these problems you considered the trajectory given by

$$\vec{r}(t) = 2[\cos(\pi t/4)\hat{i} + \sin(\pi t/4)\hat{j}]\ m.$$

In Problem H you might have noted that this trajectory is a circle of radius 2 m centered on the origin. Problem I revealed that the speed is constant and equal to $(\pi/2)$ m/s. Finally, in Problem J you found that the acceleration is constant in magnitude $[(\pi^2/8)\ \mathrm{m/s^2}]$ and always opposite to $\vec{r}$, i.e., $\vec{a}$ always points toward the center of the circle for uniform circular motion. This situation is depicted in the diagram at the right.



As you might guess, circular motion at a constant speed can be described by

$$\vec{r}(t) = R[(\cos \omega t)\hat{i} + (\sin \omega t)\hat{j}],$$

where R and ω are constants.

G(5). Using this expression for $\vec{r}(t)$, show that

(a) $r(t) = R$ (constant);

(b) $\vec{v}(t) = \omega R[-(\sin \omega t)\hat{i} + (\cos \omega t)\hat{j}]$;

(c) $v(t) = \omega R$ (constant);

(d) $\vec{a}(t) = -\omega^2 R[(\cos \omega t)\hat{i} + (\sin \omega t)\hat{j}] = -\omega^2 \vec{r}$;

(e) $a(t) = +\omega^2 R = v^2/R$ (constant).

Solution

(a) Since

$$\vec{r}(t) = \sqrt{(R \cos \omega t)^2 + (R \sin \omega t)^2} = R,$$

R (the magnitude of $\vec{r}$) = radius of circle.

(b)
$$\vec{v}(t) = d\vec{r}/dt$$
$$= R[(d/dt)(\cos \omega t)\hat{i} + (d/dt)(\sin \omega t)\hat{j}]$$
$$= \omega R[(-\sin \omega t)\hat{i} + (\cos \omega t)\hat{j}].$$

[Don't forget the chain rule, i.e., $df(u)/dx = (df/du)(du/dx)$.]

(c)
$$v(t) = \sqrt{(\omega R \sin \omega t)^2 + (\omega R \cos \omega t)^2} = \omega R.$$

Thus, the particle's speed is constant and $\omega = v/R$.

(d) $\vec{a}(t) = d\vec{v}/dt$

$$= \omega R[-(d/dt)(\sin \omega t)\hat{i} + (d/dt)(\cos \omega t)\hat{j}]$$

$$= -\omega^2 R[(\cos \omega t)\hat{i} + (\sin \omega t)\hat{j}]$$

$$= -\omega^2 \vec{r}(t).$$

Thus $\vec{a}$ is at all times opposite to $\vec{r}$; and therefore, if drawn starting on the particle, it always points toward the center of the circle.

(e) $a(t) = \sqrt{(\omega^2 R \cos \omega t)^2 + (\omega^2 R \sin \omega t)^2} = \omega^2 R = v^2/R.$

In summary, if a particle moves in a circular path of radius R at a constant speed v, its acceleration is radially inward and of magnitude v^2/R. This acceleration is usually called centripetal acceleration.

PROBLEMS

H(1). Draw the paths for each of the following trajectories:

(a) $\vec{r} = (t^2\hat{i} + 2t\hat{j})$ m, $\quad -1 \le t \le 2$ s;

(b) $\vec{r} = [2 \cos (\pi t/4)\hat{i} + 2 \sin (\pi t/4)\hat{j}]$ m, $\quad 0 \le t \le 4$ s.

I(2). (a) Determine $\vec{v}(t)$ and $v(t)$ for

$$\vec{r}(t) = [2 \cos (\pi t/4)\hat{i} + 2 \sin (\pi t/4)\hat{j}] \text{ m}.$$

Hint: Don't forget that $d(\cos \omega t)/dt = -\omega \sin \omega t$ if ω is constant. Similarly $d(\sin \omega t)/dt = \omega \cos \omega t$.

(b) What is the significance of the result for $v(t)$?

J(2). Using the position function

$$\vec{r}(t) = [2 \cos (\pi t/4)\hat{i} + 2 \sin (\pi t/4)\hat{j}] \text{ m},$$

determine $\vec{a}(t)$, $a(t)$, $a_x(t)$, $a_y(t)$. Show that for this motion the acceleration vector always points in the opposite direction to the position vector.

K(4). A cannon with a muzzle speed of 800 m/s fires at a target at a horizontal distance of 30 000 m. What initial angles of elevation will ensure success? Sketch the two paths. Hint: $2 \sin \theta \cos \theta = \sin 2\theta$.

L(5). Write an equation for a circular trajectory such that the radius is 1.5 m and the speed is 6 m/s. Determine the magnitude of the centripetal acceleration.

M(5). Consider the trajectory

$$\vec{r}(t) = 1.5[(-\sin 4t)\hat{i} + (\cos 5t)\hat{j}]\ \text{m}.$$

Show that this is circular motion at a constant speed. Determine r, v, and a. How is this trajectory different from that of Problem L?

N(5). Consider the trajectory

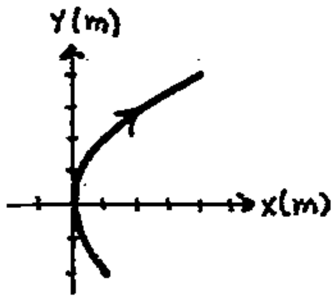
$$\vec{r}(t) = 1.5[(\cos 4t)\hat{i} - (\sin 4t)\hat{j}]\ \text{m}.$$

How is it different from those of Problems L and M?

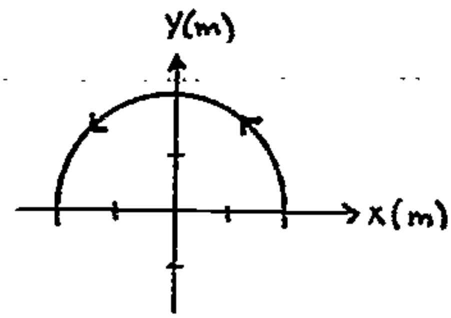
Solutions

H(1).

(a)



(b)



I(2).

$$\vec{v}(t) = (\pi/2)[-\sin(\pi t/4)\hat{i} + \cos(\pi t/4)\hat{j}]\ \text{m/s},$$

$$v(t) = (\pi/2)\ \text{m/s},$$

i.e., the speed is constant, but the velocity is not (why?).

J(2). $\vec{a} = -(\pi^2/8)[\cos(\pi t/4)\hat{i} + \sin(\pi t/4)\hat{j}]$ m/s^2,

$a = (\pi^2/8)$ m/s^2,

$a_x = -(\pi^2/8)\cos(\pi t/4)$ m/s^2,

$a_y = -(\pi^2/8)\sin(\pi t/4)$ m/s^2.

Since $\vec{a} = -(\pi^2/16)\vec{r}$, $\vec{a}$ always points in the opposite direction to $\vec{r}$.

K(4). $\theta_0 = 13.7°$; $\theta_0 = 76°$.

L(5). $\vec{r}(t) = 1.5[(\cos 4t)\hat{i} + (\sin 4t)\hat{j}]$ m,

$a = 24$ m/s^2.

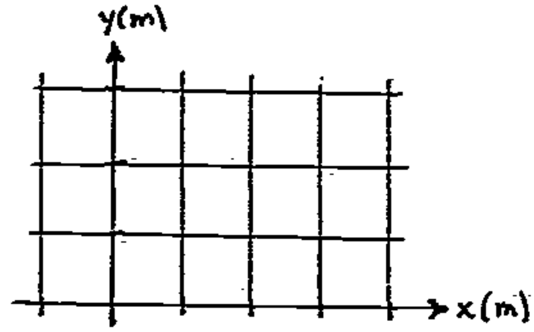
M(5). $r = 1.5$ m, $v = 6$ m/s, $a = 24$ m/s^2.

Problem M's trajectory has t = 0 displacement of $1.5\hat{i}$ m, but for this trajectory $r = 1.5\hat{j}$ m at t = 0. Thus the M trajectory is "one-quarter of a lap" ahead of the N trajectory.

N(5). The radius, speed, and acceleration (magnitude) are the same, but the particle starts at t = 0 at $\vec{r} = (1.5\hat{i})$ m and moves clockwise rather than counterclockwise as in L and M.

PRACTICE TEST

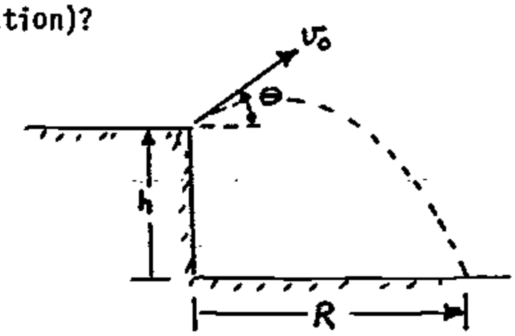
1. Draw the trajectory
   $\vec{r}(t) = [t\hat{i} + 2\sin(\pi t/2)\hat{j}]$m
   for $0 \le t \le 2$ s.



2. Determine the velocity, acceleration, and speed at $t = 1$ s for the trajectory of Problem 1.

3. At an instant when $\vec{r} = (2\hat{i} + 3\hat{j})$ m, $\vec{v} = (6\hat{i} - 4\hat{j})$ m/s, $\vec{a} = 6\hat{j}$ m/s^2:
   (a) How is $\vec{r}$ changing (magnitude and direction)?
   (b) How is $\vec{v}$ changing (magnitude and direction)?

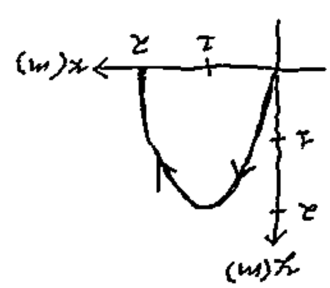
4. A ball is thrown from the cliff as shown. If $v_0 = 20.0$ m/s, $\theta = 30.0°$, and $h = 40$ m,
   (a) When does the ball strike the ground?
   (b) Calculate R.



5. Write an equation for a constant-speed circular trajectory such that
   (a) the centripetal acceleration has a magnitude of 32.0 m/s^2;
   (b) the motion is clockwise; and
   (c) at $t = 0$, $\vec{v} = 4.0\hat{j}$ m/s.

Practice Test Answers

1.



2. $\vec{v}(1\text{ s}) = \hat{i}$ m/s. $\vec{a}(1\text{ s}) = -(\pi^2/2)\hat{j}$ m/s^2.
   $v(1\text{ s}) = 1$ m/s.

3. (a) r not changing, $\vec{r}$ turning clockwise.
   (b) v decreasing, $\vec{v}$ turning counterclockwise.

4. (a) 4.1 s. (b) 70 m.

5. $\vec{r}(t) = (1/2)[-(\cos 8.0t)\hat{i} + (\sin 8.0t)\hat{j}]$ m.

Date______

pass recycle

1 2 3 4 5

PLANAR MOTION

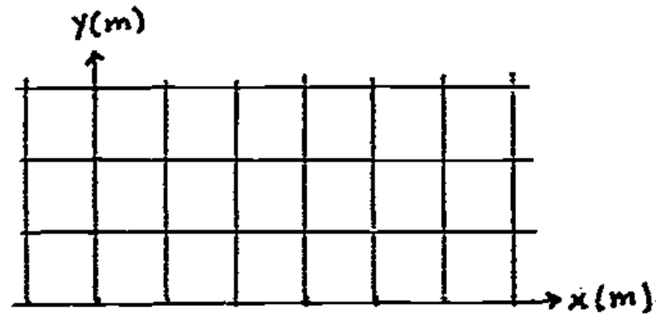
Mastery Test Form A

Name______ Tutor______

1. Draw the path

$\vec{r}(t) = [(4t/\pi)\hat{i} + (2 \sin t)\hat{j}]$ m

for $0 \leq t \leq \pi$ s.



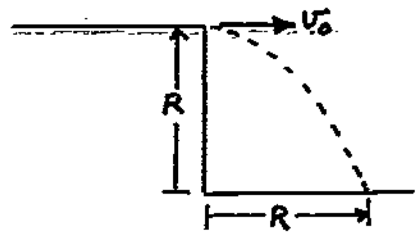
2. Determine the particle's acceleration at $t = (\pi/2)$ s (for Problem 1).

3. At an instant when $\vec{r} = 4\hat{j}$ m, $\vec{v} = (2\hat{i} - 3\hat{j})$ m/s, $\vec{a} = (-4\hat{i} + 6\hat{j})$ m/s^2:

   (a) How is $\vec{r}$ changing (magnitude and direction)?

   (b) How is $\vec{v}$ changing (magnitude and direction)?

4. A thrown rock follows the trajectory shown. Determine the initial speed v_0 if $R = 50$ m.



5. Consider the circular trajectory

   $\vec{r}(t) = 2.00[(\sin 6.0t)\hat{i} + (\cos 6.0t)\hat{j}]$ m.

   Determine
   (a) the radius,
   (b) the speed,
   (c) the magnitude of the centripetal acceleration,
   (d) the direction of motion (clockwise or counterclockwise).

Date

Pass 1 2 3 Recycle 4 5

PLANAR MOTION

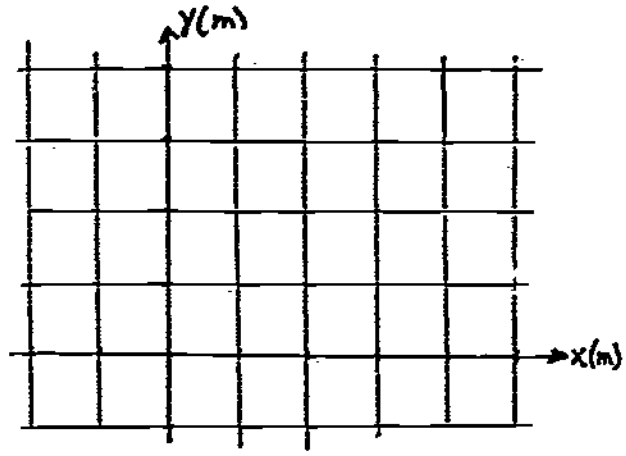
Mastery Test Form B

Name ____________________ Tutor ____________________

1. Draw the path

$\vec{r}(t) = [(1/2)t^3\hat{i} + t^2\hat{j}]$ m

for $-1 \le t \le 2$ s.



2. For the trajectory of Problem 1, determine the speed at $t = 1$ s.

3. At an instant when $\vec{r} = (-2\hat{i} + 4\hat{j})$ m, $\vec{v} = 2\hat{i}$ m/s, $\vec{a} = 6\hat{j}$ m/s^2;

(a) How is $\vec{r}$ changing (magnitude and direction)?

(b) How is $\vec{v}$ changing (magnitude and direction)?

4. A ball is thrown with an initial speed of 30.0 m/s at an angle of 60° above the horizontal.
(a) What is its acceleration when it reaches its maximum height?
(b) How far (distance) is the ball from the release point 3.00 s after release?

5. Consider the circular trajectory

$$\vec{r}(t) = 0.240[(\sin 5.0t)\hat{i} - (\cos 5.0t)\hat{j}] \text{ m.}$$

Determine
(a) the radius,
(b) the speed,
(c) the magnitude of the centripetal acceleration,
(d) the direction of motion (clockwise or counterclockwise).

Date____________________

PLANAR MOTION

Mastery Test Form C

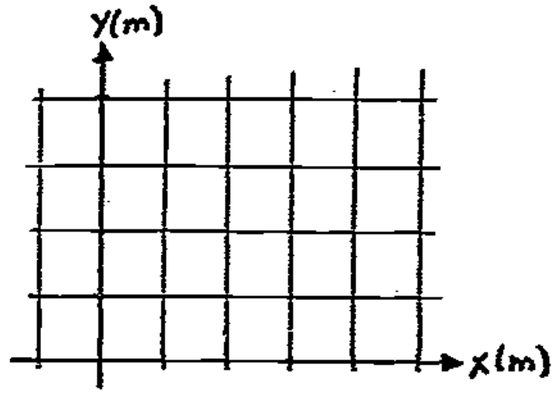
Pass 1 2 3 Recycle 4 5

Name____________________ Tutor____________________

1. Draw the path

   $\vec{r}(t) = [4\cos(\pi t/4)\hat{i} + 2t\hat{j}]$ m

   for $0 \le t \le 2$ s.



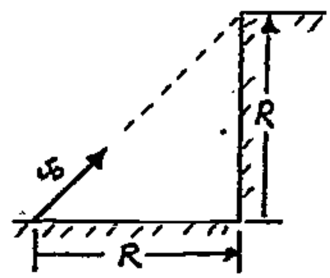
2. For the trajectory of Problem 1, determine the velocity at t = 1 s.

3. At an instant when $\vec{r} = -3\hat{i}$ m, $\vec{v} = 2\hat{j}$ m/s, $\vec{a} = (2\hat{i} - \hat{j})$ m/s^2;

   (a) How is $\vec{r}$ changing (magnitude and direction)?

   (b) How is $\vec{v}$ changing (magnitude and direction)?

4. A ball is thrown toward a wall as shown. Where will it hit if $v_0 = 20.0$ m/s and R = 20.0 m?



5. Consider the circular trajectory

   $\vec{r}(t) = 1.20[(\cos 15.0t)\hat{i} - (\sin 15.0t)\hat{j}]$ m.

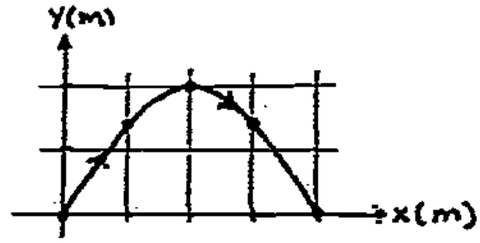
   Determine
   (a) the speed,
   (b) the magnitude of the centripetal acceleration,
   (c) the direction of motion (clockwise or counterclockwise).

MASTERY TEST GRADING KEY - Form A

What To Look For | **Solutions**

1. Correct end points; correct shape.

1. $\vec{r}(0) = 0$,
$\vec{r}(\pi/4) = (\hat{i} + \sqrt{2}\hat{j})$ m,
$\vec{r}(\pi/2) = (2\hat{i} + 2\hat{j})$ m,
$\vec{r}(3\pi/4) = (3\hat{i} + \sqrt{2}\hat{j})$ m,
$\vec{r}(\pi) = 4\hat{i}$ m.



2. Correct answer.

2. $\vec{v}(t) = d\vec{r}/dt = [(4/\pi)\hat{i} + (2\cos t)\hat{j}]$ m/s,
$\vec{a}(t) = d\vec{v}/dt = -(2\sin t)\hat{j}$ m/s^2,
$\vec{a}(\pi/2) = -2\sin(\pi/2)\hat{j} = -2\hat{j}$ m/s^2.

3. Correct answers. Ask for verbal explanation of at least one answer.

3. (a) r (magnitude) decreasing,
$\vec{r}$ turning clockwise.

(b) v (speed) decreasing,
$\vec{v}$ not turning.

4. Correct answer. v_0 in x equation, not in y equation.

4. Since $\vec{a} = -g\hat{j}$ is constant, $\vec{r} = \vec{r}_0 + \vec{v}_0 t + (1/2)\vec{a}t^2$, where $\vec{r}_0 = R\hat{j}$, $\vec{v}_0 = v_0\hat{i}$, so $x = v_0 t$ and $y = R - (1/2)gt^2$. When $x = R$, $y = 0$, thus $R = v_0 t$. $0 = R - (1/2)gt^2$. Eliminating t gives $t = R/v_0$. $R = (1/2)g(R/v_0)^2$. Thus $v_0^2 = (gR/2)$, or $v_0 = \sqrt{gR/2} = 16.0$ m/s.

5. Correct answers. Ask for verbal explanation of part (d).

5. (a) radius = r
$= \sqrt{(2.00 \sin 6.0t)^2 + (2.00 \cos 6.0t)^2}$
$= 2.00$ m.

(b) $\vec{v} = d\vec{r}/dt = 12.0[(\cos 6.0t)\hat{i} - (\sin 6.0t)\hat{j}]$ m/s.

speed $= v = \sqrt{(12.0 \cos 6.0t)^2 + (-12.0 \sin 6.0t)^2}$

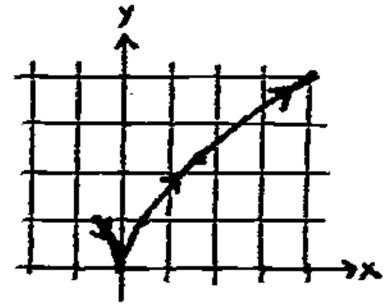
$= 12.0$ m/s.

(c) $a = v^2/r = (12.0\ \text{m/s})^2/(2.00\ \text{m}) = 72\ \text{m/s}^2$

(d) at $t = 0$, $\vec{r}(0) = 2.00\hat{j}$ m,
$\vec{v}(0) = 12.0\hat{i}$ m/s, so motion is clockwise.

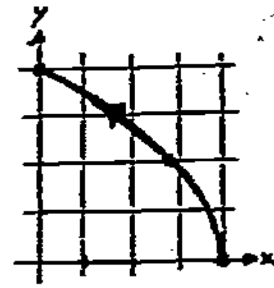
MASTERY TEST GRADING KEY - Form B

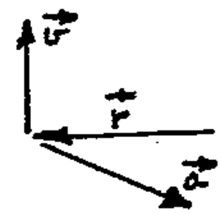
What to Look For	Solutions
1. Correct end points. Correct shape.	1. $\vec{r}(-1) = [-(1/2)\hat{i} + \hat{j}]$ m. $\vec{r}(0) = 0.$ $\vec{r}(1) = [(1/2)\hat{i} + \hat{j}]$ m. $\vec{r}(2) = (4\hat{i} + 4\hat{j})$ m. $\vec{r}(3/2) = [(27/16)\hat{i} + (9/4)\hat{j}]$ m.
2. Correct answer.	2. $\vec{v}(t) = [(3/2)t^2\hat{i} + 2t\hat{j}]$ m/s. speed $= v(1\text{ s}) = \sqrt{(9/4) + 4} = \sqrt{25/4} = (5/2)$ m/s.
3. Correct answers. Ask for verbal explanation of part (b).	3. (a) r (magnitude) decreasing, $\vec{r}$ turning clockwise. (b) v not changing, $\vec{v}$ turning counterclockwise.
4. Correct answers. You might ask student to explain why $a_y \neq 0$ when $v_y = 0$ in part (a).	4. (a) $\vec{a} = -g\hat{j} = -9.8\hat{j}$ m/s^2 at all times. (b) Since $\vec{a}$ constant, $\vec{r} = \vec{v}_0 t + (1/2)\vec{a}t^2$ ($\vec{r}_0 = 0$) or $\vec{r}(t) = (v_0 \cos\theta t)\hat{i} + [v_0 \sin\theta t - (1/2)gt^2]\hat{j}$. Thus $r(3.00\text{ s}) = \{[(30.0)(0.50)(3.00)]\hat{i}$ $+ [(30.0)(1.73)(3.00) - (1/2)(9.8)(9.0)]\hat{j}\}$ m $= [45\hat{i} + (78 - 44)\hat{j}]$ m $= (45\hat{i} + 34\hat{j})$ m. Distance $= \sqrt{(45)^2 + (34)^2} = 56$ m.
5. Correct answers. Ask for verbal explanation of part (d).	5. (a) radius $= r$ $= \sqrt{(0.240 \sin 5.0t)^2 + (0.240 \cos 5.0t)^2}$ $= 0.240$ m. (b) $\vec{v}(t) = 1.20\,[(\cos 5.0t)\hat{i} + (\sin 5.0t)\hat{j}]$ m/s. speed $= v$ $= \sqrt{(1.20 \cos 5.0t)^2 + (1.20 \sin 5.0t)^2}$ $= 1.20$ m/s. (c) $a = v^2/r = [(1.44\text{ m}^2/\text{s}^2)/(0.240\text{ m})] = 6.0$ m/s^2. (d) $\vec{r}_0 = -0.240\hat{j}$, $\vec{v}_0 = 1.20\hat{i}$ m/s. Thus the motion is counterclockwise.

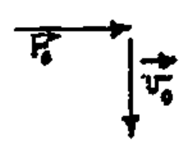


MASTERY TEST GRADING KEY - Form C

What To Look For	Solutions
1. Correct end points. Correct shape.	1. $\vec{r}(0) = 4\hat{i}$ m. $\vec{r}(1) = (2\sqrt{2}\hat{i} + 2\hat{j})$ m. $\vec{r}(2) = 4\hat{j}$ m.
2. Correct answer.	2. $\vec{v} = d\vec{r}/dt = [\,-\pi \sin(\pi t/4)\hat{i} + 2\hat{j}\,]$ m/s. $\vec{v}(1\text{ s}) = [-\pi \sin(\pi/4)\hat{i} + 2\hat{j}] = (-\sqrt{2}\pi/2\hat{i} + 2.0\hat{j})$ m/s.
3. Correct answers. Ask for verbal explanation of at least one part.	3. (a) r not changing, $\vec{v} \perp \vec{r}$. $\vec{r}$ turning clockwise. (b) v decreasing, $\vec{v}$ turning clockwise.
4. Correct answer. See that initial velocity is resolved correctly into x and y components.	4. $v_{0y} = v_{0x} = v_0 \cos 45° = v_0/\sqrt{2}$. When will x = R? $R = v_{0x}t_R = (v_0/\sqrt{2})t_R$; $t_R = \sqrt{2}R/v_0$. What is y at this time? $y = v_{0y}t - \frac{1}{2}gt^2 = \frac{v_0}{\sqrt{2}}\frac{\sqrt{2}R}{v_0} - \frac{1}{2}g\frac{2R^2}{v_0^2} = R - \frac{gR^2}{v_0^2}$ $= 20.0[1 - \frac{(9.8)(20.0)}{400}] = 20.0(0.51) = 10.2$ m. Ball hits wall 10.2 m above ground.
5. Correct answers. Ask for verbal explanation of part (c).	5. (a) $\vec{v}(t) = 18.0[(-\sin 15.0t)\hat{i} - (\cos 15.0t)\hat{j}]$ m/s. speed = v $= \sqrt{(18.0 \sin 15.0t)^2 + (18.0 \cos 15.0t)^2}$ = 18.0 m/s. (b) $a = v^2/r = (324\ m^2/s^2)/1.20\ m = 270\ m/s^2$. (c) $\vec{r}_0 = 1.20\hat{i}$ m, $\vec{v}_0 = -18.0\hat{j}$ m/s. Motion is clockwise.







NEWTON'S LAWS

INTRODUCTION

When a body is at rest, we know from experience that it will remain at rest unless something is done to change that state. A heavy box on the floor will stay in place unless it is pushed or pulled. We walk without fear beside a massive rock on level ground because we know it won't suddenly move and crush us.

Undoubtedly you have leaned against a chair only to have it move and send you scurrying for your balance. Did you then question the relationship of the interaction between you and the chair to the ensuing motion of the chair?

It was Isaac Newton who first clearly made the connection between the interactions on a body and its motion. In Newton's theory, the acceleration of every object has to be explained in terms of the interactions with other objects. Newton's laws of motion cover an enormous range of experience. At one stroke they convert what in retrospect had previously seemed chaos into a beautifully organized universe. There have been few achievements to rank with this in the history of science.

PREREQUISITES

Before you begin this module, you should be able to:	Location of Prerequisite Content
*Check the units of a given mathematical expression and show that it is dimensionally correct (needed for Objectives 2 and 4 of this module)	Dimensions and Vector Addition Module
*Add or subtract two, three, or four two-dimensional vectors given in unit-vector notation, finding the resultant (needed for Objectives 2 and 4 of this module)	Dimensions and Vector Addition Module
*Describe the position, velocity, and acceleration of an object moving in one dimension with constant acceleration (needed for Objectives 2 and 4 of this module)	Rectilinear Motion Module
*Describe the position, velocity, and acceleration of a single body moving in a plane or moving in projectile motion (needed for Objectives 2 and 4 of this module)	Planar Motion Module

LEARNING OBJECTIVES

After you have mastered the contents of this module you will be able to:

1. Free-body diagram - Draw a diagram of a particle representation of a body isolated from its environment in an inertial reference frame; and
   (a) illustrate, with vectors, all forces that act upon it; and
   (b) identify, by name, the source and each force illustrated.

2. $\vec{F} = m\vec{a}$ - Write Newton's first and second laws in mathematical form; and
   (a) choosing an appropriate coordinate system, apply the second law to a given problem involving a single massive body, solving for either a specified force or the acceleration of the body; and
   (b) use the second law to distinguish between weight and mass.

3. Action-reaction - Apply Newton's third law to a problem to relate the forces exerted and experienced by a body.

4. Motion of a particle - Solve a problem concerning the motion of a body (acceleration, velocity, and displacement) given sufficient information concerning the external forces acting on the body. (These external forces may be gravitational forces or contact forces exerted by another particle, by friction, by nonstretchable ropes, or by rigid rods.)

GENERAL COMMENTS

It should be emphasized that a particle model is used throughout this module, i.e., each body is considered as if it were concentrated at a point and had no extension in space. This should be remembered in your study of examples in the text and in working the Problem Set. All objects (blocks, cars, passengers in elevators, etc.) are to be treated as particles.

Objective 2 is stated as $\vec{F} = m\vec{a}$ because Newton's first law is implicit in the second. That is, if $\vec{F} = 0$, $\vec{a} = 0$: no force acting on the body, no acceleration (change in motion). We should also stress that the $\vec{F}$ in Newton's second law is the resultant force (vector sum) of all the forces acting on a body.

In solving problems in this module, here are some suggested rules to follow, step by step:

1. Identify the particular body to be considered.
2. Identify all interactions (forces) between the body and its environment.
3. Choose a suitable inertial coordinate system (be judicious in your choice, and you will save yourself a lot of effort).
4. Draw a diagram of the object representing it by a point; show all forces acting on the body, and show the coordinate system (free-body diagram).
5. Resolve those forces not lying along a coordinate axis into their rectangular components.
6. Apply Newton's second law.

Following these rules, especially rule 4, will save you much time in mastering Objective 4, where it is easy to confuse the various interactions, forces, and bodies.

Regarding Objective 3 (action-reaction), a more complete treatment will be given in the later module: <u>Applications of Newton's Laws</u>.

TEXT: Frederick J. Bueche, Introduction to Physics for Scientists and Engineers (McGraw-Hill, New York, 1975), second edition

SUGGESTED STUDY PROCEDURE

Read Section 2.3 in Chapter 2 and all of Chapter 5 (excluding Sec. 5.5). Objective 1 is discussed in the initial part of Section 5.3 in relation to the application of Newton's second law. Newton's first and second laws are stated in Section 5.1 (p. 54) in connection with Objective 2, and Newton's second law ($\vec{F} = m\vec{a}$) is further discussed in Section 5.2. The relation between weight and mass is given in Eq. (5.4).

BUECHE

Objective Number	Readings	Problems with Solutions		Assigned Problems		Additional Problems
		Study Guide	Text	Study Guide	Text	
1	Secs. 2.3, 5.3	A, B		I, J	Chap. 2, Prob. 7	Chap. 2, Prob. 4
2	Secs. 5.1, 5.2	C, D		K, L	Chap. 5, Prob. 7	Chap. 5, Prob. 9
3	Sec. 5.1	E, F		M, N	Chap. 5 Ques. 1	
4	Secs. 5.3, 5.4, 5.6, 5.7	G, H	Illus. 5.1, 5.2, 5.3, 5.4	O, P, Q	Chap. 5 Prob. 13, 23	R, S; Chap. 5, Probs. 10, 11, 12

Objective 4 is covered in Sections 5.3, 5.4, 5.6, and 5.7 with several good illustrations.

TEXT: David Halliday and Robert Resnick, Fundamentals of Physics (Wiley, New York, 1970; revised printing, 1974)

SUGGESTED STUDY PROCEDURE

Read Chapter 5 of your text, excluding Section 5-13 and Examples 5 and 6 (which will be covered in the module: Applications of Newton's Laws). Objective 1 is spread over four sections (5-1, 5-3, 5-8, 5-11). Table 5-1 on p. 60 gives a very helpful overview of interactions between a body and its environment that are described by forces; compare this with Table 5-4 on p. 70 for a mathematical description of the forces in each case.

Buried in the first paragraph of Section 5-11 are five rules. These rules are indispensable in working out the problems of Objective 4. Note the explicit and excellent free-body diagrams (and the particle representation) drawn for the examples, i.e., Figures 5-4(b), 5-5(b), 5-6(b), 5-8(b), and 5-11.

Newton's first law is stated and discussed in Section 5-3. Newton's second law is stated mathematically in Eq. (5-1) and discussed at length in Sections 5-2 to 5-5 and 5-8. The relationship between weight and mass is given in Eq. (5-5) and explained in Section 5-9.

Objective 3 is thoroughly explained in an excellent discussion in Section 5-6, and two very good examples are given (Examples 1 and 2). Objective 4 is presented in Sections 5-11 and 5-12 with many excellent examples.

After completing this reading, study the Problem Set and solve the assigned problems. Finally, check your understanding by taking the Practice Test.

HALLIDAY AND RESNICK

Objective Number	Readings	Problems with Solutions: Study Guide	Problems with Solutions: Text	Assigned Problems: Study Guide	Assigned Problems: Text	Additional Problems
1	Secs. 5-1, 5-3, 5-8, 5-11	A, B		I, J		
2	Secs. 5-2 to 5-5, 5-7 to 5-9	C, D	Examples 3, 4(a)	K, L	Chap. 5, Probs. 4, 11	Chap. 5, Probs. 5, 7, 14
3	Sec. 5-6	E, F	Examples 1, 2	M, N	Chap. 5, Prob. 2	
4	Secs. 5-11, 5-12	G, H	Examples 4(b), 7, 8	O, P, Q	Chap. 5 Probs. 25, 50	R, S; Chap. 5, Probs. 8, 9, 12, 13, 16, 18, 21, 35

TEXT: Francis Weston Sears and Mark W. Zemansky, University Physics (Addison-Wesley, Reading, Mass., 1970), fourth edition

SUGGESTED STUDY PROCEDURE

This particular module is spread over Chapters 1, 2, and 5. Interspersed in Chapters 2 and 5 is introductory material on Gravitation and Statics, topics that will be taken up in more depth in the modules Gravitation and Equilibrium of Rigid Bodies. We suggest that you read Sections 1-4 and 1-5 of Chapter 1, Chapters 2 and 5 in order, but skip Sections 2-4 and 5-4.

Objective 1 is discussed quite thoroughly over parts of five sections. In Section 1-5, the concept of force is introduced; however, it is introduced independent of acceleration. When particles interact (do something to each other), one measure of their interaction strength is provided by their acceleration, and this measure leads to the force concept. On p. 14 the technique for determining the resultant force from the individual forces is introduced. On p. 15 an inertial coordinate system is defined, and on p. 18 there appears an excellent discussion of the particle model and rules for constructing a free-body diagram and for solving problems. [Note an example of a free-body diagram in Figure 5-8(b) on p. 66.]

SEARS AND ZEMANSKY

Objective Number	Readings	Problems with Solutions		Assigned Problems		Additional Problems
		Study Guide	Text	Study Guide	Text	
1	Secs. 1-4, 1-5, 2-2, 2-3, 2-6	A, B		I, J		
2	Secs. 2-2, 2-3, 5-2, 5-3, 5-5	C, D	Sec. 2-6, Ex.* 1-4 Sec. 5-3, Ex. 2, 3	K, L	5-1	
3	Sec. 2-5	E, F	Sec. 2-6, Ex. 1, 2 Sec. 5-6, Ex. 4	M, L	5-3	2-1(a)-(g), 2-2
4	Secs. 2-6, 2-7, 5-6	G, H	Sec. 2-7, Ex. 1, 2, 3 Sec. 5-6, Ex. 1, 2, 3, 6, 7	O, P, Q	2-22, 5-24	2-17, 5-13, 5-17, 5-21, 5-22

*Ex. = Example

Objective 2 is spread over five sections. You will find Newton's first law stated on p. 15 with a discussion of it preceding and following this statement. Newton's second law is stated verbally and mathematically [Eq. (5-3)] on p. 58. The relation between mass and weight is given in Eq. (5-6). Disregard Eq. (5-5) since you will take this up in the module <u>Gravitation</u>.

Objective 4 is handled in Sections 2-7 and 5-6 with several examples. Disregard Examples 4 and 5 (Sec. 5-6) for now.

Rule 4 on p. 18 explicitly applies to the case where the resultant force is zero (Newton's first law), but it can be extended to the more general case (Newton's second law) by setting the algebraic sum of all the x components of the forces equal to the mass times the x component of acceleration ($\Sigma F_x = ma_x$) and the algebraic sum of all the y components of the forces equal to the mass times the y component of acceleration ($\Sigma F_y = ma_y$).

Study Problems A through H before working Problems I through Q. When you feel prepared, take the Practice Test.

TEXT: Richard T. Weidner and Robert L. Sells, Elementary Classical Physics (Allyn and Bacon, Boston, 1973), second edition, Vol. 1

SUGGESTED STUDY PROCEDURE

Your text approaches mechanics from a momentum point of view (a concept to be covered in a later module). Momentum is defined and experiments are used to establish the conservation of momentum as the empirical starting point for the development of some of the concepts to be developed in this module.

To make your text compatible with the sequence of ideas as developed in these modules, we suggest the following reading sequence: Sections 5-1 through 5-6 (excluding all the examples in these sections), 7-1 through 7-3, 7-5 through 7-6, and 8-1 through 8-3 (excluding Example 8-6). You may wish to read through Sections 5-5 and 5-6 quickly since the material in these sections is not included in any of the learning objectives of this module but is needed to facilitate the understanding of the material in this module.

WEIDNER AND SELLS

Objective Number	Readings	Problems with Solutions		Assigned Problems		Additional Problems
		Study Guide	Text	Study Guide	Text	
1	Secs. 5-1, 7-1, 7-3, 8-2	A, B		I, J		
2	Secs. 5-1, 5-2, 7-2, 7-3, 7-6, 8-1	C, D	Ex* 7-1, 8-2	K, L		
3	Secs. 7-5, 8-1	E, F		M, N	8-5	
4	Secs. 8-2, 8-3	G, H	Ex. 7-2, 8-3, 8-4, 8-5, 8-7, 8-9, 8-10	O, P, Q, R, S	8-11, 8-12	8-21

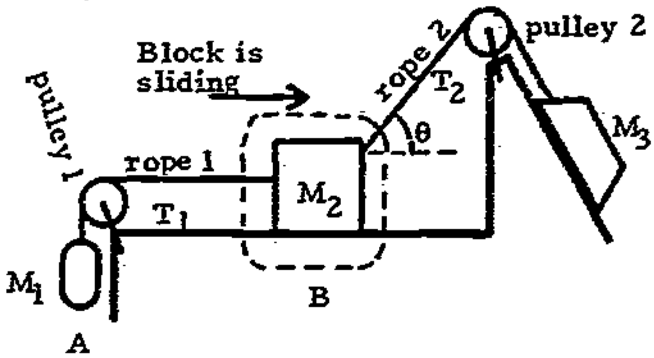
*Ex. = Example(s)

Objective 1 is spread over four sections of reading. An inertial reference frame is defined in Section 5-1. A general discussion is given in Section 7-1 of the concept that interacting particles can manifest this interaction by altering their motion, and this change in motion can be used to measure the strength of the interaction or the force. An explanation of the combination of the forces of interaction to obtain a resultant force is given in Section 7-3. The rules given on the bottom of p. 108 pertain to drawing a free-body diagram and solving problems. Carefully study some of the excellent free-body diagrams in Figures 8-5(b) and 8-6(b) and (c).

Objective 2 is covered in several sections. Newton's first law is stated on pp. 56 and 106. Newton's second law is stated verbally in Section 8-1 and mathematically in Eqs. (7-4) (for this module you will only need the simpler form $\vec{F} = m\vec{a}$) and (8-1). The relationship between mass and weight is given in Eq. (7-10). Objective 3 is developed in Section 7-5.

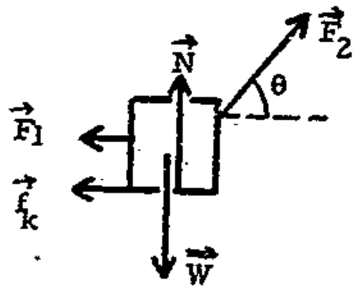
PROBLEM SET WITH SOLUTIONS*

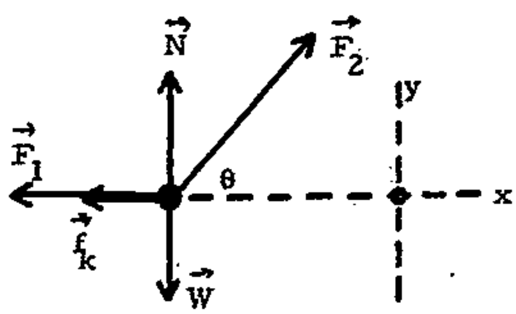
A(1). Given the situation pictured at the right, specify all the forces acting on the body B (mass M_2) and draw a free-body diagram for B.



Solution

We are interested in body B (contained in the dotted lines). $\vec{F}_1$ represents the force that rope 1 exerts on B. $\vec{F}_2$ represents the force that rope 2 exerts on B. $\vec{f}_k$ represents the frictional force on B that opposes the motion. $\vec{W}$ represents the weight or the attraction of the earth for B, $\vec{W} = m\vec{g}$. $\vec{N}$ represents the force that the supporting plane exerts on B.





Free-Body Diagram

*Each of the problems satisfying Objective 4 should also satisfy Objective 1 if worked properly.

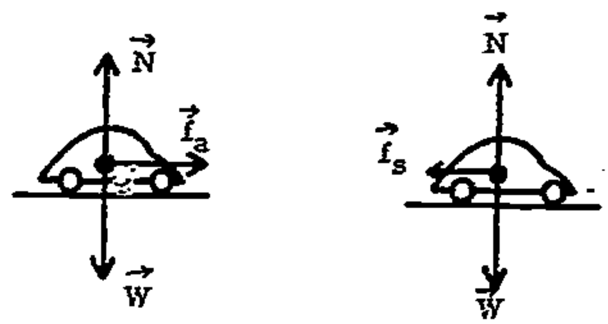
B(1). A small car has a mass of 800 kg. It can accelerate uniformly from rest to 30.0 m/s in 8.0 s, and its brakes slow it down from 30.0 m/s to 15.0 m/s in 5.0 s.

(a) Draw free-body diagrams for the car while it is accelerating and while it is decelerating (brakes applied).

(b) Express the force exerted by the road in terms of the resultant force and the weight.

Solution

(a)



Acceleration   Deceleration - Brakes Applied

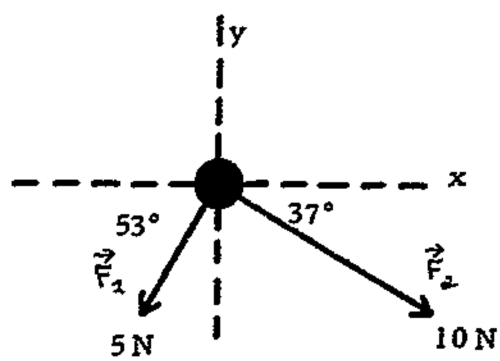
(b) The supporting road exerts a normal force $\vec{N}$ perpendicular to the point of contact and a frictional force $\vec{f}_a$ or $\vec{f}_s$ parallel to the point of contact

$$\vec{F}_c = \vec{F} - \vec{W}$$

where $\vec{F}_c = \vec{N} + (\vec{f}_a \text{ or } \vec{f}_s)$.
$\vec{F}$ is the resultant force which is parallel to the road, and $\vec{F} = \vec{F}_c + \vec{W}$.

C(2). Find the acceleration of the object (weight 5.0 N) under the action of the forces shown in the diagram.

From Newton's second law $\vec{F} = m\vec{a}$, where $\vec{F}$ is the net force. Therefore $\vec{F} = \vec{F}_1 + \vec{F}_2$. Vectorially adding $\vec{F}_1$ and $\vec{F}_2$:

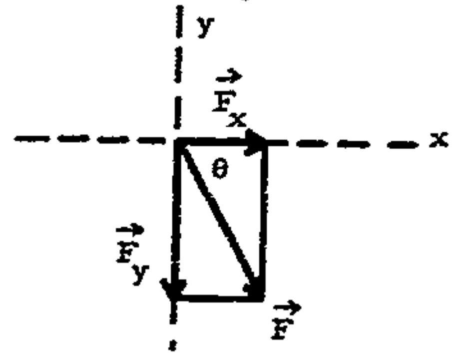


$F_x = F_{1x} + F_{2x} = -F_1 \cos 53° + F_2 \cos 37° = -3.0\ N + 8.0\ N$, $F_x = 5.0\ N$,

$F_y = F_{1y} + F_{2y} = -F_1 \sin 53° - F_2 \sin 37° = -4.0\ N - 6.0\ N$, $F_y = -10.0\ N$,

$\vec{F} = [F_x^2 + F_y^2]^{1/2} = [125\ N^2]^{1/2} = 11.2\ N$.

$\tan\theta = \frac{10\ N}{5\ N} = 2.0$, $\theta = 63.4°$ below x axis.



$\vec{F} = m\vec{a} = (\vec{W}/\vec{g})\vec{a}$ or $|\vec{a}| = Fg/W$.

$$a = \frac{(11.2\ N)(9.8\ m/s^2)}{5.0\ N} = 22\ m/s^2$$

in the same direction as $\vec{F}$, or 63.4° below the +x axis. Using unit-vector notation:

$$\vec{a} = a_x\hat{i} + a_y\hat{j} = \frac{F_x}{m}\hat{i} + \frac{F_y}{m}\hat{j} = \vec{a} = (9.8\hat{i} - 19.6\hat{j})\ m/s^2.$$

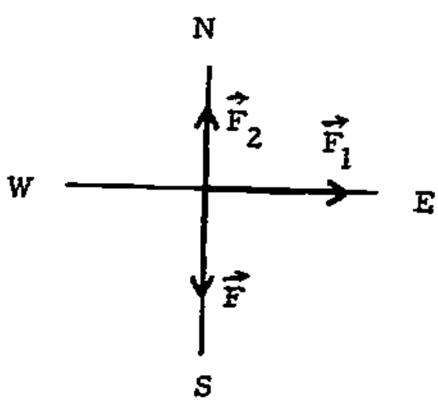
D(2). A particle of weight 19.6 N is subject to forces of 3.00 N east and 2.00 N north but travels south at a constant acceleration of 1.00 m/s^2. What third force must act on the particle?

Solution

The net force $\vec{F}$ must be south. (You should know why.) From Newton's second law:

$\vec{F} = m\vec{a}$,

$\vec{F} = \frac{\vec{W}}{\vec{g}}\vec{a} = \left(\frac{19.6\ N}{9.8\ m/s^2}\right) 1.00\ m/s^2 = 2.00\ N$ (net force).



$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$,

$F_x = F_{1x} + F_{2x} + F_{3x}$,

$F_y = F_{1y} + F_{2y} + F_{3y}$,

$F_{3x} = -3.00\ N$,

$F_{3y} = -4.00\ N$,

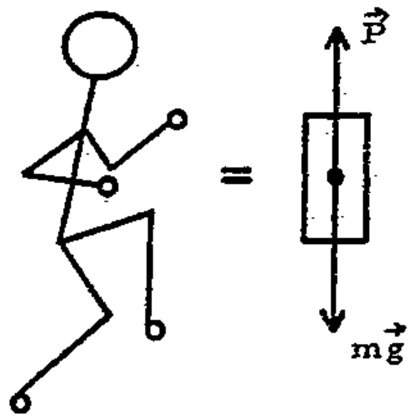
so $\vec{F}_3 = 5.00\ N$ 53° south of west, or

$\vec{F}_3 = (-3.0\ \hat{i} - 4.0\ \hat{j})\ N$.

E(3). When a high jumper leaves the ground, what object exerts the force that accelerates him upward?

Solution

Let's draw a free-body diagram and see. Considering the man as a unit, we find that the only external forces acting are his weight $m\vec{g}$ and the normal force $\vec{N}$ of the ground. Where does $\vec{N}$ come from? It's the reaction force to the downward force of magnitude $\vec{N}$ exerted by the man's foot against the ground. As long as $\vec{N} = -m\vec{g}$, the man's center of mass can accelerate neither up nor down.

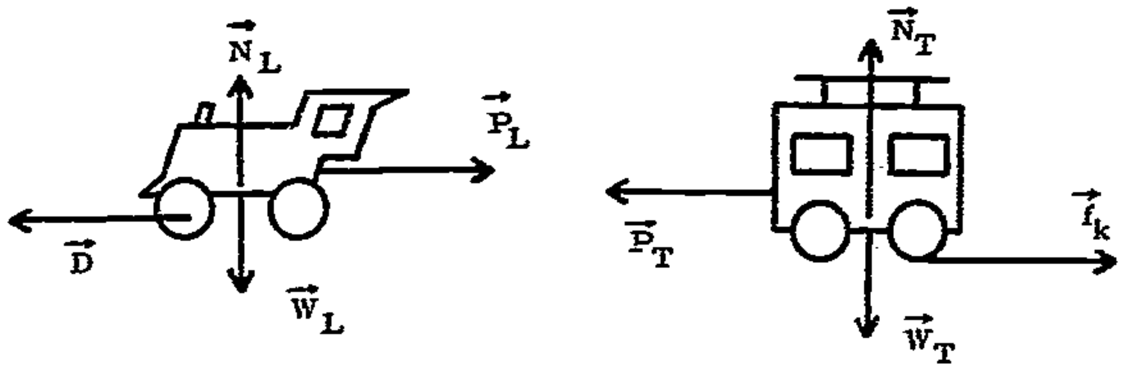


By suddenly straightening his leg, however, the high jumper can increase the force exerted against the ground. $\vec{N}$ will increase correspondingly and his center of mass will accelerate upward. If this acceleration is large enough, he can actually leave the ground. So in one sense, the force that accelerates him (as a unit) upward is exerted by the ground. However, bear in mind that part of $\vec{N}$ is simply a reaction force to forces generated inside his body - forces that allowed him to push his foot down against the ground with a force considerably greater than his own weight.

F(3). A locomotive engineer reads an excerpt from a freshman physics text and then decides to quit his job. His reason is that, according to Newton's third law, the train always pulls backward on the locomotive with a force just as great as that which the locomotive exerts on the train, and therefore the train can never move. As personnel supervisor, you are assigned the task of explaining the situation. Explain it.

Solution

These two forces form an action-reaction pair and must act on different bodies. Let's draw a free-body diagram of the locomotive and the train.



Since $\vec{P}_L$ and $\vec{P}_T$ form the action-reaction pair,

$\vec{P}_L = -\vec{P}_T$,

$\vec{D} - \vec{P}_L = M_L\vec{a}$, Locomotive where $\vec{D} - \vec{P}_L > 0$;

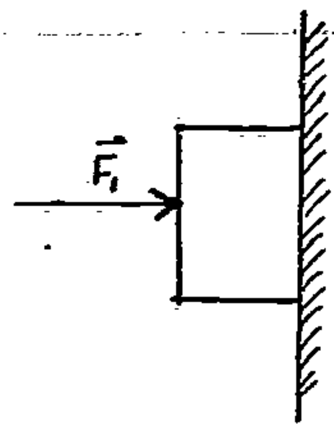
$\vec{P}_T - \vec{f}_k = M_T\vec{a}$, Train where $\vec{P}_T - \vec{f}_k > 0$.

G(4). A constant horizontal force $\vec{F}_1$ pushes a block of mass 2.00 kg against a vertical wall. The coefficient of static friction between the block and the wall is $\mu_s = 0.60$, and the coefficient of kinetic friction is $\mu_k = 0.40$.

(a) Draw a free-body diagram for the block.

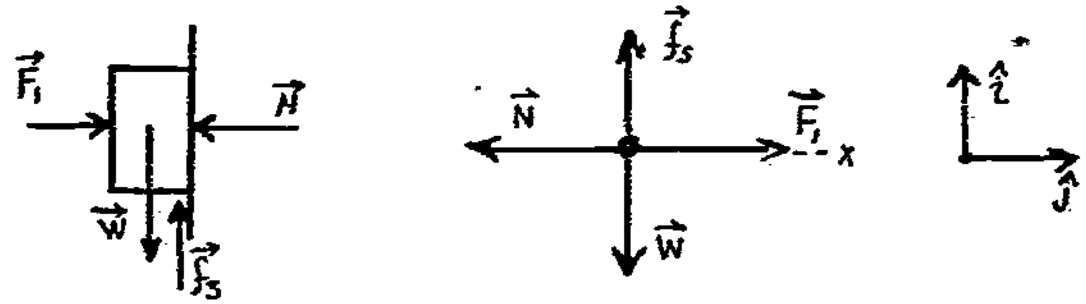
(b) What is the minimum value of $\vec{F}_1$ for which the block will not slip, if it is at rest initially?

(c) Suppose that $\vec{F}_1$ has this minimum value. The block is given a short downward push, just to start it moving. What acceleration will the block have after this push?



Solution

(a)



$\vec{N}$ is the normal force that the wall exerts on the block;

$\vec{f}_s$ is the static frictional force that the wall exerts on the block;

$\vec{W}$ is the weight of the block;

$\vec{F}_1$ is the applied force.

(b) Assume the block is stationary, i.e., zero accelerations in the x and y directions. Apply Newton's second law to the block:

$\Sigma F_x = ma_x, \quad \vec{F}_1 - \vec{N} = 0$ (1)

$\Sigma F_y = ma_y, \quad \vec{f}_s - \vec{W} = 0, \quad \vec{W} = m\vec{g}.$ (2)

The frictional force is given by

$\vec{f}_s \le \mu_s \vec{N}.$

Note: The frictional force depends on the normal, but since $\vec{N} = \vec{F}_1$, the normal depends on the applied force. Therefore the frictional force that "holds" the body up is controlled by the applied force (i.e., we are "pressing" the surfaces in closer contact and making them "bind" more).

From Eq. (2) $\vec{f}_s = \vec{W} = m\vec{g} \le \mu_s \vec{N} = \mu_s \vec{F}_1$, so $\vec{F}_1 \le m\vec{g}/\mu_s$.

The minimum value of $\vec{F}_1$ is $m\vec{g}/\mu_s$:

$$(\vec{F}_1)_{min} = \frac{(2.0\text{ kg})(9.8\text{ m/s}^2)}{(0.60)} = 33\text{ N}.$$

(c) Now $\vec{f}_k - m\vec{g} = m\vec{a}_y$ and $\vec{F}_1 = \vec{N}$:

$$\vec{f}_k = \mu_k \vec{N} = \mu_k \vec{F}_1,$$

so

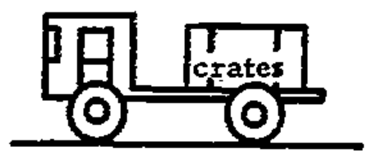
$$a_y = \frac{\vec{f}_k - m\vec{g}}{m} = \frac{\mu_k \vec{F}_1 - m\vec{g}}{m} = \frac{(\mu_k m\vec{g}/s) - m\vec{g}}{m},$$

$$a_y = \vec{g}\mu_k/\mu_s - 1 = (9.8\ \mathrm{m/s^2})\ (0.40/0.60 - 1),$$

$$a_y = 3.3\ \mathrm{m/s^2}.$$

The minus sign designates the acceleration is downward.

H(4). A truck driver has a load of crates on a flat-bed truck and is traveling along a highway at 26 m/s. If the coefficient of friction between crates and truck is 0.80, what is the shortest distance in which he can stop without letting his load slide?



Solution

Assume the deceleration is constant and the final speed is zero. Take the origin of our coordinate system to be a point where the truck begins to decelerate and the x axis along the roadway.

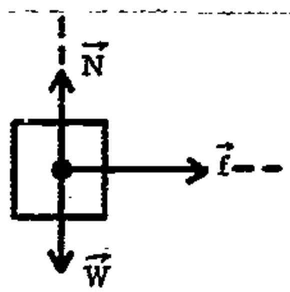
In order for the crate not to slide, what must its speed be relative to the truck bed? Therefore what must be its deceleration relative to our coordinate system?

Free-body diagram of the crate

$\vec{N}$ = normal force,

$\vec{W}$ = weight of crate = $m\vec{g}$,

$\vec{f}$ = frictional force.



Applying Newton's second law:

$$\Sigma F_x = ma_x \quad \text{and} \quad \Sigma F_y = ma_y,$$

$$\vec{f} = ma_x, \quad \vec{N} = \vec{W} = m\vec{g},$$

but $\vec{f} \le \mu_s \vec{N}$. Assume the maximum frictional force to allow the greatest deceleration before sliding begins: $\vec{f} = \mu_s \vec{N}$.

Combining equations leads to $a_x = \mu_s \vec{g}$ or $a_x = 7.8\ \mathrm{m/s^2}$.

Now we have the initial speed, final speed, and acceleration, which allows us to determine the distance:

$$x = \frac{v_x^2 - v_{x0}^2}{2a_x} = -43 \text{ m.}$$

Problems

I(1). A youngster coasts down a hill of slope θ on his sled.

(a) Identify all forces acting on the youngster.
(b) Draw a free-body diagram of the youngster.

J(1). A tractor climbs a steep grade of angle θ.

(a) Identify all forces acting on the tractor.
(b) Draw a free-body diagram of the tractor.

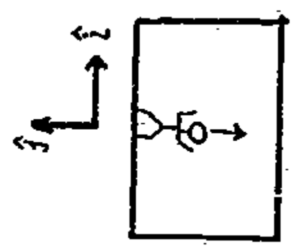
K(2). A body whose mass is 3.00 kg lies on a smooth horizontal plane. Introduce a coordinate system for answering the following questions:

(a) A horizontal force of 12.0 N is applied in a certain direction. Find the vector acceleration.
(b) A horizontal force of 5.0 N is applied, at right angles (clockwise, looking down) to the force in (a), which is removed. Find the vector acceleration.
(c) Find the vector acceleration when both forces are applied (polar and rectangular descriptions).

L(2). A skier of mass M gathers speed down a slope (angle θ to the horizontal) even though experiencing a force of friction $\vec{f}$.

(a) Make a free-body diagram for the skier.
(b) Find the acceleration of the skier.
(c) What information about the force $\vec{f}$ is implied by this problem?

M(3). A 72-kg astronaut pushes away from the side of his 700-kg space capsule with a force of 20.0 N. What happens to the space capsule?



N(3). A book with a rock lying on top of it lies in turn on a table top.

(a) Draw a free-body diagram of the book.
(b) Identify the reaction force to each force acting on the book.

O(4). The coefficient of dynamic friction of a block on an inclined plane may be determined by raising the plane just enough so the block slides with uniform speed, and measuring the angle of inclination of the plane:

(a) Make a free-body diagram for the block.
(b) Derive a relationship between the angle and the coefficient of friction for these conditions.

P(4). A block slides down an inclined plane of slope ϕ with constant velocity. It is then projected up the same plane with an initial speed v_0.

(a) How far up the incline will it move before coming to rest?
(b) Will it slide down again?

Q(4). A 29.0-kg block is pushed up a 37° inclined plane by a horizontal force of 440 N. The coefficient of friction is 0.250. Find:

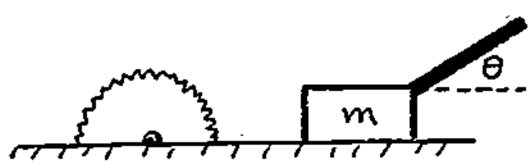
(a) the acceleration;
(b) the velocity of the block after it has moved a distance of 6.0 m along the plane, assuming it started at rest; and
(c) the normal force exerted by the plane.

R(4). A hockey puck having a mass of 0.110 kg slides on the ice for 15.2 m before it stops.

(a) Draw a free-body diagram of the puck including friction.
(b) If its initial speed was 6.1 m/s, what is the force of friction between the puck and ice?
(c) What is the coefficient of kinetic friction?

S(4). A man pushes a board of mass m through a circular saw by pushing down on it with a stick as shown:

(a) Find the force $\vec{F}$ that must be exerted to make the board slide with constant speed, in terms of the angle θ, the coefficient of friction, and the mass of the board.



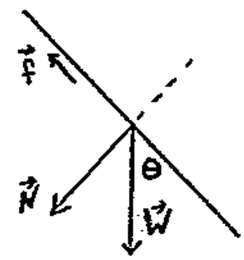
(b) Show that if the angle is too steep, the board cannot be made to slide, no matter how great a force is applied. Find this critical angle.

Solutions

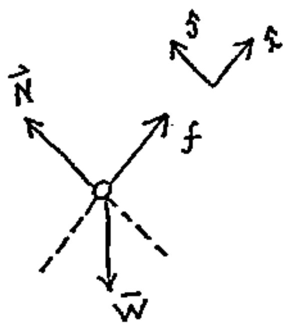
I(1). (a), (b) $\vec{N}$ = normal force that the sled exerts on the youngster (to sled),

$\vec{f}$ = frictional force that the sled exerts on the youngster (to sled),

$\vec{W}$ = weight of the body.



J(1). (a), (b) The surface of the ground exerts both $\vec{N}$ and $\vec{f}$ whereas the earth as a whole exerts $\vec{W}$.

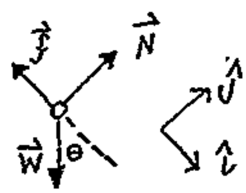


K(2). (a) $4.0\hat{j}$ m/s^2,

(b) $1.70\hat{j}$ m/s^2,

(c) $(1.70\hat{i} + 4.0\hat{j})$ m/s^2, 4.3 m/s^2, θ = 1.2 rad.

L(2). (a)

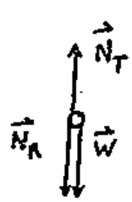


(b) g sin θ - f/M.

(c) $\vec{f} < M\vec{g}$ sin θ.

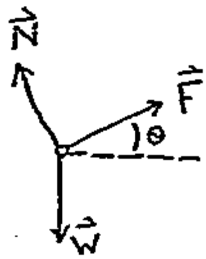
M(3). Accelerates at 0.0300 m/s^2 in -j direction until the astronaut hits the other side.

N(3).



Action	Reaction
$\vec{N}_T$	Book exerts downward force on the table ($-\vec{N}_r$)
$\vec{N}_R$	Book exerts upward force on the rock ($-\vec{N}_R$)
$\vec{W}$	Book attracts the earth with a force $-\vec{W}$

O(4). (a)



(b) $\mu = \tan\theta$.

P(4). (a) $v_0^2/4g \sin\phi$.

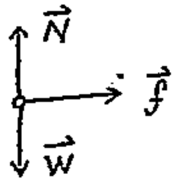
(b) No.

Q(4). (a) 2.00 m/s^2.

(b) 24.0 m/s.

(c) 490 N.

R(4). (a)



(b) 0.130 N.

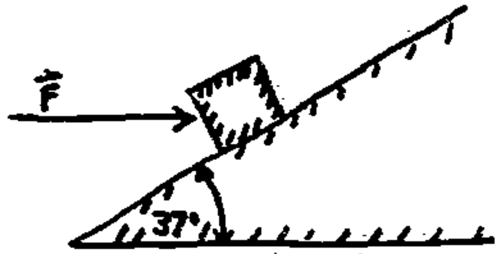
(c) 0.120.

S(4). (a) $mg/\cos\theta - \mu\sin\theta$.

(b) $\theta = \text{arc cot}\ \mu$.

PRACTICE TEST

1. A 50-kg block on an inclined plane is acted upon by a horizontal force of 50 N. The coefficient of friction between block and plane is 0.300.

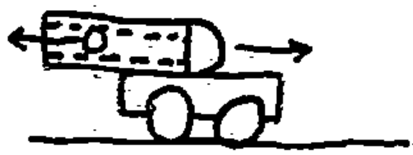


(a) Draw a free-body diagram of the block, assuming it is moving up the plane.

(b) What is the reaction force to each of the forces acting on the block?

(c) What is the acceleration of the block if it is moving up the plane?

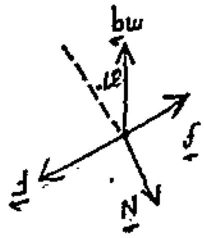
(d) How far up the plane will the block go if it has an initial upward speed of 4.0 m/s?



2. A 730-kg cannon fires a 4.0-kg cannon ball. While it is in the barrel, the cannon ball has an acceleration of 30 000 m/s^2. What is the acceleration of the cannon?

Practice Test Answers

1. (a)



(b)

	Action	Reaction
$\vec{N}$:	block exerts	$\vec{N}$ on plane
$\vec{f}$:	block exerts	$\vec{f}$ (friction) on plane
$\vec{F}$:	block exerts	$\vec{F}$ on pushing agent
$m\vec{g}$:	block attracts the earth with force $m\vec{g}$	

(c) -7.2 m/s^2.

(d) 1.10 m.

2. 160 m/s^2 to the right.

DATE____________________

NEWTON'S LAWS

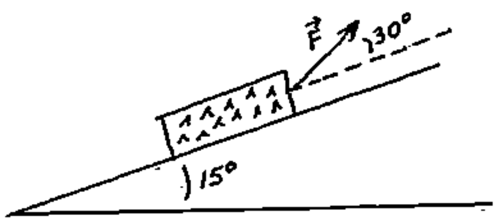
Mastery Test Form A

pass recycle

1 2 3 4

Name____________________ Tutor____________________

An elephant is being used to pull a large log up a hill having a slope of 15^0. The log has a mass of 500 kg, and the coefficient of friction between the log and the ground is 0.200. The pulling rope makes an angle of 30^0 with respect to the ground as shown in the figure.



1. Draw a free-body diagram of the log.

2. What is the reaction force to each of the forces acting on the log?

3. What force must the elephant exert ($\vec{F}$) for the log to be pulled at a constant speed?

4. If the rope breaks and the log slides down the hill, what is its acceleration?

<u>Note</u>: $\sin 15^0 = 0.258$; $\sin 30^0 = 0.50$;

$\cos 15^0 = 0.97$; $\cos 30^0 = 0.87$.

Date____________________

NEWTON'S LAWS

pass recycle

Mastery Test Form B 1 2 3

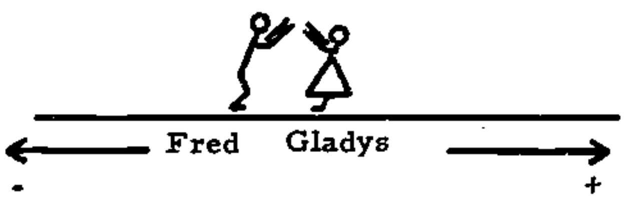
Name____________________ Tutor____________________

A horizontal force of 16.0 N will drag an 8.0-kg block along a certain horizontal surface with constant velocity. This same surface is then inclined at an angle of 37° with the horizontal, and a force parallel to the plane draws the block up the plane with an acceleration of 2.00 m/s^2.

1. Draw free-body diagrams for the block when it is on the horizontal surface and for when it is on the inclined surface.

2. What is the magnitude of the force drawing the block up the plane?

3. Fred (100 kg) and Gladys (72 kg) are standing on a smooth surface (ice). Gladys pushes Fred with a force of -260 N. What is Gladys' acceleration?



Date

NEWTON'S LAWS

pass recycle

Mastery Test Form C 1 2 3 4

Name Tutor

A student couple are filling their new waterbed for the first time on a lawn that slopes 6.0°. <u>NOTE</u>: $\sin 6.0^{\circ} = 0.104$; $\cos 6.0^{\circ} = 0.99$.

1. Draw a free-body diagram for the waterbed including friction with the lawn.

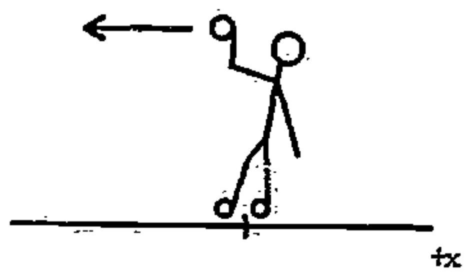
2. What is the minimum coefficient of static friction needed to keep the 680-kg waterbed from sliding away?

3. The coefficient of sliding friction is 0.300 on the wet grass. If the bed starts to move, how fast is it moving after sliding 46 m?

4. A 100-kg ice skater (no friction) throws a 0.50-kg snowball with an acceleration of 25.0 m/s^2 in the horizontal direction.

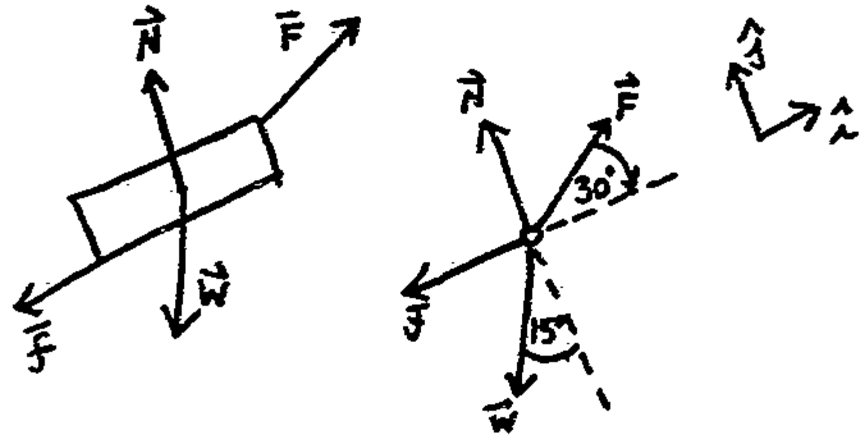
   (a) What agent exerts the force that moves the skater?

   (b) What are the magnitude and direction of the acceleration of the skater?



MASTERY TEST GRADING KEY - Form A

What To Look For	Solutions
1. Make sure all forces are present!	1.



What To Look For	Solutions
2. Each force should be given.	2.

Action Force	Reaction Force
$\vec{N}$	Log exerts normal force on the ground ($-\vec{N}$).
$\vec{F}$	Log exerts a pull on the rope ($-\vec{F}$).
$\vec{W}$	Log attracts the earth ($-\vec{W}$).
$\vec{f}$	Log exerts frictional force on the ground ($-\vec{f}$).

3. Constant speed up the plane requires $a_x = 0$.

3. Applying Newton's second law to the log:

$$\Sigma F_x = ma_x: \quad F_x \cos 30° - f - W \sin 15° = 0, \quad (1)$$

$$\Sigma F_y = ma_y: \quad N + F_y \sin 30° - W \cos 15° = 0. \quad (2)$$

Solve for N from Eq. (2) and substitute into

$\vec{f} = \mu\vec{N}$: $f = \mu W \cos 15° - \mu F \sin 30°$.

Substitute this into Eq. (1):

$$F \cos 30° - \mu W \cos 15° + \mu F \sin 30° - W \sin 15° = 0,$$

$$F(\cos 30° + \mu \sin 30°) = W \sin 15° + \mu W \cos 15°,$$

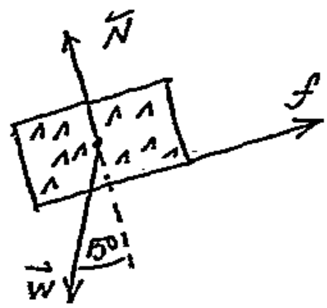
$$\vec{F} = m\vec{g}\,\frac{\sin 15° + \mu \cos 15°}{\cos 30° + \mu \sin 30°}$$

$$= \frac{(500\text{ kg})(9.8\text{ m/s}^2)[(0.258) + (0.200)(0.97)]}{0.87 + (0.200)(0.50)}$$

$$\vec{F} = 2300\text{ N}.$$

4. Free-body diagram of the log. Log is sliding down the slope, thus frictional force has changed direction.

4.



Coefficient of friction is the same as in the first part of the problem, but the normal force has changed. Minus sign means the log is accelerating in the -x (down the slope) direction.

Apply Newton's second law to the log:

$\Sigma F_x = ma_x$: $f - W \sin 15° = ma_x$, (3)

$\Sigma F_y = ma_y$: $N - W \cos 15° = 0$,

$f = \mu N = \mu W \cos 15°$

Substitute into Eq. (3), we find

$\mu W \cos 15° - W \sin 15° = ma_x$,

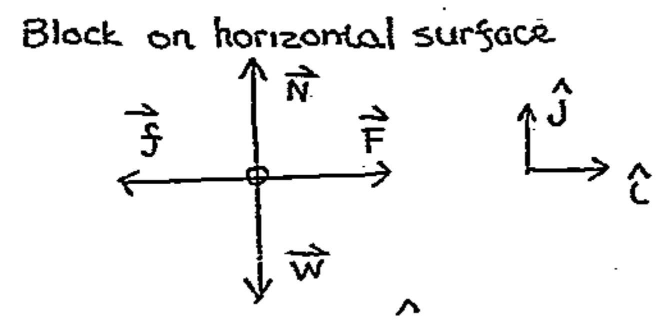
$\mu g \cos 15° - g \sin 15° = a_x$,

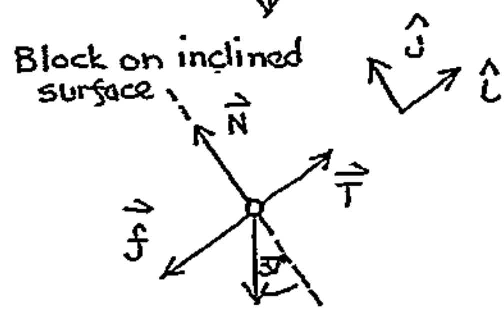
$a_x = (0.200)(9.8\ m/s^2)(0.97) - (9.8\ m/s^2)(0.258)$,

$a_x = -\ 0.65\ m/s^2$.

MASTERY TEST GRADING KEY - Form B

What To Look For	Solutions
1.	1.





2. Solve using information from the horizontal situation.

2. For the block on a horizontal surface, applying Newton's second law:

$\Sigma F_x = ma_x$: $F - f = ma = 0$,

$\Sigma F_y = ma_y$: $N - W = 0$,

$N = W = mg$, $\vec{f} = \mu\vec{N} = \mu m\vec{g}$.

Equation (1) becomes

$\vec{F} - \mu m\vec{g} = 0$

$$\mu = \frac{\vec{f}}{m\vec{g}} = \frac{16.0\ \text{N}}{(8.0\ \text{kg})(9.8\ \text{m/s}^2)},$$

$\mu = 0.20$.

For the block on the inclined surface, we apply Newton's second law:

$\Sigma F_x = ma_x$: $T - f - W \sin 37° = ma_x$, (2)

$\Sigma F_y = ma_y$: $N - W \cos 37° = 0$

Coefficient of friction doesn't change because the surfaces are the same.

$f = \mu N = \mu W \cos 37°$,

$f = \mu mg \cos 37°$.

Substitute this value for f into Eq. (2), we find

$T - \mu mg \cos 37° - mg \sin 37° = ma_x$,

$T = ma_x + \mu mg \cos 37° + mg \sin 37°$

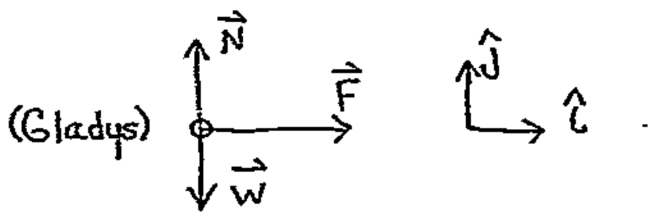
$= (80\ \text{kg})(2.00\ \text{m/s}^2) + (0.200)(8.0\ \text{kg})$

$\times (9.8\ \text{m/s}^2)(0.80) + (8.0\ \text{kg})(9.8\ \text{m/s}^2)(0.60)$

$T = 76\ \text{N}$.

3. Make sure free-body diagram is correct.

3. From Newton's third law, Fred pushes Gladys with a 260-N force:



Using Newton's second law, we find

$\Sigma F_x = ma_x$: $F = ma$,

$\Sigma F_y = ma_y$: $N - W = 0$,

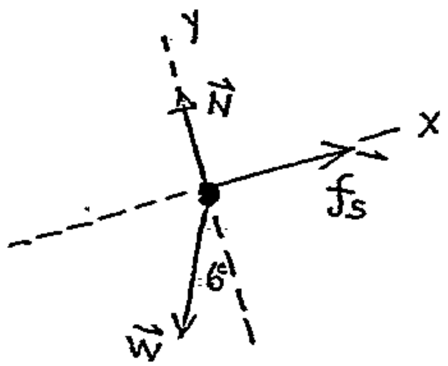
$a = \frac{F}{m} = \frac{260\ \text{N}}{72\ \text{kg}}$,

$\vec{a} = 3.6\ \text{m/s}^2$.

MASTERY TEST GRADING KEY - Form C

What To Look For	Solutions

1. Static Case:



2. Static case, the bed is not moving:

   $a_x = a_y = 0.$

   $f_s = \mu_s N$, but we want to know about this situation just before the bed slides, so $f_s = \mu_s N$.

   μ_s is unitless

2. Applying Newton's second law in component form:

   $\Sigma F_x = ma_x$: $\quad f_s - W \sin 6.0° = 0,$ (1)

   $\Sigma F_y = ma_y$: $\quad N - W \cos 6.0° = 0,$ (2)

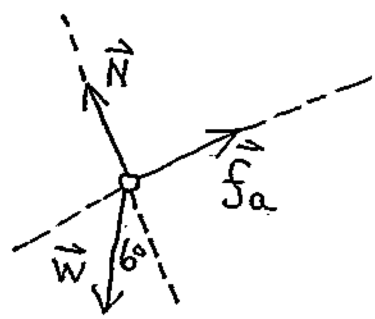
   $f_s = \mu_s N,$

   $f_s = \mu_s W \cos 6.0°$. Substitute this into Eq. (1):

   $\mu_s W \cos 6.0° - W \sin 6.0° = 0$

   $\mu_s = \dfrac{\sin 6.0°}{\cos 6.0°} = \tan 6.0° = 0.11$

3. Sliding Case:



3. a was explicitly assigned a minus sign since the bed is moving in $-x$ direction.

Using Newton's second law in component form

$$\Sigma F_x = ma_x: \quad f_R - W \sin 6.0° = -ma, \qquad (3)$$

$$\Sigma F_y = ma_y: \quad N - W \cos 6.0° = 0,$$

$$f_R = \mu_R N = \mu_R W \cos 6.0°.$$

Substitute this into Eq. (3):

$$\mu_R W \cos 6.0° - W \sin 6.0° = -ma,$$

$$W = mg,$$

$$\mu_R mg \cos 6.0° - mg \sin 6.0° = -ma,$$

$$a = g \sin 6.0° - \mu_R g \cos 6.0°,$$

$$a = g(\sin 6.0° - \mu_R \cos 6.0°),$$

$$a = (9.8\ \mathrm{m/s^2})(0.104 - 0.03),$$

$$a = 0.69\ \mathrm{m/s^2}.$$

Because of the explicit sign choice above, a should come out positive.

v_f = final speed,

v_i = initial speed,

x = distance traveled.

Referring back to the <u>Rectilinear Motion</u> module for one-dimensional uniform motion:

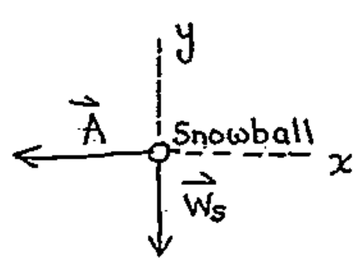
$$v_f^2 = v_i^2 + 2ax,$$

$$v_f^2 = 0^2 + 2(0.69\ \mathrm{m/s^2})(46\ \mathrm{m}),$$

$$v_f = 8.0\ \mathrm{m/s}.$$

4. (a) Acceleration of snowball is in direction of $\vec{A}$, which is in the $-x$ direction; hence the negative sign for a.

4. (a)



$\vec{A}$ is force man exerts on snowball;

$\vec{W}$ is weight of snowball.

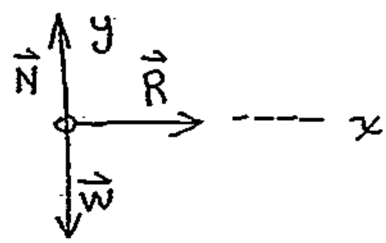
Newton's second law:

$F_x = ma_x$,

$A = ma = (0.50\text{ kg})(-25.0\text{ m/s}^2)$,

$A = -12.5$ N.

From Newton's third law: If $\vec{A}$ is action force then an equal but opposite force is exerted on the man, say $\vec{R}$:



$\vec{N}$ is normal force that ice exerts on skater;

$\vec{W}$ is weight of the skater $W = \mu g$;

$\vec{R}$ is force that snowball exerts on the man.

(b) Applying Newton's second law:

$F_x = ma_x$,

$R = Ma'$, $a' = \frac{R}{M} = \frac{12.5\text{ N}}{100\text{ kg}}$,

$a' = 0.12\text{ m/s}^2$.

a' will be in the direction of $\vec{R}$ or in +x direction.

Module _______

STUDY GUIDE

VECTOR MULTIPLICATION

INTRODUCTION

How much is A times B? This is a simple question to answer when A and B represent scalars; however, when A and B represent vectors, the answer is not obvious. In fact, on the face of it, one cannot even say whether the result should be a scalar or a vector!

Several different definitions of vector multiplication have been found useful in physics; in this module you will study two types: the scalar and vector products. Just to sharpen your interest, we point out that the vector product has the strange but useful property that $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$!

PREREQUISITES

Before you begin this module, you should be able to:	Location of Prerequisite Content
*Find values for sin θ and cos θ when θ is larger than 90° (needed for Objectives 1 and 2 of this module)	Trigonometry Review
*Transform vectors from polar form (direction and magnitude) to rectangular form (components and unit vectors) and vice versa (needed for Objectives 1 and 2 of this module)	Dimensions and Vector Addition Module

LEARNING OBJECTIVES

After you have mastered the content of this module, you will be able to:

1. Scalar product - Given two vectors (in either polar or rectangular-component form), calculate the scalar (dot) product and the angle between the vectors.

2. Vector product - Given two vectors (in either polar or rectangular-component form), calculate the vector (cross) product.

TEXT: Frederick J. Bueche, Introduction to Physics for Scientists and Engineers (McGraw-Hill, New York, 1975), second edition

SUGGESTED STUDY PROCEDURE

The text defines the scalar and vector products briefly in Section 8.1 on page 112, and in Section 11.2 on pages 178 and 179, respectively. You should look over those definitions, read General Comments 1 and 2 carefully, and work out the Problem Set. When you feel ready, try the Practice Test. If necessary reread the General Comments, and rework the appropriate problems in the Problem Set.

BUECHE

Objective Number	Readings	Problems with Solutions Study Guide	Assigned Problems Study Guide
1	Page 112 (last 3 paragraphs of Sec. 8.1), General Comment 1	A	C
2	Sec. 11.2, General Comment 2	B	D

TEXT: David Halliday and Robert Resnick, Fundamentals of Physics (Wiley, New York, 1970; revised printing, 1974)

SUGGESTED STUDY PROCEDURE

Read Section 2-4 and General Comments 1 and 2 carefully. There is some overlap, but the text does not show how to calculate dot and cross products for two vectors in unit-vector form. After the readings, work Problems A through D in the Problem Set and try the Practice Test. If you find that you need additional practice, try Problems 35, 37, 39, 45, and 47 in Chapter 2 of the text, for which answers are given.

HALLIDAY AND RESNICK

Objective Number	Readings	Problems with Solutions Study Guide	Assigned Problems Study Guide	Additional Problems
1	Sec. 2-4, General Comment 1	A	C	Chap. 2: Problems 35, 36, 37, 39, 40, 45, 47
2	Sec. 2-4, General Comment 2	B	D	

TEXT: Francis Weston Sears and Mark W. Zemansky, University Physics (Addison-Wesley, Reading, Massachusetts, 1970) fourth edition

SUGGESTED STUDY PROCEDURE

The text defines scalar and vector products briefly on pp. 95 and 139, respectively. You should look over these definitions and read General Comments 1 and 2 below. After working out Problems A through D, compare your results with the solutions given, and try the Practice Test.

SEARS AND ZEMANSKY

Objective Number	Readings	Problems with Solutions Study Guide	Assigned Problems Study Guide
1	p. 95, General Comment 1	A	C
2	p. 139, General Comment 2	B	D

TEXT: Richard T. Weidner and Robert L. Sells, Elementary Classical Physics (Allyn and Bacon, Boston, 1973) second edition, Vol. 1

SUGGESTED STUDY PROCEDURE

Read and study Sections 2-6 and 2-7 plus General Comments 1 and 2 below. There is some overlap between the text and the General Comments, but the text does not show how to calculate the cross product of two vectors in unit-vector form, as is done in General Comment 2. After the readings, work Problems A through D in the Problem Set, and try the Practice Test. If you find that you need more practice, try the additional problems in the text listed below, for which answers can be found on p. 8 of the Appendix.

WEIDNER AND SELLS

Objective Number	Readings	Problems with Solutions Study Guide	Assigned Problems Study Guide	Additional Problems
1	Sec. 2-6, General Comment 1	A	C	2-15, 2-21(a)
2	Secs. 2-6,2-7 General Comment 2	B	D	2-17, 2-19, 2-21(b), 2-23

GENERAL COMMENTS

1. Calculation of the Scalar (Dot) Product

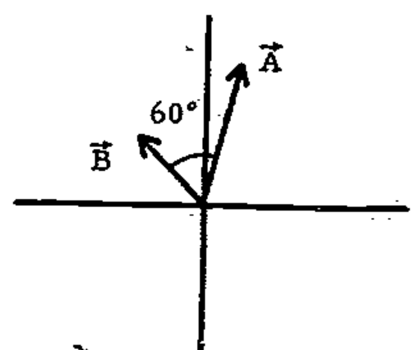
The scalar, or dot, product of two vectors is defined as follows:

$$\vec{A}\cdot\vec{B} = |\vec{A}|\ |\vec{B}| \cos\theta,$$

where θ represents the angle between the two vectors, $|\vec{A}|$ represents the magnitude of $\vec{A}$, and $|\vec{B}|$ represents the magnitude of $\vec{B}$. Thus the scalar product results in a scalar or number, rather than another vector. If the vectors have the same direction, then $\theta = 0$, $\cos\theta = 1$, and the dot product is equal to the product of the magnitudes of the two vectors, $|\vec{A}|\ |\vec{B}|$. However, if the vectors are not parallel, then $\cos\theta < 1$, and the dot product is less than the algebraic product of the magnitudes. In particular, if the vectors are perpendicular to each other, then $\theta = 90°$, $\cos\theta = 0$, and the dot product is zero. The above definition is easy to use when both vectors are expressed in polar form and one can find θ easily. However, when one or both vectors are in rectangular form, θ is not usually known, and one cannot use the above equation.

Example

Given that $\vec{A}$ has magnitude 6.0 m with $\theta_A = 80°$, and $\vec{B}$ has magnitude 3.0 m with $\theta_B = 140°$, calculate $\vec{A}\cdot\vec{B}$.

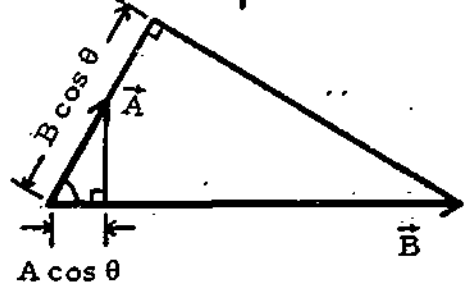


Solution

From the diagram we see that the angle between the vectors is 60°; thus

$$\vec{A}\cdot\vec{B} = (6.0\text{ m})(3.0\text{ m})\cos 60° = 9.0\text{ m}^2.$$

The dot product can also be interpreted as the component of $\vec{A}$ in the direction of $\vec{B}$ multiplied by $|\vec{B}|$, or alternatively, as the component of $\vec{B}$ in the direction of $\vec{A}$, multiplied by $|\vec{A}|$, as in the drawing on the right.



We shall now derive a general formula for the dot product of two vectors written in terms of their rectangular components. To do this, we need to know the dot products of the unit vectors with each other. First, $\hat{i}\cdot\hat{i} = 1$ since a vector is parallel to itself and the magnitude of $\hat{i}$ is 1. Similarly, $\hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$. Furthermore, since $\hat{i}$ is perpendicular to $\hat{j}$, $\hat{i}\cdot\hat{j} = 0$, and, similarly, $\hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$. Collecting these results:

$$\begin{aligned}
\hat{i}\cdot\hat{i} &= \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1,\\
\hat{i}\cdot\hat{j} &= \hat{j}\cdot\hat{i} = 0,\\
\hat{j}\cdot\hat{k} &= \hat{k}\cdot\hat{j} = 0,\\
\hat{k}\cdot\hat{i} &= \hat{i}\cdot\hat{k} = 0.
\end{aligned}$$

Now suppose we have two vectors, $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$, and we wish to calculate their dot product $\vec{A}\cdot\vec{B}$:

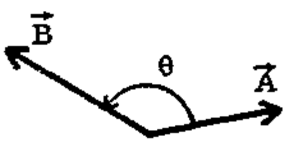
$$\begin{aligned}
\vec{A}\cdot\vec{B} &= (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})\\
&= A_xB_x(\hat{i}\cdot\hat{i}) + A_xB_y\overbrace{(\hat{i}\cdot\hat{j})}^{0} + A_xB_z\overbrace{(\hat{i}\cdot\hat{k})}^{0}\\
&\quad + A_yB_x\overbrace{(\hat{j}\cdot\hat{i})}^{0} + A_yB_y(\hat{j}\cdot\hat{j}) + A_yB_z\overbrace{(\hat{j}\cdot\hat{k})}^{0}\\
&\quad + A_zB_x\overbrace{(\hat{k}\cdot\hat{i})}^{0} + A_zB_y\overbrace{(\hat{k}\cdot\hat{j})}^{0} + A_zB_z(\hat{k}\cdot\hat{k}).
\end{aligned}$$

Thus,

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z.$$

This formula makes it easy to calculate the dot product of two vectors expressed in terms of the three unit vectors. For two-dimensional vectors, one of the components equals zero, and the above formula reduces to two terms.

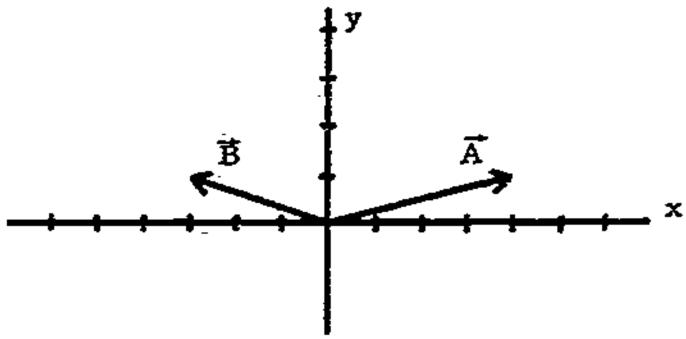
Let us consider the value of the dot product when the angle between the two vectors is greater than 90° (see figure). Using the definition



$|\vec{A}|\ |\vec{B}| \cos\theta$, one sees that $\cos\theta$ is negative, and therefore the dot product itself is negative.

To illustrate this we shall calculate a dot product for two vectors with $\theta > 90°$. Let $\vec{A} = 4\hat{i} + \hat{j}$ and $\vec{B} = -3\hat{i} + \hat{j}$; inspection of the figure below shows that $\theta > 90°$.

$$\begin{aligned}\vec{A}\cdot\vec{B} &= A_xB_x + A_yB_y\\ &= (4)(-3) + (1)(1)\\ &= -12 + 1\\ &= -11.\end{aligned}$$



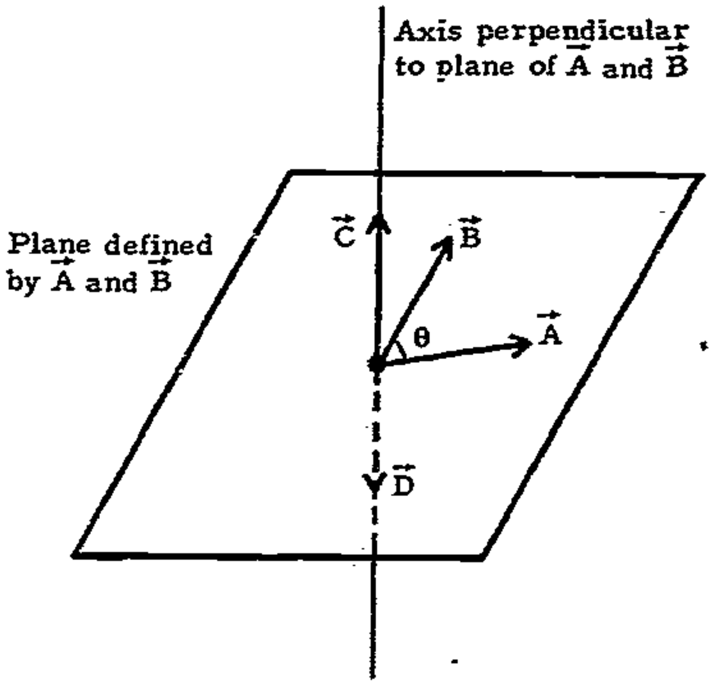
Thus the dot product is negative, as stated above.

<u>Exercise:</u> Transform $\vec{A}$ and $\vec{B}$ to polar form, calculate the dot product from the definition, and check that the result is equal to the value calculated above.

In summary, there are two ways to calculate the dot product: using $|\vec{A}|\ |\vec{B}| \cos\theta$ or $A_xB_x + A_yB_y + A_zB_z$. The first formula is useful when the vectors are expressed in polar form in two dimensions, or whenever you know the magnitudes of the vectors and the angle between them. The second formula is much easier to use when the vectors are expressed in terms of rectangular components.

2. <u>Calculation of the Vector (Cross) Product</u>

The vector product of two vectors is written as $\vec{A} \times \vec{B}$ and is equal to a third vector $\vec{C}$; the magnitude of $\vec{C}$ is defined to be $|\vec{C}| = |\vec{A}|\ |\vec{B}| \sin\theta$, where θ is the smaller angle between $\vec{A}$ and $\vec{B}$. The direction of $\vec{C}$ is defined to be perpendicular to <u>both</u> $\vec{A}$ and $\vec{B}$, or, in other words, along an axis perpendicular to the plane formed by $\vec{A}$ and $\vec{B}$ (see diagram). The direction of $\vec{C}$ along this axis is



specified by the "right-hand rule." You should imagine that the fingers of your right hand are curled around the axis, pushing $\vec{A}$ into $\vec{B}$ through the angle θ. If the thumb is held erect, it will indicate the positive direction of $\vec{A} \times \vec{B}$. Note that there are two possible angles between $\vec{A}$ and $\vec{B}$: θ and (360° - θ); ambiguity is removed by always choosing the <u>smaller</u> angle, as in the diagram.

Notice that the positive direction of the cross product depends crucially upon the order of $\vec{A}$ and $\vec{B}$ in the product. Using the right-hand rule and the diagram to find the direction of $\vec{D} = \vec{B} \times \vec{A}$ shows that $\vec{D}$ points in the opposite sense to $\vec{C}$. Thus $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$; this is an important property of the cross product. Note that, in contrast to the dot product, the cross product results in a vector; these properties are the origin of the names scalar and vector product.

<u>Example</u>

Evaluate the cross products of the various unit vectors: $\hat{i} \times \hat{i}$, $\hat{i} \times \hat{j}$, $\hat{i} \times \hat{k}$, $\hat{j} \times \hat{i}$, $\hat{j} \times \hat{j}$, $\hat{j} \times \hat{k}$, $\hat{k} \times \hat{i}$, $\hat{k} \times \hat{j}$, and $\hat{k} \times \hat{k}$. (<u>Hint</u>: Remember that $\vec{A} \times \vec{B} = |\vec{A}|\ |\vec{B}| \sin\theta$.)

<u>Solution</u>

$$\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0 \ (\sin 0° = 0),$$

$$\hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{k} = \hat{i},\ \hat{k}\times\hat{i} = \hat{j} \ (\sin 90° = 1, \text{ and right-hand rule}),$$

$$\hat{j}\times\hat{i} = -\hat{k};\ \hat{k}\times\hat{j} = -\hat{i};\ \hat{i}\times\hat{k} = -\hat{j} \ (\sin 90° = 1, \text{ and right-hand rule}).$$

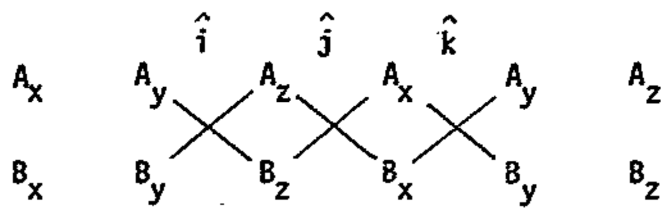
Using these results, we can calculate a general formula for the cross product of two vectors in component form. If $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$,

$$\begin{aligned}\vec{A}\times\vec{B} &= (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \times (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})\\ &= A_xB_x(\overbrace{\hat{i}\times\hat{i}}^{0}) + A_xB_y(\hat{i}\times\hat{j}) + A_xB_z(\hat{i}\times\hat{k})\\ &\quad + A_yB_x(\hat{j}\times\hat{i}) + A_yB_y(\overbrace{\hat{j}\times\hat{j}}^{0}) + A_yB_z(\hat{j}\times\hat{k})\\ &\quad + A_zB_x(\hat{k}\times\hat{i}) + A_zB_y(\hat{k}\times\hat{j}) + A_zB_z(\overbrace{\hat{k}\times\hat{k}}^{0}).\end{aligned}$$

Thus

$$\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}.$$

This result is worth preserving, but it is difficult to remember in this form. A device to assist in remembering the cross product is as follows: Write the three components of the two vectors twice as follows:



The components of the cross product can then be read from the three ×s. The first × gives the x component $(A_yB_z - A_zB_y)$; the second × gives the y component $(A_zB_x - A_xB_z)$; and the third × gives the z component $(A_xB_y - A_yB_x)$. To summarize,

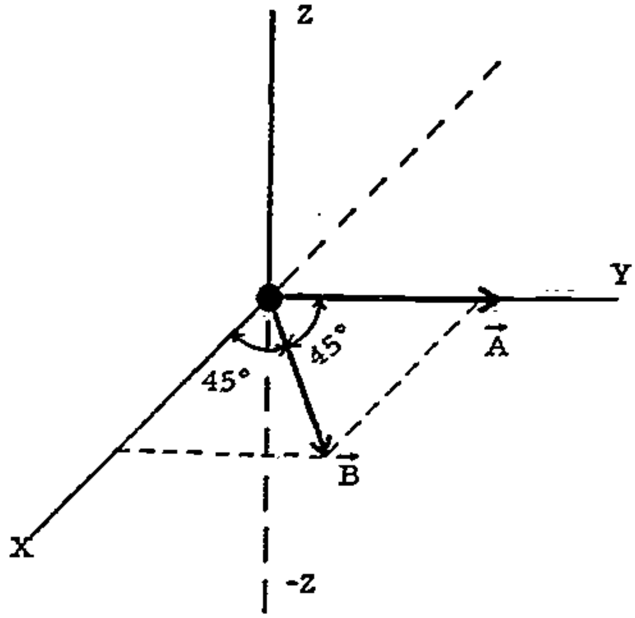
$$\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}.$$

This formula provides a way to calculate the cross product of two vectors directly when the original vectors are given in rectangular form. However, when the vectors are given in some other form, one can either transform them to rectangular coordinates, or use the alternative formula for the magnitude, $|\vec{A} \times \vec{B}| = |\vec{A}|\,|\vec{B}| \sin\theta$. With the latter method, one must find the direction of the product vector by use of the right-hand rule and by remembering that it is perpendicular to both $\vec{A}$ and $\vec{B}$.

For example, suppose that we wish to find the cross product of two vectors, <u>both</u> of which are in the xy plane. $\vec{A}$ is 4.2 cm in magnitude, along the y axis, and $\vec{B}$ has magnitude 3.6 cm and points in a direction at 45° to the positive y and positive x axes, as shown in the diagram.

The magnitude of the cross product can be found easily:

$$\begin{aligned}|\vec{A} \times \vec{B}| &= |\vec{A}|\,|\vec{B}| \sin\theta\\ &= (4.2)(3.6)\sin(45°)\\ &= (15.1)(0.707)\\ &= 11\ \text{cm}^2.\end{aligned}$$



The direction is somewhat more difficult to find; we know that $\vec{A} \times \vec{B}$ must lie along the z axis because this is the only direction perpendicular to both vectors, and the right-hand rule shows that the result must lie along the negative z axis. Therefore,

$$\vec{A} \times \vec{B} = (-11\hat{k})\ \text{cm}^2.$$

As another example, we seek the cross product of

$$\vec{A} = 3\hat{i} + \hat{j} - \hat{k} \qquad \text{and} \qquad \vec{B} = -2\hat{i} - 5\hat{j} + 6\hat{k}.$$

Since these two vectors are expressed in rectangular form, we can calculate

$$\vec{A} \times \vec{B} = (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$$

$$= [(1)(6) - (-5)(-1)]\hat{i} + [(-1)(-2) - (3)(6)]\hat{j} + [(3)(-5) - (1)(-2)]\hat{k}$$

$$= (1)\hat{i} + (-16)\hat{j} + (-13)\hat{k}.$$

Thus,

$$\vec{A} \times \vec{B} = \hat{i} - 16\hat{j} - 13\hat{k}.$$

PROBLEM SET WITH SOLUTIONS

A(1). Given $\vec{D} = \hat{i} + 3\hat{j} - 2\hat{k}$ and $\vec{E} = 4\hat{i} - 5\hat{j} + \hat{k}$, calculate the following:

(a) $\vec{D}\cdot\vec{E}$;

(b) the angle between $\vec{D}$ and $\vec{E}$.

Solution

(a) $\vec{D}\cdot\vec{E} = D_xE_x + D_yE_y + D_zE_z = (1)(4)\ m^2 + (3)(-5)\ m^2 = (4 - 15 - 2)\ m^2 = -13\ m^2.$

(b) We wish to find θ; we could use either

$$\vec{D}\cdot\vec{E} = |\vec{D}|\ |\vec{E}| \cos\theta \qquad \text{or} \qquad |\vec{D} \times \vec{E}| = |\vec{D}|\ |\vec{E}| \sin\theta.$$

Using the former, we have

$$\cos\theta = \vec{D}\cdot\vec{E}/|\vec{D}|\ |\vec{E}| .$$

From above we have that the numerator $\vec{D}\cdot\vec{E} = -13\ m^2$. Now,

$$|\vec{D}| = \sqrt{D_x^2 + D_y^2 + D_z^2} = \sqrt{1 + 9 + 4}\ m = \sqrt{14}\ m = 3.75\ m,$$

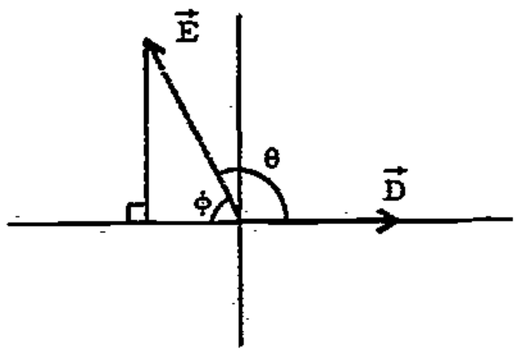
and

$$|\vec{E}| = \sqrt{E_x^2 + E_y^2 + E_z^2} = \sqrt{16 + 25 + 1}\ m = \sqrt{42}\ m = 6.48\ m.$$

(Note: We have preserved three significant figures because the calculation is not yet completed.) Therefore,

$$\cos\theta = \frac{-13\ m^2}{(3.75)(6.48)\ m^2} = 0.54 .$$

The fact that the dot product (and therefore $\cos\theta$) is negative alerts us to the fact that the vectors are at an obtuse angle with each other and $\theta > 90°$, as shown. Looking up 0.54 in a trig table gives $\phi = 58°$. But the angle we want is $\theta = 180° - \phi = 122°$.



B(2). Given the vectors from Problem A, calculate the following quantities:

(a) $\vec{D} \times \vec{E}$;

(b) $\vec{E} \times \vec{D}$ (show explicitly that $\vec{D} \times \vec{E} = -\vec{E} \times \vec{D}$).

Solution

(a) $$\vec{D} \times \vec{E} = (D_yE_z - D_zE_y)\hat{i} + (D_zE_x - D_xE_z)\hat{j} + (D_xE_y - D_yE_x)\hat{k}$$

$$= [(3)(1) - (-2)(-5)]\hat{i} + [(-2)(4) - (1)(1)]\hat{j} + [(1)(-5) - (3)(4)]\hat{k},$$

$$\vec{D} \times \vec{E} = -7\hat{i} - 9\hat{j} - 17\hat{k}.$$

(b) $$\vec{E} \times \vec{D} = (E_yD_z - E_zD_y)\hat{i} + (E_zD_x - E_xD_z)\hat{j} + (E_xD_y - E_yD_x)\hat{k}$$

$$= [(-5)(-2) - (1)(3)]\hat{i} + [(1)(1) - (4)(-2)]\hat{j} + [(4)(3) - (-5)(1)]\hat{k}$$

$\vec{E} \times \vec{D} = +7\hat{i} + 9\hat{j} + 17\hat{k} = -\vec{D} \times \vec{E}$ by comparison with (a).

Problems

C(1). Given $\vec{A}$ with magnitude 15 m pointing along the positive y axis, and $\vec{B} = (4\hat{i} - 5\hat{k})$ m, calculate the following quantities:

(a) $\vec{A}\cdot\vec{B}$;

(b) the angle between $\vec{A}$ and $\vec{B}$.

D(2). Given the vectors from Problem C, calculate the following quantities:

(a) $\vec{A} \times \vec{B}$;

(b) $\vec{B} \times \vec{A}$ (show explicitly that $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$).

Solutions

C(1). (a) 0; (b) 90°.

D(2). (a) $\vec{A} \times \vec{B} = (-75\hat{i} - 60\hat{k})\ \mathrm{m}^2$; (b) $\vec{B} \times \vec{A} = (75\hat{i} + 60\hat{k})\ \mathrm{m}^2$.

PRACTICE TEST

1. Suppose $\vec{A}$, a vector in the xy plane, can be written in polar form as (2.4 m, 310°), and $\vec{B} = -1.0\hat{i} - 1.0\hat{j}$. Calculate the following:

   (a) $\vec{A}\cdot\vec{B}$;

   (b) the angle between $\vec{A}$ and $\vec{B}$.

2. A vector $\vec{A}$ in the xy plane of magnitude 0.2 m is directed at an angle of 155° with the positive x axis. $\vec{B} = (-0.4\hat{k})$ m. Calculate $\vec{A} \times \vec{B}$.

Practice Test Answers

1. (a) $\vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}| \cos\theta = (2.4\ \mathrm{m})(\sqrt{2}\ \mathrm{m}) \cos 85° = 0.30\ \mathrm{m}^2$;

   (b) $\theta = 85°$.

2. $\vec{A} \times \vec{B} = |\vec{A}|\,|\vec{B}| \sin\theta = (0.2\ \mathrm{m})(0.4\ \mathrm{m}) \sin(90°) = 0.08\ \mathrm{m}^2$.

   $\vec{A} \times \vec{B}$ is directed at an angle of 155° + 90° = 240° with the positive x axis.

Date ____________

VECTOR MULTIPLICATION

Mastery Test   Form A

pass   recycle

1 2

Name ______________________ Tutor ______________________

1. Given $\vec{A} = (6.0\hat{i} + 1.0\hat{j} - 2.0\hat{k})$ m and $\vec{B} = (-4.0\hat{i} - 3.0\hat{k})$ m, calculate
   (a) $\vec{A}\cdot\vec{B}$;
   (b) the angle between $\vec{A}$ and $\vec{B}$.

2. Given $\vec{A} = 3.0\hat{i} + 3.0\hat{j} - 2.0\hat{k}$ and $\vec{B} = -1.0\hat{i} - 4.0\hat{j} + 2.0\hat{k}$, calculate $\vec{A} \times \vec{B}$.

Date ______________

VECTOR MULTIPLICATION   pass   recycle

Mastery Test   Form B   1   2

Name ______________________________ Tutor ______________________________

1. A certain vector $\vec{A}$ in the xy plane is 250° counterclockwise from the positive x axis and has magnitude 7.4 cm. Vector $\vec{B}$ has magnitude 5.0 cm and is directed parallel to the z axis. Calculate
   (a) $\vec{A} \cdot \vec{B}$;
   (b) the angle between $\vec{A}$ and $\vec{B}$.

2. For the vectors in Problem 1, calculate $\vec{A} \times \vec{B}$.

Date ____________

VECTOR MULTIPLICATION

pass recycle

Mastery Test Form C

1 2

Name ______________________ Tutor ______________________

1. Given $\vec{A} = -2.0\hat{j} + \hat{k}$ and $\vec{B} = 4.0\hat{i} - 3.0\hat{k}$, calculate
   (a) $\vec{A}\cdot\vec{B}$;
   (b) the angle between $\vec{A}$ and $\vec{B}$.

2. For the vectors in Problem 1, calculate $\vec{A} \times \vec{B}$.

MASTERY TEST GRADING KEY - Form A

What To Look For	Solutions
1.(a) The answer must be of the proper sign. If the units and/or the number of significant figures in the answer are not correct, remind the student of this, but do not mark it incorrect.	1.(a) $\vec{A}\cdot\vec{B} = [(6.0)(-4.0) + (1.0)(0) + (-2.0)(-3.0)]\ m^2$ $= (-24 + 6.0)\ m^2$ $= -18\ m^2.$
(b) If the student gives $\theta = 56°$, he probably picked the incorrect quadrant, and this should be marked wrong.	(b) $\lvert\vec{A}\rvert = \sqrt{(36 + 1.0 + 4)\ m^2} = \sqrt{41\ m^2}.$ $\lvert\vec{B}\rvert = \sqrt{(16 + 9)\ m^2} = \sqrt{25\ m^2} = 5.0\ m.$ $\cos\phi = \dfrac{-18\ m^2}{(\sqrt{41\ m^2})(5.0\ m)} = -0.56.$ $\phi = 56°$, and $\theta = 124°.$
2. Answer should be in unit-vector form with no units.	2. $\vec{A} \times \vec{B} = (6.0 - 8.0)\hat{i} + (+2.0 - 6.0)\hat{j} + (-12 + 3.0)\hat{k}$ $= -2.0\hat{i} - 4.0\hat{j} - 9.0\hat{k}.$

MASTERY TEST GRADING KEY - Form B

What To Look For	Solutions
1.(a) Student can see directly from statement of problem that $\theta = 90°$ and $\vec{A}\cdot\vec{B} = 0$, but converting to unit vectors and doing complete calculation is also satisfactory, although the easy way should be pointed out.	1.(a) Since vectors are perpendicular, $\vec{A}\cdot\vec{B} = 0$.
(b) See above.	(b) $\theta = 90°$.
2. If result does not have the correct units and number of significant figures, remind the student of this, but do not mark it incorrect. Results can be expressed in either component or polar form; numbers must be correct.	2. $\lvert\vec{A} \times \vec{B}\rvert = 37$ cm, $\theta = 160°$. $\vec{A} \times \vec{B} = -35\hat{i} + 13\hat{j}$.

MASTERY TEST GRADING KEY - Form C

What To Look For	Solutions
1.(a) Answer should be dimensionless with two significant figures; remind student of this if necessary, but do not grade question incorrect for this. Numbers must be correct.	1.(a) $\vec{A}\cdot\vec{B} = -3.0$.
(b) If student gives $180° - \theta$ instead of θ it should be marked incorrect, and the grader should point out how to pick quadrant when $\cos\theta$ is negative.	(b) $\cos\phi = \vec{A}\cdot\vec{B}/\lvert\vec{A}\rvert\,\lvert\vec{B}\rvert = -3.0/(\sqrt{5})(5) = -0.27$; $\phi = 74°$. $\theta = 180° - \phi = 106°$.
2. Same as 1(a) above.	2. $\vec{A}\times\vec{B} = 6.0\hat{i} + 4.0\hat{j} + 8.0\hat{k}$.