This text is one of the sequence of textbooks produced for low achievers in the seventh and eighth grades by the School Mathematics Study Group (SMSG). There are eight texts in the sequence, of which this is the sixth. This set of volumes differs from the regular editions of SMSG junior high school texts in that very little reading is required. Concepts and processes are illustrated pictorially, and many exercises are included. Similarity of triangles is the focus of the first chapter (12) in this volume. The use of ratios and scale factors is introduced, and the computation of percentages by construction of parallel lines on a grid is developed. In chapter 13 the emphasis is on computation with rational numbers in both common fraction and decimal forms. In this context exponents are introduced. Chapter 14 deals with motion geometry and perpendicularity. (SD)
SECONDARY SCHOOL
MATHEMATICS
SPECIAL EDITION

Chapter 12, Similarity

Chapter 13. More About
Rational Numbers

Chapter 14: Perpendiculars

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Chapter 12.

SIMILARITY
Chapter 12

SIMILARITY

Similar Figures

In general, we say that two figures are similar if they have the same shape but not necessarily the same size. For example,

Any two circles are similar.

Any two squares are similar.

Any two line segments are similar.

Of course, it is not true that any two triangles are similar. For example, look at the two triangles below.

They certainly do not have the same shape.

On the other hand there are triangles that are similar. The two below are.
You can think about similar triangles in terms of shrinking or stretching. We can stretch the smaller triangle so that it is the size of the larger one. Or, we shrink the larger one down to the size of the smaller one.

The class discussion exercises that follow will show you how, mathematically, you can stretch or shrink a triangle to form a similar triangle.

Class Discussion

1. On the coordinate plane below plot and label the points $A(0,0)$, $B(4,0)$, and $C(4,3)$. 
2. Draw segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$. (Use a straightedge.)

3. What kind of triangle has been formed?

4. In $\triangle ABC$:
   
   (a) The length of side $\overline{AB}$ is ______ units.
   
   (b) The length of side $\overline{BC}$ is ______ units.
   
   (c) Is there any way we can count the number of units in the length of side $\overline{AC}$? ______

5. (a) Take your compass and place the needle point at the origin (point A) and the pencil point at point C.
   
   (b) Now draw an arc intersecting the x-axis.
   
   (c) What is the coordinate of the point where the arc you just drew intersects the x-axis? ______
   
   (d) How many units in length does side $\overline{AC}$ seem to be? ______

6. The coordinates of points A, B, and C are written below. Multiply each coordinate for each point by 2, thus finding the coordinates for points $A'$, $B'$, and $C'$.

   $A(0,0)$
   $B(4,0)$
   $C(4,3)$

   $A'(__, __)$
   $B'(__, __)$
   $C'(__, __)$

7. Plot and label the points $B'(8,0)$, and $C'(8,6)$ on the same coordinate plane as you did for points A, B, and C. $A'(0,0)$ is the same as $A(0,0)$. We will refer to this point as $A'$ when discussing $\triangle A'B'C'$.

8. Draw segments $\overline{A'B'}$, $\overline{B'C'}$, and $\overline{A'C'}$. (Use a straightedge.)

9. What kind of triangle has been formed? ______

10.
10. In $\triangle A'B'C'$

(a) The length of side $A'B'$ is _______ units.
(b) The length of side $B'C'$ is _______ units.
(c) How many units in length would you guess side $A'C'$ to be? _______
(d) Use the same process you used in Problem 5 and see if your guess is correct.

11. In order for you to see more clearly the two triangles you just drew, we have taken them "off the grid" and "separated" them.

Now let us compare the lengths of corresponding sides.

(a) The length of side $A'B'$ is __________ times the length of side $AB$.
(b) The length of side $B'C'$ is __________ times the length of side $BC$.
(c) The length of side $A'C'$ is __________ times the length of side $AC$.

You can see that the length of each side of the larger triangle is 2 times the length of the corresponding sides of the smaller triangle. In other words, we can stretch $\triangle ABC$ into $\triangle A'B'C'$ by multiplying the length of each side by 2. In the same way we can shrink $\triangle A'B'C'$ into $\triangle ABC$ by multiplying the length of each side of $\triangle A'B'C'$ by $\frac{1}{2}$.
The numbers 2 and $\frac{1}{2}$ are called scale factors. In the triangles you drew, when we stretched the smaller triangle onto the larger triangle we multiplied by a scale factor of 2. If we go the other way and shrink the larger triangle onto the smaller triangle then we multiply by a scale factor of $\frac{1}{2}$.

You can see that when two figures are similar there are two scale factors, one for stretching and one for shrinking, and they are the reciprocals of each other.

**Exercises**

1. In each of the problems below the triangles are similar. Find the scale factor for stretching and the scale factor for shrinking each triangle onto the other.

   (a) ![Diagram](image)

   The scale factor for shrinking the larger triangle onto the smaller triangle is ________.
   The scale factor for stretching the smaller triangle onto the larger triangle is ________.

   (b) ![Diagram](image)

   The scale factor for stretching the smaller triangle onto the larger triangle is ________.
   The scale factor for shrinking the larger triangle onto the smaller triangle is ________.
The scale factor for shrinking the larger triangle onto the smaller triangle is ________.
The scale factor for stretching the smaller triangle onto the larger triangle is ________.

2. In each problem below the triangles are similar. Decide whether the side whose length is not known is in the smaller or larger triangle. Then find the scale factor and length of the side of the triangle not given for you.

Example.

(i) The scale factor is \( \frac{3}{5} \).
(ii) The length of side \( \overline{AB'} \) is \( \frac{3}{5} \times 5 \) or 15.

(a)
(b) The scale factor is \[ \text{scale factor} \]

(ii) The length of side \( BC \) is \[ \text{length of side } BC \] or \[ \text{length of side } BC \]
Similar Triangles

Class Discussion

Pictured below are two similar triangles, much like the ones you drew in the last lesson.

1. Look at triangle \( \triangle ACB \).
   What is the measure of \( \angle ACB \) ?

2. What is the sum of the measures of the angles of a triangle?

3. If the measure of the angle at \( A \) is 60 then what must be the measure of \( \angle x \) ?

4. Look at triangle \( \triangle AC'B' \). What is the measure of \( \angle AC'B' \) ?

5. What must be the measure of \( \angle x' \) ?

6. Fill in the blanks for the measures of each angle for \( \triangle ABC \) and \( \triangle AB'C' \).

   \[ \begin{array}{ll}
   \triangle ABC & \triangle AB'C' \\
   (a) m \angle A = \_\_\_\_ & m \angle A = \_\_\_\_ \\
   (b) m \angle ACB = \_\_\_\_ & m \angle AC'B' = \_\_\_\_ \\
   (c) m \angle x = \_\_\_\_ & m \angle x' = \_\_\_\_ \\
   \end{array} \]

(d) Complete the sentence below.

   If two triangles are similar the corresponding angles have \_\_\_\_ measure.
Before we go on, let us review exactly what we mean by corresponding sides and corresponding angles.

Look at the two similar triangles below:

\[ \triangle ABC \quad \text{and} \quad \triangle A'B'C' \]

- \( \overline{AB} \) and \( \overline{A'B'} \) are corresponding sides
- \( \overline{AC} \) and \( \overline{A'C'} \) are corresponding sides
- \( \overline{CB} \) and \( \overline{C'B'} \) are corresponding sides
- \( \angle x \) and \( \angle x' \) are corresponding angles
- \( \angle y \) and \( \angle y' \) are corresponding angles
- \( \angle z \) and \( \angle z' \) are corresponding angles

Now we know how to tell if two triangles are similar:

1. Their corresponding angles must be equal in measure,
2. each pair of corresponding sides must have the same scale factor.

**Exercises**

The triangles in each pair are similar. Mark pairs of corresponding congruent angles and pairs of corresponding sides as is shown in the example.

**Example**.
5. For each pair of similar triangles find the scale factor. Then find the missing lengths of the remaining sides.

Example.

![Diagram of similar triangles]

(i) The scale factor is \( \frac{1}{2} \)

(ii) The length of \( \overline{BC} \) is \( \frac{1}{2} \) \( \times \) \( 6 \) or \( 3 \) \( \times \) \( 8 \) or \( 4 \)

(iii) The length of \( \overline{AC} \) is \( \frac{1}{2} \) \( \times \) \( 8 \) or \( 4 \)

(a)

![Diagram of similar triangles]

(i) The scale factor is 

(ii) The length of \( \overline{C'B'} \) is 

(iii) The length of \( \overline{A'B'} \) is 
Find the scale factor and the missing lengths in the similar triangles below.

(i) The scale factor is ____.
(ii) The length of $\overline{AC'}$ is ____ or ____.
(iii) The length of $\overline{B'C'}$ is ____ or ____.

BRAINBOOSTER.

6. Find the scale factor and the missing lengths in the similar triangles below.

(i) The scale factor is ____.
(ii) The length of side $\overline{AC}$ is ____ or ____.
(iii) The length of side $\overline{EC}$ is ____ or ____.
Ratios and Scale Factors

In an earlier chapter you learned that the ratio of the number "a" to the number "b" is written as the fraction \( \frac{a}{b} \) (of course, "b" cannot be zero).

As examples, the ratio of:

- 3 to 4 is \( \frac{3}{4} \),
- 5 to 7 is \( \frac{5}{7} \),
- 7 to 10 is \( \frac{7}{10} \),
- 8 to 3 is \( \frac{8}{3} \).

Look at the similar triangles below.

You can see that to stretch \( \triangle ABC \) onto \( \triangle A'B'C' \) we would use a scale factor of 2. Or, to shrink \( \triangle A'B'C' \) onto \( \triangle ABC \) we would use a scale factor of \( \frac{1}{2} \).

An easy way to find the scale factor for two similar triangles is to compare the lengths of corresponding sides by writing them as a ratio. In the drawings above:

\[
\frac{\text{length of } A'B'}{\text{length of } AB} = \frac{10}{5} = 2
\]
\[
\frac{\text{length of } A'C'}{\text{length of } AC} = \frac{6}{3} = 2
\]
\[
\frac{\text{length of } B'C'}{\text{length of } BC} = \frac{8}{4} = 2
\]
Now this tells us that:

\[
\frac{\text{length of } A'B'}{\text{length of } AB} = \frac{\text{length of } A'C'}{\text{length of } AC} = \frac{\text{length of } B'C'}{\text{length of } BC},
\]

that is, the ratios of the lengths of corresponding sides of similar triangles are equal.

We can now use this idea of equal ratios to solve problems involving similar triangles.

**Class Discussion**

The two triangles above are similar. We want to find the lengths of sides $A'C'$ and $B'C'$.

1. The ratio $\frac{\text{length of } A'B'}{\text{length of } AB} = \frac{6}{\phantom{6}}$

2. The ratio $\frac{\text{length of } A'C'}{\text{length of } AC} = \frac{x}{\phantom{x}}$

3. Knowing that the ratios of the lengths of corresponding sides of similar triangles are equal gives $x = \frac{6}{\phantom{6}}$ (Write in the missing denominators.)
4. You know that if \( \frac{a}{b} = \frac{c}{d} \), then by multiplying as shown,

\[
\begin{align*}
\frac{a}{b} \cdot \frac{c}{d} &= \frac{ac}{bd} \\
\therefore a \cdot d &= b \cdot c.
\end{align*}
\]

This is how we compare two rational numbers and is called the **Comparison Property**. Use the Comparison Property and fill in the blanks.

\[
\frac{\_}{\_} \cdot x = \frac{\_}{\_} \cdot 6
\]

so, \( 3 \cdot x = 30 \)

5. To solve the equation \( 3 \cdot x = 30 \) we must multiply both sides of the equation by \( \frac{1}{3} \). So \( \frac{\_}{\_} \cdot 3 \cdot x = \frac{\_}{\_} \cdot 30 \) and \( x = \_ \).

6. The value for \( x \), and thus the length of side \( \overline{A'C'} \), is __________.

Let us use the same procedure to find the length of side \( \overline{B'C'} \).

7. The ratio \( \frac{\text{length of } \overline{A'B'}}{\text{length of } \overline{AB}} = \_ \)

8. The ratio \( \frac{\text{length of } \overline{B'C'}}{\text{length of } \overline{BC}} = \_ \)

9. Knowing that the ratios of the lengths of corresponding sides of similar triangles are equal gives \( y = \frac{6}{\text{missing denominators}} \).

10. Use the comparison property and fill in the missing numbers.

\[
\frac{\_}{\_} \cdot y = \frac{\_}{\_} \cdot \_
\]

so, \( 3 \cdot y = 36 \)

11. To solve the equation \( 3 \cdot y = 36 \) we multiply both sides of the equation by \( \frac{1}{3} \). So, \( \frac{\_}{\_} \cdot 3 \cdot y = \frac{\_}{\_} \cdot 36 \) and \( y = \_ \).

12. The value for \( y \), and thus the length of side \( \overline{B'C'} \) is __________.
Exercises

Use equal ratios to find the missing lengths. The triangles in each problem are similar.

1.

(Show your work in the space below.)

\[ x = \]

2.

(Show your work in the space below.)

\[ x = \]
3. At

Show your work in the space below.

\[ x = \quad \quad y = \quad \]

4.

Show your work in the space below.

\[ x = \quad \quad y = \quad \]
5. (Show your work in the space below.)

6. In Problem 3, what is the scale factor for stretching \( \triangle ABC \) onto \( \triangle A'B'C' \)?
BRAINBOOSTER.

7. We have a flagpole which we want to find the height of. We do not want to cut it down nor do we want to climb it. How do we find the height of the flagpole?

Directions:

(i) Measure the length of the flagpole's shadow, AB.

(ii) Measure the length of a friend's shadow, A'B', at the same time.

(iii) Measure the friend's height B'C.

If the two triangles are similar (and they are) you can easily solve this problem since

\[
\frac{\text{length of the pole's shadow}}{\text{length of the friend's shadow}} = \frac{\text{height of the pole} (x)}{\text{height of the friend}}
\]

If the friend is 6 feet tall and casts a 4 foot shadow, then how high is the pole if its shadow is 24 feet?
How a Photo Enlarger Works: Parallels and Similarity

If we have a triangle, such as pictured below

and we draw a segment $DE$, parallel to one of the sides.

then the triangle formed, $\triangle DEB$, is similar to the triangle we started with, $\triangle ACE$.

A photo enlarger is a fairly simple application of this idea. Basically, it is just a box with a horizontal glass shelf in the middle. The box is light-proof except for a pinhole in the top through which light may pass. (In order to admit more light, lenses are used instead of a pinhole, but the light coming through the lenses acts as though it came from a pinhole.)

Figure 1
In the dark, a photo negative is placed on the glass shelf and a piece of light-sensitive photo paper is placed on the bottom. The door is then closed and the light is turned on. At any point on the photo negative, the shading determines how much light can pass through to reach a point on the paper. Thus a correspondence is established between points on the photo negative and points on the paper.

We will say that the vertical distances from the pinhole to the two planes are 12 inches and 24 inches.

You can see that QA is parallel to Q'A', so that

\[ \triangle PQA \text{ is similar to } \triangle P'Q'A'. \]

The scale factor in stretching \( \triangle PQA \) onto \( \triangle P'Q'A' \) is

\[
\frac{\text{length of } PQ'}{\text{length of } PQ} = \frac{24}{12} = 2.
\]
Now if we take any other point $B$ on the photo negative (as in Figure 3) then $A'B'$ is parallel to $AB$.

Figure 3

and we have $\triangle PBA$ similar to $\triangle PB'A'$. Again the scale factor in stretching $\triangle PBA$ onto $\triangle PB'A'$ is

$$\frac{\text{length of } PA'}{\text{length of } PA} = \frac{24}{12} = 2$$

You can see that the distance between any two points in the enlargement is always twice the corresponding distance on the photo negative.
Class Discussion

Suppose we use the same photo enlarger and want an enlargement that will be three times as large as the photo negative. How far from the top should the shelf be placed in order to do this?

Solution.

1. In Figure 3, $\triangle PBA$ is similar to $\triangle P'B'A'$.

2. Then

$$\frac{\text{length of } PA'}{\text{length of } PA} = \text{scale factor}$$

is the ratio that will give us a scale factor.

3. Since the length of $PA'$ is 24 and the desired scale factor is 3, we can then write

$$\frac{24}{\text{(length of } PA)} = \frac{3}{1}$$

4. Using the comparison property

$$1 \cdot 24 = 3 \cdot (\text{length of } PA)$$

5. Multiplying both sides of the equation by $\frac{1}{3}$

$$\left(\frac{1}{3}\right) \cdot 24 = \left(\frac{1}{3}\right) \cdot 3 \cdot (\text{length of } PA)$$

$$8 = \text{(length of } PA)$$

Thus, the shelf with the photo negative on it should be 8 inches from the top.
Exercises

The glass shelf in the photo enlarger is movable. How far from the top should the shelf be placed in order to get enlargements with the following scale factors? Use the fact that

\[
\frac{24}{\text{length of } PA} = \frac{\text{scale factor}}{1}
\]

1. A scale factor of 4.

Distance from top = _______ inches.


Distance from top = _______ inches.

3. A scale factor of 12.

Distance from top = _______ inches.
BRAINBOOSTER.

4. A scale factor of 5.

Distance from top = blank inches.
How to Divide Up a Line Segment

Pictured below is a number line with perpendicular lines drawn through the points 1 through 5.

Because \( \ell_1, \ell_2, \ell_3, \ell_4, \) and \( \ell_5 \) are all perpendicular to the same line, you know that they are parallel to each other. Furthermore, as they pass through the points 1, 2, 3, 4, and 5 which are equally spaced on the number line, you know that these parallel lines are equally spaced.

We have drawn segment \( \overline{OA} \) which intersects \( \ell_1, \ell_2, \ell_3, \) and \( \ell_4 \) at points \( D_1, E_1, F_1, \) and \( G_1. \)
Class Discussion

1. Take your compass and place the needle point at 0 and the pencil point at \( D_1 \).

2. Without changing your compass setting, place the needle point at \( D_1 \) and the pencil point at \( E_1 \).

3. Is \( \overline{OD_1} \cong \overline{D_1E_1} \)?

4. Place the needle point at \( E_1 \) and the pencil point at \( F_1 \). Is \( \overline{D_1E_1} \cong \overline{E_1F_1} \)?

5. Again, without changing your setting, use your compass to find out if \( \overline{OD_1} \), \( \overline{D_1E_1} \), \( \overline{E_1F_1} \), \( \overline{F_1G_1} \), and \( \overline{G_1A} \) are all congruent to each other. Are they?

6. Use a straightedge to draw segment \( \overline{OB} \).

7. Use your compass to find out whether the segments cut off by the parallel lines in \( \overline{OB} \) are all congruent to each other. Are they?

8. Use a straightedge to draw \( \overline{OC} \).

9. Use your compass to find out whether the segments cut off by the parallel lines in \( \overline{OC} \) are all congruent to each other. Are they?

10. Do you agree that, no matter what line segment is drawn so that it passes through these equally spaced parallel lines, the segments they cut off will all be congruent to each other?
Now we will show you how to use these ideas to divide a line segment into smaller congruent line segments.

Suppose we want to divide $\overline{AF}$ into four smaller congruent line segments.

Draw a ray $\overrightarrow{AT}$.

We take a compass and lay off four congruent segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DE}$ on ray $\overrightarrow{AT}$. 
Next, we draw $EF$.

Now, if we construct the lines through points $B$, $C$, and $D$ parallel to $EF$, they will intersect $AF$ in three points, $x$, $y$, and $z$.

and $AX$, $XY$, $YZ$, and $ZF$ will be congruent to each other. Thus, we have divided $AF$ into four smaller congruent segments. We have also formed four triangles -- $\triangle AEF$ and three smaller triangles all similar to $\triangle AEF$.

The problem with using the above method is that constructing lines that are parallel is not an easy task. Fortunately, you always have equally spaced parallel lines available to you. For example, the lines on your notebook paper are parallel and equally spaced.

We will use the equally spaced parallel lines on a piece of notebook paper to divide a line segment into smaller congruent line segments.
Problem. Divide line segment $\overline{AB}$ into three smaller congruent line segments.

The equally spaced parallel lines below are much like those in a piece of notebook paper.

1. First, we pick a point $B_1$ on one of the top lines.
2. We place the needle point of the compass on point $B$ and the pencil point on point $A$.
3. Without changing the compass setting we place the needle point on $B_1$ and draw an arc intersecting the lines. Any line segment drawn from $B_1$ to a point on the arc will be congruent to $\overline{AB}$.
4. We want to divide $\overline{AB}$ into three smaller congruent line segments so we find the point where the arc intersects the third line down from the line that contains $B_1$ and label that point $A_1$.
5. Now we draw $\overline{A_1B_1}$. Now the parallel lines clearly divide segment $\overline{A_1B_1}$ in three smaller congruent segments. As $\overline{A_1B_1} \cong \overline{AB}$ all you need to do is take your compass and lay off these congruent segments on $\overline{AB}$. 
Exercises

Use the equally spaced parallel lines below to divide the given line segments into smaller congruent segments.

1. Divide \(AB\) into three smaller congruent line segments.

\[\text{Diagram: } A \quad \overline{AB} \quad B\]

2. Divide \(CD\) into five smaller congruent line segments.

\[\text{Diagram: } C \quad \overline{CD} \quad D\]

3. Divide \(EF\) into seven smaller congruent line segments.

\[\text{Diagram: } E \quad \overline{EF} \quad F\]
Ratios and Similar Triangles

Earlier in this chapter you found that the ratios of corresponding sides of similar triangles are equal. Knowing this, you were able to solve problems involving similar triangles. For example, to find the missing length of $\triangle A'C'$ in the two similar triangles below.

You first wrote

\[
\frac{\text{length of } A'B'}{\text{length of } AB} = \frac{6}{3}
\]

then

\[
\frac{\text{length of } A'C'}{\text{length of } AC} = \frac{x}{8}
\]

You know that these two ratios are equal so

\[
\frac{x}{8} = \frac{6}{3}
\]

By the comparison property

\[
3 \cdot x = 8 \cdot 6
\]

or

\[
3 \cdot x = 48
\]

Multiplying both sides of the equation by $\frac{1}{3}$, you get

\[
\left(\frac{1}{3}\right) \cdot 3 \cdot x = \frac{1}{3}(48)
\]

or

\[
x = 16
\]
Thus, the length of $A'C'$ is 16. Of course, if you saw that the scale factor was 2 then all you had to do was multiply 2.8 and you would have had the answer. Often, though, it is not easy to see what the scale factor is.

Class Discussion

In the last lesson you learned that if you have a triangle, 

![Diagram](a.png)

and you drew a line segment, $DE$, parallel to one of the sides then the triangle formed, $\triangle DCE$, is similar to $\triangle ACB$.

Suppose you have two similar triangles and the lengths of certain sides of the triangles are given, as is shown below:

![Diagram](b.png)

Can we find the length of the side $x$? Your knowledge of equal ratios and the comparison property will let you find the length of $x$. 

35
1. Suppose we want the scale factor for shrinking the **larger** triangle onto the smaller triangle. Then, using the sides of the triangles that are horizontal, the scale factor, written as a ratio, would be ________.

2. If we use the sides of the triangle that are **vertical**, then the **same** scale factor, written as a ratio, would be ________.

3. Now we know the scale factors are equal and therefore the ratios are equal, so
   \[ x = \__ \]

4. Using the **comparison** property we get
   \[ x = \__ \cdot \__ \]
   So, \( 100 \cdot x = 1600 \)

5. To solve this equation we multiply both sides by ________.

6. After doing the arithmetic we find that
   \[ x = \__ \]

**Exercises**

The triangles in each problem are similar. Use equal ratios to find a value for \( x \).

1. \[
\begin{align*}
\text{60} \\
\text{x} \\
\text{50} \\
\text{100}
\end{align*}
\]

(a) Write the equal ratios here. ______________________

(b) Use the comparison property. ______________________

(c) Multiply both sides of the equation by the correct number, which is ______________________.

(d) Do the arithmetic to get the answer. \( x = \__ \)
2. \[ \begin{align*}
\text{(a) Write the equal ratios here.} & \quad \text{______________} \\
\text{(b) Use the comparison property.} & \quad \text{______________} \\
\text{(c) Multiply both of the equations by the correct number, which is} & \quad \text{______________} \\
\text{(d) Do the arithmetic to get the answer.} & \quad x = \quad \text{______} \\
\end{align*} \]

(Do your work below.)

3. \[ \begin{align*}
\text{(Do your work below.)} & \\
\end{align*} \]

\[ x = \quad \text{______} \]
4. 

\[ \triangle \]

\[ \begin{align*}
80 \\
20 \\
x
\end{align*} \]

\[ \begin{align*}
100
\end{align*} \]

(Do your work below.)

\[ x = \_ \]

5. 

\[ \triangle \]

\[ \begin{align*}
60 \\
80 \\
x
\end{align*} \]

\[ \begin{align*}
100
\end{align*} \]

(Do your work below.)

\[ x = \_ \]
(Do your work below.)

\[ x = \]
In the last lesson you may have noticed that in every problem the larger triangle of the two similar triangles had a side of length 100.

In shrinking the larger triangle onto the smaller one you wrote a ratio in which the 100 was in the denominator. For example:

\[
\frac{80}{100}
\]

Using the horizontal sides of the two triangles you wrote

\[
\frac{50}{100}
\]

and then using the vertical sides you wrote the ratio

\[
\frac{x}{80}
\]

As these two ratios represent the shrinking scale factor and as scale factors in similar triangles are equal, you then could write

\[
\frac{x}{80} = \frac{50}{100}
\]

Your knowledge of the comparison property and your ability to solve equations then let you do as follows:

(a) \[\frac{x}{80} = \frac{50}{100}\]
(b) \[100 \cdot x = 50 \cdot 80\]
(c) \[100 \cdot x = 4000\]
(d) \[\left(\frac{1}{100}\right) \cdot 100 \cdot x = \left(\frac{1}{100}\right) \cdot 4000\]
(e) \[x = \frac{4000}{100}\]
(f) \[x = 40\]
Although you may not have realized it, when you worked these problems you were also solving a percent problem.

Whenever a ratio is written with 100 in the denominator you are writing that ratio as a percent. For example,

\[
\frac{50}{100} \quad \text{may be read as 50 percent.}
\]

As the symbol for percent is \%, we can then write

\[
\frac{50}{100} = 50 \cdot \frac{1}{100} = 50\%.
\]

You can see that \(\frac{1}{100}\) is 1%.

Class Discussion

Let us now use the idea of similar triangles to help us solve a percent problem.

Problem. 30\% of 80 is what number?

Picture Solution:

(a) On the grid on the next page, count over 10 squares on the x-axis and mark it 100. This will represent the horizontal leg of the larger triangle.

(b) Now count over 3 squares on the x-axis and mark it 30. This will represent the horizontal leg of the smaller triangle. Notice that you now have pictured the ratio \(\frac{30}{100}\) or 30\%.

(c) Count up 8 squares on the y-axis and mark that point 80. Use a straightedge to draw the segment that connects the point marked 100 to the point marked 80. You now have drawn the larger of the two similar triangles.
(d) Now, put your straightedge on the point marked 30 and as nearly as you can draw a line parallel to the segment you drew from 100 to 80, so that it cuts the y-axis. Mark this point t. You now have drawn the smaller of the two similar triangles.

(e) Make a guess as to where the point t lies on the y-axis.

Arithmetic Solution:

\[
\frac{30}{100} = \frac{t}{80}
\]

\[30 \cdot 80 = 100 \cdot t\]

\[2400 = 100 \cdot t\]

\[(\frac{1}{100}) \cdot 2400 = (\frac{1}{100}) \cdot 100 \cdot t\]

\[24 = t\]

Therefore, 30% of 80 is 24. How close was your guess to this answer?
Exercises

Each of the problems in this exercise set is a percent problem pictured on a grid. We have drawn one of the two similar triangles for you. You are to draw the other triangle by drawing a line through the marked point parallel to the given line and then estimate the answer.

Example. What % of 80 is 20?

You are given this type of picture.

You complete the picture like this.

What % of 80 is 20?
About 25 %.

Estimate the value of this point.
1. What is 50% of 80?

50% of 80 is about _______.
2. What % of 40 is 10?

10 is about _____% of 40?

3. 40% of what number is 20?

40% of about _____ is 20.
4. 15% of 60 is what number?

$15\% \text{ of } 60 = \frac{15}{100} \times 60 = 9$

5. What % of 150 is 30?

$30 = \frac{30}{150} = \frac{1}{5} = 20\%$

30 is about 20% of 150.
6. 25\% \text{ of what number is } 40 \text{?}

Here are the solutions to each of the problems you just finished doing. If your estimates are within 5 of these solutions you have done well.

1. 50\% \text{ of } 80 \text{ is } 40 .
2. 10 \text{ is } 25\% \text{ of } 40 .
3. 40\% \text{ of } 50 \text{ is } 20 .
4. 15\% \text{ of } 60 \text{ is } 9 .
5. 30 \text{ is } 20\% \text{ of } 150 .
6. 25\% \text{ of } 160 \text{ is } 40 .
Solving Percent Problems

Suppose you are given the problem, 

"25% of 120 is what number?"

You know that you can picture this problem on a grid and estimate the answer. In most cases your estimate will be close. But we need to be more than close. We need to be able to get an exact answer. Picturing percent problems on a grid not only lets you estimate the answer but it also lets you see the ratios so that you can get an exact answer.

Let us picture the problem, "25% of 120 is what number?", on a grid and see if we can arrive at the exact solution, not just an estimate.

Looking at our picture we can estimate that the answer should be close to 30, but we are really not sure.

We do know that the ratios of corresponding sides of similar triangles are equal, so we can write:

$$\frac{t}{120} = \frac{25}{100}$$

Using our knowledge of the comparison property and our ability to solve equations, we then write:

$$100 \cdot t = 25 \cdot 120$$

$$100 \cdot t = 3000$$

$$t = \frac{3000}{100}$$

$$t = 30$$

We now have found the solution to the problem and we see that our estimate was a good one.
Exercises

For each problem, make an estimate of the solution by drawing similar triangles on the grid. Then use the comparison property and your ability to solve equations to find the arithmetic solution.

1. 25% of 72 = ?

(a) Write the equal ratios here:

(b) Use the comparison property:

(c) Multiply both sides of the equation by the correct number:

(d) Do the arithmetic to get the answer:

25% of 72 = ___
2. 32% of 75 = ?

32% of 75 = ________

3. What % of 125 is 65?

_____ % of 125 is 65.

(Show arithmetic solution below.)
What % of 75 is 15?

\[
\frac{15}{75} = \frac{1}{5} = 20\%
\]

5. 32% of what number is 24?

\[
\frac{24}{32}\% = \frac{24}{0.32} = 75
\]
6. 20% of what number is 27?

(Show arithmetic solution below.)

20% of _____ is 27.
Pre-Test Exercises

These exercises are like the problems you will have on the chapter test. If you don't know how to do them, read the section again. If you still don't understand, ask your teacher.

1. (Section 12-1.)
Find the scale factor for stretching and the scale factor for shrinking each triangle onto the other.

(a) 3

(b) 5

2. (Section 12-1.)
Find the scale factor and then find the length of the side of the triangle not given to you.

(a) Scale factor is

(b) \( m_{A'C'} = \)
3. (Section 12-2.)

The triangles below are similar. Mark pairs of corresponding sides and pairs of corresponding angles.

4. (Section 12-3.)

Use equal ratios to find the missing length. The triangles are similar.

(a) \[ \frac{3}{5} = \frac{4}{x} \]

(b) \[ \frac{3}{8} = \frac{6}{x} \]
5. (Section 12-5.)
In the drawing below, $l_1$, $l_2$, $l_3$, and $l_4$ are parallel and equally spaced. They are cut by a transversal, $t$.

What do you know about the measures of segments $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$?

6. (Section 12-6.)
Use equal ratios to find the value of $x$.

$x = \underline{\phantom{0000}}$
7. (Section 12-7.)
\[
\frac{25}{100} = 25 \cdot \frac{1}{100} = \_\% 
\]

8. (Section 12-7.)
Estimate the value of \( t \).

\[ \begin{array}{c}
80 \\
20 \\
\_ \\
100 \\
\_ \\
\end{array} \]

\( t \) is about \_\_

9. (Section 12-8.)
On the grid below use similar triangles to estimate the answer to
"20\% of 90 is what number?".

\[ \begin{array}{c}
20\% \text{ of } 90 \text{ is } \_ \_ \\
\_ \\
\_ \\
\_ \\
\_ \\
\_ \\
\_ \\
\end{array} \]
10. (Section 12-8.)

Solve for $x$.

(a) $\frac{x}{120} = \frac{25}{100}$

\[ x = \frac{25 \times 120}{100} \]

(b) $\frac{10}{100} = \frac{9}{x}$

\[ x = \frac{10 \times 100}{9} \]

(c) $\frac{x}{100} = \frac{25}{50}$

\[ x = \frac{25 \times 100}{50} \]
11. (Section 12-8.)

(a) 3 is what % of 12?

3 is _____ % of 12.

(b) 15% of 80 is what number?

15% of 80 is ______.

(c) 30% of what number is 36?

30% of ______ is 36.
1. Find the scale factor for stretching and the scale factor for shrinking each triangle onto the other.

(a) The scale factor for stretching is \( \frac{C''}{C} \).
The scale factor for shrinking is \( \frac{C}{C''} \).

(b) The scale factor for stretching is \( \frac{C'}{C} \).
The scale factor for shrinking is \( \frac{C}{C'} \).
2. Find the scale factor and then find the length of the side not given.

(a) Scale factor is \( \frac{C}{26} \) \( \frac{10}{24} \)

(b) \( m \overline{A'C'} = \) ________

3. The triangles below are similar. Mark pairs of corresponding sides and pairs of corresponding angles.
4. Use equal ratios to find missing lengths. The triangles are similar.

(a) \[x = \frac{3}{5} \times \frac{9}{12} = \frac{9}{20}\]

(b) \[x = \frac{4}{6} \times \frac{2}{4\frac{1}{2}} = \frac{4}{13} \times \frac{2}{9} \approx \frac{1}{3}\]
5. In the drawing below, \( \ell_1, \ell_2, \ell_3, \) and \( \ell_4 \), are parallel and equally spaced. They are cut by a transversal, \( t \).

What do you know about the measures of segments \( AB, BC, \) and \( CD \)?

6. Use equal ratios to find the value of \( x \).
7. \( \frac{50}{100} = 50 \cdot \frac{1}{100} = \frac{1}{2} \)

8. Estimate the value of \( t \).

\[ \begin{array}{c}
\text{t is about} \\
\end{array} \]

9. On the grid below, use similar triangles to estimate the answer to "30% of 40 is what number?".

\[ \begin{array}{c}
30\% \text{ of } 40 \text{ is about} \\
\end{array} \]
10. Solve for $x$.

(a) \( \frac{x}{160} = \frac{25}{100} \)

\[ x = \ldots \]

(b) \( \frac{10}{100} = \frac{18}{x} \)

\[ x = \ldots \]

(c) \( \frac{x}{100} = \frac{15}{60} \)

\[ x = \ldots \]
11. (a) 5 is what % of 20?

\[ \text{5 is } \underline{\phantom{0000}} \% \text{ of 20.} \]

(b) 25% of 80 is what number?

\[ \underline{25\%} \text{ of 80 is } \underline{\phantom{0000}}. \]

(c) 40% of what number is 36?

\[ \underline{40\%} \text{ of } \underline{\phantom{0000}} \text{ is 36.} \]
Check Your Memory: Self-Test

1. (Section 8-4.)
   Fill the blanks.
   (a) To undo multiplying by 4', you multiply by __________.
   (b) To undo adding 6, you __________.
   (c) To undo adding $\frac{3}{5}$, you __________.
   (d) To undo multiplying by $\frac{3}{8}$, you __________.

2. (Section 8-6.)
   Solve these equations.
   (a) \(7x = 35\)
   \(x = \) __________
   (d) \(\frac{x}{8} = 7\), \(x = \) __________
   (b) \(x - 11 = 5\)
   \(x = \) __________
   (e) \(\frac{2}{3}x - 2 = 4\)
   \(x = \) __________
   (c) \(\frac{3}{4}x = 6\)
   \(x = \) __________

3. (Section 10-5.)
   Put a decimal point in the following numbers so that:
   (a) the 5 is in the tenths place
   \(346185\)
   (b) the 8 is in the ones place
   \(70893\)
   (c) the 4 is in the hundredths place
   \(245671\)
   (d) the 2 is in the thousandths place
   \(5624\)
   (e) the 3 is in the tens place
   \(308\)

4. (Section 10-11.)
   Multiply.
   (a) \(2.8 \times 100 = \) __________
   (b) \(.372 \times 1000 = \) __________
   (c) \(.48 \times .25 = \) __________
   (d) \(2 \times .125 = \) __________
In each of the following figures, find the measure of \( \angle x \).

(a) \( m \angle x = \) ____________

(b) \( m \angle x = \) ____________

(c) \( m \angle x = \) ____________
6. (Section 11-4.)

Sketch a figure to show what this looks like.

\[ EF \parallel HG \]
\[ EH \perp EF \]
\[ HG \cong EF \]
\[ FG \perp HG \]

What kind of figure is this?

Now check your answers on the next page. If you do not have them all right, go back and read the section again.
Answers to Check Your Memory: Self-Test

1. (a) \( \frac{1}{4} \)
   (b) add \(-6\)
   (c) add \(\frac{3}{2}\)
   (d) multiply by \(\frac{8}{3}\)

2. (a) \(-5\)
   (b) 6
   (c) 8
   (d) 56
   (e) 9

3. (a) 34618.5
   (b) 708.93
   (c) .245671
   (d) .5624
   (e) 30.8

4. (a) 280
   (b) 372
   (c) \(.1200 = .12\)
   (d) \(.250 = .25\)

5. (a) \(m \angle x = 140\)
   (b) \(m \angle x = 50\)
   (c) \(m \angle x = 60\)
6. The figure is a rectangle. (It is also one kind of parallelogram, and you may have drawn a square.) It may be in many positions, like these:

```
H
G F
```

etc.

However, the angle at H should be opposite the angle at F no matter which way you drew it.
Chapter 13

MORE ABOUT RATIONAL NUMBERS
Chapter 13
MORE ABOUT RATIONAL NUMBERS

Introduction

In your very first experience with numbers, you used numbers to count things. You could always find an answer to problems about things (and therefore about the numbers you used) simply by moving the things themselves and counting again.

Later you learned to add, subtract, multiply, and divide the numbers themselves, and you found that they obeyed certain rules. When you worked with whole numbers, for instance, you knew you could choose two numbers, like 5 and 3, and add them, and that you would get just one answer, no matter whether you added 5 + 3 or 3 + 5.

When you chose two numbers and tried to subtract, however, you found that 5 - 3 = 2, but for 3 - 5 = ? , you did not get any answer at all.

In Chapter 5 you learned that the problem 3 - 5 = ? does have an answer, -2, in the numbers called integers. You learned where these numbers are located on the number line.

In the same way, you knew that when you multiplied two whole numbers, the answer was always a whole number, and it didn't matter in which order you multiplied them. 6 x 3 = 18 and 3 x 6 = 18.

When you tried to divide, however, you found that 6 divided by 3 is a whole number, but 3 divided by 6 is not.

In Chapter 6 you learned about rational numbers and you found that 3 divided by 6 is a rational number which may be called \( \frac{3}{6} \) or \( \frac{1}{2} \), or \( \frac{4}{8} \) or many other names, and you learned where the rational numbers are located on the number line.

For a long time you have known how to "compute" using whole numbers. You know how to go about getting the answer when you add and multiply whole numbers, and how to subtract and divide them when an answer is possible.
You also know how to add, subtract, and multiply integers, but you have not yet studied division with negative integers. With rational numbers, you can multiply and divide, and you know that you can add and subtract all rational numbers, but there are some kinds of problems you still have to learn how to do.

In this chapter, then, you will "fill the gaps" in your ability to compute with integers and rational numbers. You will learn to divide with negative integers and to add or subtract any rational numbers, using fraction names as well as decimal names.
Another Look At Integers

Before you go on to division with negative integers, let's be sure you remember what you learned earlier about these numbers.

Class Discussion

1. On the number line above, the only points that are labeled correspond to whole numbers. You have seen that the numbers to the right of zero are also called integers. To the left of zero are the integers, which are the opposites of the positive ones. You have called these "opp 1", "opp 2", and so on. You write 1, 2, ... On the number line above, find the point that is one unit to the left of 0 and label it -1. Label the point that is two units to the left of 0 as -2, etc.

2. The point that is 5 units to the right of 0 corresponds to . The point that is 6 units to the left of 0 corresponds to .

3. The number is neither positive nor negative.

4. Every integer may be represented by an arrow above the number line. The direction the arrow points shows whether the integer is positive or negative. The arrow below shows the integer 3.

How many units long is the arrow? 

Which direction does the arrow point?
5. The arrow above is _______ units long.
   It points to the ________.
   It represents the integer ________.

6. The arrow above is _______ units long.
   It points to the ________.
   It represents the integer ________.

7. To solve $6 + (-3)$ using an arrow, we start at 6 and draw an arrow 3 units long pointing to the left.
   So $6 + (-3) = ________$. 
8. Each of these number lines can be used to solve an addition problem.

(a) 

(b) 

(c) 

(d) 

Write the letter of the number line that goes with each problem, and then give the answer.

(□) 1 + 4 = □□□□

(□) 4 + ▴5 = □□□□

(□) ▴1 + 5 = □□□□

(□) ▴5 + ▴4 = □□□□
MULTIPLE CHOICE.

9. The sum of two positive integers is always
   (a) positive
   (b) negative
   (c) zero
   (d) none of the above

10. The sum of two negative integers is always
    (a) positive
    (b) negative
    (c) zero
    (d) none of the above

11. The sum of an integer and its opposite is always
    (a) positive
    (b) negative
    (c) zero
    (d) none of the above

12. The sum of a positive integer and a negative integer is
    (a) positive if the integer farthest from zero is positive
    (b) negative if the integer farthest from zero is positive
    (c) positive
    (d) negative
    (e) zero

13. (a) In working with whole numbers, you learned that 9 - 7 = ________
    (b) In the set of integers, 9 + -7 = ________
    (c) Subtraction problems may be rewritten as addition problems so that you __________ the __________ of the subtrahend.
Rewrite each of the following subtraction problems as an addition problem and find the answer.

(a) \(4 - 3 = \) 
(b) \(-3 - 2 = \)
(c) \(4 - (-3) = \)
(d) \(-3 - (-2) = \)
On the number line above, letters are used to stand for positive and negative integers. Zero is shown on the line. In the blank beside each letter below, write the letter that stands for its opposite.

1. (a) A
   (b) V
   (c) P
   (d) B

2. The function $f: x \rightarrow \text{opp } x$ is often called the opposite function. Complete the table of inputs and outputs, and graph this function on the coordinate plane below.

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>opp $x$</td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>-3</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td></td>
</tr>
</tbody>
</table>
3. Fill the blanks with < (is less than) or > (is greater than) to make each statement true.

(a) 3 ___ 5
(b) 7 ___ 2
(c) ___ 4
(d) 5 ___ 8
(e) ___ 3
(f) ___ 1

4. Find the answer to each problem below. If it is an addition problem, you will not need the blank below the problem. If it is a subtraction problem, rewrite the problem to show that you add the opposite of the subtrahend.

(a) 3 + 5 =
(b) 3 + ___ =
(c) ___ - 3 =
(d) 4 - ___ =
(e) 6 - ___ =
(f) 8 + ___ =
(g) ___ + 7 =
(h) ___ - 5 =
(i) ___ - 7 =
(j) ___ - 6 =
(k) 14 + ___ =
(l) 7 - 7 =
(m) ___ + 9 =
(n) ___ - 9 =
Multiplying and Dividing with All the Integers

Because you can think of multiplication as repeated addition, you know that:

\[ -3 + -3 + -3 + -3 = 4 \cdot -3 \]
\[ -3 + -3 + -3 + -3 = -12 \]

so \[ 4 \cdot -3 = -12 \]

With integers, just as with whole numbers, it doesn't make any difference in which order you multiply two numbers. So you see that since \[ 4 \cdot -3 = -3 \cdot 4 \], and \[ 4 \cdot -3 = -12 \], it must be true that \[ -3 \cdot 4 \] also is \[ -12 \].

Class Discussion

1. When you multiply two positive integers, the answer is ________.
2. When you multiply a negative integer by a positive integer, the answer is ________.
3. When you multiply a positive integer by a negative integer, the answer is ________.
4. When you multiply two negative integers, the answer is ________.

In Chapter 5 you learned that the answer, in Question 4, is "positive". You found this out by graphing the "doubling" function and by making a multiplication table. You may have found it hard to believe, however.

Now let's make this clearer and at the same time learn how to divide with negative integers.

In all your work with numbers, you have found that a number may have many different names. The number 12, for instance, may be named using addition \((9 + 3)\), subtraction \((13 - 1)\), multiplication \((6 \times 2)\), or division \(\frac{24}{2}\). However, each one of these names "belongs to" the number 12 only. If you want to name some number that is not 12, you can't call it \(9 + 3\), because \(9 + 3\) is always 12. (It would be very awkward if this were not so. Think about it.)
You also know that multiplication statements about numbers can be rewritten as division statements using exactly the same numbers.

\[3 \times 4 = 12\] so \[\frac{12}{3}\] must be 4

\[3 \times -4 = -12\] so \[\frac{-12}{3}\] must be _____

\[-3 \times 4 = -12\] so \[\frac{-12}{3}\] must be _____

The answer to this question

\[-3 \times -4 = ?\]

can't be -12, because then we would have to say that

\[\frac{-12}{3} = -4\],

and in that case \[\frac{12}{3}\] would name two different numbers, 4 and -4.

We know that this can't be true, so

\[-3 \times -4 = 12,\]

and \[\frac{-12}{3} = _______.\]

From the last problem, you are ready to fill the blanks in these statements.

5. When the integers you multiply are both positive or both negative, the answer is _______.

6. When you divide integers that are both positive or both negative, the answer is _______.

7. When you divide integers, if one is positive and the other is negative, the answer is _______.
Exercises

1. $7 \cdot 15 = \underline{\text{______}}$

2. $\frac{105}{7} = \underline{\text{______}}$

3. $-8 \cdot 2 = \underline{\text{______}}$

4. $\frac{-16}{2} = \underline{\text{______}}$

5. $9 \cdot -3 = \underline{\text{______}}$

6. $\frac{-27}{3} = \underline{\text{______}}$

7. $-5 \cdot -6 = \underline{\text{______}}$

8. $\frac{30}{5} = \underline{\text{______}}$

9. $-5 \cdot 9 = \underline{\text{______}}$

10. $\frac{-72}{12} = \underline{\text{______}}$

11. $\frac{14}{-4} = \underline{\text{______}}$

12. $\frac{-81}{27} = \underline{\text{______}}$

13. $\frac{-64}{16} = \underline{\text{______}}$

14. $5 \cdot -19 = \underline{\text{______}}$

15. $\frac{48}{12} = \underline{\text{______}}$

16. $4 \cdot 28 = \underline{\text{______}}$

17. $-15 \cdot -15 = \underline{\text{______}}$

18. $\frac{-625}{25} = \underline{\text{______}}$

19. $\frac{900}{30} = \underline{\text{______}}$

20. $\frac{169}{13} = \underline{\text{______}}$

Do your work in the space at the right.

21. $\frac{352}{2} = \underline{\text{______}}$

22. $-25 \cdot 46 = \underline{\text{______}}$

23. $\frac{513}{3} = \underline{\text{______}}$

24. $\frac{-729}{9} = \underline{\text{______}}$

25. $-54 \cdot 13 = \underline{\text{______}}$
Another Look at Rational Numbers

Now that you can divide both positive and negative integers when the answer is an integer, you can apply what you know to other rational numbers.

Class Discussion

1. \( \frac{16}{4} = \) and \( -\frac{16}{4} = \).

The integer named by both \( \frac{16}{4} \) and \( -\frac{16}{4} \) is the same. So you can say that

\( \frac{16}{4} = -\frac{16}{4} \).

Parentheses ( ) mean, "Do this first". If you have a problem like 3 + \((\frac{16}{4})\) = ? , you think, "\( \frac{16}{4} = \frac{1}{4} \), so I add 3 + \( \frac{1}{4} \) = ?".

We use parentheses in the same way with the raised dash on the outside: "( )". In this case, you think, "Take the opposite of the number that is inside the parentheses."

\( \frac{16}{4} = \) so \( -(\frac{16}{4}) = \).

You see that \( \frac{16}{4} \), \( \frac{16}{4} \), and \( -(\frac{16}{4}) \) are all names for ___.

Therefore, you know that

\( \frac{16}{4} = \frac{16}{4} = -(\frac{16}{4}) \).

2. Rewrite each of the following as a fraction in two different ways. The first one is done for you.

(a) \( -\frac{1}{3} \) \( -\frac{1}{3} \) \( -(\frac{1}{3}) \)

(b) \( -\frac{9}{2} \)  

(c) \( -\frac{6}{3} \)  

(d) \( -\frac{5}{6} \)  

(e) \( \frac{8}{9} \)  

(f) \( -\frac{4}{5} \)  

(g) \( -\frac{2}{3} \)
Although we know that a negative rational number can be named in the ways we have discussed, it is often important in solving problems to use only one certain way. For instance, you have added rational numbers like this:

\[
\frac{6}{3} + \frac{6}{3} = \frac{6 + 6}{3} = \frac{12}{3} = 4
\]

Suppose, now, you must add \( \frac{6}{3} + \frac{6}{3} \). Of course, in this problem you could use the integer name for each number and add: \( 2 + (-2) = 0 \).

Let's see how you can do it using the fraction names.

\[
\frac{6}{3} + \frac{6}{3} = ?
\]

What denominator would you use? Instead of writing \( \frac{6}{3} \), write \( \frac{-6}{3} \), and then you can find the answer in the usual way.

\[
\frac{6}{3} + \frac{-6}{3} = \frac{6 + (-6)}{3} = \frac{0}{3} = 0
\]

3. Show how to rewrite each of the following problems so that the denominator is the same. Then find the answer in simplest form.

(a) \( \frac{1}{4} + \frac{1}{4} = \frac{+}{4} \)

(b) \( \frac{4}{9} + \frac{2}{9} = \)

(c) \( \frac{9}{10} + \frac{7}{10} = \)

\[
= \]

\[
= \]

\[
= \]
4. Another kind of problem comes up when you try to compare two rational numbers. Let's recall what you did with positive numbers in Chapter 6. Example. \( \frac{1}{2} \) and \( \frac{1}{4} \)

\[
\begin{array}{c}
\frac{1}{2} \quad \frac{2}{4} \\
\frac{1}{4} \quad \frac{1}{2}
\end{array}
\]

\( \frac{2}{4} \) is greater than \( \frac{1}{2} \), so you know that \( \frac{1}{2} \) is greater than \( \frac{1}{4} \).

Let's try this method to compare \( \frac{4}{2} \), which is another name for \( \frac{2}{1} \), and \( \frac{2}{1} \), which is another name for \( \frac{2}{2} \).

You know that \( \frac{2}{2} \) is greater than \( \frac{2}{1} \), so \( \frac{4}{2} > \frac{2}{1} \).

However, if you write

\[
\begin{array}{c}
\frac{4}{2} \quad \frac{2}{1}
\end{array}
\]

it looks as if it were the other way around, because \( \frac{2}{1} \) is less than \( \frac{4}{2} \).

You get the correct result when you rename \( \frac{2}{1} \) as \( \frac{2}{2} \).

\[
\begin{array}{c}
\frac{4}{2} \quad \frac{2}{1}
\end{array}
\]

To avoid mistakes when you work with fraction names for negative numbers, write the negative symbol in the numerator rather than in the denominator. The form used should be \( \frac{-a}{b} \).

5. (a) \( \frac{4}{2} = \) 

(b) \( \frac{-4}{2} = \) 

(c) Therefore, \( \frac{4}{2} = \frac{-4}{2} \). \( \frac{-4}{2} \) is the name of a positive rational number. Certainly you would use \( \frac{4}{2} \) instead of \( \frac{-4}{2} \) if you were going to add:

\[
\frac{4}{2} + \frac{-4}{2} = ?
\]

\[
\frac{4 + (-4)}{2} = \frac{4 - 4}{2}
\]

\[
= \frac{0}{2}
\]

\[
= 0
\]
6. If you want to compare $\frac{4}{2}$ and $\frac{-4}{1}$, you could write:

\[
\begin{array}{c}
\frac{4}{2} \\
\frac{-4}{1}
\end{array}
\]

$-4$ is greater than $-8$, so it looks as if $\frac{4}{2}$ is greater than $\frac{-4}{1}$.

But this is impossible, because $\frac{4}{2} = \frac{-4}{-1}$ and $\frac{-4}{1}$ is not greater than $4$. If you use the name $\frac{-4}{1}$ instead of $\frac{4}{1}$, you get the right answer:

\[
\begin{array}{c}
\frac{4}{2} \\
\frac{-4}{1}
\end{array}
\]

To be sure you get the right answer in such problems, write all rational numbers which are positive without any negative symbols. Write $\frac{a}{b}$, not $\frac{-a}{b}$.
Exercises

1. Compare these rational numbers. Rewrite the fractions if necessary. Then put < or > in the blank.

   (a) \( \frac{-15}{13} \) \( \quad \frac{3}{4} \)
   (b) \( \frac{-7}{8} \) \( \quad \frac{6}{11} \)
   (c) \( \frac{18}{5} \) \( \quad \frac{4}{3} \)
   (d) \( \frac{-9}{8} \) \( \quad \frac{2}{3} \)
   (e) \( \frac{-1}{3} \) \( \quad \frac{1}{4} \)

2. Use what you know about rewriting rational numbers to find the sums of the two numbers in each problem below. Give your answer in simplest form.

Example. \( \frac{-2}{5} + \frac{4}{5} = ? \)

\[ \frac{-2}{5} + \frac{4}{5} = \frac{-2 + 4}{5} = \frac{2}{5} \]

\[ = \frac{-6}{5} \quad \text{(answer)} \]

   (a) \( \frac{7}{8} + \frac{-1}{8} = \)
   (b) \( \frac{-3}{5} + \frac{2}{5} = \)
   (c) \( \frac{-1}{6} + \frac{7}{6} = \)
Multiples

Much of what you can do with any kind of number depends on what you know about whole numbers. You depend on the whole numbers to help you work with rational numbers. Learning about multiples of whole numbers, for instance, will help you add rational numbers.

Your multiplication table for whole numbers can also be called a table of products, because it tells you the product of any two numbers from 0 through 30. We can use still another name for this useful table. It is a table of multiples.

If you multiply two whole numbers, the result is a multiple of each of them.

You get 14 when you multiply 2 by 7, so 14 is a multiple of 2 and a multiple of 7.

You also get 14 when you multiply 1 by 14, so 14 is also a multiple of 1 and a multiple of 14.

As you see, 14 is a multiple of four different numbers: 1, 2, 7, and 14.

**Class Discussion**

Look at the "2" row in your multiplication table. On the first page, you see

0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30

Each of these numbers is a multiple of 2, because

0 = 2 \times 0
2 = 2 \times 1
4 = 2 \times 2
6 = 2 \times 3
8 = 2 \times 4
10 = 2 \times 5
and so on.
On the second page, you find the multiples of 2 that you get from multiplying 2 by each number from 16 to 30.

1. The multiples of _____ are all even numbers. If your table had a third page, you could find more multiples of 2. The next multiple after 60 is 62.

2. Which multiples of 2 come after 62?

3. What multiple of 2 do you get when you multiply 2 by 400? What multiple of 2 do you get when you multiply 2 by 10,000? What multiple of 2 do you get when you multiply 2 by 3,000,000,000?

As you know, there is no last or largest whole number, so there is no last or largest multiple of 2.

4. List the first ten multiples of 3.

5. List the first ten multiples of 7.

6. List the first ten multiples of 5.

7. The number that appears in every row of multiples is _______. Because any number times zero is zero, it is a multiple of every number, and because it's always in the list, we usually don't bother to write 0 when we list the multiples of a number.

8. After 0, what is the first multiple of 1? _______


10. The first multiple of any number is the ________ of itself. Every number is a _________ of itself.
Now look through your table carefully.

11. How many times do you find the number 2 as a multiple?

12. The number 2 is a multiple of only two numbers, 2 and ____.

13. How many times do you find the number 3 as a multiple? ______
    3 is a multiple of only two numbers, ____ and ____.

14. In Chapter 4, you learned about prime numbers. 2 and 3 are prime numbers. They are multiples only of themselves and 1.
   List five other prime numbers from your table. (Note: They are numbers that appear as multiples exactly twice.)
   ______    ______    ______    ______    ______

Exercises

1. Draw a ring around each number that does not appear at all on the first page of your table.
   31 33 37 39 41 43 47 49 51 53 57 59
   61 63 67 69 71 73 77 79 81 83 87 89

2. You have circled the prime numbers between 30 and 90. Write each number that you did not circle and show that it is a multiple of two numbers besides itself and 1.
   Example. 33 = 3 x 11
   ______    ______
   ______    ______
   ______    ______
   ______    ______

3. What number is a multiple of every number? _____
4. Show that \(24\) is a multiple of \(1, 2, 3, 4, 6, 8, 12, 24\).

(a) \(24 = 1 \times \) \_
(b) \(24 = 2 \times \) \_
(c) \(24 = 3 \times \) \_
(d) \(24 = 4 \times \) \_
(e) \(24 = 6 \times \) \_
(f) \(24 = 8 \times \) \_
(g) \(24 = 12 \times \) \_
(h) \(24 = 24 \times \) \_

5. Show that \(16\) is a multiple of \(1, 2, 4, 8, 16\).

(a) \(16 = 1 \times \) \_
(b) \(16 = 2 \times \) \_
(c) \(16 = 4 \times \) \_
(d) \(16 = 8 \times \) \_
(e) \(16 = 16 \times \) \_

6. Show that \(12\) is a multiple of \(1, 2, 3, 4, 6, 12\).

(a) \(12 = 1 \times \) \_
(b) \(12 = 2 \times \) \_
(c) \(12 = 3 \times \) \_
(d) \(12 = 4 \times \) \_
(e) \(12 = 6 \times \) \_
(f) \(12 = 12 \times \) \_

7. Show that \(15\) is a multiple of \(1, 3, 5, 15\).

(a) \(15 = 1 \times \) \_
(b) \(15 = 3 \times \) \_
(c) \(15 = 5 \times \) \_
(d) \(15 = 15 \times \) \_
Common Multiples

In your multiplication table, the numbers in the "1" row are just like the numbers at the top of the table. From this you know that 0 is a multiple of 1, 1 is a multiple of 1, 2 is a multiple of 1, 3 is a multiple of 1, and so on. Every whole number is a multiple of 1.

Class Discussion

1. (a) A number that is a multiple of _____ is called an even number.
   (b) Even numbers are numbers which have, as their last digit, ___, ___, ___, or ___.
   (c) List the first ten multiples of 2. _____ _____ _____

2. List the first ten multiples of 3. _____ _____ _____

3. Which numbers are in both lists? _____, _____, and _____.

Because these numbers are common to both lists, they are called common multiples of 2 and 3. What is the smallest common multiple of 2 and 3? _______. This is called the least common multiple of 2 and 3, and is usually shortened to L.G.M.

Every common multiple of 2 and 3 is also a multiple of 6. Is there a greatest common multiple of 2 and 3? _______.

4. Is 24 a common multiple of 3 and 4? _______. What is the smallest number that is in both the "3" row and the "4" row? The L.C.M. is _______.


Suppose you want to find a common multiple of two numbers like 6 and 8 when you don't have your multiplication tables with you. The adding method is an easy way.

Write down both numbers.

\[
\begin{array}{c}
6 \\
8
\end{array}
\]

6 is less than 8. Add 6 to 6.

\[
\begin{array}{c}
6 \\
(6 + 6) = 12
\end{array}
\]

Look at the bottom numbers. 8 is less than 12. Add 8 to 8.

\[
\begin{array}{c}
6 \\
8 \\
(6 + 6) = 12 \\
16 = (8 + 8)
\end{array}
\]

Compare the bottom numbers again. The one in the 6 column is smaller than the one in the 8 column. Add 6 to 12.

\[
\begin{array}{c}
6 \\
8 \\
(6 + 6) = 12 \\
16 = (8 + 8) \\
(6 + 12) = 18
\end{array}
\]

Compare. 16 < 18. Add 8 to 16.

\[
\begin{array}{c}
6 \\
8 \\
(6 + 6) = 12 \\
16 = (8 + 8) \\
(6 + 12) = 18 \\
24 = (8 + 16)
\end{array}
\]

Compare. 18 < 24. Add 6 to 18.

\[
\begin{array}{c}
6 \\
8 \\
(6 + 6) = 12 \\
16 = (8 + 8) \\
(6 + 12) = 18 \\
24 = (8 + 16) \\
(6 + 18) = 24
\end{array}
\]

24 = 24, so 24 is a common multiple -- in fact, the least common multiple of 6 and 8. All the common multiples of 6 and 8 are multiples of 24; that is, 48, 72, 96, 120, 144, etc.
5. Use this method to find a common multiple of 9 and 12.

\[
\begin{array}{cc}
9 & 12 \\
(9 + 9) = & (12 + 12) \\
(9 + \_\_) = & (12 + \_\_) \\
(9 + \_\_) = & \_ \\
\end{array}
\]

The least common multiple of 9 and 12 is \_. All of the common multiples of 9 and 12 are multiples of \_.

Exercises

1. Find the least common multiple for each pair of numbers. (You do not have to write \(10 + 10\), etc., at the side to show what you add.)

(a) 10 15

\[
\begin{array}{cc}
\_ & \_ \\
\_ & \_ \\
\_ & \_ \\
\end{array}
\]

L.C.M. is \_.

(b) 8 10

\[
\begin{array}{cc}
\_ & \_ \\
\_ & \_ \\
\_ & \_ \\
\_ & \_ \\
\end{array}
\]

L.C.M. is \_.

(c) 15 20

\[
\begin{array}{cc}
\_ & \_ \\
\_ & \_ \\
\_ & \_ \\
\_ & \_ \\
\end{array}
\]

L.C.M. is \_.

(d) 9 15

\[
\begin{array}{cc}
\_ & \_ \\
\_ & \_ \\
\_ & \_ \\
\_ & \_ \\
\end{array}
\]

L.C.M. is \_.

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2. (a) All of the common multiples of 10 and 15 are also multiples of _____.
   List the next 3 common multiples of 10 and 15.
   ____  ____  ____

(b) All of the common multiples of 8 and 10 are also multiples of _____.

(c) Give the first three common multiples of 9 and 15.
   ____  ____  ____

(e) \[ \begin{array}{cc} 12 & 8 \\ \hline \hline \end{array} \]
   L.C.M. is _____

(f) \[ \begin{array}{cc} 6 & 9 \\ \hline \hline \end{array} \]
   L.C.M. is _____

(g) \[ \begin{array}{cc} 21 & 14 \\ \hline \hline \end{array} \]
   L.C.M. is _____
Class Discussion

1. If you use the adding method to find the least common multiple of 2 and 8, it goes like this:

   \[
   \begin{array}{c}
   2 \quad 8 \\
   2 < 8, \text{ so you add } 2 \text{ to } 2. \\
   4 \\
   4 < 8, \text{ so you add } 2 \quad 8 \\
   6 \\
   6 < 8, \text{ so you add } 2 \quad 8 \\
   8 \\
   \end{array}
   \]

   You now have \( 8 = 8 \), and you know that the L.C.M. of 2 and 8 is 8.

   It is easier to use a different method to find the least common multiple of 2 and 8.

   You have probably realized that if, in a pair of numbers, one of them is a multiple of the other, then the larger one is the least common multiple of the pair.

2. The least common multiple of 2 and 8 is \[______\], because \[______\] is a multiple of 2.

3. The least common multiple of 12 and 6 is \[______\], because \[______\] is a multiple of 6.

4. The least common multiple of 20 and 4 is \[______\], because \[20\] is a multiple of \[______\].

5. The least common multiple of 8 and 24 is \[______\], because \[______\] is a multiple of \[______\].
6. Now let's use the adding method to find the least common multiple of 5 and 7.

\[
\begin{array}{ccc}
5 & 7 \\
\hline
\hline
\hline
\hline
\hline
\end{array}
\]

The least common multiple of 5 and 7 is _____, but \(5 \times 7 = _____\)!

7. What is the least common multiple of each of these pairs?

2 and 3? _____  (And \(2 \times 3 = _____\))
2 and 5? _____  (And \(2 \times 5 = _____\))
2 and 7? _____  (And \(2 \times 7 = _____\))
3 and 5? _____  (And \(3 \times 5 = _____\))
3 and 7? _____  (And \(3 \times 7 = _____\))

What special kind of numbers are 2, 3, 5, and 7? _____

When you want to find the least common multiple of two prime numbers, the easiest way is just to multiply the two numbers.

8. In the pair of numbers 9 and 4, neither number is prime.
Find the least common multiple of 9 and 4.

\[
\begin{array}{ccc}
9 & 4 \\
\hline
\hline
\hline
\hline
\hline
\end{array}
\]

The least common multiple of 9 and 4 is _____, and \(9 \times 4 = _____\).
To see why this happens, think what prime numbers are multiplied together to get 9 and to get 4. In Chapter 4 you learned that the prime factorization of 9 is $3 \times 3$ and the prime factorization of 4 is $2 \times 2$. The numbers 9 and 4 do not have any factors that are alike, so we say that they are relatively prime. That is, they are not prime themselves, but they are prime in relation to each other. Relatively prime numbers do not have any common factors except 1.

On the other hand, 9 and 12 are not relatively prime because 3 is a factor of both. $9 = 3 \times 3$ and $12 = 2 \times 2 \times 3$, so the least common multiple of 9 and 12 is not $9 \times 12$.

The L.C.M. of 9 and 12 is $2 \times 2 \times 3 \times 3 = 36$.

**Exercises**

For the following pairs of numbers, use the method that is easiest for you to find the least common multiple.

1. 5 and 10  \hspace{1cm} \text{L.C.M.} \hspace{1cm} 

2. 4 and 12  \hspace{1cm} \text{L.C.M.} \hspace{1cm} 

3. 3 and 4  \hspace{1cm} \text{L.C.M.} \hspace{1cm} 

4. 8 and 9  \hspace{1cm} \text{L.C.M.} \hspace{1cm} 

5. 9 and 24  \hspace{1cm} \text{L.C.M.} \hspace{1cm} 

6. 2 and 9  \hspace{1cm} \text{L.C.M.} \hspace{1cm} 

7. 15 and 18  \hspace{1cm} \text{L.C.M.} \hspace{1cm} 

8. 7 and 9  \hspace{1cm} \text{L.C.M.} \hspace{1cm} 

9. 16 and 12  \hspace{1cm} \text{L.C.M.} \hspace{1cm} 

10. 10 and 24  \hspace{1cm} \text{L.C.M.} \hspace{1cm} 

11. 11 and 3  \hspace{1cm} \text{L.C.M.} \hspace{1cm} 

12. 14 and 21  \hspace{1cm} \text{L.C.M.} \hspace{1cm} 

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Adding Rational Numbers

Now you will use common multiples to add any two rational numbers, even if their denominators are not the same.

Class Discussion

When you add \( \frac{1}{4} + \frac{1}{4} \), you write \( \frac{1}{4} + \frac{1}{4} = \frac{1+1}{4} \), and then simplify your answer if possible, like this:

\[
\frac{1}{4} + \frac{1}{4} = \frac{1+1}{4} = \frac{2}{4} = \frac{1}{2}
\]

1. To make sure you remember how to add rational numbers, find the sums for each of the problems below.

(a) \( \frac{2}{8} + \frac{3}{8} = \frac{2+3}{8} = \frac{5}{8} \)

(b) \( \frac{2}{3} + \frac{1}{3} = \frac{2+1}{3} = \frac{3}{3} = 1 \) (Simplify your answer: 1)

(c) \( \frac{5}{8} + \frac{-1}{8} = \frac{5+(-1)}{8} = \frac{4}{8} = \frac{1}{2} \) (Simplify your answer: \( \frac{1}{2} \)).
2. In Chapter 7, you saw that sometimes you had to use a different name for a rational number in order to add it to another number.

In this spinner, $P(\text{red}) = \frac{1}{4}$ and $P(\text{blue}) = \frac{1}{2}$. To find $P(\text{either red or blue})$ you did not know how to add $\frac{1}{4} + \frac{1}{4}$, so you used a different name for $\frac{1}{2}$. You saw that $P(\text{blue})$ could be written as $\frac{3}{4}$, so you added

\[
\frac{2}{4} + \frac{1}{4} = \frac{3}{4}
\]

So, $P(\text{either red or blue}) =$ ________.
Here is a flow chart you may use to add any two rational numbers. Use inputs of $\frac{3}{8}$ and $\frac{3}{4}$ and show the output below.

Output: $\frac{3}{8}, \frac{3}{4}$.
You have added $\frac{3}{8} + \frac{3}{4}$ and found that the sum is $\frac{36}{32}$, although $\frac{36}{32}$ is not in simplest form.

You can see how this was done if you rename $\frac{3}{8}$ and $\frac{3}{4}$:

$$\frac{3}{8} \cdot \frac{4}{4} = \frac{12}{32} \quad \text{(write the numerator)} \quad \text{and} \quad \frac{3}{4} \cdot \frac{8}{8} = \frac{24}{32} \quad \text{(write the numerator)}.$$

So you add $\frac{12}{32} + \frac{24}{32} = \frac{36}{32}$.

Now you can see how to use common multiples. When you add two rational numbers, they must have the same denominator. The problem, then, is to find some common denominator and use it to find new names for the numbers.

When you multiply any two numbers together, the answer is a multiple of each of them. In the box $D \leftarrow b \cdot d$ you got a common denominator by finding a common multiple of the two denominators, 8 and 4:

$$8 \cdot 4 = 32$$

In $F \leftarrow a \cdot d$, you were completing the job of renaming the first number:

$$\frac{3}{8} \cdot \frac{4}{4} = \frac{12}{32}$$

In $S \leftarrow b \cdot c$ you were renaming the second number:

$$\frac{3}{4} \cdot \frac{8}{8} = \frac{24}{32}$$

Finally, you found the numerator of your answer: $N \leftarrow F + S$, so

$$\frac{12 + 24}{32} = \frac{36}{32}$$

As you saw, the output was not the simplest form of the answer, because

$$\frac{36}{32} = \frac{103}{32}.$$
4. Can you get the simplest form of the answer by some other method?
Look at the denominators of \( \frac{3}{8} \) and \( \frac{3}{4} \). 8 is a ______ of 4, so the least common multiple of 8 and 4 is ______.

You can rename \( \frac{3}{4} \) so that the new fraction has the denominator 8.

\[
\frac{3}{4} = \frac{\text{________}}{8} \quad \text{(write the numerator)}
\]

Now you can add.

\[
\frac{3}{8} + \frac{3}{4} = \frac{3 + 6}{8} = \frac{\text{________}}{8} = \frac{9}{8}
\]

Is your answer in simplest form? ______

5. (a) Use the flow chart to find the answer.

\[
\frac{2}{3} + \frac{1}{6} = \frac{\text{________}}{6}
\]

(b) Is the output, \( \frac{\text{________}}{6} \), in simplest form? ______

(c) What is the least common multiple of 3 and 6? ______

   Rename \( \frac{2}{3} \):

\[
\frac{2}{3} = \frac{\text{________}}{6}
\]

(d) Do the problem the usual way, using the new name for \( \frac{2}{3} \).

\[
\frac{4}{6} + \frac{1}{6} = \frac{\text{________}}{6} = \frac{\text{________}}{6}
\]

(e) Is your answer in simplest form? ______

6. (a) Use your flow chart to find the answer: \( \frac{2}{3} + \frac{3}{4} = \frac{\text{________}}{12} \)

(b) Is your answer in simplest form? ______

(c) What is the least common multiple of 3 and 4? ______

As you see, sometimes your flow chart gives you the answer in simplest form and sometimes it doesn't.
7. (a) Use the flow chart to add: \[ \frac{2}{3} + \frac{5}{6} = \]

(b) Simplify your answer.

(c) Since \( \frac{2}{3} = \frac{4}{6} \), you can write:

\[
\frac{2}{3} + \frac{5}{6} = \frac{4}{6} + \frac{5}{6} = \frac{9}{6} = \frac{3}{2}
\]

(d) Is this answer in simplest form?

(e) Simplify it.

You may not always get the answer in simplest form even when you use the least common denominator, but you are more likely to do so than when you use the flow chart.

Exercises

Use the flow chart or find the least common denominator to find the sums in the following problems. Write your answers in simplest form. You may want to use both methods a few times in order to check your answers.

1. \[ \frac{1}{2} + \frac{1}{3} = \]
2. \[ \frac{1}{4} + \frac{1}{3} = \]
3. \[ \frac{2}{5} + \frac{1}{4} = \]
4. \[ \frac{1}{2} + \frac{5}{6} = \]
5. \[ \frac{3}{4} + \frac{1}{2} = \]
6. \[ \frac{5}{4} + \frac{2}{3} = \]
7. \[ \frac{1}{8} + \frac{1}{16} = \]
8. \[ \frac{5}{9} + \frac{1}{3} = \]
9. \[ \frac{4}{7} + \frac{1}{8} = \]
10. \[ \frac{5}{16} + \frac{3}{8} = \]
11. \( \frac{7}{12} + \frac{2}{3} = \) \\
12. \( \frac{1}{3} + \frac{3}{4} = \) \\
13. \( \frac{1}{2} + \frac{5}{12} = \) \\
14. \( \frac{5}{8} + \frac{1}{10} = \) \\
15. \( \frac{7}{16} + \frac{3}{4} = \) \\
16. \( \frac{9}{8} + \frac{3}{4} = \) \\
17. \( \frac{2}{5} + \frac{1}{10} = \) \\
18. \( \frac{1}{6} + \frac{1}{5} = \) \\
19. \( \frac{6}{5} + \frac{3}{10} = \) \\
20. \( \frac{5}{2} + \frac{5}{4} = \)
Subtracting Rational Numbers

When you have a subtraction problem using rational numbers, you rewrite the problem as an addition problem and add the opposite of the subtrahend.

Example. \[ \frac{1}{2} - \frac{1}{4} = \frac{1}{2} + (-\frac{1}{4}) \]

\[ = \frac{2}{4} - \frac{1}{4} \]

\[ = \frac{2 - 1}{4} \]

\[ = \frac{1}{4} \]

So \[ \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \]

Exercises

Rewrite each subtraction problem as an addition problem and find the answer. You may use your flow chart if you wish.

1. \[ \frac{5}{8} - \frac{1}{2} = \]

2. \[ \frac{3}{4} - \frac{1}{3} = \]

3. \[ \frac{1}{2} - \frac{5}{6} = \]

4. \[ \frac{2}{3} - \frac{1}{6} = \]
5. \( \frac{3}{6} + \frac{1}{2} = \) \\

6. \( \frac{1}{4} - \frac{1}{3} = \) \\

7. \( \frac{1}{8} - \frac{1}{16} = \) \\

8. \( \frac{5}{8} - \frac{1}{16} = \) \\

9. \( \frac{2}{3} - \frac{5}{6} = \) \\

10. \( \frac{5}{6} - \frac{2}{3} = \) \\

11. \( \frac{7}{8} - \frac{1}{2} = \) \\

12. \( \frac{7}{8} - \frac{3}{4} = \)
Rational Numbers That Are Powers of Ten

In Chapter 4 you saw that $10^2$ means $10 \times 10$, and $10^3$ means $10 \times 10 \times 10$, or 1000. We can put this information in a chart.

<table>
<thead>
<tr>
<th></th>
<th>10,000</th>
<th>1000</th>
<th>100</th>
<th>10</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^4$</td>
<td>$10^3$</td>
<td>$10^2$</td>
<td>$10^1$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In the first row of boxes, each number is 10 times the one to the right of it.

1. (a) 1 is 10 times what number? In the chart above, write $\frac{1}{10}$ in the box to the right of 1.

(b) $\frac{1}{10} = 10 \times \ ?$

If you don’t know, think of the way multiplication and division are related. That is:

$3 \times 8 = 24$, so $\frac{24}{3} = 8$.

In the problem, $\frac{1}{10} = 10 \times \ ?$, you can find the answer by dividing $\frac{1}{10}$ by 10. Remember that when you divide you just multiply by the reciprocal of the divisor, so

$\frac{\frac{1}{10}}{10} = \frac{1}{10} \times \frac{1}{10}$

Therefore, $\frac{1}{100} = 10 \times \ ?$

Write $\frac{1}{100}$ in the box to the right of $\frac{1}{10}$. 
2. Now look at the second row of boxes. Each exponent is 1 greater than the exponent of the number to the right of it.

$$\begin{align*}
10^4 & \quad 10^3 & \quad 10^2 & \quad 10^1 \\
\end{align*}$$

Also, notice that the exponent shows how many zeros there are after the 1 in each number.

(a) You can write the other numbers from the top row as powers of ten also. What exponent do you think will be used for the number 1? 

Mathematicians agree that $10^0 = 1$.
Write $10^0$ in the box below 1.

(b) 0 is 1 greater than what number? 

We agree that $10^{-1}$ is another name for $\frac{1}{10}$. Write $10^{-1}$ in the box below $\frac{1}{10}$. Notice that the "1" part of the exponent shows how many zeros there are in $\frac{1}{10}$. The negative symbol shows that the 10 is below the bar in the fraction, $\frac{1}{10}$.

(c) What exponent will be used with 10 to show $\frac{1}{100}$? 

Write $10^{-2}$ below $\frac{1}{100}$.

(d) What power of 10 is $\frac{1}{1000}$? 

Write $10^{-3}$ below $\frac{1}{1000}$.

3. Using exponents makes work with powers of 10 very easy. You know that $10^5 \times 10^3 = 100,000 \times 1000 = 100,000,000$.

You get this answer easily by adding the exponents 5 and 3.

$$10^5 \times 10^3 = 10^{5+3} \quad = 10$$ (Write the exponent.)
Division is also easy with powers of 10. To divide $10^5$ by $10^2$, you can write $\frac{100000}{100} = 1000$ or you can just subtract the exponent of the divisor.

$$\frac{10^5}{10^2} = 10^{5-2} = 10^{3+2} = 10$$ (Write the exponent.)

This is especially helpful in problems like $100,000,000$ divided by $\frac{1}{1000}$.

The "old" way

$$\frac{100,000,000}{\frac{1}{1000}} = 100,000,000 \times 1000$$

(multiply by the reciprocal of the divisor.)

$$= 100,000,000,000$$

The "exponent" way

$$\frac{10^8}{10^{-3}} = 10^{8+3} = 10$$ (Write the exponent.)

Exercises

1. In the following problems, use the method of adding or subtracting exponents first. Then, at the right, check your answer by doing the problem without exponents.

Example.

$$10^2 \times 10^{-1} = 10^{2-1} \quad \text{and} \quad 100 \times \frac{1}{10} = 10$$

$$= 10^1$$

(Check: $10^1 = 10$.)

(a) $10^3 \times 10^2 = \underline{\hspace{2cm}}$ and $\underline{\hspace{2cm}} \times \underline{\hspace{2cm}} = \underline{\hspace{2cm}}$

= \underline{\hspace{2cm}}

(Check: \underline{\hspace{2cm}} = \underline{\hspace{2cm}}.)
(b) \(10^5 \times 10^{-3} = \quad\) and \(\quad \times \quad = \quad\)
\[\frac{10^5}{10^2} = \quad\]
(Check: \(\quad = \quad\))

(c) \(10^2 \times 10^4 = \quad\) and \(\quad \times \quad = \quad\)
\[\frac{10^2}{10^5} = \quad\]
(Check: \(\quad = \quad\))

(d) \(10^{-3} \times 10^{-2} = \quad\) and \(\quad \times \quad = \quad\)
\[\frac{10^{-3}}{10^{-5}} = \quad\]
(Check: \(\quad = \quad\))

(e) \(\frac{10^5}{10^4} = \quad\) and \(100,000 \times \frac{1}{10,000} = \quad\)
\[\frac{10^5}{10^4} = \quad\]
(Check: \(\quad = \quad\))

(f) \(\frac{10^{-5}}{10^4} = \quad\) and \(\quad \times \quad = \quad\)
\[\frac{10^{-5}}{10^{-4}} = \quad\]
(Check: \(\quad = \quad\))
2. Write these problems using the exponent form and write the answers with exponents.

(a) \[ \frac{10,000,000}{100} = \quad = \]

(b) \[ \frac{1}{100} \times \frac{1}{10,000} = \quad = \]

(c) \[ \frac{1}{1,000} \times 10,000 = \quad \times \quad = \]

(d) \[ \frac{1}{10,000} \quad = \quad = \]

(e) \[ \frac{1}{100,000} \quad = \quad = \]
In Chapter 10 you learned how to rewrite fractions as decimal numerals. If a fraction has a denominator that is a power of 10, there are just as many decimal places in the decimal numeral as there are zeros in the denominator.

\[ \frac{1}{1,000} = .001 \quad \text{and} \quad .01 = \frac{1}{100} \]

**Class Discussion**

Since another name for \( \frac{1}{10} \) is \( 10^{-1} \), and \( \frac{1}{10} = .1 \), you can see that \( 10^{-1} \) is also a name for .1.

Since \( 10^{-2} = \frac{1}{100} \), you can see that \( 10^{-2} \) is also a name for ________ (decimal numeral). The digit 2 in the exponent \( -2 \) tells the number of decimal places in the decimal numeral.
Finish this chart.

<table>
<thead>
<tr>
<th>Power of 10</th>
<th>Decimal Numeral</th>
<th>Fraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^6$</td>
<td>1,000,000</td>
<td></td>
</tr>
<tr>
<td>$10^5$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$10^4$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$10^3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$10^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$10^1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$10^0$</td>
<td></td>
<td>$\frac{1}{10}$</td>
</tr>
<tr>
<td>$10^{-1}$</td>
<td></td>
<td>$\frac{1}{100}$</td>
</tr>
<tr>
<td>$10^{-2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$10^{-4}$</td>
<td></td>
<td>$\frac{1}{1,000}$</td>
</tr>
</tbody>
</table>
(Of course you can write all the decimal numerals as fractions using a power of 10 as the denominator. For instance, \(1 = \frac{10}{10}\) or \(\frac{100}{100}\) or \(\frac{1}{1}\).)

To multiply 100,000 by .001, you can use the decimal numerals:

\[
\begin{align*}
100,000 \times 0.001 &= 100,000 \\
100,000 \times \frac{1}{1000} &= 100,000 \\
\end{align*}
\]

You can use the fraction form:

\[
100,000 \times \frac{1}{1000} = \frac{100,000}{1000} = \quad \text{\underline{\text{fraction form:}}} \quad \frac{100,000}{1000}
\]

Or you can use the exponent form: \(10^5 \times 10^{-3} = 10^{5-3} = \quad \text{\underline{\text{exponent form:}}} \quad 10^2\)

Using exponents is especially helpful when you have very small or very large numbers:

\[
\begin{align*}
1,000,000 \times 1,000,000 &= 1,000,000,000,000 \\
10^6 \times 10^6 &= \quad \text{\underline{\text{exponent form:}}} \quad 10^{6+6}
\end{align*}
\]

You can use exponents when you divide by a power of 10 written as a decimal numeral.

Decimal form: \[
\frac{100}{0.001} = \frac{100}{0.001} \times \frac{10,000}{10,000}
\]

\[
= \frac{1,000,000}{1}
\]

\[
= 1,000,000
\]

or exponent form: \[
\frac{10^2}{10^{-4}} = 10^{2-(-4)} = 10^2 + 4 = \quad \text{\underline{\text{exponent form:}}} \quad 116
\]
Rewrite each problem below using exponents with $10$. Write your answer with an exponent.

1. (a) $10,000 \times 0.0001 = \underline{\hspace{2cm}} \times \underline{\hspace{2cm}}$
   
   (b) $100 \times 100,000 = \underline{\hspace{2cm}} \times \underline{\hspace{2cm}}$
   
   (c) $0.001 \times 0.01 = \underline{\hspace{2cm}} \times \underline{\hspace{2cm}}$
   
   (d) $0.0001 \times 0.000001 = \underline{\hspace{2cm}} \times \underline{\hspace{2cm}}$
   
   (e) $\frac{1}{100} \times 1000 = \underline{\hspace{2cm}} \times \underline{\hspace{2cm}}$
   
   (f) $\frac{1}{1000} \times \frac{1}{10,000} = \underline{\hspace{2cm}} \times \underline{\hspace{2cm}}$
   
   (g) $\frac{0.1}{0.001} = \underline{\hspace{2cm}} \times \underline{\hspace{2cm}}$
   
   (h) $\frac{0.0001}{0.01} = \underline{\hspace{2cm}} \times \underline{\hspace{2cm}}$
2. Rewrite each problem below using decimal numerals. Write the answer as a decimal numeral.

(a) \(10^5 \times 10^3 = \underline{\hspace{2cm}} \times \underline{\hspace{2cm}} = \underline{\hspace{4cm}}\)

(b) \(10^{-2} \times 10^{-3} = \underline{\hspace{2cm}} \times \underline{\hspace{2cm}} = \underline{\hspace{4cm}}\)

(c) \(10^6 \times 10^{-4} = \underline{\hspace{2cm}} \times \underline{\hspace{2cm}} = \underline{\hspace{4cm}}\)

(d) \(10^0 \times 10^{-2} = \underline{\hspace{2cm}} \times \underline{\hspace{2cm}} = \underline{\hspace{4cm}}\)
Scientists often have to use very large or very small numbers. They need very large numbers when they talk about the distance to the stars and they need very small numbers when they talk about the size of an atom. Some of these numbers, when written down, require a great many zeros. Writing a great many zeros is not only a lot of trouble but it also can lead to mistakes. If you have to multiply a number with many zeros, it is easy to write one too many or one too few and get the wrong answer.

For example, light travels at a speed of about 186,000 miles per second! The distance of stars from the earth is measured in light years, or the distance light travels in a year. This is a very large number indeed.

To find out, in the usual way, what a light year is, we first multiply 186,000 by 60 to find how far light travels in 1 minute:

\[
\begin{align*}
186,000 \times 60 &= 11,160,000 \\
&= \text{miles in 1 minute}
\end{align*}
\]

We multiply 11,160,000 by 60 to find how far light travels in one hour:

\[
\begin{align*}
11,160,000 \times 60 &= 669,600,000 \\
&= \text{miles in 1 hour}
\end{align*}
\]

We multiply 669,600,000 by 24 to find how far it travels in one day:

\[
\begin{align*}
669,600,000 \times 24 &= 2678400000 \\
&= 13392000000 \\
&= 16070400000 \\
&= \text{miles in 1 day}
\end{align*}
\]
Last, we can multiply 16,070,400,000 by 365 to find how far light travels in one year:

\[
\begin{array}{c}
16,070,400,000 \\
\times 365 \\
\hline
80,352,000,000 \\
964,224,000,000 \\
4,821,120,000,000 \\
\hline
5,865,696,000,000
\end{array}
\]

(miles in 1 year)

Suppose you wanted to find the distance in miles of a star that was 492 light years away from the earth. Think how many zeros you would have to write in multiplying

\[5,865,696,000,000 \times 492\]

(We won't do this here!)

Because it is so clumsy to work with so many digits, scientists use powers of 10 to make their work easier.

Class Discussion

1. You can think of any number as 10 times some other number.

\[
\begin{align*}
35 &= 10 \times 3.5 \\
48.9 &= 10 \times 4.89 \\
56.4 &= 10 \times 5.64
\end{align*}
\]

2. You can think of any number as 100 times another number.

\[
\begin{align*}
650 &= 100 \times 6.5 \\
785.7 &= 100 \times 7.857 \\
966.4 &= 100 \times 9.664
\end{align*}
\]

3. You can think of any number as \(\frac{1}{10}\) times another number.

\[
\begin{align*}
.459 &= \frac{1}{10} \times 4.59 \\
.1561 &= \frac{1}{10} \times 1.561 \\
.20723 &= \frac{1}{10} \times 2.0723
\end{align*}
\]
In scientific notation, every number is written so that there is only one digit to the left of the decimal point in the number to be multiplied by a power of 10. (Naturally, you wouldn't write 1,000,000 as $1 \times 10^6$ or 5 as $5 \times 10^0$. You'd just write $10^6$ and 5.)

This is not as complicated as it sounds. Let's go back to the light year problem, and let scientific notation help us do the work.

To begin with, we write 186,000 as $1.86 \times 10^5$. To find out what power of 10 we need, we start at the right of 186,000 (at the decimal point) and count back until there is just one digit left. That is five places, so $186,000 = 1.86 \times 10^5$.

To multiply that number by 60, we write 60 as $6 \times 10^1$.

$$186,000 \times 60 = (1.86 \times 10^5) \times (6 \times 10^1) = (1.86 \times 6) \times (10^5 \times 10^1)$$

because it doesn't matter in what order you multiply.

Then multiply 1.86 by 6:

$$\begin{array}{c}
1.86 \\
\times 6 \\
\hline
11.16
\end{array}$$

But 11.16 has two places to the left of the decimal point. We rewrite it as $1.116 \times 10^1$.

Our miles-in-one-minute number now is

$$(1.116 \times 10^1) \times (10^5 \times 10^1)$$

and that is $1.116 \times 10^7$ (miles in 1 minute).

Next we multiply that number by 60 to find how far light travels in one hour, but again we use $6 \times 10^1$ as the name for 60.

$$(1.116 \times 10^7) \times (6 \times 10^1) = (1.116 \times 6) \times (10^7 \times 10^1)$$

Then we multiply:

$$\begin{array}{c}
1.116 \\
\times 6 \\
\hline
6.696
\end{array}$$

and we know the miles-in-one-hour can be written

$$6.696 \times 10^8$$
Next $6.696 \times 10^8$ is multiplied by 24 to find how far light travels in one day.

$$(6.696 \times 10^8) \times (2.4 \times 10^1) = (6.696 \times 2.4) \times (10^8 \times 10^1)$$

Multiply:

\[
\begin{array}{c}
6.696 \\
\times 2.4 \\
\hline
2.6784 \\
13.3920 \\
\hline
16.0704
\end{array}
\]

16.0704 can be written $1.60704 \times 10^1$, so the miles-in-one-day can be written

$$(1.60704 \times 10^1) \times (10^8 \times 10^1)$$ or $1.60704 \times 10^{10}$.

Last, we multiply by 365, which is $3.65 \times 10^2$.

\[
\begin{array}{c}
1.60704 \\
\times 3.65 \\
\hline
803520 \\
9642240 \\
48211200 \\
\hline
5.8656960
\end{array}
\]

So a light year is

$$5.8656960 \times 10^{12}$$ miles

Suppose we had found this number in the first place. How would we know what it looked like in the usual form? To change from scientific notation to our "regular" notation, we just move the decimal point twelve places to the right. We have to put in 5 zeros to do this, and we see:

$$5.8656960 \times 10^{12} = 5,865,696,000,000$$
Exercises

1. Use the flow chart on page 13-10c to rewrite each number in scientific notation. Remember that every whole number can be written with a decimal point after the ones place.

Example 1. \(3,450,600 = \) 

Step (a): 3. (See Boxes 3 and 4 in the flow chart.)
Step (b): \(3.4506\) (See Box 5 in the flow chart.)
Step (c): You moved the point 6 places to the left. (See Boxes 6 and 7.)
Step (d): Then \(3,450,600 = 3.4506 \times 10^6\) (From Box 9.)

Example 2. \(.0134567 = \) 

Step (a): 1. (See Boxes 2 and 4.)
Step (b): \(1.34567\) (See Box 5.)
Step (c): You moved the decimal point 2 places to the right. (See Boxes 6 and 7.)
Step (d): Then \(.0134567 = 1.34567 \times 10^{-2}\) (From Box 8.)

(a) \(450 = \) \(f) \(457,000,000 = \) 

(b) \(2407.35 = \) \(g) \(.00846 = \) 

(c) \(.0001968 = \) \(h) \(99,452,760 = \) 

(d) \(.045735 = \) \(i) \(38.4093 = \) 

(e) \(574821.6 = \) \(j) \(.75369 = \)
Flow Chart for Writing Scientific Notation

START

No

Is the number greater than 1?

Yes

Copy the first non-zero digit.

Copy the first digit.

Put a decimal point after it.

Copy the rest of the digits to the right in order until there are no digits left except zeros.

Count the number of places between the old decimal point and the new one.

Did you move the decimal point to the left?

No

Use the opposite of the number from Box 6 as the exponent with 10.

Yes

Use the number from Box 6 as the exponent with 10.

STOP
Flow Chart for Changing
Scientific Notation to Ordinary Numerals

START

1
Look at the exponent with 10.

No

Is the exponent positive? Yes

2
Move the decimal point that many places to the left.

STOP

3
Move the decimal point that many places to the right.
2. These members are written in scientific notation. Use the flow chart on page 13-10f to write them the usual way.

Example 1. \(3.45 \times 10^{-3} = \) ?

Step (a): \(3\) (See Box 1.)

Step (b): \(0.00345\) (See Box 2.)

So \(3.45 \times 10^{-3} = 0.00345\)

Example 2. \(2.019 \times 10^5 = \) ?

Step (a): \(5\) (See Box 1.)

Step (b): \(201900\) (See Box 3.)

So \(2.019 \times 10^5 = 201900\)

(a) \(6.15 \times 10^{-5} = \) 

(b) \(1.001 \times 10^3 = \) 

(c) \(5.492 \times 10^2 = \) 

(d) \(2.8875 \times 10^{-4} = \) 

(e) \(9.6264 \times 10^6 = \) 

(f) \(7.3405 \times 10^{-2} = \)
Rational Numbers on the Number Line

Class Discussion

Take pages 13-11d and 13-11e out of your notebook.

The length of the unit segment used on each line is the same.

On line A, the unit is divided into hundredths. Find the point that corresponds to ten-hundredths. Label it \( \frac{10}{100} \). Then find the points that correspond to \( \frac{20}{100} \), \( \frac{30}{100} \), \( \frac{40}{100} \), and so on, and label them.

On line B the unit is divided into hundredths again. Find the point that corresponds to \( \frac{10}{100} \) on this line. Label that point 10%.

Then find and label the points that correspond to 20%, 30%, 40%, and so on.

On line C, the unit is divided into \( \frac{1}{10} \), \( \frac{2}{10} \), \( \frac{3}{10} \), and so on.

On line D, the unit is again divided into \( \frac{1}{10} \) parts. Label the points \( \frac{1}{10} \), \( \frac{2}{10} \), \( \frac{3}{10} \), and so on.

On line E, the unit is divided into \( \frac{1}{8} \), \( \frac{2}{8} \), \( \frac{3}{8} \), and so on.

Label the points on lines F, G, and H to show what rational number each point corresponds to. Do not simplify the fractions. (That is, write \( \frac{2}{4} \), \( \frac{3}{6} \), etc. instead of \( \frac{1}{2} \).)

Carefully cut along the dashed lines so that you have eight separate number lines. Keep line H apart for awhile, but stack the others up on your desk with line G on top.

Your teacher will give you three pins. Make sure the number lines are stacked up so that when you stick a pin through the point labeled 0 on line G it goes through the point labeled 0 on all the other lines, too. Leave the pin there.
Stick another pin through the stack so it goes through the point labeled 1 on every number line.

Stick the last pin through the point labeled \( \frac{1}{4} \) so that it goes straight down through all the number lines.

1. Now carefully take out the pins.

   On line E, the pin went through the point labeled _____.
   On line F, the pin went through a point between _____ and _____, and closer to _____.
   On line D, the pin went through the point half-way between _____ and _____.
   On line C, the pin went through the point half-way between _____ and _____.
   On line B, the pin went through the point half-way between _____ and ___. This point corresponds to _____%.
   On line A, the pin went through the point half-way between _____ and ____. Label this point as a fraction with the denominator 100.

Stack up lines A through F again, with F on top. Stick pins through the points labeled 0 and 1 as before. Stick the third pin through the point labeled \( \frac{3}{5} \). Take out the pins.

2. Show the point where the pin went through the other number lines.

   A _____
   B _____
   C _____
   D _____
   E between _____ and _____, and closer to _____.
Stack up the number lines with A on the bottom in this order: A, B, C, D, G, F. Stick pins through points 0 and 1. Stick the third pin through the point labeled \( \frac{7}{8} \). Take out the pins.

3. Show where the pin went through the other number lines.

A halfway between \( \frac{100}{100} \) and \( \frac{100}{100} \) (Write the numerators.)

B between _______ and _______.

C between _______ and _______ and closer to _______.

D between _______ and _______ and closer to _______.

G halfway between _______ and _______.

Stack up all the lines with H on top. Stick the third pin through the point labeled \( \frac{1}{3} \). If you have been careful every time, the pin went through a point not marked on any of the other lines.

4. Show where the pin went through on each line.

A between _______ and _______.

B between _______ and _______.

C between _______ and _______.

D between _______ and _______.

E between _______ and _______.

F between _______ and _______.

G between _______ and _______.

Since the unit segment is the same length on each number line, you can use this method to find different names for some other rational numbers, too.
Exercises

Think of a pin going through each point named. Show where it goes through each of the number lines given. (Use the lines and pins if you need them.)

1. \(\frac{3}{4}\)
   E: _____  B: _____  A: _____

2. \(\frac{2}{5}\)
   D: _____  C: _____  B: _____  A: _____

3. \(\frac{4}{8}\)
   H: half-way between _____ and _____
   G: _____  C: _____  B: _____

4. \(\frac{7}{10}\)
   A: _____  B: _____  D: _____
   F: half-way between _____ and _____
Fractions, Decimals and Percents

Suppose somebody offered you a choice like this:

1. You can have \( \frac{2}{5} \) of 400 dollars.
2. You can have \( 0.4 \times 400 \) dollars.
3. You can have 40% of 400 dollars.

Which would you choose?

You know it wouldn’t make any difference because \( \frac{2}{5} \), 0.4, and 40% all name the same number. You would get $160 no matter which you choose.

Any rational number can be written as a fraction, a decimal numeral, or a per cent. \( \frac{2}{5} = 0.4 \) and \( \frac{2}{5} = 40\% \).

In Chapter 10 you learned how to rename fractions as decimals and decimals as fractions. In Chapter 12 you learned that whenever a ratio is written with 100 in the denominator, you are writing a percent. \( \frac{50}{100} = 50\% \). Since you know that \( \frac{50}{100} = \frac{1}{2} \), you know that \( 50\% = \frac{1}{2} \). And since \( \frac{1}{2} = 0.5 \), \( 50\% = 0.5 \).

Class Discussion

1. To rewrite a percent as a fraction, the first step is to copy the number, leaving off the % sign. Use this as the numerator of a fraction. The denominator is _______.

\[
75\% = \frac{75}{100} \quad \text{(Write the numerator.)}
\]

\[
20\% = \frac{20}{100} \quad \text{(Write the denominator.)}
\]

\[
62\% = \frac{62}{100} \quad \text{(Write the fraction.)}
\]
The next step is to simplify the fraction if possible.

\[
\frac{75}{100} = \frac{25}{25} \times \quad \text{so} \quad \frac{75}{100} = \quad \text{(Write the fraction.)}
\]

\[
\frac{20}{100} = \frac{20}{20} \times \quad \text{so} \quad \frac{20}{100} = \quad \text{(Write the fraction.)}
\]

\[
\frac{62}{100} = \frac{2}{2} \times \quad \text{so} \quad \frac{62}{100} = \quad \text{(Write the fraction.)}
\]

2. To rewrite a percent as a decimal, write the percent as a fraction with the denominator 100.

\[30\% = \frac{30}{100} \quad \text{(Write the denominator.)}\]

\[150\% = \frac{150}{100} \quad \text{(Write the numerator.)}\]

\[5\% = \quad \text{(Write the fraction.)}\]

Next, rewrite the fraction as a decimal numeral.

\[\frac{30}{100} = \quad \text{(Write the fraction.)}\]

\[\frac{150}{100} = \quad \text{(Write the fraction.)}\]

\[\frac{5}{100} = \quad \text{(Write the fraction.)}\]

3. Sometimes a percent has a fraction or a decimal in front of the % sign. Here are two examples: \(1 \frac{1}{2}\%\), \(0.5\%\). Again, write a fraction with the denominator 100.

\[1 \frac{1}{2}\% = \frac{1 \frac{1}{2}}{100}\]

You know how to rewrite \(1 \frac{1}{2}\) as a decimal. \(1 \frac{1}{2} = \quad \text{.}\)

So \(\frac{1 \frac{1}{2}}{100} = \frac{1.5}{100}\)
To divide by 100, you move the decimal point _____ places to the _____, so 1.5 divided by 100 = _____.

Therefore

\[ \frac{1}{2} \% = \boxed{.5\%} \]

Divide .5 by 100.

\[ .5\% = \boxed{.005} \] (Write the decimal.)

4. Here are two ways to change a fraction to a percent. One way is to rename the fraction with the denominator 100.

\[ \frac{1}{4} = \frac{25}{100} \] (Write the numerator.)

Rewrite the new fraction as a percent.

\[ \frac{25}{100} = \boxed{25\%} \]

For some fractions, the numerator of the new fraction may not be a whole number.

\[ \frac{5}{8} = \frac{5 \times 125}{8 \times 125} = \frac{625}{1000} = \frac{62.5}{100} \]

\[ = \boxed{62.5\%} \] or \[ \frac{62 \frac{1}{2}}{100} \]

\[ = \boxed{62 \frac{1}{2}\%} \] or \[ \frac{62 \frac{1}{2}}{100} \]
Sometimes when you rename a fraction, you get a repeating decimal. This brings us to the second way to change a fraction to a percent, and this is a way you can use with all fractions.

\[ \frac{1}{3} = \ldots .3 \]

You know that \( \frac{1}{3} \) is written \( .3 \) as a decimal. Since percent means hundredths, you can divide the numerator of the fraction by the denominator, but use only two places (tenths and hundredths place) to the right of the decimal point.

To find \( \frac{1}{3} \), divide: \( 3 \div 1.00 \)

So \( \frac{1}{3} = .33 \frac{1}{3} \) or \( \frac{33 \frac{1}{3}}{100} \) or \( 33 \frac{1}{3} \% \).

To change a decimal to a percent, write the decimal as a fraction with a power of ten as the denominator.

\[ .875 = \frac{875}{1000} \quad (\text{Write the denominator.}) \]

Now rename this fraction so that the denominator is 100.

\[ \frac{875}{1000} = \frac{?}{100} \quad (\text{To find the numerator, think } 875 \times 100 = 1000 \times \ldots) \]

\[ \frac{87.5}{100} = 87.5\% \]

So \( .875 = .875\% \).

Remember that when you rename a decimal as a percent, you move the decimal point two places to the right and put a percent sign: \( .375 = \ldots 37.5\% \) and \( .2 = \ldots 20\% \).
1. Fill in the chart below so that each line gives three different names for the same number.

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Decimal</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}$</td>
<td>.5</td>
<td>50%</td>
</tr>
<tr>
<td>$\frac{1}{8}$</td>
<td>.125</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{5}$</td>
<td>.2</td>
<td></td>
</tr>
<tr>
<td>$\frac{2}{3}$</td>
<td>.63</td>
<td></td>
</tr>
<tr>
<td>14%</td>
<td>.14</td>
<td>14%</td>
</tr>
<tr>
<td>150%</td>
<td>.15</td>
<td>150%</td>
</tr>
<tr>
<td>12 $\frac{1}{2}$%</td>
<td>.125</td>
<td></td>
</tr>
<tr>
<td>130%</td>
<td>.13</td>
<td></td>
</tr>
<tr>
<td>.45</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{4}$%</td>
<td>.025</td>
<td></td>
</tr>
<tr>
<td>5.5%</td>
<td>.055</td>
<td></td>
</tr>
<tr>
<td>.001</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
2. Solve these problems. Use the way that's easiest for you. (Remember that "of" means "times").

Example. \[ \frac{12.5}{100} \times 128 = \frac{1600.0}{100} \]

One way:

\[ \frac{12.5}{100} \times 128 = 16 \]

Another way:

\[ .125 \times 128 = 16,000 \]

A third way:

\[ \frac{1}{8} \times 128 = \frac{128}{8} \]

\[ = 16 \]

(a) \( \frac{1}{4} \) of 60 = _______  
(b) \( \frac{3}{8} + \frac{1}{3} = \) _______  
(c) \( .25 \% \) of 480 = _______  
(d) \( .375 \times 72 = \) _______  
(e) \( 60 \% \) of 100 = _______  
(f) \( 66 \frac{2}{3} \% \) of 150 = _______  
(g) \( .25 \times .875 = \) _______  
(h) \( .39 \times 200 = \) _______  
(i) \( \frac{276}{10} = \) _______  
(j) \( 55 = 50\% \) of _______  
(k) \( \frac{3}{125} = \) _______
Pre-Test Exercises

These exercises are like the problems you will have on the chapter test. If you don't know how to do them, read the section again. If you still don't understand, ask your teacher.

1. (Section 13-1.)
Find the answer to these addition and subtraction problems. Use the line below each subtraction problem to rewrite it as an addition problem.

(a) $4 + 6 = \underline{\hspace{1cm}}$
(b) $13 - 4 = \underline{\hspace{2cm}}$
(c) $10 - 8 = \underline{\hspace{1cm}}$
(d) $-5 + 3 = \underline{\hspace{1cm}}$
(e) $-2 - 7 = \underline{\hspace{1cm}}$
(f) $3 - 6 = \underline{\hspace{1cm}}$
(g) $-2 + 5 = \underline{\hspace{1cm}}$
(h) $4 - 2 = \underline{\hspace{1cm}}$
(i) $-6 - 9 = \underline{\hspace{1cm}}$
(j) $-7 - 3 = \underline{\hspace{1cm}}$
(k) $-7 + 8 = \underline{\hspace{1cm}}$
(l) $-15 + 5 = \underline{\hspace{1cm}}$
(m) $479 + 479 = \underline{\hspace{1cm}}$

2. (Section 13-2.)
Multiply or divide as shown.

(a) $4 \cdot 6 = \underline{\hspace{1cm}}$
(b) $7 \cdot 9 = \underline{\hspace{1cm}}$
(c) $\frac{14}{2} = \underline{\hspace{1cm}}$
(d) $\frac{15}{3} = \underline{\hspace{1cm}}$
(e) $-8 \cdot 4 = \underline{\hspace{1cm}}$
(f) $-5 \cdot 5 = \underline{\hspace{1cm}}$
(g) $\frac{-13}{13} = \underline{\hspace{1cm}}$
(h) $-\frac{6}{2} = \underline{\hspace{1cm}}$

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3. (Section 13-3.)
Rewrite the numbers below if necessary. Then compare them.
(Use < or >).
(a) \( \frac{17}{3} \) \( \frac{14}{2} \)
(b) \( \frac{2}{6} \) \( \frac{5}{3} \)
(c) \( -\frac{1}{2} \) \( \frac{2}{7} \)

4. (Section 13-3.)
Rewrite if necessary and add.
(a) \( \frac{1}{6} + \frac{3}{6} = \frac{1}{2} \)
(b) \( \frac{3}{4} + \frac{1}{4} = 1 \)

5. (Section 13-4.)
Fill the blanks.
(a) The number \( \frac{1}{2} \) is a multiple of every number.
(b) Every number is a multiple of \( \frac{1}{2} \).
(c) The first multiple of every number is the \( \frac{1}{2} \) of itself.
(d) Every number is a \( \frac{1}{2} \) of itself.
(e) The first five multiples of \( 4 \) (except 0, of course) are \( 4, 8, 12, 16, 20 \), and \( \frac{1}{2} \).
(f) The multiples of \( 2 \) are called \( \frac{1}{2} \) numbers.
(g) The first five prime numbers are \( 2, 3, 5, 7, 11 \), and \( \frac{1}{2} \).
(h) Prime numbers are multiples of exactly \( \frac{1}{2} \) numbers.
6. (Section 13-5.)
Fill the blanks.
(a) If, in a pair of numbers, one of them is a multiple of the other, then the __________ number is the least common multiple of the pair.
(b) The least common multiple of 4 and 16 is _______.
(c) The least common multiple of two prime numbers is the __________ of the two.
(d) The L.C.M. of 5 and 7 is _______.
(e) If two numbers have no common factors except 1, their L.C.M. is the __________ of the two numbers.
(f) The L.C.M. of 9 and 4 is _______.
(g) The L.C.M. of 15 and 18 is _______.
(h) Every common multiple of 2 and 3 is also a multiple of _______.

7. (Section 13-6.)
Add the following rational numbers. (You may use the flow chart on Page 7 of your tables if you need it.) Use the space at the right for your work.
(a) $\frac{1}{2} + \frac{1}{3} =$ _______

(b) $\frac{2}{3} + \frac{3}{4} =$ _______

(c) $\frac{1}{4} + \frac{3}{8} =$ _______

(d) $\frac{1}{4} + \frac{1}{9} =$ _______

(e) $\frac{3}{4} + \frac{5}{6} =$ _______
8. (Section 13-7.)
Rewrite each subtraction problem as an addition problem and then find the answer. Use the space at the right for your work.

(a) \( \frac{3}{8} - \frac{1}{4} = \) 

(b) \( \frac{3}{4} - \frac{2}{3} = \) 

(c) \( \frac{9}{8} - \frac{5}{6} = \) 

(d) \( \frac{4}{5} - \frac{1}{2} = \) 

(e) \( -\frac{1}{3} - \frac{1}{6} = \) 

9. (Section 13-8.).
Use exponents to multiply or divide.

(a) \( 10^{-1} \times 10^{-4} = \) 

(b) \( \frac{10^5}{10^3} = \) 

(c) \( 10^5 \times 10^{-3} = \)
10. (Section 13-9.)
Rewrite each problem using 10 and an exponent. Write your answer with an exponent.

(a) \( \frac{10^4}{10^{-1}} = \) 

(b) \( \frac{1}{1,000} \times \frac{1}{100} = \)

(c) \( \frac{1,000,000 \times 0.01}{.0001} = \)

(d) \( \frac{100,000}{.0001} = \)
11. (Section 13-10.)
Write these numbers using scientific notation. Use the flow chart on page 13-10e if you need it.

(a) $456,000,000 = \underline{\hspace{2cm}} \times \underline{\hspace{2cm}}$

(b) $297.457 = \underline{\hspace{2cm}} \times \underline{\hspace{2cm}}$

(c) $0.001642 = \underline{\hspace{2cm}} \times \underline{\hspace{2cm}}$

(d) $54600.7 = \underline{\hspace{2cm}} \times \underline{\hspace{2cm}}$

(e) $0.0087523 = \underline{\hspace{2cm}} \times \underline{\hspace{2cm}}$

12. (Section 13-11.)
Fill in the chart.

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Decimal Numeral</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}$</td>
<td>.5</td>
<td>50%</td>
</tr>
<tr>
<td>$\frac{1}{6}$</td>
<td></td>
<td>87.5%</td>
</tr>
<tr>
<td></td>
<td>.8</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
1. Addition and Subtraction. Use the line below the problem to rewrite it if necessary.
   (a) \( 5 + (-11) = \) 
   \[ \underline{\text{Answer: }} \]
   (b) \( 4 - 7 = \) 
   \[ \underline{\text{Answer: }} \]
   (c) \( -8 + 16 = \) 
   \[ \underline{\text{Answer: }} \]
   (d) \( 3 - (-4) = \) 
   \[ \underline{\text{Answer: }} \]
   (e) \( -9 - 2 = \) 
   \[ \underline{\text{Answer: }} \]

2. Multiply.
   (a) \( 3 \cdot 9 = \) 
   \[ \underline{\text{Answer: }} \]
   (b) \( -15 \cdot -2 = \) 
   \[ \underline{\text{Answer: }} \]
   (c) \( -8 \cdot 3 = \) 
   \[ \underline{\text{Answer: }} \]
   (d) \( 4 \div 7 = \) 
   \[ \underline{\text{Answer: }} \]

3. Divide.
   (a) \( \frac{36}{4} = \) 
   \[ \underline{\text{Answer: }} \]
   (b) \( -\frac{9}{3} = \) 
   \[ \underline{\text{Answer: }} \]
   (c) \( -\frac{8}{8} = \) 
   \[ \underline{\text{Answer: }} \]
4. Find the L.C.M. of each pair of numbers.
   (a) 3 and 12: L.C.M. is __________
   (b) 15 and 8: L.C.M. is __________
   (c) 9 and 15: L.C.M. is __________

5. Add. Use the space at the right for your work.
   (a) \( \frac{2}{3} + \frac{1}{4} = \) __________

   (b) \( \frac{3}{5} + \frac{1}{10} = \) __________

   (c) \( \frac{9}{8} + \frac{1}{6} = \) __________

6. Use the line below each subtraction problem to rewrite it as an addition problem. Then find the answer. The space at the right is for your work.
   (a) \( \frac{3}{4} - \frac{1}{3} = \) __________

   (b) \( \frac{5}{6} - \frac{1}{3} = \) __________

   (c) \( \frac{5}{8} - \frac{3}{4} = \) __________
7. Rewrite the following using exponents. Use an exponent in your answer.

(a) \[1000 \times 1000 = \_ \times \_ = \_\]

(b) \[.001 \times 1000 = \_ \times \_ = \_\]

(c) \[\frac{1000}{1,000,000} = \_ = \_\]

(d) \[\frac{1}{10} \times \frac{1}{1000} = \_ = \_\]

8. Write these numbers using scientific notation.

(a) \[40.6 = \_\]

(b) \[274,820,000 = \_\]

(c) \[.00006821 = \_\]

(d) \[.00567 = \_\]


<table>
<thead>
<tr>
<th>Fraction</th>
<th>Decimal Numeral</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>_</td>
<td>_</td>
<td>_</td>
</tr>
<tr>
<td>[\frac{2}{3}]</td>
<td>_</td>
<td>_</td>
</tr>
<tr>
<td>_</td>
<td>_</td>
<td>50%</td>
</tr>
<tr>
<td>_</td>
<td>1.125</td>
<td>_</td>
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<td>_</td>
<td>_</td>
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<td>_</td>
<td>_</td>
<td>_</td>
</tr>
<tr>
<td>_</td>
<td>.25</td>
<td>_</td>
</tr>
<tr>
<td>[\frac{1}{1,000}]</td>
<td>_</td>
<td>_</td>
</tr>
<tr>
<td>_</td>
<td>147 [\frac{147}{53}]</td>
<td>_</td>
</tr>
</tbody>
</table>
Check Your Memory: Self-Test

1. (Sections 9-5 and 9-8.)

BC is the perpendicular bisector of \( \overline{AD} \). Therefore, 
\[ \triangle ABC \cong \triangle \text{_______} \]
by the ________ property of congruence.

2. (Section 9-4.)
Using the line segment below as one side, construct a triangle with all three sides congruent.

3. (Section 9-7.)
Use the triangle you just drew as one half of a rhombus and complete the rhombus.
4. (Section 10-6.)

Compare these numbers. Show which is smaller by putting < or > between them.

(a) .06 _____ .0395
(b) .2001 _____ .096.
(c) 3.014 _____ .6
(d) .397 _____ .3971

5. (Section 10-13.)

Add or subtract. Watch the signs.

(a) .0321 + .76 = 
(b) 2.751 + 13.528 = 
(c) .309 - .27 = 
(d) .0126 - .00451 = 

6. (Section 12-2.)

For the following pair of similar triangles, write the scale factors and then find the lengths that are missing.

Scale factor: _____ and _____

x = 

y = 
7. (Section 12-8.)

Write the missing numerators and denominators. Then solve the problem.

(a) To find 40% of 40, you write:

\[
\frac{x}{40} = \text{ (Use the space below for the arithmetic.)}
\]

40% of 40 is ________.

(b) To find what percent of 60 the number 45 is, you write:

\[
\frac{x}{100} = \frac{45}{60}
\]

______% of 60 is 45.

(c) To find what number 18 is 50% of, you write:

\[
\frac{18}{x} = \frac{50}{100}
\]

18 is 50% of ________.

Now check your answers on the next page. If you do not have them all right, go back and read the section again.
Answers to Check Your Memory: Self-Test

1. $\triangle ABC \cong \triangle DBC$ by the SAS congruence property.

2. Your triangle should have looked like this:

   ![Triangle A](image1)

   or like this:

   ![Triangle B](image2)

3. Your rhombus should be congruent to this one, but it may be turned around.

   ![Rhombus](image3)
4. (a) \(0.06 > 0.0395\)
   (b) \(0.2001 > 0.096\)
   (c) \(3.014 > 2.6\)
   (d) \(0.397 < 3.971\)

5. (a) \(0.7921\)
   (b) \(16.279\)
   (c) \(0.039\)
   (d) \(0.00809\)

6. Scale factor: \(\frac{1}{2}\) and \(2\).
   \(x = 5\)
   \(y = 12\)

7. (a) \(\frac{40}{100} = \frac{x}{40}\)
   \[40\% \text{ of } 40 \text{ is } 16\]
   (b) \(\frac{x}{100} = \frac{45}{60}\)
   \[75\% \text{ of } 60 \text{ is } 45\]
   (c) \(\frac{18}{x} = \frac{50}{100}\)
   \[18 \text{ is } 50\% \text{ of } 36\]
Chapter 14
PERPENDICULARS
You have learned that two geometric figures are congruent to each other if they have exactly the same size and shape. Two figures that are congruent may be made to fit together by a motion. In this part of the chapter we will look at three types of motion. For example, the figure A on the left may be made to fit onto the figure B by a sliding motion in the direction of the arrow. We say that figure B is the slide image of figure A, or that the slide maps figure A onto figure B. Likewise, figure B' is a slide image of figure A' because a sliding motion in the direction of the right-hand arrow carries A' onto B'.

Look at the congruent figures shown below. There is no slide that will map figure R onto figure S.

But figure R may be made to map onto figure S by a turn about point X as a center of turn. We say that figure S is the turn image of figure R, or that the turn maps figure R onto figure S.
The center of turn may even be a point common to the two figures. Look at the congruent figures shown below.

The third basic motion that we will consider is the flip. You can think of a flip as the motion involved when you turn over a sheet of paper. In the process of flipping a sheet of paper, the plane of the paper is rotated about a line as shown below.

In this case, the line $AB$ is called the flip axis. Look at the congruent pair of figures, $E$, and $F$, shown below.
Figure E may be made to map onto figure F by a flip about \( l \) as the flip axis. We say that figure F is the flip image of figure E or that the flip maps figure E onto figure F.

In the figure below, note how the figures are related by flips about \( n \) and \( m \).

It is interesting to see that sometimes two congruent figures can be made to map onto each other by different motions. For example, figure H can be made to map onto figure G by a flip or by a turn/slide or by other ways. Find one other way.
Exercises

1. In each case, the given pair of figures is congruent. For each congruent pair, show exactly how the figure marked (1) may be mapped onto the figure marked (2) by using the motions of slide, turn, or flip, or a combination of these methods. Use arrows to show slides or turns and dotted lines to show flip axes. Try to show more than one method of making the figures map onto each other.

Example:

Possible Solution:

Possible Solution:

Possible Solution:

Possible Solution:

turn about Point P

turn/slide

\$\text{ turn/flip}$

\$\text{ slide/flip}$

Can you find any others? (Hint: There are some!)
2. The figures below are to be flipped on the dotted lines as axes. In which cases does the figure map onto itself?

(a) \(\quad\) (b) \(\quad\) (c) \(\quad\) 

(d) \(\quad\) (e) \(\quad\) (f) \(\quad\)

3. For each figure below, draw all flip axes such that the figure maps onto itself.

(a) \(\quad\) (b) \(\quad\) 

(c) \(\quad\) (d) \(\quad\)

4. Can you think of a figure that has infinitely many flip axes?
BRAINBOOSTER.

5. Think of each figure below as a wall design that extends forever in both directions. What motion or motions will carry the design onto itself?

(a) 

(b) Name two types of motion that will carry this design onto itself.

(c) Name two types of motion that will carry this design onto itself. Can you find a third type of motion?
Congruence of a Figure with Itself

Before you start this lesson your teacher will give you a 3 x 5 card. Label the corners of this rectangular card with capital letters, as shown below. Now turn the card over and label the corners on the back so that A is in back of A, B is in back of B, C is in back of C, and D is in back of D.

Remove Page 14-2e from your notebook and place the card in the center of the page. Trace along the edges of the card so that you draw a rectangle. Label the vertices of the rectangle as shown below.
Now, in how many ways can you fit the card back onto the rectangle? One way would be to slide the card back onto the rectangle in the exact position it was when you made the tracing.

This fitting shows the identity congruence

\[ \text{ABCD} \cong \text{ABCD} \]

Notice that there is a correspondence between each corner of the card and the vertices of the rectangle that looks like this:

\[ \text{ABCD} \cong \text{ABCD} \]

We always write congruences so that we can tell, even without looking at the picture, which parts of the figures correspond. The picture below is an example of what we mean by this correspondence.
Class Discussion

1. (a) Place your card back on the tracing so as to show the identity congruence, \( \triangle ABCD \cong \triangle ABCD \).

(b) Now make a half-turn with the card so that the corner \( A \) takes position at \( C \), the corner \( B \) takes a position at \( D \), the corner \( C \) takes a position at \( A \), and the corner \( D \) takes a position at \( B \).

(c) This fitting shows the congruence \( \triangle ABCD \cong \triangle ABCD \).

2. (a) Place your card back on the tracing so as to show the identity congruence, \( \triangle ABCD \cong \triangle ABCD \).

(b) Now take the card and flip it about its vertical axis so that corner \( C \) takes a position at \( D \), corner \( B \) takes a position at \( A \), corner \( D \) takes a position at \( C \), and corner \( A \) takes a position at \( B \).

(c) This fitting shows the congruence \( \triangle ABCD \cong \triangle ABCD \).
3. (a) Place your card back on the tracing so as to show the identity congruence, \( \text{ABCD} \cong \text{ABCD} \).

(b) Flip your card about its horizontal axis

(c) This fitting shows the congruence

\[ \text{ABCD} \cong \ldots \]

You can see that with a rectangle there are four different positions in which we can show congruence, and in each case the correspondences are different.

**Exercises**

1. The figure below is called an **isosceles trapezoid** with \( \overline{AB} \parallel \overline{DC} \) and \( \overline{AD} \cong \overline{BC} \). Write two correspondences that show that \( \text{ABCD} \) is congruent with itself.

(a) \( \text{ABCD} \cong \ldots \)  

(b) \( \text{ABCD} \cong \ldots \)
2. The triangle below is an equilateral triangle. In an equilateral triangle all three sides are congruent. Write the correspondences that show that \( \triangle ABC \) is congruent to itself. There are six such correspondences.

![Diagram of an equilateral triangle](image)

(a) \( \triangle ABC \cong \triangle ABC \)
(b) \( \triangle ABC \cong \triangle ABC \)
(c) \( \triangle ABC \cong \triangle ABC \)
(d) \( \triangle ABC \cong \triangle ABC \)
(e) \( \triangle ABC \cong \triangle ABC \)
(f) \( \triangle ABC \cong \triangle ABC \)

3. The figure below is a square. Write eight correspondences that show that \( RSTV \) is congruent to itself.

![Diagram of a square](image)

(a) \( RSTV \cong RSTV \)
(b) \( RSTV \cong RSTV \)
(c) \( RSTV \cong RSTV \)
(d) \( RSTV \cong RSTV \)
(e) \( RSTV \cong RSTV \)
(f) \( RSTV \cong RSTV \)
(g) \( RSTV \cong RSTV \)
(h) \( RSTV \cong RSTV \)
Right Angles and Perpendicular Lines

In an earlier chapter you learned that if two lines intersect and the four angles formed are all congruent, then the lines are perpendicular to each other.

The exercises that follow will give you a chance to review some of the ideas of perpendicularity.

Exercises

1. (a) To the right, make a sketch showing that a ray pointing north and a ray pointing east are perpendicular to each other, thus forming a right angle.

   (b) A ray pointing north is also perpendicular to a ray pointing in what other direction? __________

   (c) A ray pointing east is also perpendicular to a ray pointing in what other direction? __________
2. The circle to the right is marked every 10 degrees. One pair of perpendicular rays is \( \overrightarrow{CA} \) and \( \overrightarrow{CG} \). Name another pair of perpendicular rays: \( \overrightarrow{\text{and}} \) \( \overrightarrow{\text{and}} \)

3. \( \angle AOC \) and \( \angle BOD \) are right angles, \( \angle COD = 30 \).

(a) \( m \angle COB = \) \( \)  
(b) \( m \angle AOB = \) \( \)  
(c) \( m \angle DOA = \) \( \)

BRAINBOOSTER.

4. Take a piece of paper and try to fold and crease it twice so that when you open it back up the creases will be perpendicular to each other.
Sets of Equidistant Points in a Plane (Perpendicular Bisectors)

In the drawing below the distance from point \( P \) to point \( A \) is the same as the distance from point \( P \) to point \( B \). Therefore \( P \) is said to be equidistant from \( A \) and \( B \).

\[ \bullet P \]

\[ \bullet A \bullet B \]

Class Discussion

1. Use a straightedge to draw \( AP \), \( PB \), and \( AB \).

2. What kind of figure has been formed? __________________________

3. This particular type of triangle is called an isosceles triangle.
   (a) What do you know about \( m \overline{PA} \) and \( m \overline{PB} \)?

   __________________________

   (b) If two sides of a triangle are congruent, then the triangle is an ___________ triangle.

4. Suppose we take the isosceles triangle \( \triangle APB \) and, using the line that passes through point \( A \) and \( B \) as a flip axis, we flip over the triangle so that we form a figure like the one on the right.

   (a) Is \( \overline{PA} \cong \overline{P'A} \)? __________

   (b) Is \( \overline{PA} \cong \overline{P'B} \)? __________

   (c) Is \( \overline{P'B} \cong \overline{BP} \)? __________

   (d) Is \( \overline{BP} \cong \overline{AP} \)? __________

   (e) Then \( \overline{PA} \), \( \overline{P'A} \), \( \overline{P'B} \) and \( \overline{BP} \) are all ___________ to each other.

   (f) What kind of figure is \( APBP' \)?

[Diagram of a triangle and its reflection]
5. In the figure of Problem 4, draw the diagonal $PP'$ and label the point where the diagonals intersect, $M$. By using the SSS property of congruence, we can show that $\Delta APM \cong \Delta BPM$. See if you can follow this reasoning:

(1) $AP \cong BP$ This was a given fact.
(2) $MA \cong MB$ The diagonals of a rhombus bisect each other.
(3) $PM \cong PM$ Identity congruence.

Then, by the SSS property of congruence,

$\Delta APM \cong \Delta BPM$.

Exercises

1. Below is a line segment $AB$ and its perpendicular bisector $l$.

(a) Pick any point on $l$ above $AB$ and label it $P$.
(b) Draw $PA$ and $PB$.
(c) Show that any point $P$ that lies on $l$ is equidistant from $A$ and $B$. To do this, use the SAS congruence property.

(1) $MA \cong MB$ because ____________________________
(2) $\angle AMP \cong \angle BMP$ because ____________________________
(3) $MP \cong MP$ because ____________________________
(4) Therefore, $\Delta APM \cong \Delta _____$

Now, since point $P$ was any point on $l$, this tells you that each point on the perpendicular bisector $l$ is equidistant from $A$ and $B$. 
2. You are given points A, B, and C.

(a) Draw \( \overline{BC} \) and \( \overline{AC} \).

(b) Construct the perpendicular bisectors of \( \overline{BC} \) and \( \overline{AC} \) such that they intersect.

(c) Label the point of intersection of the perpendicular bisectors, \( D \).

(d) Are points A, B, and C equidistant from point \( D \)?

(e) Place the needlepoint of your compass on point \( D \) and the pencil point on point A.

(f) Now draw a circle.

(g) Do the points B and C lie on the circle?

(h) If you are given any three points of a circle, can you always find the center of the circle?
3. Points A, B, and C are all equidistant from a certain point.

(a) Find that certain point. Label it O.

(b) Make a drawing that shows all the points equidistant from point O.

(c) What is this figure called?
Think of the drawings above as representing a wheel going downhill, on level ground, and then uphill.

In each case, line \( \ell \) touches the circle at exactly one point and is therefore called a tangent. Point \( P \) is called the point of contact or point of tangency.

Class Discussion

1. For each of the circles above, draw \( OP \).

2. (a) What do you think is the relation between the tangent line \( \ell \) and the radius \( OP \)?

(b) Mathematicians can prove that:

(i) A line tangent to a circle is perpendicular to a radius at the point of tangency.

(ii) A line perpendicular to a radius at its outer endpoint is tangent to the circle.
3. Pictured below is a circle with center $O$ and diameter $AB$. (A line segment through the center of a circle is the diameter if its endpoints lie on the circle.) Point $P$ is a point on the circle.

(a) Draw $\overrightarrow{AP}$. Be sure the ray extends past point $P$.
(b) Use your compass and straightedge to construct a line perpendicular to $\overrightarrow{AP}$ passing through point $P$.
(c) Does the line you just constructed also pass through point $B$? ________ It should.
(d) What kind of an angle is $\angle APB$? __________
(e) What kind of a triangle is $\triangle APB$? __________
If you were to do the same construction for any other point on the semi-circle, the result would be the same.

Any angle formed by two rays which have a common vertex on a circle and which go through the endpoints of a diameter of that circle is a right angle.

Exercises

1. (a) In the circle below draw ray $OP$.
   (b) Construct a line $\ell$ perpendicular to ray $OP$ passing through point $P$.
   (c) You know that the line $\ell$ you just constructed is perpendicular to the radius $OP$; therefore, line $\ell$ is ________ to the circle.
2. You will now construct two tangents from point \( P \) to the circle drawn below.

(a) Draw \( OP \).

(b) Find the midpoint \( M \) of \( OP \) by bisecting \( OP \).

(c) Put the needle point of your compass on point \( M \) and the pencil point on point \( P \).

(d) Now draw a circle intersecting the other circle in two points.

(e) Label these points of intersection \( Q \) and \( R \).

(f) Draw \( PQ \) and \( PR \). Both of these lines are tangent to the circle at points \( Q \) and \( R \).
BRAINBOOSTER.

3. Two nails are driven in at A and B, and a carpenter's square is pressed against the nails as shown.

Describe the path of point C as the carpenter's square is moved around but kept pressed against the two nails.

\[ \text{Diagram with points A, B, and C marked.} \]
Class Discussion

1. Take Page 14-6c out of your notebook and place it on your desk.

Fold the paper so that \( C \) falls on top of \( A \), and crease the paper along the dotted line \( \ell \).

Line \( \ell \) is the perpendicular bisector of \( \overline{AC} \).

Now fold \( A \) on top of \( B \), and crease the paper to show the perpendicular bisector of \( \overline{AB} \).

Now fold \( B \) on top of \( C \), and crease the paper to show the perpendicular bisector of \( \overline{BC} \).

(a) Use a straightedge and draw the lines represented by the creases.

(b) Look at the three perpendicular bisectors of the three sides of the triangle. Do they intersect at a point?

(c) Place the needle point of your compass at the point of intersection and the pencil point on vertex \( A \). Now draw a circle. Do the points \( B \) and \( C \) lie on the circle you drew?

(d) Is the point of intersection of the three perpendicular bisectors of the three sides of the triangle equidistant from each vertex?
2. Take Page 14-6d out of your notebook. Fold B on A and pinch the paper at M to mark the midpoint of AB. Then fold the paper so that the crease passes through C and M as shown by the dotted line \( \ell \). 

\( \overline{CM} \) is called a median of the triangle.

Using the same method, find the median from B to \( \overline{AC} \) and the median from A to \( \overline{BC} \).

(a) Use a straightedge and draw the lines represented by the creases.

(b) Do the three medians intersect in a point?

3. Take Page 14-6e out of your notebook. Fold the paper so that B falls on \( \overline{BC} \) and the crease passes through A as shown.

\( \overline{AP} \) is called an altitude of the triangle.

Use the same procedure to find the altitude from B to \( \overline{AC} \) and the altitude from C to \( \overline{AB} \).

(a) Use a straightedge and draw the lines represented by the creases.

(b) Do the three altitudes intersect at a point?

(c) Is each altitude perpendicular to a side of the triangle?
4. Take Page 14-6f out of your notebook.

Fold the paper so that $\overline{AB}$ falls along $\overline{AC}$ and crease the paper along the dotted line $\ell$ as shown.

Line $\ell$ is the bisector of $\angle A$.

In the same way fold and crease the paper to show the bisectors of $\angle B$ and $\angle C$.

(a) Use a straightedge and draw the lines represented by the creases.
(b) Do the three angle bisectors intersect at a point?
Exercises

1. Use the figure below to answer the questions that follow. 
\[ \overline{MA} = \overline{MB} \text{ and } \angle ACF = \angle BCF \].

Name a segment that shows:
(a) a perpendicular bisector of a side of the triangle.
(b) a median of the triangle.
(c) an altitude of the triangle.
(d) an angle bisector of an angle of the triangle.

2. \( \triangle ABC \) is an isosceles triangle with \( \overline{AB} \equiv \overline{AC} \). Use compass and straightedge to construct:
(a) the perpendicular bisector of \( \overline{BC} \).
(b) the altitude from \( A \) to \( \overline{BC} \).
(c) the median from \( A \) to \( \overline{BC} \).
(d) the bisector of \( \angle A \).
Pre-Test Exercises

These exercises are like the problems that will be on the chapter test. If you don't know how to do them, read the section again. If you still don't understand, ask your teacher.

1. (Section 14-1.)
The figures below are congruent. Using arrows to show slides or turns and dotted lines to show flip axes, show two methods of making triangle (1) map onto triangle (2).

![Diagram of congruent triangles](image)

2. (Section 14-1.)
In the figure below, draw a flip axis such that the figure maps onto itself.

![Diagram of flip axis](image)
3. (Section 14-1.)
How many flip axes does the figure below have? 

4. (Section 14-2.)
The two triangles below are congruent. List the corresponding parts for each triangle.

---

- \(A\) corresponds to \(E\)
- \(B\) corresponds to \(F\)
- \(C\) corresponds to \(G\)
- \(D\) corresponds to \(H\)
- \(\angle A\) corresponds to \(\angle E\)
- \(\angle B\) corresponds to \(\angle F\)
- \(\angle C\) corresponds to \(\angle G\)
5. (Section 14-2.)
The figure below is an isosceles right triangle where \( \overline{CA} \equiv \overline{CB} \).
Write two correspondences that show that \( \triangle ACB \) is congruent to itself.

\[
\begin{align*}
\text{(a)} & \quad \triangle ACB \cong \quad \text{(b)} & \quad \triangle ACB \cong \\
\end{align*}
\]

6. (Section 14-3.)
Name a pair of rays that are perpendicular to each other.

\[
\begin{align*}
\text{(a)} & \quad \angle A \cong \quad \text{(b)} & \quad \angle A \cong \\
\end{align*}
\]

7. (Section 14-4.)
In the figure below line \( \ell \) is the perpendicular bisector of \( \overline{AB} \) and point \( P \) is on line \( \ell \). Draw \( \overline{PA} \) and \( \overline{PB} \). What kind of triangle have you drawn?

\[
\begin{align*}
\text{(a)} & \quad \triangle \cong \quad \text{(b)} & \quad \triangle \cong \\
\end{align*}
\]
8. (Section 14-4.)
All points on a circle are equidistant from the ________ of the circle.

9. (Section 14-5.)
In the drawing below, line $\ell$ touches the circle in exactly one point, $P$; therefore, the line is ________ to the circle.

10. (Section 14-5.)
The drawing below shows that a line tangent to a circle is ________ to a radius of the circle.
11. (Section 14-6.)
In the triangle below, draw in:
(a) the perpendicular bisector of $\overline{AB}$
(b) the altitude from $C$ to $\overline{AB}$
(c) the median from $A$ to $\overline{CB}$
(d) the bisector of $\angle B$
1. The figures below are congruent. Using arrows to show slides or turns and dotted lines to show flip axes, show two methods of making the figure (1) map onto figure (2).

2. In the figure below, draw a flip axis such that the figure maps onto itself.

3. How many flip axes does the figure below have?  [Blank]
4. The two triangles below are congruent. List the corresponding parts for each triangle.

Corresponding parts:

- \( \angle B \) corresponds to \( \angle D' \)
- \( \angle L \) corresponds to \( \angle L \)
- \( \angle \) corresponds to \( \angle \)
- \( \angle \) corresponds to \( \angle \)
- \( \angle \) corresponds to \( \angle \)

5. The figure below is an isosceles triangle where \( CA \cong CB \). Write two correspondences that show that \( \triangle ACB \) is congruent to itself.

(a) \( \triangle ACB \equiv \) 

(b) \( \triangle ACB \equiv \)
6. Name a pair of rays that are perpendicular to each other.

7. In the figure below, line \( \ell \) is the perpendicular bisector of \( \overline{AB} \). Points \( P \) and \( Q \) are on line \( \ell \) and are equidistant from \( M \). Draw \( \overline{PA}, \overline{PB}, \overline{QA}, \) and \( \overline{QB} \). What kind of a figure have you drawn?

8. All points on a circle are equidistant from the center of the circle.
9. If a line is tangent to a circle, then the line touches the circle in exactly one point.

10. The drawing below shows that a line perpendicular to a radius of the circle.

11. In the triangle below $\overline{AM} \cong \overline{BM}$ and $\angle CBF \cong \angle ABF$:

   (a) Segment $\overline{DM}$ is called the ______________ of $\overline{AB}$.

   (b) Segment $\overline{CE}$ is called an ______________.

   (c) Segment $\overline{CM}$ is called a ______________.

   (d) Ray $BF$ ______________ the angle at $B$.
Check Your Memory: Self-Test

1. (Section 11-3.)
Using a compass and straightedge, construct a line through point P parallel to line $\overline{AB}$.

2. (Sections 12-2 and 11-8.)
Are the triangles shown above similar? _______ If so, what similarity property did you use? ___________
3. (Section 12-7.)
Use ratios to solve these problems.

(a) 25% of 48 is

(b) 150% of 60 is

(c) What percent of 75 is 15?

4. (Section 13-2.)
Divide.

(a) \( \frac{35}{7} = \)

(b) \( \frac{108}{12} = \)

(c) \( \frac{45}{15} = \)

(d) \( \frac{64}{16} = \)

(e) \( \frac{125}{25} = \)

5. (Section 13-7.)
Subtract. (Use the right side of the page for your work. Write the answer in simplest form on the blank.)

(a) \( \frac{1}{4} - \frac{1}{2} = \)

(b) \( \frac{3}{5} - \frac{1}{10} = \)

(c) \( \frac{5}{8} - \frac{1}{3} = \)

(d) \( \frac{9}{4} - \frac{11}{16} = \)

(e) \( \frac{5}{3} - \frac{5}{6} = \)

Now check your answers on the next page. If you do not have them all right, go back and read the section again.
4. (a) -5
(b) -9
(c) 3
(d) -4
(e) 5

5. (a) $\frac{1}{4}$
(b) $\frac{5}{10} = \frac{1}{2}$
(c) $\frac{7}{24}$
(d) $\frac{25}{16}$
(e) $\frac{5}{6}$