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ABSTRACT

This yearbook presents many aspects of mathematics learning by children between the ages of three and eight. Addressed to teachers of primary school children, the book begins with chapters discussing learning and cognition, the primary curriculum, and research on mathematics learning at this age level. Eight subsequent chapters deal with the teaching of specific mathematics content: problem solving; mathematical experiences; number and numeration; operations on whole numbers; fractional numbers; geometry; measurement; and relations, number sentences, and other topics. The final chapter discusses curricular change. A major theme throughout the book is the importance of experience to learning, and the building of new knowledge on the foundation these experiences provide. The book is designed to provide easy reference to both general information, such as answers to research questions, and suggested classroom activities related to specific topics. Many illustrations, the use of two-color printing, and a detailed index aid the user in this regard. (SD)

mathematics learning in early childhood

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

mathematics learning in early childhood

thirty-seventh yearbook

national council of teachers of mathematics

1975

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contents

about the book

Children, ages three to eight: their mathematics, their learning, their teacher, their achievement, assessing their knowledge and giving guidance to their thoughts, planning their experiences, and cultivating their ability

... to solve problems

... to learn mathematics

... to be successful with mathematics

... to see uses of mathematics

... to have fun and satisfaction with mathematics

—that's what this yearbook is about.

It is a book for teachers—for teachers of young children—designed to help them in making important decisions about teaching and about curriculum. It was planned to be a source of knowledge and a source of activities.

Knowledge about children's general cognitive development, about the way they learn mathematics, and about curriculum design is reflected in chapters 1, 2, and 3. In these early chapters is established a general framework for learning and for curriculum. Knowledge about learning and teaching specific mathematical content is found in chapters 4 through 11 Knowledge of special curriculum projects and direction for change are outlined in chapter 12.

Activities for children permeate the "content" chapters, chapters 4 through 11. These activities are designed to engage children and to elicit thoughtful responses to produce effective learning. Most will stimulate teachers to go further in designing their own activities.

Three themes are stressed: problem solving, relating mathematics to the real world of the child, and a developmental point of view toward learning.

"Problem Solving" was deliberately placed first among the content chapters. This prime placement suggests both the importance of problem solving as a major goal and the use of problem solving in all the content areas.

Relating mathematics to the real world of the child requires at least two things: (1) relating the mathematics to be learned to that which a child already knows; and (2) relating the mathematics to be learned to the child's own perceptions and views of concrete manipulative objects and situations in the real world. All content chapters stress the need to begin with concrete objects. Questions are then asked about the features of the objects, leading to the abstraction of the mathematical idea. Suggestions are given for helping children relate symbols and verbal language to the mathematical ideas in a meaningful way.

The developmental nature of learning should be evident throughout the book. Each author took seriously the charge to try to show how given content relates to, and grows from, that which a child already knows. A careful analysis of each child's learning is emphasized. The teacher is viewed as a planner of experiences—experiences executed with a balance between teacher direction and child initiation. Incidental experiences are to be used, but they are not to be the only learning experience.

The informality of the learning atmosphere is reflected in the activities and in the suggestions to teachers. Furthermore, it was a major factor considered in the overall design of the book. The design, the large pages, the use of a second color all should combine to

reflect an attractive and inviting atmosphere, which should be sought with children in the classroom.

The variety of the content chapters suggests a wide range of objectives for children aged three to eight. These chapters contain a careful analysis of familiar ideas as well as fresh views of some content areas.

Special concern was noted for the mathematics learning of children at an early age. To give emphasis to this concern, chapter 5 was devoted to mathematics experiences for young children. Most of the other content chapters also have specific suggestions that are useful with young children. What should be striking to the reader of chapter 5 and the other chapters is that the mathematics learning of very young children is much more than just counting (too often the only goal of early childhood education).

The book is designed to be used by teachers on an independent basis, by teachers engaged in continuing education, by preservice teachers in courses on teaching mathematics to children, and by graduate students in seminars on mathematics learning. The book should be helpful to school systems active in curriculum development at the primary school level and to preschool programs such as Head Start. It is a hope that the practical suggestions will make the book of continuing use to a teacher in the classroom. Further, it is hoped that the research and the theoretical work will be of continuing use to researchers and scholars in mathematics education.

The time to edit and produce this book has meant that the two chapters stressing research and the chapter on newer curricula reflect the knowledge up to two years ago. Since research and curricular results do change over time, the reader is alerted to seek subsequent publications in these areas.

Many people deserve special thanks for their contribution to the book. As editor, I saw a great array of talent and energy given freely by many people. Among this large group, I wish to thank these especially:

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It has been fun to work on the book. I hope that some parts of it elicit for the reader the same feelings of excitement and satisfaction that have gone into its production.

Joseph N. Payne, *Editor*

learning & cognition

harold stevenson

IN MANY ways this is an opportune time to write about current research in children's learning and cognitive development as it may be related to the teaching of mathematics. The past fifteen years have been productive ones for psychologists interested in how children learn and think, and some of the information they have gathered should prove to be of interest to those who face the practical problem of teaching mathematics to young children. Not too many years ago a chapter like this would necessarily have been based on ideas rather than research, on speculation rather than observation. Psychologists in the past tended to be interested less in doing research with children than in studying learning and problem solving using lower animals and college sophomores. Today there is such an abundance of studies dealing with children that the problem is how to synthesize the most important and useful data. Literally thousands of articles on cognitive development and children's learning have been published during the past two decades.

A comprehensive picture of this prodigious research cannot be presented in a short chapter. The approach taken here is to describe some of the major conclusions and to illustrate the basis of these conclusions with typical experiments. The reader should keep in mind that the research has been conducted in the laboratory rather than in the classroom and that few efforts have been made to evaluate these findings in teaching situations. However, we believe that the applicability of this information is not restricted to the laboratory setting and thus suggest that it is not too soon to try to make conscious use of these findings in real-life situations.

major conclusions of the research

The following generalizations are rather well substantiated by experimental data. Some are implicit in the teaching techniques already in practice in most schools; others may be new to educators. Still others may contradict—or may appear to contradict—assumptions underlying current teaching practices.

individual differences

Wide individual differences exist in the abilities of children to learn and to solve problems, and these differences are complex and difficult to determine—an obvious statement, perhaps, and one that most teachers would confirm. Yet its infallibility is basic to any consideration of children's learning. Whatever the group, whatever the task and its presentation, children tend to learn at different speeds. The rate of acquiring a conditioned response differs among newborns just as the rate of learning differential equations differs among twenty-year-olds. The pervasiveness of individual differences precludes the possibility of producing equally rapid progress by all children through any set of materials. The more we discover about individual differences, the more complex the problem seems to become.

A commonly held hypothesis about the basis of differential rates of learning is that learning ability is determined by the child's level of intelligence. Everyday experience indicates that brighter children tend to learn more rapidly, especially in the early grades. The difficulty in reaching conclusions about the relation between learning and intelligence from what goes on in classrooms is that children do not start learning with equal amounts of information and experience The child who has a good vocabulary is more likely to be able to use language effectively in school, the child who has broad experience and can identify a large number of common objects will be able to relate this experience to his classroom studies. Because there is so much transfer from the child's everyday experience to what occurs in school, the classroom is probably not the best place to try to determine the relationship between learning and intelligence.

Perhaps a more revealing approach would be to use tasks in which there is less transfer from everyday life. This can be done with many of the materials used in laboratory studies. We can test children's ability to learn to associate the names of two animals, to remember the location of cards of different colors, to learn a new code, or to learn a new principle. Although differences in past experience are not eliminated, these tasks are less dependent on what children have already learned. The results of a number of studies using such tasks have been reported, and the correlation between IQ and rate of learning in these laboratory tasks of learning and problem solving is rarely more than .50 (i.e., $r = .50$). This means that intelligence is related to children s ability to learn, but that rarely is more than 25 percent ($r^2 = .50 \times .50$, or 25 percent) of the variability in performance on the learning task attributable to variability in intelligence. It is not appropriate, therefore, to discuss learning and intelligence as identical functions. Many factors besides intelligence play an important role in determining the rate of learning. For example, the widely differing levels of anxiety with which children enter learning situations have been found to be significantly related to their rate of learning. The style with which they approach problems also differs—some respond rapidly and impulsively, others are more cautious and reflective—and some learning situations require one or another style of approach more or less exclusively. Furthermore, the nature of children's motivation to achieve and their level of aspiration have been found to play an important role in determining how well they will perform in learning tasks.

It would, therefore, seem that the teacher must be a psychodiagnostician to comprehend all the factors that may contribute to differences in performance among children in a classroom. This suggestion is impracticable, of course, but it does emphasize the importance of the teacher's close attention to the characteristics of each child as well as to the material

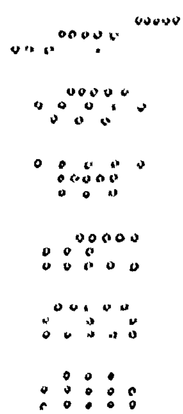
presented. On the basis of what is known about individual differences in learning, it seems that the teacher's efforts should be directed increasingly toward presenting to children materials that will capitalize on their individual strengths.

attention to irrelevances

Children may make errors because they attend to the irrelevant attributes of a situation. Some of the most frequently quoted examples of inadequate cognitive development are found in the responses of young children to the conservation problems developed by Piaget. The child is shown two identical beakers filled with equivalent amounts of liquid. After he has judged the amounts to be the same, the liquid in one beaker is poured into a third, narrower beaker. The child is asked whether the two beakers now contain the same amounts of liquid, and young children typically say that they do not. Their explanation is that the water level differs in the two beakers. Their judgment is based, therefore, on the attribute of height rather than volume. In another example the child is asked to count out ten beans and place them in two parallel rows, with each bean in one row directly opposite its counterpart in the other row. Is the number of beans in each row the same? Yes, the child concludes. Then the beans in one row are placed close together so that they occupy a shorter length than the beans in the other row. Now are there equal numbers of beans in each row? The young child is likely to answer no. Why? Because one row is longer than the other. He appears unable to comprehend that differences in length do not always produce differences in quantity.

Do results like these mean that the child believes that variations in one attribute imply changes in a second attribute? Or are the results more readily explained by the manner in which young children deploy their attention? Can young children be taught to ignore changes in an irrelevant attribute and base their judgment on what happens to the relevant attribute? Gelman (1969) has presented evidence that they can.

From the adult's point of view only the amount of liquid is relevant in problems dealing with the conservation of liquid quantity. But on the test trials, children notice that the beakers may differ in size, shape, height, and width. Gelman attempted to force children to concentrate only on the quantitative relations and to ignore other attributes. Standard tests of conservation of length, number, liquid, and mass were given to five-year-olds, and only those who failed to demonstrate conservation were retained for the study. For training, these children were given experience with oddity problems in which they had to select the odd stimulus in a set of three. Number was the relevant attribute, and spatial configuration was irrelevant; that is, the child was to learn to choose the stimulus that differed in number from the other two members of the set. The stimuli were carefully constructed so that on one trial there may have been two different patterns of five dots with a third, still different pattern of only three dots (■). On the next trial there may have been one set of five dots arranged in a short line but two sets of four dots spread out in different arrangements.



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The children soon learned to select the stimulus that was "odd" in its number of elements, regardless of its spatial configuration. They were then retested for conservation of length and number, and nearly all showed perfect conservation. In addition, when they were retested for conservation of liquid and mass.

which they also had failed earlier, over half the children demonstrated conservation. Therefore, both specific and generalized transfer occurred as a result of the oddity training. When the children were later presented with many different examples of quantitative equalities and differences and were required to respond on the basis of these examples, they performed effectively in tests of conservation. Their inadequacies appeared, not in their thinking, but in their attending.

We know that attention is a prerequisite for learning—if the child is not attending to the material being presented, it is impossible for learning to occur. We should be equally aware that his attention must be directed specifically to those aspects of the situation that are critical for the solution of the problem. For the teacher, what is relevant and what is irrelevant is obvious. We cannot assume, however, that what is relevant for the teacher will attract the child's attention. What appears to be poor conceptualization by the child may simply be a result of the teacher's failure to present the problem so that its critical features are highlighted.

distraction by irrelevant information

Young children are easily distracted by the presence of irrelevant information. It is very difficult for us to see the world as it is perceived by young children. Vast amounts of experience have enabled us to respond to critical features of our environment and to ignore those aspects that have no immediate significance. Young children do this only with difficulty; for them, incidental features of the environment may be as salient as those that have some importance. The ability to observe selectively—to categorize the environment into what is critical and what is not—develops rather late; evidence indicates that not until the child is ten or twelve years old is he able to do this spontaneously. The younger child, however, can do it with help or special training, but in designing materials for him, we run into a paradox. In an effort to make the material interesting, unessential details are often included—scenes contain more than the central figures, workbook formats vary from page to page. The net effect is that the additional details introduced to heighten interest often act at the same time as distracters and lead the child to fail to note the central information that is being imparted.

This point is illustrated in a study by Lubker and Small (1969), who presented third and fourth graders with an oddity task of the type described earlier. The problem looked simple. All the child had to do was choose from a set of four stimuli the one that differed from the others in color. But in some of the tasks irrelevant information was present—for example, the forms differed in brightness, size, or thickness. Adults would rapidly learn to ignore the irrelevant information, for they could quickly determine that it was of no help in attaining the correct response. For children, however, the presence of the irrelevant information acted as a distraction, and their performance suffered. When tested with stimuli that contained one or two irrelevant dimensions, they performed only slightly above chance at the end of the training. When no irrelevant dimensions were present, over 90 percent of their responses were correct.

We can make learning easier if we eliminate as much irrelevant information as possible, for young children have a hard time doing this by themselves. At the same time, we can be helpful if we heighten the differences among stimuli by having them differ consistently in more than one respect. That is, irrelevant information may be deleterious to learning, but redundant information may be helpful. Learning to discriminate a large black square from a small white circle would be easy, but learning to discriminate a square that was sometimes black and sometimes large from a circle that was also black at times and large at times would be very difficult.

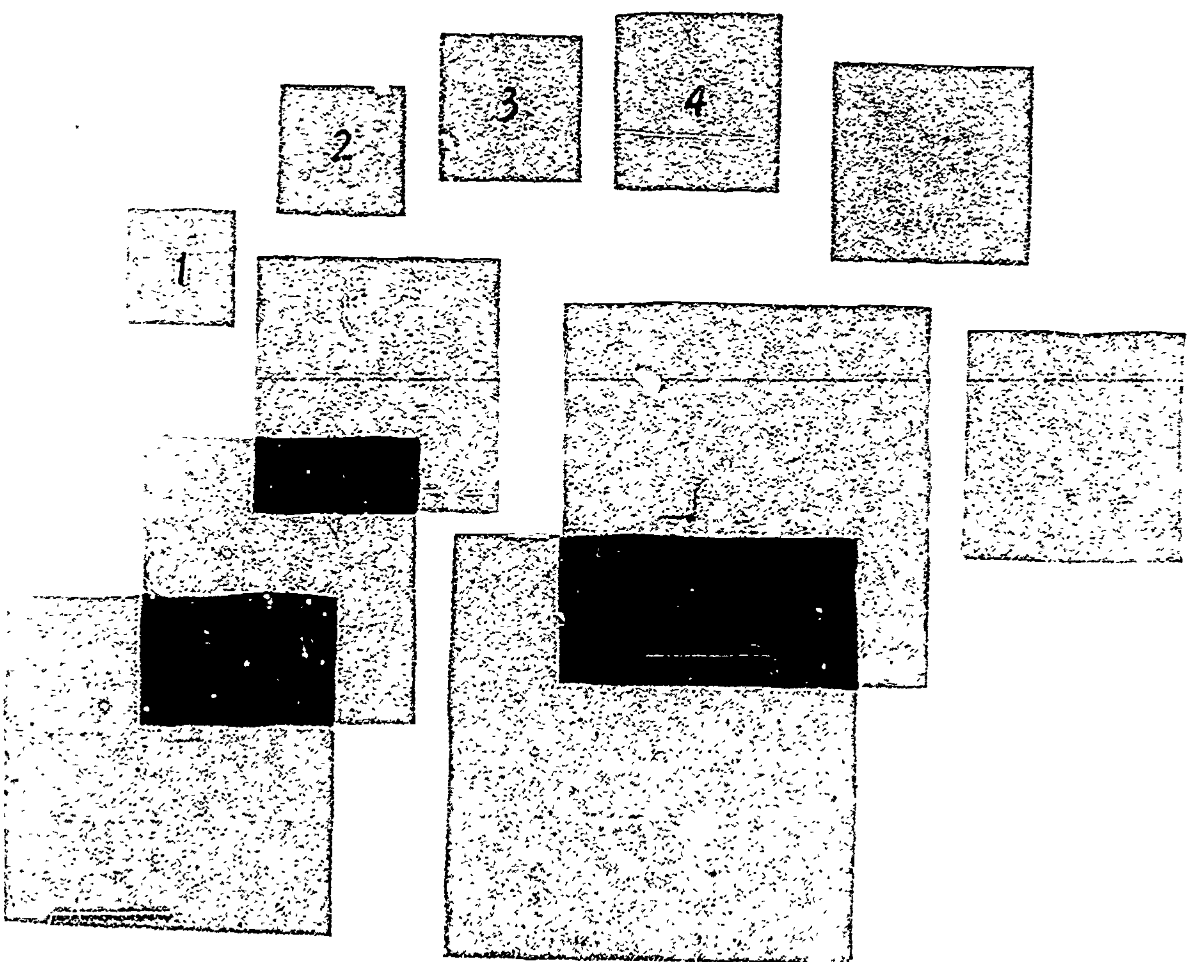
transfer of information

Transfer is facilitated for the child if each rule has a wide variety of examples covering a wide range of extremes. The goal of most teaching is

to provide information that can be used appropriately in new settings. Therefore, we are interested in improving the child's ability to transfer information from one context to another. We commonly teach the child a rule and then ask him to use the rule with new materials, a task that is often difficult for young children. Although they are able to learn the rule for solving the original problem, they approach the changed situation as if it posed a new problem and fail to apply the rule.

An example of this difficulty can be seen in the behavior of young children in problems of transposition, where the child is asked to learn a relational rule and transfer it to new sets of stimuli. The child is taught to choose from a set of stimuli the one that differs in a property such as size. Original learning can occur in a trial-and-error fashion. Each time the child chooses, say, the middle-sized stimulus in a set of three, he is rewarded; choices of the smallest or largest stimuli do not lead to reward. When the middle-sized stimulus is chosen consistently, the experimenter introduces a new set of stimuli that differ in absolute size from the training set but bear the same within-set relation. Young children have difficulty with this type of problem and fail to apply the rule they have just learned. Could it be that they fail to demonstrate transfer, not because they did not learn the rule well, but because they failed to understand that a response learned in one situation is applicable to other situations? If this is true, it should be possible to demonstrate to the child during the original training that the response is not restricted to stimuli with certain absolute properties. Beatty and Weir (1966) have done this with three- and four-year-olds, children who typically have difficulty in transferring concepts such as largest, smallest, and middle-sized.

The stimuli were sixteen squares with area ratios of 1.3 to 1. Training was conducted with stimuli 4-5-6 and 14-15-16 (the squares are

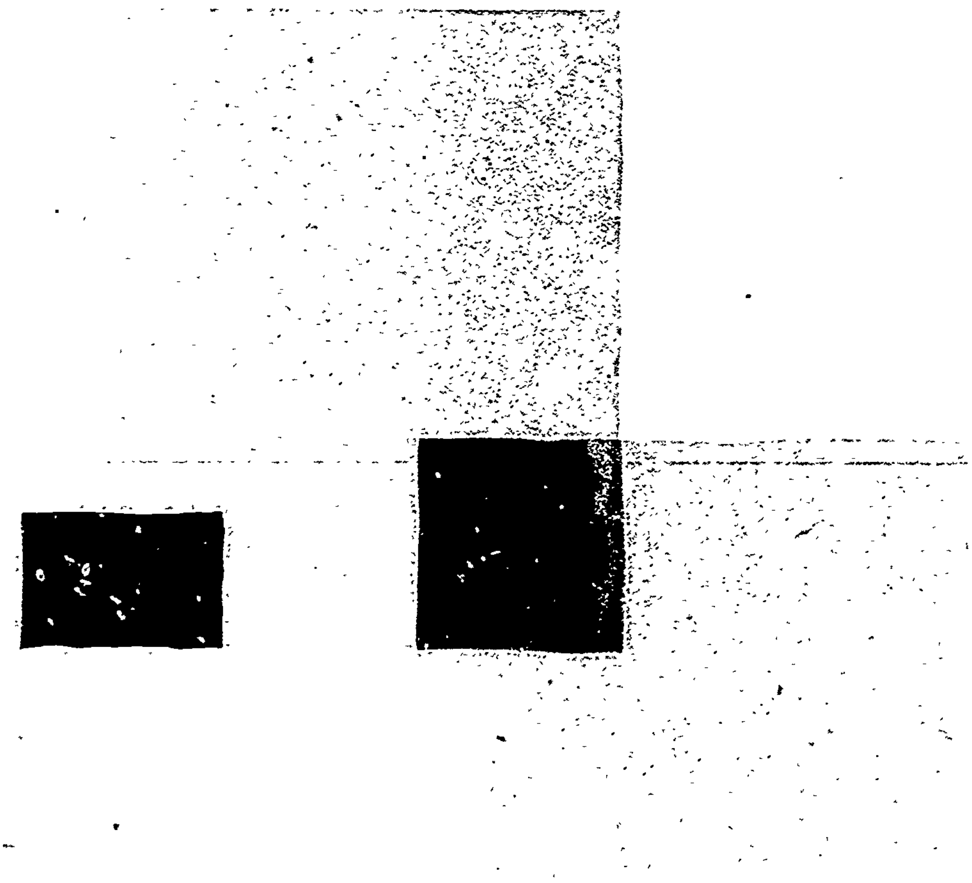


numbered from 1 to 16 in order of increasing size for convenience of description). The sets of stimuli were presented in random order. This type of training should increase the child's understanding that the choice of intermediate size is appropriate over a broad range of stimuli. After they learned the correct response to the training stimuli, a new set, 9-10-11, was presented without comment. Would the children choose at random, or would their first choice be that of the stimulus of intermediate size? Over three-fourths of the children demonstrated transfer—they chose the stimulus of intermediate size.

This study offers an interesting insight into how we can improve the child's ability to transfer information. If we teach the child from the beginning that the same rule is applicable to widely divergent instances within the same class of problems, we should increase the likelihood that the child will be able to use the rule with still different instances. We usually try to select similar examples for use during the child's first exposure to a problem in an effort to aid him in his original learning, but this may have the unexpected effect of restricting his ability to use the information in other situations.

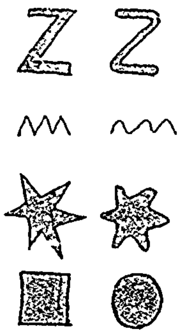
language and abstract thought

Language may be helpful, but it is unnecessary for abstract thought. An enormous amount of effort has been expended in attempts to understand the relationship between them. Some psychologists view abstract thought as a product of language; others believe that language



development and cognitive development are parallel but not interdependent processes. There is no question that language may be of help in conceptualization. but can concepts be learned and used without the intervention of language? For older children and adults it is nearly impossible to separate the two processes. Language is such a highly practiced skill that one can translate nearly all experience into words. Young children, however, are still in the process of learning language. Are there ways in which we can demonstrate that they are capable of abstract thinking, as exemplified in their correct application of certain concepts, but are unable to tell us how they solved the problem or to describe the concepts they employed?

We can use a study by Caron (1968) as a reference. Three-year-olds do not know words to describe the concepts of roundness or angularity. Furthermore, without prior training it is extremely difficult for them to use these concepts. Caron sought to develop pretraining experiences that might lead these young children to employ the concepts correctly. Many sets of figures were constructed in which the differentiating attribute was the roundedness or pointedness of a portion of the figure. The figures were paired in a discrimination problem where the correct choice was dependent on the consistent selection of a figure that contained one of these characteristics (■). Some children consistently had to pick the stimulus with a rounded portion, and others had to pick the stimulus with a pointed portion. For some of the children the figures were initially presented only in part. Rather than use the fully represented figure, only the portion of each figure that contained the distinctive attribute was visible. Very gradually, over a long series of trials, the full figure was faded in. By seeing at first only the critical feature, three-year-olds were able to learn the discrimination. They gave clear evidence of having used the concepts but there was no indication that the concepts had been represented in words. The children could not tell the experimenter at the end of the study how they had solved the problem, nor could they pick out the "round" and "pointed" figures when they were directed to do so.



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The same results were obtained with a different pretraining procedure. Other groups of children were asked to fit the stimulus figures into a hollow **V** shape. The figures with an angular portion fitted into the **V**, but the others did not. The child was to go through the stimuli, placing the figures that fitted the shape into one pile and those that did not into another. When the children later were required to learn the discrimination task, they were highly successful. Again, they could not give a verbal explanation of how they solved the problem, nor were they able to identify the figures that possessed the attribute described by the adult. From studies such as these we can conclude that children are capable of using concepts they cannot verbalize.

We may be expecting too much of children, especially during the early years when language is still undergoing rapid development, if we require that they should always be able to tell us how they solved a problem. Words are the natural means for transmitting knowledge among adults, but they are not always the most effective medium for instructing children. Yet classrooms are highly verbal environments, a fact that may indicate why scores on the verbal portion of IQ tests are better predictors of school success than scores on the nonverbal portions of the tests. Perhaps our classrooms in the early elementary years are too full of words. Learning may be aided if children are given greater opportunities to learn in other ways.

learning by observation

Children learn well through observation. In most formal learning situations the child is expected to act rather than observe. We tend to think that we are not performing effectively as teachers unless something is being actively taught. unless the child is making some form of verbal or motor response throughout the lesson Each child is given a workbook and expected to learn by solving each successive problem. Our children come to expect that while they are solving problems at the board they are supposed to be learning, but otherwise they are supposed to be waiting their turn passively This is in direct contrast with what happens in everyday life. where a significant amount of learning occurs through the observation of other persons in the home, at play, or on television Children learn styles of dress by observing what others are wearing: they learn certain types of speech by hearing what other people say Young children learn complex games by watching older children perform them. Active participation by the child is of great importance in producing many types of behavioral change. but we know that the child can also learn effectively by observing the behavior of other persons. In fact, in some situations learning may be more effective through observation than through direct participation.

A study by Rosenbaum (1967) illustrates many of the features of observational learning. An observer and a performer (children from grades 1 through 6) participated in each experimental session. The performers solved twenty position discriminations in which one of four positions was correct. They responded by inserting a stylus into holes of an 80-hole matrix (20 rows, 4 columns). They were required to locate the correct hole before proceeding to the next row. Both performers and observers were then given a printed duplicate of the matrix and asked to mark the position that was correct in each row The observers demonstrated a significant degree of learning, even though their experience had been limited to viewing another person's efforts and the consequence of his response The scores of the observers not only were above chance at all grade levels but exceeded those obtained by the performers themselves. Spared the chores faced by the performers of following directions, inserting the stylus at the correct times, and remembering which holes had and had not been tried, the observers apparently were able to stand back and view the performer's efforts in a casual, but effective, manner.

Although great concern is shown over how young children learn the wrong things and wrong values by passively watching television, not enough regard is shown for the value of observational learning in the classroom. (Eventually, of course, we must ask the child to perform, for otherwise we have no indication of how much he has learned.) During the acquisition process, especially in topics that may be difficult for some children, we could use observational learning in creative and constructive ways. For example, in the early phases of teaching addition could we not arrange conditions so that the child could learn through observation rather than solely through performance? Using games and educational materials in the classroom offers good opportunities for children to learn through the observation of their own efforts and the efforts of others. Formal instruction that "a cube is a figure with six sides" or that "two one-fourths equal one-half" may cause young children much more difficulty than having an opportunity to observe the relations and to hear incidental verbalizations.

inappropriate hypotheses

Children use hypotheses, but their hypotheses may be inappropriate for the problem being presented. At one time we were led to believe that children functioned pretty much in a stimulus-response manner. get the response to occur in the presence of the relevant stimulus, reinforce the response, and learning will take place. According to this view, children are passive respondents to their environment, controlled by the contingencies between response and reinforcement. There is no doubt that we can exert a strong influence over what children learn and think when we have full control of the

resources that are available to them, as in institutions, or when we are able to offer them their only access to highly desired rewards. In most everyday environments, however, we have few opportunities to exert this kind of influence.

Many child psychologists currently view the child as an active participant in the construction of his environment—an individual who responds with hypotheses and expectations, preferences and biases. Children behave in a highly systematic manner, even at very early ages; they appear to act on their environment, not merely to respond to it. At times, however, children do not seem to operate in such a sophisticated fashion—their behavior appears to be uncomprehending and inappropriate. It is easy to attribute this behavior to dullness or a lack of interest. Careful scrutiny of what children are doing, however, may lead us to different conclusions. They may be responding in the best way they know, developing hypotheses and strategies that to them seem to be reasonable avenues to successful performance. The problem may be that the hypotheses and strategies they devise are too simple or too complex for the task at hand.

For example, Weir and Stevenson (1959) studied children of ages three, five, seven, and nine, who were required to learn the correct member of each of five pairs of pictures of common animals. Each pair of pictures appeared once in each block of 5 trials and a total of 140 trials was given. This is not a difficult task, and we would expect increasingly rapid learning with increasing age. All the child had to do was discover and remember which animal in each of five pairs the experimenter had arbitrarily assigned as correct. For three-year-olds the problem should have been difficult, and it was; only about half of their total number of responses were correct. Performance improved for five-year-olds—they were able to name the animals and use the names as cues for remembering which animal in each pair was correct. But at age seven performance dropped, and by age nine the children were making approximately the same number of correct responses as the three-year-olds. The problem had not changed and the children were older, but performance was unbelievably poor.

We could conclude that the older children were not paying attention, were uninterested in the task, or were an unrepresentative sample of nine-year-olds. None of these explanations seemed to hold up. The children did appear to be interested in the problem and evidenced disappointment when their choices were incorrect. Why, then, should they have performed so poorly? A clue to the cause of their difficulty was found in what they had to say at the end of the task when they were asked how they had known which animal was correct. The older children made statements like "I thought it was going to be a pattern" or "I thought you were going to change them all around." They had formed complex hypotheses, and these hypotheses had hindered their progress in reaching the simple, correct solution.

We can make erroneous interpretations about children's abilities if we do not take the time to investigate the basis of their mistakes. If they do not behave the way we think they should, we may consider them far less capable than they really are.

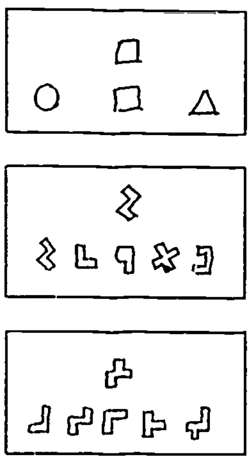
recognition of components

Difficult problems can be solved more readily if they are broken down into successively simpler components. One of the great contributions of programmed instruction and of behavior modification has been the demonstration of how complex problems can be mastered, often without error, if the problem is broken down into its components. Elementary components are presented first, and after each of these is mastered, successively more complex components are introduced.

Just as children make assumptions about what the teacher expects for an answer, teachers make assumptions about what children already know—and in both cases the assumptions may be inappropriate. For example, we give children a problem in which they are to make judgments of "same" or "different" about geometrical figures. The children confuse squares and rectangles, circles and ellipses are frequently judged to be the same. We repeat the instructions, but errors continue. Finally, we

realize that our request has been misinterpreted The children had been defining "same" to mean *having a common characteristic*, but we had meant *identical*. Had we attempted to analyze the task, we would have seen that our first step should have been to demonstrate what we meant by "same" and "different." Once this step is understood, the problem becomes much clearer. This may be a trivial example, but it illustrates the importance of examining our assumptions concerning what children know before proceeding to more complex problems.

Perhaps a better example is found in a study by Bijou, reported in Bijou and Baer (1963), which required children of ages three to six to make matching-to-sample responses that would be difficult even for adults. The child was asked to match a complicated geometric figure appearing at the top of a screen with one of five alternative figures that differed in structure and in angle of rotation. (See ■.)



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How were children this young brought to such a remarkable level of proficiency? Bijou broke the problem down into its simplest components and required that each step be understood before the next step was introduced. At first only a simple figure, such as a circle, appeared as the sample. Directly below it was a matching circle along with a square and a triangle. The identity of both the sample and the correct alternative was emphasized by presenting simple geometric figures in close proximity. Gradually, in successive trials, the matching figure was moved from directly below the sample, the figure became more complex, and the number of choices was increased to five. The children proceeded through the training at their own pace, but each problem had to be solved correctly before the next, slightly more difficult problem appeared.

Many other examples could be used, but the point is the same: we can often produce superior performance if we identify the components of a task, present the components in successively more complex combinations, and assure ourselves that the child has mastered each of the steps. We should make no assumptions about what the child is capable of doing until after he has demonstrated his level of competence. However obvious this may seem, it takes a study such as Bijou and Baer's to impress us with how much can be accomplished with this procedure.

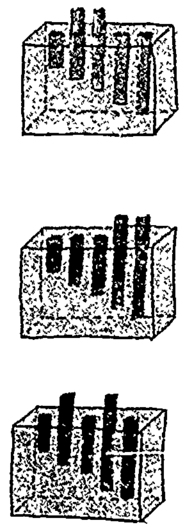
remembering the components of the problem

Children may fail to solve problems because they cannot remember the components of the problem. When a child is presented with a simple word problem and fails to come up with the correct answer, we usually conclude that he is incapable of making the appropriate inferences from the information contained in the problem. In other words, he cannot solve the problem even though he understands its components. This conclusion is common among teachers and psychologists. For example, in his theory of cognitive development Piaget has made much of the fact that children under the ages of seven or eight are incapable of making transitive inferences. In his studies, children were given problems of the following, familiar type: If Mary is older than Jane, and Jane is older than Sue, who is older, Mary or Sue? To solve this problem, it is necessary to combine two separate pieces of information, the relations between the ages of Mary and Jane and

the ages of Jane and Sue From these two relations the child must infer a third relation that is not directly specified. If it is true, as Piaget holds, that young children have great difficulty in making such inferences before close to the end of the first grade, we may wonder how they can comprehend elementary principles of measurement where comparisons require this kind of transitive thinking.

A recent study by Bryant and Trabasso (1971) has cast doubt on the validity of claims that transitive inference does not occur at early developmental periods. They asked whether children's poor performance on inference problems was due to their cognitive immaturity or to their failure to remember the first and second relations. If the difficulty was due to problems of memory rather than of thought, it should have been possible to find transitive inference at much younger ages, provided that appropriate care was taken to insure that the child remembered the components of the problem.

Bryant and Trabasso used five wooden rods of different lengths (A through E in order of decreasing length), and the subjects were four-, five-, and six-year-olds. The rods protruded one inch from the top of a box and were color coded The child was asked the color of the rod that was longer (or shorter) within a pair. After he responded, the rods were removed and the child could observe whether his response was correct. (See ■.) Training was given on four comparisons, A > B, B > C, C > D, and D > E, presented repeatedly in this fixed order or its reverse Training continued in this manner until the child was correct at least eight of ten times, after this, the four pairs were presented in random order until the child responded correctly six times in succession This extensive procedure was adopted to insure that the child had thoroughly learned the initial relations. Immediately following this training, each child was tested four times on all ten possible pairs of colors Memory was assessed from the responses to the four original comparisons, ability to make inferences was assessed from the responses to the six other comparisons. Transitive inference was found at all ages. Four-year-olds, for example, gave correct responses on at least 90 percent of the trials (except in the comparison B > D, where they were correct on 78 percent of the trials).



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This whole procedure, with one modification, was repeated with a different sample of children. Rather than allowing them to see which of the rods was longer after each response during the training period, the adult simply told them the correct answer. The results were practically identical to those of the first study.

These studies lead to an important point. Before concluding that children have failed to reach a level of cognitive development where certain types of mental operations can take place, one must be sure that basic conditions for successful response have been met. One of these conditions is that the child remember the content of the problem.

The same line of argument has been presented by Kagan and Kogan (1970). It has been asserted that children under the ages of seven or eight are incapable of performing adequately on class-inclusion problems. Typically, a problem of the following type is presented orally. "See these beads? These are all wooden beads. Some of them are brown and some of them are white. Tell me, are there

more wooden beads or more brown beads?" Young children typically say that there are more brown beads. Such results have been interpreted as evidence that the child is unable to think simultaneously of both the whole and the parts, of classes and subclasses, and that this is an index of cognitive immaturity. Kagan and Kogan report that when six-year-olds were asked a problem of this type orally, only 10 percent were able to give the correct answer. When the problem was presented in written form and all elements of the problem were available to the child as long as he wished, 70 percent gave the correct response. This is another example where poor performance is a result, not of inadequacies in thought, but of an inability to retain all the required information.

teacher/child relationship

The relationship between teacher and child is an important determinant of the child's performance. Every teacher faces a basic dilemma: With so little time and so much material to cover, how can the teacher worry about relating to each individual child? Is the teacher's primary responsibility to impart information and to teach skills, with the building of positive interpersonal relationships merely an incidental goal? The answer must always be no. The kind of relationship that exists between a teacher and a pupil will determine, in part, how much and how well the child will learn. Whatever effort that is spent in developing sound teaching procedures can lead to only partial success unless the child perceives the teacher as a potentially interested and supportive person This is especially true in teaching young children, for they have not reached the point where formal school material itself begins to have sufficient inherent interest to insure their making persistent efforts to learn.

Many studies have demonstrated that the adult's effectiveness in influencing children's behavior depends on the quality of their interaction. We know that children will try harder and that learning will be aided if the adult rewards the child's efforts with praise or other forms of supportive response. Even praise, however, may have differing degrees of influence, depending on the role the adult establishes with the child. McCoy and Zigler (1965) have demonstrated this experimentally. An adult experimenter attempted to establish different roles with six- and seven-year-old boys. She took the boys in groups of six to a classroom where they were given attractive art materials with which to work. With one group she attempted to be as neutral as possible, keeping busy at her desk and trying not to elicit bids for social interaction. In a more positive setting, a second group of boys was again allowed to work with the art materials, but this time the adult was diligent in her efforts to interact with each boy, and she tried to be complimentary, helpful, and responsive. Three sessions, held one week apart, were conducted in this manner. One week after the third session these boys, along with a third group who had had no prior interaction with the experimenter, played a game with her in which she made supportive comments about their performance twice a minute. The boys were allowed to play the game as long as they wished. The boys for whom the adult was a stranger terminated the game after an average of 2.5 minutes. The boys with whom the experimenter had behaved in a neutral manner in the earlier sessions remained at the game for an average of 9.6 minutes, and those with whom she had interacted positively remained for an average of 13.4 minutes.

Other studies show that adults differ greatly in their ability to influence the behavior of young children and that these differences increase as children grow older. Still other studies offer examples of how adults can be trained to adopt roles that reduce these differences. In general, adults can heighten their effectiveness with children if they are enthusiastic, involved, and responsive. The critical, punitive, aloof adult may be effective in some situations but rarely in interacting with young children. Much of the motivation of young children to learn subjects such as mathematics depends on the human factor—children work in part to please the teacher. Ideally, we hope for situations where learning is directed by the interests of the child, but in the early years these interests

are usually not developed sufficiently to motivate the child without a teacher's enthusiasm and encouragement.

concluding remarks

Some of the ideas that can be derived from recent research on children's learning and thought have been presented here. Undoubtedly, a great many more could be developed. Because of limited space, however, it has been necessary to present these ideas in skeletal outline. Further information about research on children's learning can be found in Stevenson (1972); more on children's cognitive development can be found in Flavell (1970) and Rohwer (1970). We hope that in the future, psychological research with children will be linked more closely to education. From such mutual efforts it should be possible to develop a sound scientific basis for teaching practices.

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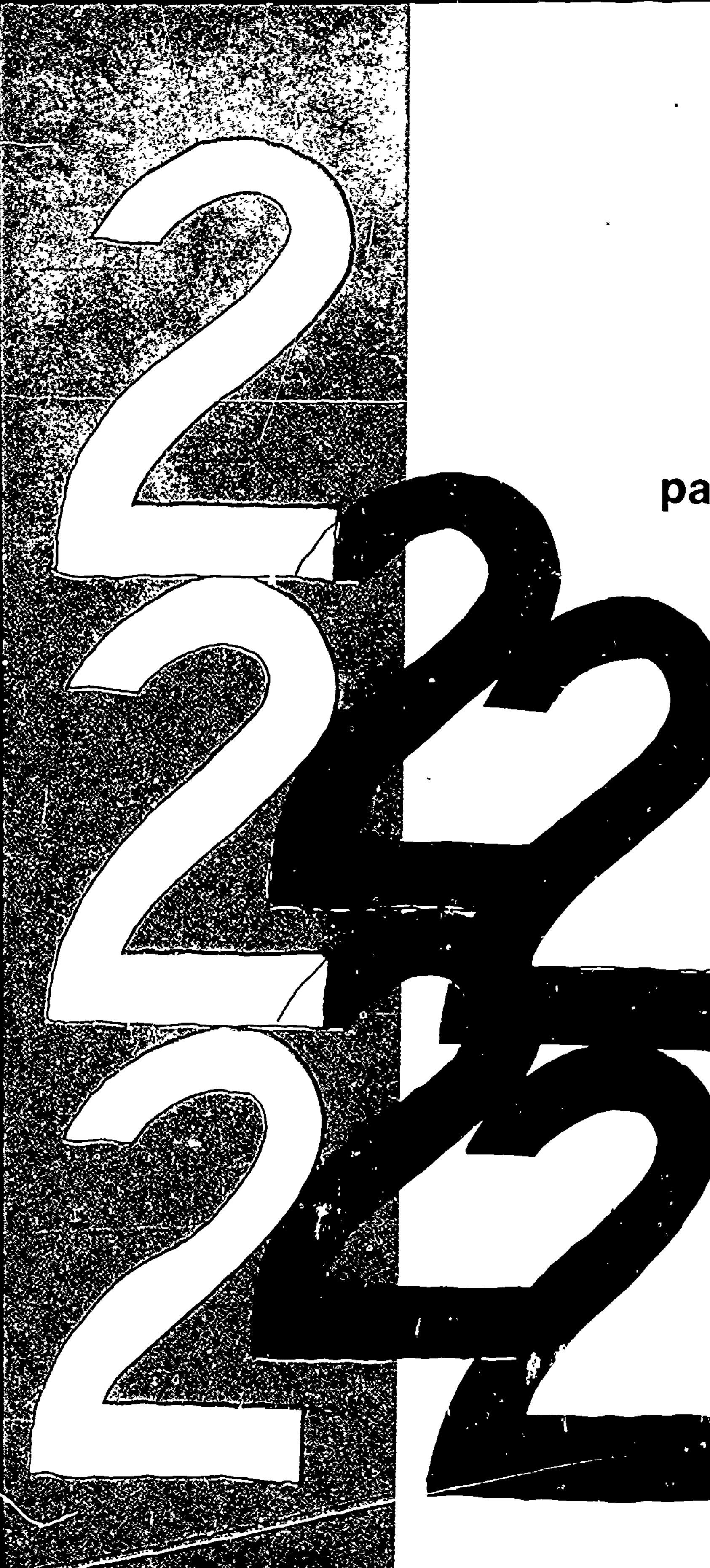
the curriculum



paul trafton

What are the goals or purposes of instruction in mathematics?

What mathematical concepts and skills should children learn?

What factors contribute to the successful learning of mathematics?

How can reflective, analytical thinking be developed in mathematics?

How can children be helped to use their knowledge in recognizing, managing, and interpreting the mathematical aspects of their world?

The continuing process of developing well-balanced, effective programs of instruction for young learners involves many variables, including mathematics, children, and ways of relating mathematics to children. Raising broad, basic questions, such as those above, is an integral part of the process. Our perceptions on such questions greatly affect the nature of any mathematics program for children. The positions taken affect such matters as the range of content included, the allocation of priorities, curriculum emphases, ways of sequencing content and learning experiences, types of instructional procedures employed, and ways of organizing instruction for adjusting learning to individuals.

The list of questions fundamental to the learning of mathematics and the development of appropriate programs can be greatly expanded: What kinds of experiences, and in what amounts, are necessary to help children deal with the abstractions inherent in the learning of mathematics? What is the relative importance of mathematical concepts and specific skills? How can these two aspects of learning

be related so that each complements and supports the other? How much emphasis should be placed on the learning of computational skills and how is computational proficiency developed? What are appropriate problem-solving experiences and how can they be made an integral part of the curriculum?

The rapid development of programs for the preschool child in recent years has given rise to a new set of questions from those persons working at this level: Should explicit emphasis be placed on certain mathematical notions? Is such emphasis consistent with broader child-development goals at this level? What mathematical ideas should be explored? What types of foundational experiences in mathematics promote desired thinking patterns and understandings in the young child? How can the mathematical aspects of more general experiences be put to maximum use?

Many important curriculum questions are complex, and simple, conclusive answers to them are difficult to provide. Prior to making curriculum decisions, however, those interested in the learning of young children should discuss such questions extensively. More specific questions are posed by those responsible for the learning of children on a daily basis: What is the significance of developmental experiences with basic ideas and relationships such as classification and matching? Should such work receive precedence in preprimary work over direct emphasis on counting and recognizing numerals? Is counting still an important skill and how does it fit into work with sets stressing cardinality? How does understanding the meaning of addition and its properties relate to developing proficiency in finding sums? Is it important to develop the relationship between addition and subtraction? What are effective techniques for helping children learn basic facts? Why is it important to employ a developmental approach to computational procedures such as the subtraction algorithm? Why should informal experiences with nonstandard content such as geometry be included and how can such work be fitted into a crowded curriculum? Such specific questions require pragmatic, albeit tentative, answers if the helpful guidance sought by teachers is to be provided.

Understanding the implications of curricular questions and curricular emphases rests heavily on an understanding of the nature and purposes of a well-balanced mathematics program. Yet communicating the nature of such programs has been difficult for several reasons. Many educators see only the arithmetic aspect of the program, not comprehending the purpose or value of other mathematical components. Even the arithmetic aspects are viewed diversely. Some view arithmetic in terms of computational skills, with proficiency in them an end in itself, they see proficiency as the result of an emphasis on procedural aspects of computing alone. Others focus on the use of arithmetic understandings and skills in familiar practical situations, they argue that number aspects of applied situations are sufficient to develop these understandings and skills. At times the curriculum is organized primarily around one of these viewpoints despite the lack of evidence supporting either position.

The relative importance of sequential, systematic instruction and exploratory or problem-solving experiences is difficult to determine. Frequently the meaningful learning of new content is dependent on prior understandings and skills, a fact that implies a role for sequential instruction focusing on specific ideas. But this aspect of instruction is often ignored by teaching pieces of mathematics in isolation and not teaching specific content with care, and it is often abused by imposing too rigid a structure on the curriculum. Specific content can often be put in a broader curriculum setting. The role of intuitive, exploratory work, as well as opportunities to apply and extend learning in a variety of ways, could be increased. Carefully sequenced learning is often extended to areas that need not be highly structured, such as geometry.

It is often difficult to see the long-range implications of early experiences with mathematical ideas and relationships. Preprimary- and primary-level programs include experiences that may be unfamiliar to many educators. Pupils have early experiences with the language of logical thinking, with mathematical relations, with aspects of informal geometry, and with structural properties of operations. In addition to their present value for children, these experi-

ences also have strong implications for future work in mathematics. For example, implicit in classification experiences is the notion of an equivalence relation. Informal work with this idea at an early age builds a foundation for future applications and work.

Broadening the mathematical component of the curriculum through an infusion of new content and placing the familiar arithmetic program in a broader mathematical context have clouded the goals of the curriculum. Consequently, teachers have felt an increasing sense of inadequacy or insecurity toward mathematics, a feeling that has led many of them to avoid new content. Sometimes, however, the content, language, and symbolism of a new program are clear, but their underlying purposes, as well as the relation of familiar ideas to new approaches, remain obscure. Furthermore, a limited mathematics background may restrict the teacher's ability to modify or extend his pupils' learning experiences. Thus, although an understanding of the purposes of a mathematics curriculum is essential for responding adequately to significant curriculum questions, several factors may cause educators to misunderstand or misinterpret the curriculum.

Today the mathematics curriculum is under intense scrutiny, with particular attention being given to broad questions related to the teaching and learning of mathematics. This is due to several factors. There is abundant evidence that the learning of mathematics is still not a successful experience for many children. It has been difficult to implement promising classroom practices on a widespread basis. Weaknesses of the reform movement of the 1960s are now more apparent. The continuing development of experiences appropriate for children and representative of many areas of mathematics causes stress over the allocation of instructional priorities and emphases. Findings about children's intellectual development raise interesting but difficult questions about how those findings may affect instruction and, in turn, the content and structure of the curriculum. (The fact that the implications of present findings are not immediately obvious, together with the continuous input of new evidence from investigations of children's thinking and learning, make it hazardous to base firm decisions on the present state of knowledge.) Finally, questions about the curriculum and the nature of meaningful learning in mathematics are a natural consequence of the current broader discussion of goals and purposes in preschool and elementary education.

This chapter will discuss many of the questions and issues dealing with curriculum planning and classroom instruction. The purpose is to provide a broad basis for discussions of current issues, to identify important elements of effective instruction and curriculum development, and to provide a framework that can guide daily practice for those who work with children.

The chapter is developed in two major sections. In the first section, it is suggested that current issues are not unique to the present era but are continuing ones. Issues basic to two major reform periods in mathematics education are identified and related to the present situation, for a knowledge of past attempts to deal with important concerns can be beneficial to present discussions. In the second section, the importance of balance is established in the building of effective programs for young learners, with balanced approaches discussed for several areas. It is often difficult to keep a sense of perspective regarding the complexities of learning and instruction at a time when a bewildering array of approaches and alternatives are being advocated.

The content presented is pertinent to both preprimary and primary levels, although the chapter at times may speak more directly to issues at the primary level. Chapter 5 contains a comprehensive discussion of preprimary programs.

continuing issues in mathematics curriculum development

The direction of the elementary school mathematics program has undergone many changes in this century. In the past forty years alone there have been two major periods of re-

form. The first of these periods began in the early 1930s and extended into the 1950s and is frequently referred to as the "meaningful arithmetic" era. The second reform period, known as the "modern mathematics" movement, had its beginnings in the decade of the fifties.

In each of these periods, many of the issues that are of concern today were an integral part of the discussions then and influenced the direction of the mathematics curriculum. An analysis of past discussions can set current debate in a broader perspective and provide helpful input for understanding pervasive issues more clearly.

issues in the development of meaningful arithmetic programs

The setting for reform. For an extended period beginning in the early 1930s, the arithmetic program of the elementary school was the subject of intense scrutiny, the result of which was a great transformation in the emphasis of instruction. Initially the curriculum centered on the learning of skills with a strong reliance on drill as the principal mode of instruction, but by the end of the period, strong support had developed for presenting arithmetic in a "meaningful" way. The issues and discussions of this period can be traced by the interested reader through the three yearbooks devoted to the teaching of arithmetic at that time (NCTM 1935; NCTM 1941; NSSE 1951) as well as through the analysis of this era in the Thirty-second Yearbook of the National Council of Teachers of Mathematics (1970).

When the period opened, the prevailing viewpoint was that arithmetic was a tool subject consisting of a series of computational skills. The rote learning of skills was paramount, with rate and accuracy the criteria for measuring learning. This approach became labeled as the "drill theory" of arithmetic and was described by William Brownell as follows (1935, p. 2):

> Arithmetic consists of a vast host of unrelated facts and relatively independent skills. The pupil acquires the facts by repeating them over and over again until he is able to recall them immediately and correctly. He develops the skills by going through the processes in question until he can perform the required operations automatically and accurately. The teacher need give little time to instructing the pupil in the meaning of what he is learning. . . .

This approach, as well as the psychology on which it was often based, was attacked throughout the period. Numerous examples were cited of its weaknesses: pupils performed poorly, neither understanding nor enjoying the subject; they were unable to apply what they had learned to new situations; forgetting was rapid; learning occurred in a vacuum, since the curriculum related only rarely to the real world; and little attention was paid to the needs, interests, and development of the learner. At the same time, a movement to abandon systematic instruction occurred as a reaction against the sterile, formal programs of the day. Whereas the economic conditions of the depression supported the "tool subject" conception of the drill approach, the movement away from programs of instruction in arithmetic had its roots in the child-development movement of the times. The thesis was that systematic instruction was unnecessary, since pupils would learn arithmetic "incidentally" through informal contact with number in daily life and through the quantitative aspects of activity-oriented or experience-oriented units. It was felt that as pupils worked on broad units dealing with such topics as transportation or communication, situations requiring arithmetic would arise. In such settings pupils would learn skills readily, with the applied situation providing the motivation for learning. The notion of "meaning" was of central importance to the incidentalists, compared to the slight attention it received from the advocates of drill approaches. The incidentalists saw meaning being supplied by the socially significant settings in which number work originated. This point of view was a popular one, fitting well into the prevailing educational thought of the day. In many schools systematic instruction was abandoned, at least officially, and in other schools "formal" approaches (i e , drill approaches) were deferred for the first few years of schooling.

The nature of "meaningful" arithmetic. It was in this setting of widely divergent conceptualizations of the nature of an arithmetic curricu-

lum that the "meaning" theory of arithmetic instruction was proposed. The crucial ingredient under meaningful instruction was that pupils should understand or see sense in what they learned. Also, a new view of arithmetic itself came to be associated with meaningful instruction. In the Tenth Yearbook of the NCTM Brownell wrote, "The 'meaning' theory conceives of arithmetic as a closely knit system of understandable ideas, principles, and processes" (1935, p. 19). He further explained that now the true test of learning was having "an intelligent grasp upon number relations and the ability to deal with arithmetical situations with proper comprehension of their mathematical as well as their practical significance" (p. 19).

The meaningful approach, representing a reasonable alternative to two extremes, thus proposed a view of meaning that was related to ideas inherent in the subject matter. The approach received support and nurture from the growth of a new psychology of learning that emphasized the role of insight in learning.

Throughout the era, however, the notion of meaningful arithmetic retained an elusiveness and a level of generality that obscured discussion and increased the difficulty of both implementing and assessing the effects of meaningful programs. Eventually, attempts were made to describe more explicitly what meaningful arithmetic involved. In 1947 Brownell, the most articulate spokesman for the approach, provided a more detailed explanation (p. 257).

> "Meaningful" arithmetic, in contrast to "meaningless" arithmetic, refers to instruction which is deliberately planned to teach arithmetical meanings and to make arithmetic sensible to children through its mathematical relationships. . . . The meanings of arithmetic can be roughly grouped under a number of categories. I am suggesting four.
>
> 1. One group consists of a large list of basic concepts. Here, for example, are the meanings of whole numbers, of common fractions, of decimal fractions, or per cent, and most persons would say, of ratio and proportion. . . .
>
> 2. A second group of arithmetical meanings includes understanding of the fundamental operations. Children must know when to add, when to subtract, when to multiply and when to divide. They must possess this knowledge and they must also know what happens to the numbers used when a given operation is employed. . . .
>
> 3. A third group of meanings is composed of the more important principles, relationships, and generalizations of arithmetic, of which the following are typical: when 0 is added to a number, the value of that number is unchanged. The product of two abstract factors remains the same regardless of which factor is used as multiplier. The numerator and denominator of a fraction may be divided by the same number without changing the value of the fraction.
>
> 4. A fourth group of meanings relates to the understanding of our decimal number system and its use in rationalizing our computational procedures and our algorisms.

Dealing with instructional issues. In the twenty years following the publication of the NCTM's Tenth Yearbook in 1935, the notion of meaningful programs in mathematics was kept before educators and slowly gained acceptance as a valid approach to the teaching and learning of arithmetic. Spokesmen for a meaningful approach continued to point out the similarities to the other approaches, although they still noted the differences. The extreme emphasis on the mechanical learning of skills, prevalent under a drill theory, was consistently rejected under meaningful programs. Also rejected was the conception of arithmetic as simply a tool or skills subject, since this notion seemed to promote an emphasis on teaching an isolated set of skills apart from the understandings inherent in the subject and without recognizing the factors involved in making skill learning meaningful.

Nonetheless, the learning of skills and the use of drill was accorded a place. Computational skills were accepted as a major component of the program because of their usefulness in the real world. The complexity of skill learning was recognized in the careful attention given to the development of children's thinking and to the sequencing of instruction in this area. This was due to the realization that meaningful development of computational skills would result in more effective initial learning and retention. Thus skills were carefully related to concepts of the basic operations and rationalized through reliance on physical models

and properties of the decimal numeration system. A limited use of drill techniques was accepted as valid for helping pupils fix learning after understanding had first been established.

During this period the drill approach became discredited (although it remained alive in many classrooms). and the discussion shifted to how the curriculum should be organized to support meaningful instruction in arithmetic. The key question was. "Shall arithmetic be taught as a systematic subject. or should the pupils acquire arithmetical abilities incidentally, i.e., in connection with other subjects. or only as they become a part of purposeful life activities?" (McConnell 1941, p. 282). Although proponents of meaningful approaches were generally supportive of many of the aims of incidental programs, they rejected the abandonment of systematic instruction inherent in such programs. The central contention was that incidental approaches did not provide the careful, sustained development required for pupils to understand mathematical concepts. To support this claim several factors were cited: that number aspects of a project would be treated superficially in the immediacy of the situation, that opportunities for quantitative thinking would not be recognized owing to a lack of content knowledge, that heavy demands were placed on teachers to recognize and develop the quantitative aspects of a situation, that most units contained insufficient number experiences, and that arithmetic was again treated as a tool subject consisting of a series of isolated skills.

Underlying the support for planned programs was the viewpoint expressed by the NCTM Committee on Arithmetic. The committee, echoing Brownell, stated that "arithmetic is conceived as a closely knit system of understandable ideas, principles, and processes . . ." (Morton 1938, p. 269). This led to the argument that systematic, sequenced instruction was necessary in order to achieve the required understanding, reasoning ability, and skills. The report of the Commission on Post-War Plans of the NCTM supported this position when it stated, "Mathematics, including arithmetic, has an inherent organization. This organization must be respected in learning. Teaching, to be effective, must be orderly and systematic; hence, arithmetic cannot be taught informally or incidentally" (1945, p. 202). Buswell noted that "competence in quantitative thinking warrants purposeful teaching of the most effective type that can be devised" (Buswell 1951, p. 3). However, systematic instruction was distinguished from formal instruction, and it was carefully shown how systematic instruction made provisions for the needs, interests, and developmental level of the child. This was to be accomplished through building learning on experiences familiar to the child, providing many informal experiences at early stages of learning, involving the child actively, and providing many opportunities for using arithmetic in socially significant situations.

The debate regarding the organization of the curriculum involved the more fundamental question of the source of meaning in learning. Is meaning found within mathematics or does meaning emerge from the mathematical aspects of an applied situation? In the eyes of mathematics educators, there was little doubt on this issue. According to Brownell, "Meaning is to be sought in the structure, the organization, the inner relationships of the subject itself" (1945, p. 481).

As implied earlier, proponents of planned programs recognized that the curriculum had to relate to the real world. The report of the Commission on Post-War Plans also noted, "If arithmetic does not contribute to more effective living, it has no place in the elementary curriculum" (1945, p. 200). Thus recognition was given to two aims of instruction, the mathematical and the social. The question of the source of meaning was broken down into the dual ideas of meaning and significance. Meaning dealt with the mathematical aims of instruction, whereas significance came from relating learning to the real world. Buckingham, in a key article, put it this way (1938, p. 26):

> The teacher who emphasizes the social aspects of arithmetic may say that she is giving meaning to numbers. I prefer to say that she is giving them significance. In my view, the only way to give numbers meaning is to treat them mathematically. I am suggesting, therefore, that we distinguish these two terms by allowing, broadly speaking, significance to be social and meaning to be mathematical. I hasten to say, however, that each idea supports the other.

This viewpoint became accepted and provided a workable resolution of a difficult issue. It resulted, nonetheless, in a static view of what content should comprise the curriculum. The statement of the NCTM Committee on Arithmetic in 1938 reflected the prevailing point of view (Morton 1938, p. 269):

> Arithmetic is an important means of interpreting children's and adults' quantitative experience and of solving their quantitative problems. Consequently, the content should be determined largely on the basis of its social usefulness and should consist of those concepts and number relationships which may be effectively used.

This position limited both the range of content and the range of applications included. It also had the effect of limiting the mathematical orientation of the curriculum. Operations with whole numbers and fractional numbers comprised a large portion of the curriculum. Although emphasizing the mathematical aspects of such work was encouraged, in practice this was frequently not done, and the definition of "mathematical aspects" would today be considered quite limited. However, the position did give the work with computation a broader mathematical and social perspective, which provided a stronger motivation for such learning.

Assessing the impact. This first period of reform was a rich one in which today's educator can find much literature that is still significant. Two of the key issues were obviously the interrelated ones of the origin of meaning and the organization of the curriculum. Recognizing and establishing roles for mathematical and applied aspects was an important contribution. Although the resolution was not ideal, it must be noted that educators still seek an appropriate balance between these two aspects of instruction. The recognition of the importance of direct, planned instruction in achieving certain instructional objectives was another contribution. Such a position was central to the reform movement of the 1960s, where it was often articulated primarily for mathematical reasons. Yet at this earlier time the support was based on the recognition that such instruction was necessary to develop the level of quantitative thinking and computational proficiency essential for a functional use of mathematics in the daily lives of people. Those stating the position were often general educators and seldom had strong connections to the field of mathematics.

During the period little change occurred in the content of the curriculum. Computation remained of central importance, with a greater rationalization of procedures and a greater social orientation sometimes accompanying it. However, the question of how to make learning more effective was extensively and significantly treated, with major contributions being made regarding children's learning of mathematical ideas. The influence of this work is still felt today. Strong emphasis was placed on the psychology of the learner and, as a result, on the developmental aspects of learning—a fact that had several effects on curriculum and instruction. The progressive nature of learning was recognized and implemented through the spiraling of content. The use of physical materials in learning received great support, as did the active involvement of pupils in formulating ideas and discovering relationships. Concern for children's learning drew attention to the problem of providing for individual differences among learners. As guiders of learning, teachers were encouraged to be aware of each pupil's development and to provide necessary individual guidance. It should also be noted that the focus on broader purposes of instruction had an impact on standardized testing. At this time problem-solving sections on tests occurred, and later, items on concepts appeared.

The literature of this period should hold particular interest for those concerned with the early aspects of children's learning of arithmetical ideas. Many writers centered their discussions of meaningful learning on children's early schoolwork with number and operations. The many insightful observations regarding children's learning and the perceptive suggestions for working with children give the literature a richness and practicality that make it useful even today. The roots of an active approach to readiness are also found at this time, along with recommendations that pupils have experiences with number ideas from their earliest exposure to schooling so that they can use and build on the knowledge they already possess.

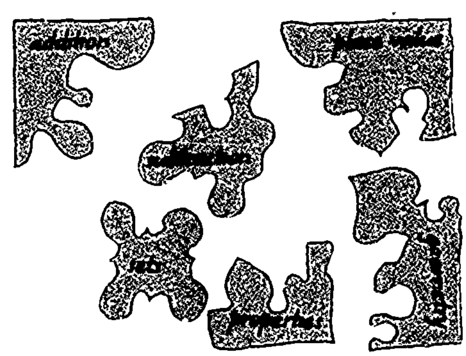
In summary, the meaningful arithmetic period is a significant one. The issues considered were important and held major and continuing implications for children's learning. They dealt with fundamental questions that are of concern today. Discernible progress was made in formulations about curriculum emphases and the nature of learning in arithmetic. A better-balanced curriculum was the result. It is obvious that the issues were not totally resolved and that recommended practice did not result in the widespread change in classroom practice that was necessary. However, a foundation was established on which future reformers could build new approaches to continuing issues.

issues in the development of modern programs

The setting for reform. The period beginning around 1955 was one of extensive reform in mathematics education, with the greatest changes occurring at the elementary school level in the early-to-mid-1960s. In many respects, this era continues up to the present, but without the continuing major content reorganization and the spotlight of national attention that was part of the initial years. An intense period of curriculum reform seldom occurs without an accompanying complex of forces that create and support the climate of change. In the previous period, the impetus was the reaction against a narrow conception of arithmetic, together with the rejection of a psychology of learning. After 1955, the contributing factors were an awareness of the need for greater mathematical substance in the curriculum to meet the nation's scientific and technological needs, a dissatisfaction with the quality of instruction in mathematics, and a general dissatisfaction with the quality of the mathematics itself. Many viewed the movement as a first attempt to bring meaning to the learning of mathematics. This was not so much a reflection on what had been advocated in past years; rather, it was a reaction against prevailing instructional practices, pointing up the continuing problem of translating change into widespread practice.

The nature of reform. The heart of the movement was a new conception of meaningful programs and meaningful learning in mathematics. Now the idea of meaning was closely linked to the mathematical validity of the content of the curriculum. There was no dichotomy between meaning and mathematics, a mathematically valid program would be a meaningful one. In the previous period, the focus on meaning had been primarily methodological; that is, the content was given a new setting rather than changed. Now, however, both the content of the curriculum and ways of presenting it effectively were being considered.

Content considerations centered on the idea of structure, with "structure" being interpreted in two ways. First, structure was applied to the idea of building the curriculum around the broad, unifying themes of mathematics. Instead of being a collection of individual topics only slightly related (as in ■), the curriculum would be organized around a series of mathematical themes woven or spiraled throughout the curriculum (●).



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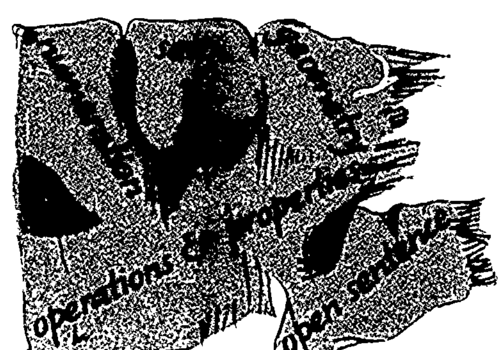

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As a result, mathematical ideas could be expanded and extended as pupils had continuing experiences with them. It was argued that this would result in a more natural flow in the development of mathematical content. Specifics of mathematics could be related to basic ideas, and mathematical skills in particular could be put into a more conceptual framework. One obvious result of this conception was an expansion of the content of the curriculum. At an early age pupils were exposed to elements of sets, open sentences, properties, geometry, and graphing.

There were pedagogical as well as mathematical purposes for this approach to content organization. It was felt that learning would be more natural and continuous, since new understanding could easily be built on previous learning. New approaches sometimes came to have significant pedagogical value apart from mathematical considerations. For example, the early use of primitive set ideas was often supported because of their significance as a foundational, unifying role in mathematics and the more logical basis they created for the development of number and operation ideas. In addition, set ideas have provided a way to look with more care at the idea of cardinal number, with the result that greater precision can be brought to the development of this notion. Set and subset ideas underlie general classification activities, which are often developed with young learners prior to classifying sets on the basis of "same number." Similarly, the early introduction of geometry has come to be seen as a basis for helping a child interpret an inherently geometric environment.

The second application of the idea of structure was the more mathematically recognized use of the term, namely, the explicit use of structural properties of the real-number system. It was assumed that an early development of such ideas as commutativity, associativity, and the identity elements would contribute to a deeper comprehension of the nature of operations on sets of numbers and to an awareness of the distinction between operations and techniques for computing answers. Not only were properties identified early, but they formed the basis for justifying techniques and rationalizing algorithms. One example occurs in the use of grouping by ten to solve addition facts with sums greater than ten. As shown in ■, the process is justified in step-by-step fashion by using the associative property. In ●, a formal rationalization of the addition algorithm at the primary-grade level is presented. This method relies heavily on structural properties as well as place-value ideas.

$$\begin{aligned} 8 + 5 &= 8 + (2+3) \\ &= (8+2) + 3 \\ &= 10 + 3 \\ &= 13 \end{aligned}$$

■

$$\begin{array}{r} 20 + 8 \\ + \; 40 + 7 \\ \hline 60 + 15 \end{array} \quad \begin{aligned} &= 60 + (10 + 5) \\ &= (60 + 10) + 5 \\ &= 75 \end{aligned}$$

●

Although this use of structure had wide support initially, doubts were eventually raised about the appropriateness of such deductive approaches. Some felt that this approach was too formal and involved too much symbolization. They wondered whether such techniques might even contribute to problems with learning skills rather than resolve them.

A strong pedagogical thrust to the reform movement called much attention to ways of developing ideas meaningfully. The phraseology was often similar to that of the previous era. Pupils were to be actively involved in the learning process through their use of materials, through

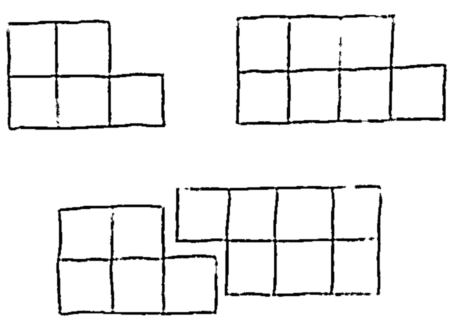
the discovery of relationships, and through discussion with their peers and teachers. One notable aspect was the emphasis on exploratory, discovery-oriented experiences in which the thinking process used by the pupils was considered of equal or greater importance than the particular content under consideration.

For example, pupils might be asked to find a pattern in adding pairs of odd numbers. After discovering that each sum was an even number (□), some pupils generalize that this would always happen. Then they engage in a discussion and demonstration of why this must be so, as in

Odd Numbers

1, 3, 5, 7, 9, 11,	$3 + 5 = 8$
13, 15, 17, 19,	$7 + 11 = 18$
21, 23 . . .	$13 + 17 = 30$

□



The mastery of the generalization "The sum of two odd numbers is an even number" would not be stressed in this experience, although naturally it was hoped that some would remember and be able to use the generalization. The lesson was justified on the basis of the experience it provided in discovering relationships. From such experiences, pupils would develop insight into their use of numbers.

The pedagogical thrust had other aspects. Initial developmental work was again highlighted—lessons were to begin at an informal level, making use of materials when appropriate and connecting the new elements in a learning situation to ideas with which the pupils were familiar. Such an approach to learning was considered essential if the use of symbols was to have meaning. Mathematics educators also examined how content might be more effectively sequenced to promote learning.

Curriculum and instructional issues. It has been established that this movement dealt in a major way with issues of content and methodology. Other aspects of curriculum and instruction also received consideration, as well as those elements of the new movement causing concern. The learning of computational skills was one of these (as it had been before). The significant position that computational skills continued to occupy is attested to by the great emphasis in the early 1960s on explaining and implementing this aspect of reform.

Again there was the concern that computational skills occupied too central a position in the curriculum. Although the goal of reasonable computational proficiency was still supported, reformers placed skill work in a broader mathematical framework and built a case for exploratory work in the early stages of learning facts. Essentially the approach to skills consisted of two basic tenets: first, that allowing pupils to discover basic facts and the relationships between them would result in quick, efficient fact learning; and second, that extensive mathematical rationalization of algorithms would result in their being effectively learned. Some thought that it was not even necessary for children to learn basic facts.

As a result of these tenets and of the assertion that pupils could learn more mathematics earlier, computational skills came to be introduced too early in many programs. Many first-grade programs included all the addition and subtraction facts, and renaming in two-and three-digit addition and subtraction examples became standard topics in second-grade text materials. Although the intent may have been to provide earlier exploratory work, thus allowing greater time prior to teaching for mastery, in practice this increase in content resulted in significantly less time being devoted to developmental work. Unfortunately, the assumptions were overstated, and the learning of computa-

tional skills remained a problem. The result was that computation skill continued to occupy too great a share of the curriculum. This was a curriculum area about which teachers were sensitive, and many, failing to receive answers to their questions, stopped listening and reverted to old ways.

Another issue considered again was the role of applications and real-world problem-solving situations. In the earlier reform movement, the learning of mathematics was considered purposeful only in its social applications. This gave mathematics its significance and its reason for existence. During the modern mathematics era, a broader conception of "problem solving" was articulated. Problem solving could occur totally within the setting of mathematics, as in discovering a pattern or learning a new idea through reliance on previous learning. Further, even though it was important for mathematics to relate to the physical world, such applications were not considered necessary to lend purpose or significance to the study of mathematics This was contained within the mathematics itself. It was also argued that an understanding of mathematical relationships would result in a high level of transfer to applied situations Concern was expressed about this position, and it continued to undergo modification. Today, two aspects of a necessary and proper role of applications are considered. (1) the relation of number ideas and skills to real-world settings, and (2) the role of real-world experience in the initial learning of concepts. The first is particularly applicable at the primary level, the second is of concern at both the preprimary and the primary levels.

Questions were eventually raised regarding the appropriateness of certain topics and approaches for young children. Thought was given to how ideas might be treated in a manner more congruent with the thinking of children. Examples of such concern were the formal application of properties and a formal point-line-plane approach to geometry. Some felt that the approach to structure, together with the earlier introduction of topics, might have intensified existing problems. Also, findings from developmental psychology did not appear to support the level of formalism sometimes found in the reform programs.

Educators began to make a distinction between content that was necessary to know and content that could be considered "nice" to know. From a curriculum that had once been labeled intellectually impoverished, the problem now was determining which experiences should be included and to what extent particular content should be emphasized. This was of particular concern to classroom teachers, who ultimately had to make the decisions. Unfortunately, teachers seldom received the specific, practical guidance they sought.

Assessing the impact. The modern mathematics era had its greatest impact during the 1960s. It was accompanied by a highly optimistic feeling that the changes advocated would resolve many of the problems of teaching and learning mathematics. Yet the message of reform was frequently distorted, with many educators focusing on the "new" content and its accompanying language and symbolism. Often the spirit of the content did not come through. The heavy emphasis on the mathematics when training teachers was seldom accompanied with imaginative ways of developing it with children. Some approaches to the content seemed to be more appropriate for middle or high achievers than for the total spectrum of learners. When it became apparent that problems still existed, many became discouraged, unaware of the scope of the task involved. Some labeled the movement a failure—a premature and shortsighted conclusion—and suggested that educators must start anew the task of creating effective programs.

Yet some significant, positive changes did occur, and considerable progress was made in dealing with recurring issues. The curriculum became revitalized in its content and pedagogy. A better conception of the role of content in programs for children was created. New content was introduced, and it appears to have established a firm place for itself in the curriculum. In particular, geometry has finally been accepted, after many statements supporting its potential over a 160-year period and after several earlier, unsuccessful attempts to implement programs in it. The momentum in this direction encouraged mathematics educators to investigate and introduce other appealing new

content as the movement progressed. Examples of this activity at the early childhood level include social graphs and probability. Also at this time informal, planned programs of instruction gained acceptance at the kindergarten level The blend of content and methodology resulted in a great number of teachers and pupils finding mathematics more exciting to teach and to learn. Further, the climate of innovation promoted creative efforts on the part of teachers and encouraged a host of individuals representing many kinds of input to become interested in the curriculum and to make many contributions to curriculum development. Investigations into several aspects of children's thinking and learning in mathematics grew at a rapid rate. Finally, unprecedented attention by educators nationwide was focused on fundamental questions of curriculum and learning. By the late 1960s, however, the attention of the schools had turned to other major problems, and this climate no longer exists today.

That problems remain is not so much a mark of failure in the reform attempt as it is a reminder of the difficulty of creating a change that takes into account all the variables that determine effective learning, the difficulty of even identifying those factors. and the difficulty of translating promise into practice.

issues of current concern

Diverse questions from a variety of sources are being raised today about school mathematics. Many of the issues identified in the preceding sections should seem familiar. Concern about curriculum content and about ways of developing ideas in a manner appropriate to a child's level of development is ever present. Three particular sources of concern are apparent at the present time.

One such concern is the learning of computational skills Most individuals would still accept reasonable proficiency with computational techniques as one important aspect of elementary school programs. Yet the changing emphases of the past era probably caused more confusion in this aspect of the program than in any other area. In an attempt to define and attain goals in the area of computation today, educators show a disturbing tendency to ignore concepts and instead focus on the end products of learning without recognizing the process by which children develop proficiency. Some would again define the curriculum as a series of isolated skills and employ drill as the primary technique for teaching these skills.

A second concern or issue involves the role of applications and problem-solving experiences in mathematics. Some urge that systematic instruction be abandoned and replaced with general problem-solving experiences drawn from the real world. At times this position closely resembles that of the incidentalists earlier. Some advocate emphasizing experiences selected from mathematical areas that closely relate to the physical world and lend themselves to manipulative experiences, such as measurement and geometry. Although proponents of this viewpoint recognize the importance of the strands of numbers and operations on numbers, they are often unclear on how an understanding of them and a competency with them are to be developed.

A third current issue concerns ways of organizing instruction to provide for individual differences. At one extreme, individualized instruction is equated with self-paced instruction. The same goals are set for all students, with the important variable being the rate at which pupils master tasks. At the other extreme, an approach is advocated in which pupils select learning experiences on the basis of interest. Much of the discussion of this third issue appears to take place without consideration of what content is worth knowing and the variables that affect meaningful learning. In particular, little consideration is given to the role of guided instruction in making learning enjoyable and successful and in developing insightful patterns of thought.

Each of these issues is a valid one and deserves extended and comprehensive discussion. Each deals with a real problem. Each has a bearing on the learning of children. Yet none of the issues is new. It is interesting to relate current discussions to past attempts at dealing with the same issues. Whereas it is true

that issues must be interpreted in the context of the periods in which they arise, it is nonetheless unfortunate that a greater awareness of the historical counterparts of present issues does not exist Such awareness could not only sharpen the focus of the debate but also reveal weaknesses inherent in extreme formulations of the issues.

In this section, several issues and their influence on curriculum development have been identified In the following section, several aspects of these and other issues are developed more fully. The discussion will center on the importance of balanced approaches to key concerns. As one becomes aware of the many factors that must be considered in creating effective programs, the complexity of the task becomes obvious. Looking at a single variable in isolation is not enough. Desired goals can be realized only by deliberate, sensitive consideration of many variables. The analysis of previous attempts provides at least partial support for this notion.

considerations in building a balanced curriculum

Each of the two reform efforts cited previously can be characterized as a search for balance—balance in the content, emphases, and methodology of the curriculum, and balance among the needs of the discipline, the needs of society, and the needs of the individual. Current curricular efforts partially reflect the need to establish better balance in programs.

Although the reform of the 1930s and 1940s resulted in a better-balanced curriculum at that time, the strong emphasis on social applications did not result in balance with respect to the role of mathematical content. In the 1960s the reform movement attempted to achieve a better balance by highlighting content and formulating a broader conception of problem solving. In so doing, the reformers may have created an imbalance in the opposite direction.

Frequently, new efforts seem to be constructed solely on the rationale of being different from existing "bad" practices rather than being the result of broadly based considerations of the many components involved in making learning effective. Today, a new attempt is being made to find a workable balance in the curriculum. Evidence indicates that focusing on a few variables or on narrow interpretations of them will not result in the desired progress. Rather, the attempt must take into account the many factors involved, their complexity and interdependence, the many kinds of learning in mathematics, and the full implications of what it means to learn mathematics. Even though ultimate solutions to continuing issues may never be realized, some factors that contribute to effective programs can be identified, and this section will discuss some of the considerations that can guide curriculum planners and classroom teachers.

a broadened view of curriculum content

A balanced program with respect to content involves broadening the scope of the content beyond whole-number ideas and reexamining the treatment of whole numbers.

Number ideas have long received the major attention at the kindergarten and primary levels. It is true that young children become aware of number ideas at an early age and have a continuing, expanding need to be able to use numbers in a variety of ways. Perceptive teachers are well aware of the gradual, developmental process of building understanding and competency with whole numbers. Thus it is not unusual for work with whole numbers to comprise the entire mathematics curriculum.

In recent years, many new content areas have been introduced into the curriculum in order to expose children to the many facets of mathematics and the many connections between mathematics and the real world. Much of this new content has a strong potential for providing productive and satisfying learning experiences for children. For example, many geometric ideas have been found to be quite appropriate at the early childhood level. Ideas of symmetry and congruence can be explored.

Solid shapes can be discussed and classified in a variety of ways: some solids roll, and others slide; some have corners, some have straight edges, and others have curved edges. Geometry provides a vehicle for showing the close connection between mathematics and the real world.

In turn, work with geometry provides a better way to conceptualize measurement. Many developmental experiences with initial measurement ideas are possible while building a foundation for understanding measurement. Another source of experience is found in collecting and organizing data. Graphing ordered pairs of numbers is productive for upper-primary levels and relates to science activities in which quantitative relationships are graphed. Preliminary work with probability shows promise of being another appropriate area for informal exploration.

Broadening the curriculum content can make several contributions to children's learning. First, early informal experience with important mathematical ideas can contribute to greater success in future treatments of them—for example, an understanding of certain basic fractional-number ideas at the early childhood level would provide a firmer basis for later work Second, mathematical ideas have an inherent interest for children and are the source of imaginative activities appropriate to children at this level Third, an exposure to a wide range of mathematical ideas makes children aware of the pervasiveness and usefulness of mathematics in daily life Finally, these topics contribute to general instructional goals, such as developing a problem-solving mind set, showing the relationship between mathematics and the real world, creating an orientation toward finding patterns and relationships, and making children active participants in learning experiences Goals in this latter area are particularly appropriate ones around which preprimary programs can be constructed. Thus the new content emphases have relevance for the full range of early childhood years.

Balancing program content involves reexamining most aspects of the important work with number, particularly at the primary level. That teachers consider this aspect of the curriculum overcrowded is, in part, the result of the recent downward shift in the grade placement of computational topics. Reexamining the work with numbers could alleviate several problems and increase the time available for broadening the range of the content. Factors to be considered include (1) reassessing the appropriateness of the current grade placements of topics, (2) finding more effective ways of developing number ideas, (3) identifying those aspects of number work that are basic and those that can be reserved for advanced students at each level, and (4) helping teachers determine which experiences should be considered exploratory and which should result in mastery.

a balanced view of learning and instruction

The curriculum should include a variety of types of learning and thinking strategies. Each has implications for the organization of learning experiences and instructional strategies.

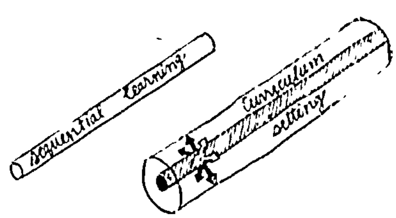
The study of mathematics involves many kinds of learning. learning content. learning specific techniques, learning relationships and generalizations, learning problem-solving strategies. learning to relate new ideas to previous knowledge, and learning to apply and extend ideas. The process of learning mathematics cannot be separated from the thinking employed in the various types of learning. It is important to recognize that our view of the nature of learning in mathematics can influence the mental set that children develop toward learning mathematics.

The broad variety of types of learning implies the use of different instructional emphases to attain goals. For purposes of discussion they may be partitioned into two categories. One category centers on a range of guided learning techniques designed to help learners identify and focus on specific elements. The other category includes several techniques in which less guidance is provided. Obviously it is important to match the requirements of learning situations with the strategies employed.

The range of types of learning has implications for the organization of the curriculum. At

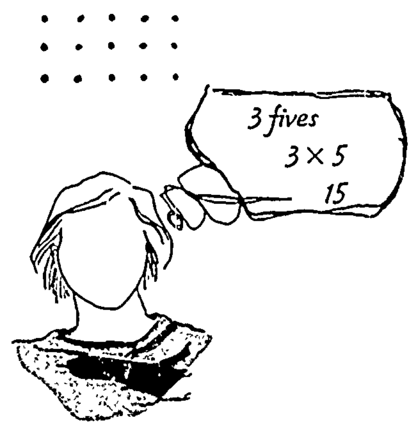
times carefully sequenced instruction is needed, involving related series of lessons designed to build understanding and help the learner "fit the pieces together." At other times less sequencing is required, and more unstructured approaches may be appropriate. Often pupils need a broad background of experience on which to construct more precise notions. In both situations a clear sense of purpose and careful planning are necessary to maximize the potential for learning.

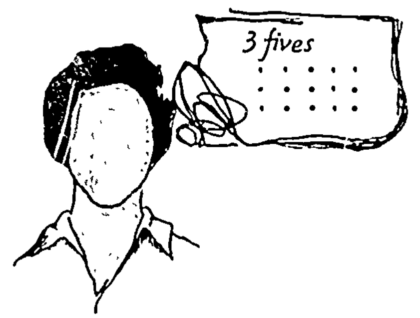
The distinction between the sequential, hierarchical aspects of learning mathematics and the less structured dimensions of the subject is difficult for many to perceive. The diagram here provides a focus for further discussion. The narrow cylinder represents the sequential aspect; the wide cylinder represents broader aspects of learning, or the curriculum setting. Placing the narrow cylinder inside the other shows how the two relate to each other; that is, learning can be put into a broader context even while building understanding systematically. However, establishing a role for each aspect and keeping them in balance is one of the more difficult tasks in curriculum planning and instruction.



Exploratory Experiences
Problem Solving
Pattern Finding
Applications and Extension

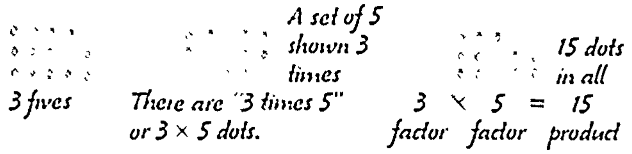
Several illustrations of learning situations are now provided in order to focus on the types of learning and the roles of instruction. A major goal in early instruction on a new operation is to help the learner relate mathematics to particular models from which the mathematics can be abstracted and to enable him to move readily from one realm to the other. This is illustrated with multiplication—extended work at the "translation" level builds a firm foundation on which other mathematical ideas and computational techniques, such as division, can be meaningfully constructed.



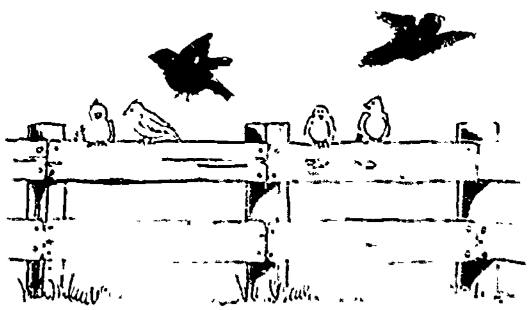


In building a connection between the model and the mathematics, the teacher must keep in mind that extended discussion and careful guidance is necessary. Three steps in conceptualizing multiplication are shown next. The first step is to help children describe the array, using the informal language "3 fives." Next, the connection between this step and the fact that a set of 5 dots is shown three times is made. This leads to the description "3 times 5," and then "3 × 5." Finally, these experiences need to be related to the complete number sentence and the ideas of factor and product. Without such guidance, symbols such as 3 × 5 may have little meaning. The importance of such guidance is indicated by the student who remarked, "The three 5s I get, but what does 'times' mean?" After a redevelopment of the first two stages, focusing on *a set of 5 shown one time, two times,*

three times, he exclaimed, "Oh, is that what multiplication is all about!"



Many early primary pupils need guidance in symbolizing subtraction. A child may understand the idea of subtraction but be confused about how to write the number sentence. The teacher may need to help him recognize that the numeral for the sum is written in front of the minus sign.



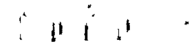
How many in all? (6)
Where do you show the 6 in the number sentence?

Another type of learning involves the use of strategies for solving numerical situations. Children frequently develop intriguing methods of their own. The following dialogues show the types of thinking that children employ:

Six-year-old: I know what 100 and 100 is. . . . 200.
Adult: How do you know?
Six-year-old: Because 1 and 1 is 2.

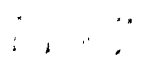
A six-year-old is determining the total shown on a pair of dice.

Child: 5 . . . 6, 7, 8.



Child: 8.

Child: 11.



Adult: How did you know it was 11?
Child: Because 6 and 6 is 12.

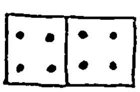
Eight-year-old: How many days is 4 weeks?
Adult: Can you figure it out?
Eight-year-old: . . . 28.
Adult: Why?
Eight-year-old: Well, 7 and 7 is 14, and if you double 14, you get 28.

Seven-year-old: What's 16 and 16?
Adult: Why do you want to know?
Seven-year-old: I'm doubling numbers. You know—1 and 1 is 2, 2 and 2 is 4, 4 and 4 is 8, 8 and 8 is 16.
Adult. Can you figure out 16 and 16 by yourself?
Seven-year-old: . . . it's 32.
Adult: How do you know?
Seven-year-old: Well, 16 and 16 is two 10s and two 6s Two 10s is 20: 20 and 6 is 26 . . . 27, 28, 29, 30, 31, 32.

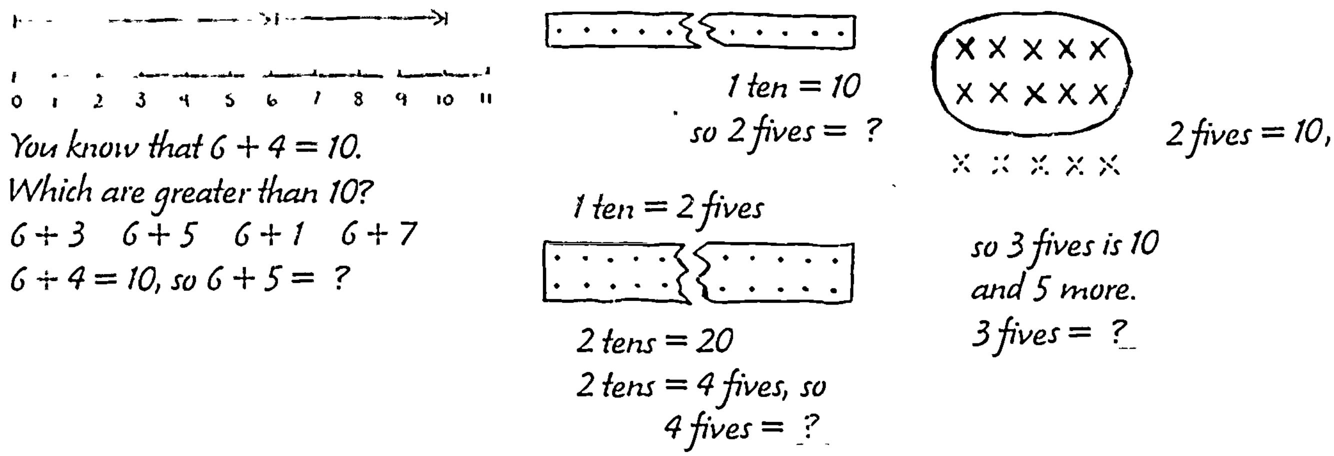
A parent and a six-year-old boy are discussing a swimming class that will meet twice a week for 5 weeks.

Parent: How many times would you go in all?
Child. pausing and looking at each finger on one hand as he counts to himself, answers: 10 times.
Parent: How did you figure it out?
Child: Well, I counted 1 on my fingers and 2 in my head. The child demonstrates, pointing to a finger as he says each number: 2, 4, 6, 8, 10.

One of the contributions teachers can make to children's learning is to encourage such patterns of thinking and provide instruction designed to develop a variety of thinking strategies. Such instruction will appeal to the natural interest children have in number ideas. Work with basic facts provides an excellent opportunity to do this, but in striving for mastery, the teacher frequently overlooks this aspect of the work In fact, such activity needs to be viewed as one element in building proficiency with facts. Four illustrations of thinking strategies that can be developed are shown next. Emphasizing various approaches to thinking about facts helps children organize their learning and builds an awareness that this work is a rational, meaningful activity.



$4 + 4 = 8$
so $4 + 5 = ?$

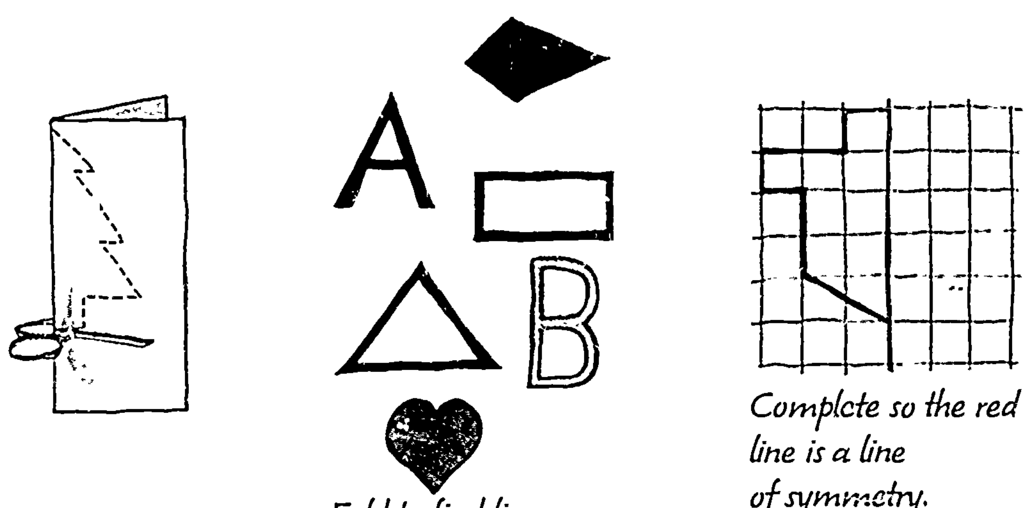


Many instances of exploratory and problem-solving experiences are given in chapter 4 and chapter 5, but one additional situation is provided in ■ . Symmetry is very much a part of the child's world and is not a difficult notion. This idea can be explored in a great number of interesting activities, many of which require little guidance and can be extended in a variety of ways.

The intent of this section has been to demonstrate that learning in mathematics is not all of the same kind. The variety of types of learning, ranging from the specific to the very broad, indicates a corresponding variety of approaches in teaching mathematics. The nature of many mathematical ideas requires some guidance if they are to have clear, continuous development. Therefore, a well-balanced curriculum makes provisions for many kinds of learning and matches the type of guidance given to the nature of the learning.

a balanced emphasis on concepts, skills, and applications

Establishing a balance among these three elements has important implications for the quality of mathe-



Fold to find lines of symmetry.

Complete so the red line is a line of symmetry.

■

matics programs. A balanced approach to computation is particularly important.

The teaching of mathematics involves working with children in three distinct, yet frequently related, areas. The first area involves the learning of mathematical content—the ideas, concepts, and relationships that are the heart of the subject. Teachers are responsible for helping children form concepts and for guiding the development of concepts from hazy, tentative notions into more mature, productive understandings. The second area involves learning specific techniques and skills that are necessary for being able to move facilely in the world of mathematical ideas and in the world of everyday experience. Finally, from mathematical ideas and skills come applications that relate mathematics to a variety of other situations. Such situations can provide a motivation for the learning of mathematics as well as stimulate reflective thought.

Developing a well-balanced view of these three dimensions has proved difficult. Many individuals have seen only the skills dimension of the subject. The mathematics program then becomes a curriculum of computational and measurement skills studied in isolation from its other aspects. One result is a premature focus on skill work At other times, as a result of trying to counterbalance the undue focus on skills, the content or applications aspects become the lens through which the entire subject is viewed.

Teachers should therefore be aware that the study of mathematics involves learning in all three areas and that distorting the curriculum by emphasizing only one phase of it hinders children's learning. Since a curriculum necessarily deals with number concepts and operations on numbers, it is particularly important here to maintain an overall balance among content, skills, and applications. The interdependence of these elements and the significance of a balanced approach for effective learning should be recognized.

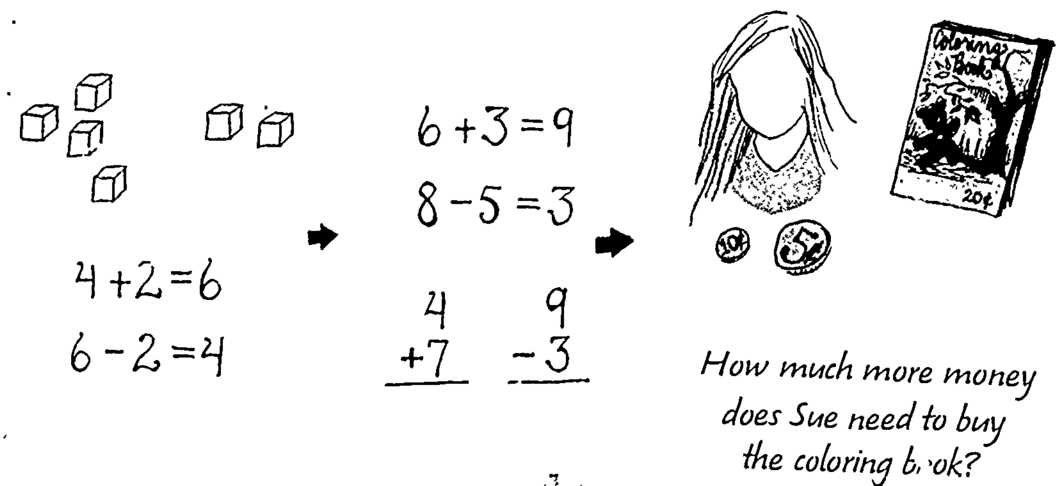
Two facets deserve further comment. First, instruction may emphasize particular aspects at a given time. For example, preprimary programs develop a cluster of concepts related to the idea of cardinal number, and little is gained by emphasizing such skills as forming numerals or reciting addition facts. And in geometry, too, few specific skills or techniques are to be mastered. Thus instruction can maximize the conceptual and application aspects.

Second, it is important to recognize the relationships that exist between these elements, particularly in the area of number. One way of relating them is pictured here, showing the importance of basic understanding for future work.

Concepts → Skills → Applications

An example of this relationship is provided in the next figure. Early work should center on establishing an understanding of the operations of addition and subtraction. Pupils learn to construct number sentences to describe situations. Next, children learn to compute and to apply their knowledge to real-world situations.

However, the interrelationship of these elements needs to be interpreted more broadly—



their interactions can operate in more than one direction:

Concepts

Skills ⇄ *Applications*

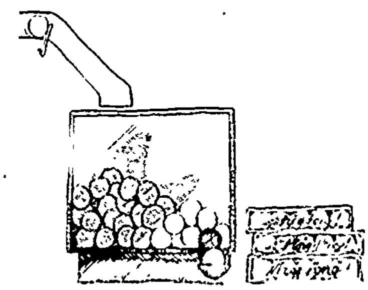
For example, physical situations frequently give rise to number concepts. The early "counting on" experiences found in some games or in using dice provide an informal background for the study of addition. Pairing shoes or pairing children for a classroom game suggests the ideas of "coming out even" or "one left over." Such activities can be the basis for developing even and odd numbers. The hypothetical problem of packing Ping-Pong balls in boxes leads to finding the greatest number of fives and the remainder. This in turn would lead to interpreting division as finding a quotient and a remainder.

A rich conceptual background does facilitate the learning of arithmetic skills, but it is also true that skills can contribute to the learning of ideas, a point that is sometimes overlooked. An example is provided in the pattern developed below. Skill in adding may be an important factor in successfully discovering and using such patterns; possessing necessary skills can free children to concentrate on the conceptual aspects of a situation.

No formula can direct the proper mixing of these three aspects. Additional evidence is needed before more explicit guidance can be provided. Nonetheless, an awareness of the importance of each of the three aspects and their mutually supporting roles can bring greater sensitivity and flexibility to the instructional process.

PAUL'S PING-PONG BALL PACKING COMPANY

Put 5 in each box



13 Ping-Pong balls.

What is the greatest number of boxes Paul can fill?

How many balls left over?

$13 = (? \times 5) + ?$

$5\overline{)13}$

A closer look at computation. Much of the discussion in these last two sections has direct relevance for the teaching of computation. Many persons feel that computation should be more than the mere manipulation of numbers. They argue that computation should be taught so that—

1. computational techniques are related to basic concepts and properties (computation is one aspect of the larger topic of operations on numbers);

● ● ● ● ● ○ ○ ○ ○ ○

$5 + 5 = 10$

● ● ● ● ● ● ○ ○ ○ ○

$6 + 4 = 10$

$8 + 8 = 16$	$20 + 20 = 40$
so $9 + 7 = \underline{?}$	$21 + 19 = \underline{?}$
$10 + 10 = 20$	$300 + 300 = 600$
so $11 + 9 = \underline{?}$	$305 + 295 = \underline{?}$

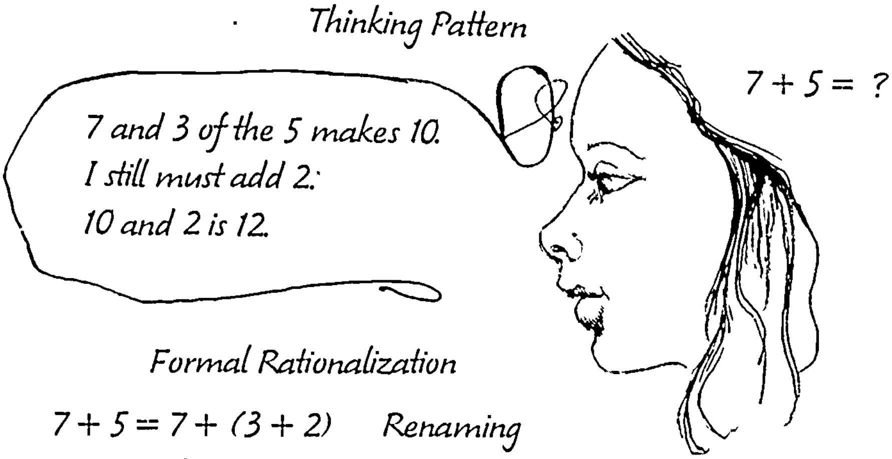
You know that $40 + 40 = 80$.

Which equals 80: $\underline{42 + 28}$ $\underline{42 + 36}$ $\underline{42 + 38}$?

2. computation is used as a vehicle for developing mathematical thinking and specific thinking strategies;
3. computational techniques are related to component concepts;
4. a variety of experiences in applying computation are provided.

Several aspects of computation have been treated already. Two additional elements will now be discussed. The first deals with the role of formal rationalization in developing informal thinking strategies. Two thinking patterns and formal rationalizations of them are presented in the next figures.

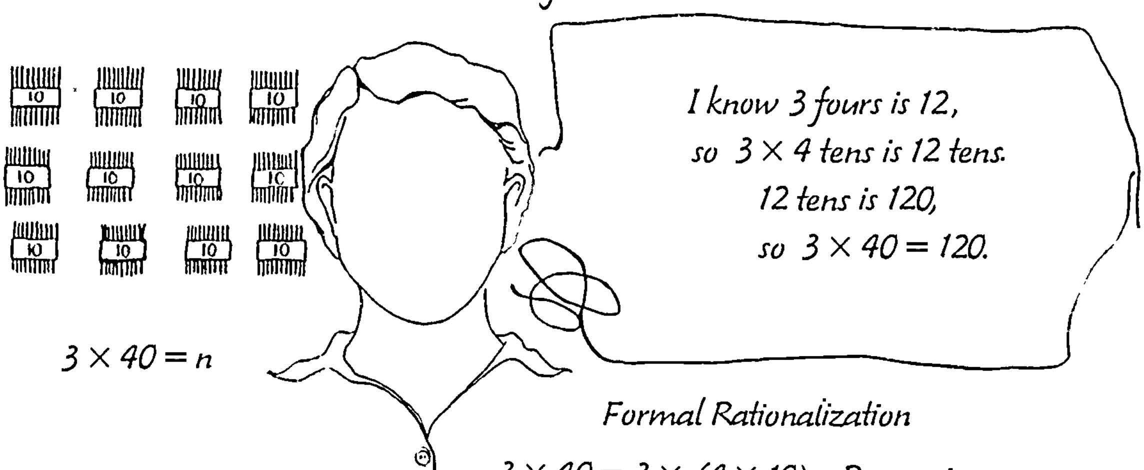
Thinking Pattern



Formal Rationalization

$$\begin{aligned} 7+5 &= 7+(3+2) && \text{Renaming} \\ &= (7+3)+2 && \text{Associative Property} \\ &= 10+2 \\ &= 12 \end{aligned}$$

Thinking Pattern



Formal Rationalization

$$\begin{aligned} 3\times 40 &= 3\times(4\times 10) && \text{Renaming} \\ &= (3\times 4)\times 10 && \text{Associative Property} \\ &= 12\times 10 \\ &= 120 \end{aligned}$$

$\begin{array}{r}3\\+4\\\hline\end{array}$ $\begin{array}{r}34\\+40\\\hline\end{array}$	→ $\begin{array}{r}34\\+42\\\hline\end{array}$	→ $\begin{array}{r}38\\+48\\\hline\end{array}$	→ ··· → $\begin{array}{r}683\\+244\\\hline\end{array}$
Adding Tens	*Adding Tens & Ones*	*Renaming Ones as Tens*	*Renaming Tens as Hundreds*

In recent years, formal rationalizations based on a deductive use of structural properties have been used to *develop* approaches to computation The assumption has been that these approaches build a thinking pattern. It is argued, however, that it is important to distinguish between establishing an informal, verbal pattern and demonstrating mathematically at a later time why the pattern works. Children are capable of grasping relationships intuitively and informally. Frequently they are able to apply a pattern when they cannot appreciate the value of a formal rationalization A systematic. step-by-step application of properties may even prevent children from seeing the pattern itself. Although at some point it may be helpful for them to see how certain approaches can be justified logically on the basis of a few unifying ideas, establishing and applying patterns of thinking must be the primary aim. Formal rationalizations should occupy a subordinate role with a carefully defined use, and the distinction between the two approaches needs clearer recognition than it has received in recent years.

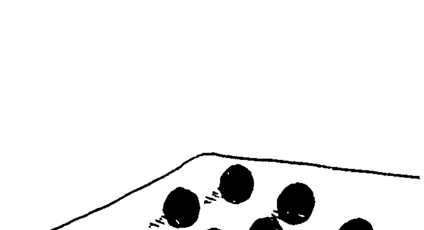
A second aspect of computation is the role of sequencing instruction in developing computational algorithms. Since the learning of algorithms has many complexities, it is important to provide systematic instruction designed both to relate this work to previous learning and to clarify the ideas involved. Sensitive, perceptive teaching that builds on basic conceptual understanding is required, and instruction should proceed carefully and deliberately, with a recognition of the child's level of maturity.

In the first section of the chapter, the importance of setting systematic instruction in a broader framework was discussed. This can be done when teaching computational algorithms. Such work can continually be related to problem situations, opportunities can be provided to explore various number patterns, and pupils can be encouraged to use informal computational techniques built around a variety of estimation and mental arithmetic approaches. Such a setting gives a more holistic framework to the focus on algorithmic details.

There are two ways in which sequencing is involved in work with computation. First, the hierarchical ordering of skills should be recognized. For example, learning to solve 38 + 48 follows lower levels of operations, this skill in turn precedes renaming with three-digit numerals (■). Recognizing and building instruction around the various levels of difficulty in computing can help learning to proceed more smoothly. Understanding at one level is important for learning at the next.

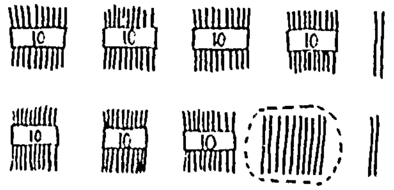
Second, sequencing is also an important factor in developing a particular algorithm. Initial instruction on the renaming algorithm in subtraction might proceed as follows:

1. Development of the concept that subtraction is not always possible in the set of whole numbers (●)



6 marbles on a table. Can you pick up 5 marbles? 6 marbles? 7 marbles? 8 marbles? What is the greatest number you can pick up? Is there a whole number that makes 6 − 7 = n true?

2 Development of renaming around the principle that a number may be renamed as "1 less ten and 10 more ones"

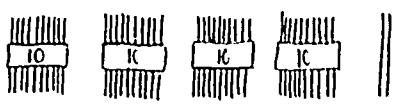


4 tens 2 ones

↓ ↓

3 tens 12 ones

3. Use of a developmental algorithm that highlights these two ideas as well as more general grouping and place-value notions



3 12
~~4~~ tens ~~2~~ ones
– 2 tens 8 ones

Does 2 – 8 name a whole number?

How do you rename?

4. A transition to the final, compact form

3 12 → *3 12*
~~4~~ tens ~~2~~ ones → *~~42~~*
– 2 tens 8 ones → *– 28*

5 A focus on when it is appropriate to rename

In which do you need to rename?

65	74	87	81	60	32
– 48	– 38	– 35	– 36	– 42	– 20

An unhurried, careful development closely links this procedure to physical and mathematical ideas and provides an opportunity for their integration in the final algorithm.

Developing proficiency with computational skills is not an easy task; however, ignoring the complex problems of instruction in this area or adopting simplistic positions will not produce more effective learning. In the development of more effective programs, all the many elements involved must be considered.

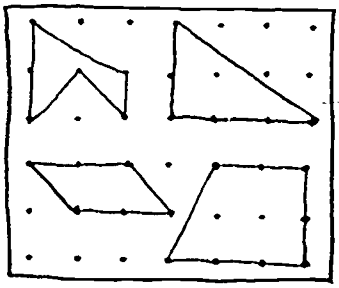
a balanced view of the use of physical materials

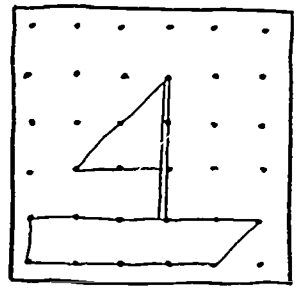
The use of physical materials can make many contributions to early learning. Yet physical materials are not a panacea. Understanding the instructional purposes they may serve can help them to be used productively.

The judicious use of physical materials at the early childhood level plays an important role in the learning of mathematics, particularly in the initial formation of some mathematical content. Today the use of a wide variety of materials is receiving strong support—support that sometimes has led to such statements as "Learning cannot occur without the physical manipulation of materials." Although this probably overstates the case, it is unfortunately true that mathematics is often taught with little or no use of materials or other perceptual supports.

Even though there is general support for the use of materials, a careful description of how they influence learning is difficult at this time for many reasons. The amount of experience necessary to effect learning is hard to determine. More must be known about bridging the gap between the materials and the ideas they develop. Knowing which materials are the most appropriate for particular concepts is difficult. The relative roles of physical experiences and pictorial representations is not clear. Some see the manipulation of materials as an end in itself; others concentrate on their use as a means to an end. Additional knowledge in this area would be invaluable in providing a guide to daily practice.

In the meantime, it is important to stress that pupils' involvement in physically oriented activ-





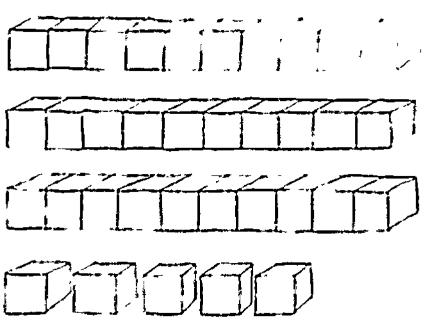
■

ities can contribute both to learning and to the establishment of a positive learning climate. One guide to the productive use of such experiences is an awareness of the purposes for which the manipulation of materials can be employed:

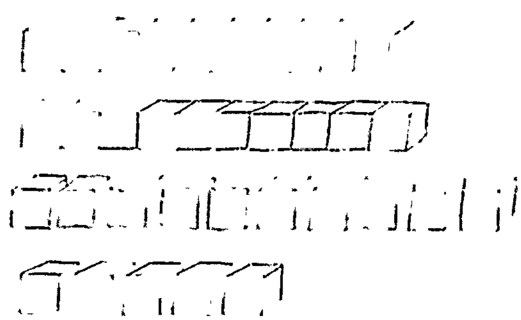
1. Materials can be used in exploratory experiences where the purpose is to provide readiness for later work. For example, kindergarten children can explore the idea of shape by making familiar figures on the geoboard as well as examples of various polygons (■).

2. Materials can be used in the development of a specific mathematical idea. To develop the pattern of renaming used in the subtraction algorithm, teachers should have pupils regroup physical materials and record the results. At this level specific guidance is needed if pupils are to see the connection with the idea being developed. The activity should lead to the generalization that a number can be renamed as *1 less ten and 10 more ones* (●).

3. Materials can be used to provide added experience with an idea that has already been developed. Many preprimary experiences with early number work are of this type. For example, in dealing with grouping and place-value ideas, the teacher might place around the room envelopes containing sets of buttons, beads, paper clips, toothpicks, and so on. Pupils move about the room and empty the contents of each envelope, group them by tens, record the number of tens and the number of ones left over, and then write the standard numeral (□).



3 tens, ? ones



2 tens, ? ones

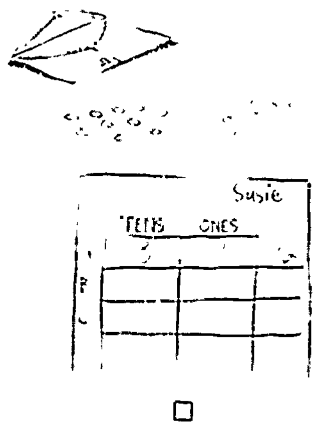
You can rename by thinking 3 5

3 tens 5 ones

1 less ten ——→ 2 tens 15 ones

and 10 more ones - - - - - - ↑

●



4. Materials can be used as the source of experiences in which pupils extend their knowledge or apply it in a variety of ways. The experiences with symmetry cited earlier in the chapter illustrate this use.

An awareness of the many uses to which activities can be put can enable a teacher more effectively (*a*) to determine the purpose for a given activity, (*b*) to provide help in selecting activities appropriate for a given instructional purpose, and (*c*) to give guidance in how the activity should be structured. Activities that are carefully and purposefully integrated into instructional programs can be of great value, but they are not cure-alls. An intelligent and balanced use of activities, as well as an awareness of when they contribute to learning mathematics, is required.

a balanced approach to the organization of instruction

Effective instruction for all pupils involves the consideration of many approaches to organizing instruction. Such consideration must take into account many variables related to learning and instruction. A variety of approaches may need to be employed.

Adjusting instruction to the needs and abilities of individuals is receiving renewed and appropriate interest today. A concern for the development of each child, for the quality of his learning, and for the level of success he attains must be at the heart of the educational process. Basically, individualizing instruction implies a point of view or attitude toward students that is independent of an organizational pattern. A basic element of this point of view is the acceptance of each child as an individual worthy of adult respect. Other important ingredients include an acceptance of the child's ideas, a provision of opportunities for pupil input in developing and selecting learning experiences, a concern for the quality of the child's intellectual development, and a willingness to take time to know the child as an individual.

Concern for pupils also involves concern for the quality and extent of their learning. Closely related to this concern is the attempt to find appropriate content and to develop it appropriately. Factors that promote learning must be recognized and considered when fitting instruction to children. In order to help children learn effectively and enjoyably, teachers must have a knowledge of the learning process in mathematics and use it in making decisions about organization for instruction.

Individualizing instruction—that is, personalizing instruction—and adjusting it in order to maximize the probability of successful learning necessarily involves organization for instruction. Yet plans for classroom management are a means to an end, they are techniques for promoting more effective learning in a pleasant, accepting atmosphere.

Individualized instruction is frequently equated with self-paced instruction. It involves learners moving independently through predetermined lessons, receiving guidance as requested, and progressing at their own rates. At times the curriculum emphasis is on the skill dimensions of learning, with little development work provided and with the measures of learning and understanding limited to correct responses to written exercises. Although independent progress approaches may be used at appropriate times, identifying individualized instruction with a single organizational pattern may cause significant considerations about the learning process to be ignored.

Balance in this area can be achieved by using techniques appropriate to particular situations. When wisely implemented, group developmental lessons take into account individual capabilities and provide means for individual input. Interest in learning in general can be in-

creased by stimulating interest in the learning of new content and presenting it informally. Children can be helped to organize their ideas and see the important elements in new learning. Opportunity can be provided for discussion, with child-initiated questions providing additional input for a lesson or altering its direction. Interacting with peers and concerned adults is satisfying to children and is an important part of learning.

In the process of developing a topic, pupils might work with materials either in small groups or individually for certain periods of time. At times, a variety of groupings may be employed. Pupil performance with specific content is often the basis for such groupings. Sometimes pupils need to work independently on particular weaknesses or on more advanced work with a topic. A flexible program would often provide the opportunity for pupils to select their own learning experiences.

Classroom organization for instruction is one of the elements that must be considered when attempting to provide for individual differences in learning. Yet having organizational patterns serve more basic goals requires consideration and sensitive judgment about the nature of particular content, . purposes of particular instruction, the learner, and the learning process. Instruction organized in a flexible and balanced manner is likely to produce effective progress.

summary

This chapter has identified some of the ideas and issues related to effective instructional programs for young children. It has been argued throughout that curriculum deliberations should focus on basic questions that deal with the learning of mathematics in a comprehensive and realistic fashion. Several issues of curriculum and instruction have been discussed in the context of their influence on curriculum over an extended period of time. They have also been discussed in terms of the notion that balanced emphases in several areas are important for learning. Each of the five aspects of a balanced curriculum should receive careful attention, each is an important aspect of providing better perspective on the mathematics curriculum.

The task of building and implementing effective programs is not an easy one. It is never completed. Yet through thoughtful consideration of significant questions, careful development of new approaches, wise implementation of ideas ...th demonstrated potential, and a commitment to excellence can come progress that will result in meaningful learning by the children we serve.

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research on learning mathematics

marilyn suydam

fred weaver

HOW do children learn mathematics? This is the basic question that must be answered in making decisions about *what* mathematics to teach, *when* it can be taught, and *how* it should be presented to children. Research studies have been directed at facets of this basic question about learning, and taken as a whole, these studies provide some guidance for teachers in making necessary decisions:

- What is it possible and feasible for teachers to teach and for children to learn?
- Which of several specified alternatives may be preferable to teach, and for which children are they appropriate?
- When may certain specified content be taught?
- How may content be taught, considering both school-controlled and teacher-controlled factors?
- What materials may be used to teach certain specified content?

The major emphasis in this chapter is on research done on the learning of mathematics in school settings through grade 3. The research conducted only with older children is not cited, even though some of the findings may be applicable to younger children. It may come as some surprise (as it did to the reviewers) that so little actual research has been done on early childhood education. Many aspects have been explored only minimally, and so the reviewers have occasionally extrapolated and given their own reactions.

The major emphasis in this chapter is on the learning of mathematical ideas and skills. Interwoven with mathematics learning are affective

factors that may be learnable but not directly teachable, yet those who work with children must be continually aware of their vital importance. The limited number of studies in the affective area are reviewed first, followed by studies on the developmental aspects of mathematics learning and mathematics learning from systematic instruction.

affective factors of mathematics learning

The affective domain pertains to attitudes, interest, self-concepts, and factors such as anxiety and frustration, all of which influence or interact with motivation. These various concomitants that children carry with them to each mathematics lesson affect the way they feel about mathematics. Research on mathematics education, however, has looked carefully at very few of these concomitants.

what is the role of attitudes in mathematics?

Teachers of young children realize, possibly even more than other teachers, the importance of attitudes in learning. Young children tend to indicate overtly their feelings of enjoyment, interest, and enthusiasm; they also readily indicate their dislike or apathy. Yet surprisingly little research has been done concerning their attitudes toward mathematics. Probably this fact is due in large part to (1) the difficulty of measuring attitudes when dealing with young children (e.g., their responses on group tests tend to be of questionable validity and (2) the instability of young children's attitudes from day to day.

A concern with attitudes and other affective factors has developed because it seems reasonable to assume that the way children feel about, and react to, mathematical content and ideas is related to the quantity and quality of their learning. This logical relationship has led to continual concern with "motivation," though exactly what motivation is has been the subject of some debate. However, it may be assumed that motivation includes whatever the teacher does to increase children's interest in learning. One hopes that increased interest will lead to increased achievement, though research has not provided evidence that this relationship is as strong as might be expected (see, for example, Deighan [1971]). Numerous reports have been made about games, materials, and techniques that teachers have used successfully to increase interest. It seems reasonable that the manner in which the teacher presents an activity or idea is closely related to the reaction of children to that activity or idea. What teachers say—and how they say it—has been found to be particularly important. Not surprisingly, praise has been found to be a highly effective way to motivate. For instance, Masek (1970) studied the behavior of children when teachers provided such positive reinforcement as verbal praise, physical contact, and facial expressions indicating approval. Significant increases in mathematical performance and in task orientation occurred when such reinforcement was frequent; lower performance was noted when reinforcement was stopped, and performance increased once more when reinforcement was again used.

Teachers and other mathematics educators generally believe that children learn more effectively when they are interested in what they learn and that they will achieve better in mathematics if they like mathematics. Therefore, continual attention should be directed toward creating, developing, maintaining, and reinforcing positive attitudes.

what are the attitudes of young children toward mathematics?

Researchers who are interested in measuring attitudes toward mathematics have tended

to begin their study in grade 3. A large percentage of both boys and girls in grades 3 through 6 have indicated that they liked mathematics (Greenblatt 1962; Stright 1960). The evidence on whether boys like it better than girls, or vice versa, is conflicting.

Very little is known from research about children's attitudes toward mathematics prior to grade 3. Generally, teachers find that the attitudes of children in the early grades are positive toward both school and mathematics. To most observers, it is apparent that the ideas of number and geometry hold a fascination for these children. It is commonly believed that from grade 3 to grade 6, however, children's attitudes toward mathematics become less positive, and research provides some support for this belief (for example, see Deighan [1971]). Yet, other investigations have shown that children's attitudes toward mathematics become *more* positive from grade 4 to grade 6 (for example, see Crosswhite [1972]).

It seems likely that a relationship would exist between the teacher's attitude toward mathematics and his pupils' attitudes toward mathematics. Yet only a weak relationship between such attitudes has been observed in grades 3 through 6 (Caezza 1970, Deighan 1971; Wess 1970). This finding, which conflicts with common sense, may be accounted for by the way in which the attitudes were measured in these studies. Other rationales could also be developed, of course.

developmental aspects of mathematics learning

Children begin to acquire mathematical ideas almost from the time they become aware of the world around them. Researchers interested in the development of children have attempted to ascertain both *what* ideas are developed and *how* they are developed.

what are the findings of research related to children's development and their implications for mathematics education?

Ilg and Ames (1965), with others at the Gesell Institute, have long been engaged in longitudinal studies of how children develop and the implications of these studies for the school. They propose that a thorough analysis of the child's developmental level would aid teachers in providing an individualized program for each child. Their tests are designed to ascertain developmental factors (including the development of mathematical ideas) rather than knowledge acquired from school instruction. They point out that educators must keep in mind the meshing of three factors: the child at a certain age or level of growth, the child as a unique individual, and the child in a certain environment.

A vast amount of research has been directed toward testing various aspects of the theory of Piaget. In his theory, Piaget depicts intelligence as an active, organizing process whereby the child attempts to structure his world and give meaning to it. Piaget has identified four important factors involved in the development of intellectual capacity: (1) maturation, (2) encounters with the physical environment, (3) social experience, including both interaction with, and instruction by, adults and peers; and (4) a process he calls *equilibration*, by which the child incorporates new knowledge. Not only has Piaget identified these four factors, but he has also—

1. identified certain stages of development and established approximate chronological age boundaries for each stage;
2. ascertained that a child's development is invariant in terms of sequencing—that is, he passes from one stage to another in much the same order as all other children;
3. developed certain procedures, or *protocols*, for ascertaining the child's stage of development, involving a specific set of questions to be asked about a specific set of materials.

An overwhelming amount of Piagetian-inspired research has been concerned with three questions:

1. Are the stages proposed by Piaget valid? Most research has confirmed the existence of these stages: that is, they do occur, and they occur by and large in the order Piaget suggested Piaget maintains that a major shift in the nature of children's thinking occurs at about age seven. At this point, the child becomes less prone to distractions than he was earlier, can manipulate ideas more readily, is aware of contradictions, and can correct himself when he makes mistakes. Thus the period of early childhood marks the transition from thought that is perceptual and subjectively oriented to thought that is conceptual, objective, and systematic The child develops increasing trust in reasoning rather than reacting almost entirely on the basis of what he perceives.

Research findings on when the stages occur have indicated more variability than Piaget suggested. For instance, Sawada and Nelson (1967) reported that the threshold age for conservation of length appeared to be between ages five and six, not ages seven and eight as Piaget found.

2. What is the effect of training on advancing the age at which the stages are reached? Piaget's interest was in determining how children develop spontaneously and in interaction with their environment, but without planned changes in that environment. He contends that the child develops mathematical concepts not only from instruction but also independently and spontaneously from his own experiences (Piaget 1953) Other research indicates that by the use of certain procedures, children can be trained to evidence a stage earlier than would be expected if they were untrained. However, there is no evidence that training changes the overall developmental level of the child. The positive effect of training is rarely retained *unless* the child is already in transition from one level to another (see, for example, Coxford [1964]).

3. Are Piaget's protocols accurate measures? Research generally indicates, with little variation, that unless the questions and materials of Piaget's protocols are used, results may differ. (For example, Sawada and Nelson [1967] used a nonverbal method of assessment rather than a verbal one, a fact that may have contributed to their findings, contradicting those of Piaget.)

The key question that we as teachers of mathematics find of greatest concern—What are the implications for the teaching of mathematics?—has been the subject of much discussion. Sinclair succinctly states the conclusion of many (1971, p. 2):

> Educational applications of Piaget's experimental procedures and theoretical principles will have to be very indirect—and he himself has given hardly any indication of how one could go about it. His experiments cannot be modified into specific teaching methods for specific problems, and his principles should not be used simply to set the general tone of an instructional program.

Comparatively little research has explored the classroom implications of Piaget's theory. The topic on which the most research has been done is conservation—the realization of the principle that a particular aspect of an object or a set of objects may remain invariant under changes in other, irrelevant aspects of the situation. Conservation of number involves awareness that the number of objects in a set remains unchanged in spite of changes in the arrangement of the objects. Piaget believes that the child must grasp the principle of conservation before he can develop a reliable concept of cardinal number. He found that the development of this concept takes place at about seven to eight years of age. He believes that number concepts involving cardination and ordination, which are aspects of classification (the grouping of objects according to their similarities) and seriation (the ordering of objects according to their differences), cannot be taught to younger children, since the child is conceptually unable to understand.

Conservation of number has been shown by several researchers to be related to work with counting and with addition and subtraction. Almy, Chittenden, and Miller (1966) noted that children who conserve at an early age do better in beginning arithmetic than those who are late in acquiring conservation. Robinson (1968)

also found significant relationships between a child's ability to conserve, seriate, and classify and his level of achievement in grade 1. He concluded that conservation may be necessary but not sufficient for mathematics achievement. Others have found that first graders who are conservers tend to do better at certain mathematical problem-solving tasks. Steffe (1967) reported that the tests of conservation of numerousness used in his study provided an excellent prediction of success in solving addition problems and learning addition facts for children entering first grade. Using the same test in a companion study on subtraction, LeBlanc (1968) reported a similar conclusion. Dodwell (1961) and Wheatley (1968) have also indicated that the tests of number conservation that they used may be meaningful measures of arithmetic readiness.

Ambiguous findings were reported by Almy and her associates (1970). Second-grade children who had no prescribed mathematics lessons in either kindergarten or first grade performed about as well on certain Piaget-derived tasks as second-grade children who did have prescribed mathematics lessons beginning in kindergarten But the latter groups performed better than the second-grade children for whom prescribed lessons did not begin until first grade The following conclusions were noted (Almy et al. 1970, p. 170):

> The notion that the instruction should match or pace the children's cognitive development appears logical, but much is still to be learned about how to assess such development as well as about pacing instruction from it.
>
> The problem is not only that development appears to be uneven, and that progress as the teacher observes it may be a matter of advance and retreat rather than the forward-moving sequence so often inferred from Piaget's theory, it is also that individual children differ so much not only in cognitive level, but also in their attitudes, their interests, and their concerns.

It may also be true that the type of teaching the child has had may affect the pace of his development, and the type of materials he and his teacher use may be of importance. The learning style of the child may also be a factor to consider For instance, Simpson (1971) found that the child's ability to conserve substance, weight, and volume was affected by whether he had a reflective or an impulsive learning style.

Although an overwhelming quantity of the Piagetian-oriented research has concentrated on conservation, attention is gradually shifting to other facets of Piaget's theory, such as classification and seriation. Piaget stated that the child must first construct simple collections of materials before he is able to manipulate objects and classes within hierarchical systems. Nowak (1969) found that third graders benefited from instruction on two simple hierarchical operations. Johnson (1970) reported that the kindergarteners he studied were able to categorize by using almost exclusively a perceptual focusing strategy that involved a seemingly random choice of a focus picture and a one-to-one search for identical pictures. The task was extremely simplified when the focusing picture was supplied. Bonney (1970) assessed the seriation skills of first graders and found that they did not tend to generalize across types of content (concrete or verbal, quantitative or interpersonal).

Thus it appears that a child's mathematical achievement may be aided by instruction that is based on an assessment of his developmental level and that includes content that helps him to focus on relevant aspects of such factors as conservation, seriation, and classification. Piaget's protocols can aid in assessment, but they are not intended to be used for instruction. Although they could provide training guides, there seems to be little long-range payoff from training. Instead, the emphasis should be on providing experiences that help the child gain in understanding, most of the chapters in this yearbook focus on these various experiences.

what has been ascertained about the mathematical knowledge children have when they enter school?

Through the years, many surveys have assessed the mathematical knowledge acquired by children during their first five or six years (see, for example, Buckingham and MacLatchy [1930], Brownell [1941], Ilg and Ames [1951];

Bjonerud [1960]; Rea and Reys [1970]). An analysis of these surveys provides information on both the extent and the variability of attainment Unfortunately, the generalizations that can be made from such studies are limited for several reasons:

1. Usually the groups were relatively small, and readily available groups, rather than groups representative of the total population, were used.
2. Few surveys have been replicated, most were conducted at only one point in time, as well as in only one locality. As with most surveys, the question arises whether the findings would be accurate for other children in other places.
3. The methods of collecting information varied. Some used individual interviews; others used group tests. Some used one question to affirm a particular aspect; others asked many. Also the depth of the questions varied from those requiring single-word responses to those requiring manipulation of objects or, rarely, reasons for answers.

But most important of all in analyzing the usefulness of these surveys is the fact that what teachers really need to know is not what *other* children know, but what *their* children know. As Brownell stated, "Research findings tell the teacher . . . little about the class as a whole, but they tell very much less about the number abilities of particular children" (1941, p. 61). Each teacher must plan experiences and base instruction on an accurate knowledge of the mat[illegible] ideas of each child [illegible] ventory of [illegible] concepts and skills of each child

Perhaps one of the greatest contributions of surveys assessing the knowledge of young children has been the development of procedures and instruments for assessment. Tests such as those developed by Rea and Reys (1970) or Schwartz (1969) can serve the teacher as valuable tools in assessing children's mathematical knowledge.

Tests designed to ascertain the child's status in relation to different aspects of Piaget's theory are also being developed. If it is true, for instance, that the child must conserve number before he can deal meaningfully with counting, addition, subtraction, and other mathematical ideas, then it becomes important that teachers ascertain his level of conservation as they take inventory of his mathematical ideas.

Some conclusions *can* be drawn from the research. It is clear that children have acquired many mathematical ideas before they begin formal instruction in school (Rickard 1967, Heimgartner 1968). Moreover, in the range and extent of their knowledge, there is relatively little difference between children today and children decades ago. Also, as the age, intelligence, or socioeconomic level of his parents increases, the child's knowledge of mathematical ideas appears to increase.

The variability of the methods of assessing the knowledge of children entering kindergarten makes it difficult to affirm precisely what children in general know. A summary of the data on five-year-olds without prior schooling indicates that—

1. many children can count and find the number of objects to ten, and some are able to count to at least twenty;
2. some can say the number names for tens in order (that is, ten, twenty, thirty . . .), but far fewer can say the names when counting by twos and fives;
3. most know the meaning of "first," and many can identify ordinal positions through "fifth";
4. many can recognize the numerals from 1 to 10, and some can write them;
5. most can give correct answers to simple addition and subtraction combinations presented verbally either with or without manipulative materials;
6. most have some knowledge about coins, time, and other measures, about simple fractional concepts, and about geometric shapes.

It is not only when a child enters kindergarten (or first grade) that teachers need information on his mathematical knowledge—the beginning of each year is a particularly important point of assessment, for it provides the teacher with a point of beginning and a base line-guide

for planning the year's work. It indicates to the teacher what prerequisite skills and understandings the child has and thus indicates what he is probably ready to learn. But assessing readiness is an almost continuous process. The teacher needs to know when the child is ready to proceed with, and profit from, experiences or instruction at each step in the mathematical program. Decisions must be reached about when to introduce a new process or a new aspect of a process. A teacher cannot expect to be effective unless readiness is continually assessed.

mathematics learning from systematic instruction

Not unexpectedly, the bulk of the research on mathematics education has been concerned with the effects of systematic instruction. This fact can be considered under two headings. (1) the organization of the program and other general methodological considerations and (2) the content and how it may be taught.

organization and method

what are the effects of various types of kindergarten experiences?

Research has indicated that children who have kindergarten experience score higher in later grades than those who do not have kindergarten experience (Haines 1961). Most, but not all, studies indicate that children who were older when they entered kindergarten or grade 1 achieved more than their younger classmates.

The specific type of program for kindergarten has had, and will continue to have, much research focused on it. One type of research is concerned with creative ways of developing mathematical ideas. For instance, Fortson (1970) taught beginning mathematics to five-year-olds through multiple-stimuli techniques using concrete materials, bodily movement, and sound. The children saw, heard, and felt patterned numerical relationships. They scored higher after this program than pupils not having it.

Another type of study is concerned with more formal processes. Carr (1971) studied the effects of a program modeled after the Bereiter-Engelmann preschool program (see chapter 12). Bereiter-Engelmann materials were used in kindergarten with children who had had zero, one, or two years of the special training. No significant differences were found on four tasks of the Piaget type involving ability to conserve number, to discriminate, to seriate, and to enumerate. A significant correlation was found, however, between performance on these tests and on achievement tests. It was concluded (1) that instruction of this type is most effective when the child has reached a certain level in his cognitive development that might be described as number readiness and (2) that tests of the Piaget type can be used to assess a child's readiness for systematic instruction in arithmetic.

Kindergarten programs vary greatly. It is frequently averred that a planned program does not need to be a formal program. The teacher can provide experiences and materials that enable the child both to explore mathematical ideas in his environment and to resolve his mathematical questions as they arise. Textbooks and workbooks do not need to be part of such a program.

what factors need to be considered in planning for instruction?

It may seem to many educators that two factors of particular importance in planning for instruction are how the school is organized and how the classroom is organized. Certainly an extensive amount of research (particularly at intermediate-grade levels) has been devoted to attempts to ascertain the "best" patterns of school and classroom organization. But the clearest generalization that can be drawn from this research is that apparently no one pattern per se will increase pupil achievement in mathematics. Perhaps the most important implication of the various studies is that good teachers

are effective regardless of the nature of the pattern of school or classroom organization.

The methods of instruction that the teacher uses are of particular importance, though probably no one method or procedure is effective for all teachers with all content. In general, however, the importance of teaching with meaning has been confirmed. Dawson and Ruddell (1955) summarized studies of various aspects of meaning. They concluded that meaningful teaching generally leads to greater retention, greater transfer, and increased ability to solve problems independently. They suggested that teachers should (1) use more materials, (2) spend more class time on development and discussion, and (3) provide short, specific practice periods. More recent studies have supported these findings. (For a more extensive discussion of the role of meaning in mathematics, see Weaver and Suydam [1972].)

Many teachers have noted that children fail to retain knowledge well over the summer vacation. The amount of loss varies with the child's ability and age, but the length of time between the presentation of the material and the start of the vacation is of particular importance. Practice during the summer and review concentrated on materials presented in the spring have been shown to be especially important.

Transfer, the ability to apply something learned from one experience to another, appears to be facilitated by instruction in generalizing, that is, by teaching children to see patterns and to apply procedures to new situations. In most studies there is the implication that transfer is facilitated when teachers plan and teach for it. And children need to know that transfer is one of the goals of instruction.

Many studies have provided evidence that the type of test is important in measuring outcomes. This point was emphasized during the 1960s when there was much concern about the effect of "modern" programs compared with "traditional" programs. Generally, it was found that pupils who had had a "modern" program scored higher on a "modern" test and that those who had had a "traditional" program scored higher on a "traditional" test. In addition to published, standardized tests, Ashlock and Welch (1966), Flournoy (1967), and Thompson (1969), among others, have reported paper-and-pencil tests designed to measure achievement with contemporary programs, with their stress on concepts and not merely on skills.

what factors associated with the learner influence achievement in mathematics?

Research has shown that the ability to learn specific mathematical ideas is highly related to age and intelligence. Socioeconomic level also is related to achievement, but not as strongly. Achievement does increase as the socioeconomic level of the parent increases (see, for example, Unkel [1966]).

The majority of studies with the environmentally disadvantaged are status studies, providing descriptive information on how students were achieving at the time of the study. Some studies, however, have compared the achievement of pupils from two or more levels. Thus, Montague (1964) reported that kindergarten children from a high socioeconomic area scored significantly higher on an inventory of mathematical knowledge than pupils from a low socioeconomic area. Dunkley (1965) similarly reported that the achievement of pupils from disadvantaged areas was generally below that of children from middle-class areas. Differences were greater in first grade than in kindergarten, which suggests that being disadvantaged may have a cumulative effect.

Johnson (1970) noted that kindergarten children from low socioeconomic backgrounds demonstrated less ability to categorize consistently on attribute resemblance. They showed similar *stages* of development regardless of socioeconomic level, however, they apparently proceed through the stages at a slower pace.

In other studies, it has been suggested that personality factors or emotional difficulties may be more relevant to a lack of success in mathematics than intelligence. In one of the few existing studies that deals specifically with the relationship of cognitive style to mathematical achievement, Cathcart and Liedtke (1969) tested fifty-eight children in second and third grades. They reported that children in second grade tended to be impulsive (to jump quickly

to conclusions), whereas those in third grade were comparatively reflective (more inclined to reserve judgment) The relationship between intelligence and cognitive style was not statistically significant, although there was a trend for those with high intelligence to be more reflective than those with lower intelligence. No significant interaction was noted between cognitive style and sex, grade, age, intelligence, or conservation However, reflective pupils attained higher mathematics achievement scores than impulsive pupils.

Although some researchers have reported that boys and girls score differently both on tests of reasoning and on tests of fundamentals (for example, see Wozencraft [1963]), most have concluded that what little difference exists is not sufficient to influence curriculum decisions.

what is the role of diagnosis in mathematics instruction?

The purpose of diagnosis is to identify strengths as well as weaknesses and in the case of weakness, to identify the cause and provide appropriate remediation. As part of this process, there have been many studies that ascertained the errors that pupils make. For instance, Roberts (1968) analyzed computation items from a third-grade standardized achievement test and classified errors into four major categories wrong operation, computation error, "defective" algorithm, and undiscernible errors The inaccurate use of algorithms accounted for the largest number of errors; errors due to carelessness or lack of familiarity with addition and multiplication facts were fairly constant for all levels. Roberts suggested that teachers must carefully analyze the child's method and give *specific* remedial help.

Ilg and Ames (1951) concluded that less emphasis should be put on whether answers are right or wrong and more on the *kinds* of errors the child makes The kind of error provides a good clue to the intellectual process the child is using, since errors often are not idiosyncratic, the same types of errors occur over and over.

The individual-interview technique has been used with success in many studies. The procedure, as described by Brownell (1944), is to face a child with a problem, let him find a solution, and then, in order to elicit his highest level of understanding, challenge him. The intent is to ascertain not only what the child knows but how he thinks about the mathematical procedure he is using.

Computer-assisted instruction is presently being used in some elementary school mathematics classes. Suppes has reported extensively on the use of both tutorial and drill-and-practice programs (see, for example, Suppes and Morningstar [1969]). The computer readily collects data on how children are responding, thus facilitating a diagnosis of their difficulties as well as increasing one's knowledge of how they learn. The drill-and-practice materials, with some provision for individual needs, result in at least equivalent achievement in less time than it would take the classroom teacher using conventional methods.

how necessary is the use of concrete materials?

Materials have generally been categorized into three types. concrete, or manipulative; semiconcrete, or pictorial, and abstract, or symbolic. Much research has focused on ascertaining the role of each of these, the three studies cited here illustrate three approaches to the question.

Ekman (1967) compared the effectiveness of three ways of presenting addition and subtraction ideas to children in third grade. (1) presenting the ideas immediately in algorithm form, (2) developing the ideas with pictures before presenting the algorithms, or (3) developing the ideas with cardboard disks, manipulated by the pupils, before presenting the algorithms. On the "understanding" and "transfer" scales, there was some evidence that the third group performed better. On the "skill" scale, both the second and the third groups, each using some form of materials, performed better immediately after learning, although no significant differences were noted in the three groups by the end of the retention period.

In a study with more precision, Fennema (1972) investigated the relative effectiveness of a meaningful concrete and a meaningful symbolic model in learning a mathematical principle at the second-grade level. Learning was

measured (1) by tests of recall, in which problems were stated in the symbols used during instruction; (2) by two symbolic transfer tests, which included problems that were untaught symbolic instances of the principle and on which pupils could use either the model they had learned or familiar concrete aids, and (3) by a concrete transfer test, which measured the ability to demonstrate the principle on an unfamiliar concrete device. She found no significant differences on any of the tests, the children were able to learn a mathematical principle by using either a concrete or a symbolic model when that model was related to knowledge the children had achieved. Making the teaching meaningful appeared to be as important as the materials used.

Knaupp (1971) studied two modes of instruction and two manipulative models for presenting addition and subtraction algorithms and the ideas of base and place value to second-grade classes He found that both teacher-demonstration and student-manipulation modes with either blocks or sticks resulted in significant gains in achievement.

What types of manipulative materials have been found to be effective?

Earhart (1964) used an abacus to teach first-, second-, and third-grade children whose teachers received in-service help, other groups received instruction without the use of an abacus. On tests of reasoning no significant differences were noted. but on tests of fundamentals. the group using the abacus performed significantly better. It is difficult, however, to tell whether the abacus or the in-service help was the basis for this difference.

In a study by Harshman, Wells, and Payne (1962), first-grade children were taught for one year with programs of varying content, using either (1) a collection of inexpensive commercial materials, (2) a set of expensive commercial materials, or (3) materials provided by the teacher Teachers in the first two groups received in-service training in the use of the materials. When significant differences in achievement were observed, they were always in favor of the third program It was concluded that high expenditure for manipulative materials does not seem justified and that different materials should perhaps be used with different IQ groups.

Weber (1970) studied the effect of the reinforcement of mathematics concepts with first graders. She used (1) paper-and-pencil follow-up activities or (2) manipulative and concrete materials for follow-up activities. No significant differences were noted on a standardized test, however, a definite trend favored the groups using manipulative materials, and these groups scored significantly higher on an oral test of understanding.

In another investigation in grade 1, Lucas (1967) studied the effects of using attribute blocks, which are varied in shape, color, and size. He found that the children using the blocks showed a greater ability both to conserve number and to conceptualize addition-subtraction relations than those children not using them.

Much research has been focused on the effectiveness of the Cuisenaire materials. In these studies, the materials were used as the entire program instead of being used in conjunction with some other program. They served as a manipulative material, and it is not always clear whether students in the comparison groups used *any* manipulative materials.

In studying a group of first graders using the Cuisenaire program, Crowder (1966) reported that (1) the children learned more conventional subject matter and more [illegible] concepts and skills than those taught in the conventional program, (2) average and above-average pupils profited most from the Cuisenaire program, and (3) sex was not a significant factor in relation to achievement. but socioeconomic status was. Working with first and second graders, Hollis (1965) also compared the use of a Cuisenaire program with a conventional approach. He concluded (1) that children learned conventional subject matter as well with the Cuisenaire program as they did with the conventional program and (2) that pupils taught by the Cuisenaire program acquired additional concepts and skills beyond the ones taught in the conventional program.

Brownell (1968) used tests and extensive interviews to analyze the effect of three mathematical programs on the underlying thought

processes of British children who had studied those programs for three years. He concluded that (1) in Scotland, the Cuisenaire program was in general much more effective than the conventional program in developing meaningful mathematical abstractions and that (2) in England, the conventional program had the highest overall ranking for effectiveness in promoting conceptual maturity, with the Dienes and Cuisenaire programs ranked about equal to each other. Brownell inferred that the quality of teaching was decisive in determining the relative effectiveness of the programs.

Other studies have been concerned with the effect of using Cuisenaire materials on a particular topic for shorter periods of time. Lucow (1964) and Haynes (1964) studied the use of these materials in teaching multiplication and division concepts for six weeks in third grade. Lucow attempted to take into account the effect of prior work in grades 1 and 2. He concluded that the Cuisenaire materials were as effective as regular instruction. Haynes used pupils who were unfamiliar with the materials, he found no significant differences in achievement between pupils who used the Cuisenaire materials and those who did not.

Prior background, length of time, and the specific topic may account for differences in the success of the Cuisenaire materials. It has been suggested that the program might be more effective in grades 1 and 2, with its effectiveness dissipating during third grade. At any rate, the findings of these studies tend to reinforce the findings of other studies that the use of manipulative materials aids primary-grade children in developing mathematical ideas.

are mathematics programs that are designed for specific groups successful?

Much attention has been directed in recent years to the use of special programs for the environmentally disadvantaged. It is not at all surprising to find studies reporting that programs designed to provide special treatments and emphases for such pupils result in higher achievement when compared with "regular" programs that include no special provisions for such pupils.

A mathematics program designed for Mexican-American children in first grade capitalized on the intrinsic motivation of success by taking into account their need and desire to progress from their low levels of understanding and knowledge (Castaneda 1968). The group using this program showed greater gains than a group using a program with no such special provisions. In another study with Spanish- and English-speaking children, Trevino (1968) found that among first-grade children tested in arithmetic fundamentals and among third-grade children tested in arithmetic reasoning, those taught bilingually did significantly better than those taught exclusively in English.

In a pilot project undertaken to evaluate the use of School Mathematics Study Group (SMSG) materials by disadvantaged pupils in kindergarten and grade 1, Liederman, Chinn, and Dunkley (1966) reported wide variability in achievement. The variability was great within any given class and between classes. Similar variability is well documented for all groups of children, a persistent reminder of the fact of individual differences.

Heitzman (1970) used extrinsic motivation, in the form of plastic tokens that were exchangeable for toys and candy, to reward skills-learning responses of a group of primary-grade pupils from migrant families. These children achieved significantly higher scores on a skills test than a group not receiving the tokens.

Watching the television program Sesame Street had a measurable effect on the achievement of kindergarten children in recognizing and using numerical symbols and on their knowledge of geometric form (Carrico 1971).

Although most Head Start programs were not by intention academically oriented, some studies attempted to measure the effects of such programs on later achievement. For instance, Mackey (1969) reported that children who participated in a Head Start program generally scored significantly higher on arithmetic tests at the end of first grade than qualified pupils who did not participate in Head Start. Emanuel (1971) found no differences for first-grade children, but those with Head Start experience had higher achievement scores in grade 2 and higher marks in both grades 2 and 3.

what is the role of vocabulary and language?

The arithmetic vocabulary used by kindergarten children was analyzed by Kolson (1963). He identified 229 words, which represented 6 percent of the total number of different words used by the children. Quantitative vocabulary accounted for 70 percent of the total arithmetic vocabulary. Johnston (1964) analyzed the vocabulary of four- and five-year-olds and found that number words comprised approximately 50 percent of all the mathematical vocabulary; measurement words, 10 percent; position words, 30 percent, and form words, less than 10 percent. No relationship was found between aptitude and mathematical vocabulary use.

Smith and Heddens (1964), in a report on the readability of mathematical materials used in the elementary school, reported that average reading levels tended to be considerably higher than the assigned grade level of the materials, particularly at grade 1.

In another type of study comparing the vocabularies of reading and mathematics textbooks. Stevenson (1971) identified 396 mathematical words that were used in third-grade mathematics textbooks. Of these, 161 were common to all the mathematics books, but only 51 were used in both the reading and the mathematics textbooks studied. Willmon (197[illegible]) found that the 473 words she listed from twenty-four textbooks used in grades 1 through 3 were generally used less than twenty-five times. It seems evident that teachers must spend some time teaching the vocabulary used in textbooks.

Beilin and Gillman (1967) studied the relationship between a child's number-language status and his ability to deal with problem-solving tasks in grade 1. Number language was found to be related to the cardinal-ordinal number task Knight (1971) reported that using subculturally appropriate language to teach a unit on nonmetric geometry enabled pupils to perform more successfully than those taught and assessed by means of standard language. Teachers should consider the needs of children even to the point of thinking carefully about the words they use when talking about mathematical ideas.

content

how should operations with whole numbers be conceptualized?

Research pertaining to this question has focused principally on the operations of subtraction, multiplication, and division. Although instruction on addition is vital, its conceptualization has not concerned researchers to the extent that the conceptualization of the other operations has. Addition appears to be the easiest operation for most children, especially since it is so readily conceptualized as a "counting on" process and since children encounter so many situations in their daily lives in which they combine groups of objects.

Subtraction. Gibb (1956) explored ways in which pupils think as they attempt to solve subtraction problems. In interviews with thirty-six children in second grade, she found that pupils did best on "take away" problems and poorest on "comparative" problems. For instance, when the question was "How many are left?" the problem was easier than when it was "How many more does Tom have than Jeff?" "Additive" problems, in which the question might be "How many more does he need?" were of medium difficulty and took more time. She reported that the children solved the problems in terms of each situation rather than conceiving that one basic idea appeared in all applications.

This same relative order of difficulty among subtraction situations was observed by Schell and Burns (1962) for twenty-three pupils in second grade. However, performance levels on the three types of subtraction problems were not significantly different (in a statistical sense), even though the pupils themselves considered "take away" problems to be the easiest.

Coxford (1966) and Osborne (1967) found that an approach using set partitioning, with emphasis on the relationship between addition and subtraction, resulted in greater understanding than the "take away" approach. It is

important that those who want to develop set-subset concepts as a strand in the curriculum consider this finding.

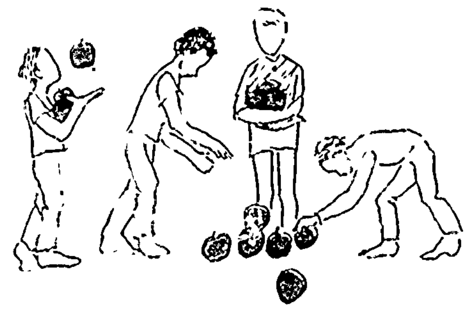
Multiplication. Traditionally the multiplication of whole numbers has been conceptualized for children in terms of the addition of equal addends. For instance, "4 × 7" has been interpreted to mean "7 + 7 + 7 + 7." But logical difficulties are inherent in this interpretation when the first factor in a multiplication example is 0 or 1.

Some recent research has investigated the feasibility of using other conceptualizations of multiplication. Hervey (1966) reported that second-grade pupils had significantly greater success in conceptualizing, visually representing, and solving equal-addend problems than Cartesian-product problems. Cartesian-product problems were conceptualized and solved more often by high achievers than by low achievers, more often by boys than by girls, and more often by pupils with above-average intelligence than by pupils with below-average intelligence. Hervey was not able to determine the extent to which her findings might have been influenced by the nature of prior instruction or by differences inherent in the mathematical nature of the two conceptualizations.

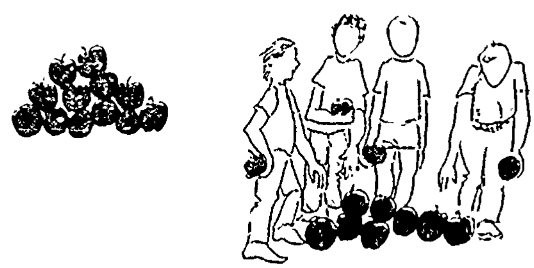
Another conceptualization of multiplication may be associated with rectangular *arrays*—either independent of, or in conjunction with, Cartesian products. At the third-grade level Schell (1964) investigated the achievement of pupils who used only array representations for their introductory work with multiplication as compared with pupils who used a variety of representations. He found no conclusive evidence of a difference in achievement levels.

Division Two kinds of problem situations for division need to be distinguished:

1. *Measurement* problems—Find the number of equivalent subsets.
   Example: If each boy is to receive 3 apples, how many boys can share 12 apples?



2. *Partition* problems—Find the number of elements in each equivalent subset.
   Example: If there are 4 boys to share 12 apples equally, how many will each boy receive?
   Each boy begins by taking an apple. This is repeated until all the apples are taken.



Zweng (1964) found that partition problems were significantly more difficult for second graders than measurement problems. She further reported that problems in which two sets of tangible objects were specified (as in the preceding problems) were easier than those problems in which only one set of tangible objects was specified.

Generally speaking, the conceptualization of an operation is closely associated with real-world problem situations to which that operation may be applied. Since the same operation may be applied to problem situations that differ in significant respects, a single interpretation may not be adequate for children until they are able to conceptualize in less particular ways.

Is learning facilitated by introducing "inversely" related operations or algorithms at the same time?

Considering how frequently this question arises (especially in relation to the Piagetian

concept of reversibility), it is somewhat surprising to find that little recent research has been done on it.

Spencer (1968) reported that some intertask interference may occur when the addition and subtraction operations are introduced more or less simultaneously but that emphasis on the relationship between the operations facilitates understanding.

Wiles, Romberg, and Moser (1972) investigated both a sequential and an integrated approach to the introduction of two currently used algorithms for addition and subtraction examples that involve renaming:

$$\begin{array}{rcl} 37 & = & 30 + 7 \\ +\underline{16} & = & \underline{10 + 6} \\ & & 40 + 13 = \\ & & 50 + 3 = 53 \end{array}$$

$$\begin{array}{rcccl} 53 & = & 50 + 3 & = & 40 + 13 \\ \underline{-16} & = & \underline{-(10 + 6)} & = & \underline{-(10 + 6)} \\ & & & & 30 + 7 = 37 \end{array}$$

There was no evidence to support any advantage of an integrated approach (introducing the two algorithms more or less simultaneously) over a sequential approach (introducing first the addition algorithm, then the subtraction algorithm).

Research pertaining to the (more or less) simultaneous introduction of multiplication and division is lacking.

what procedures are effective in developing basic number facts?

Basic number facts (e.g., $6 + 3 = 9$, $11 - 4 = 7$, $5 \times 4 = 20$, $42 \div 6 = 7$) are important for two quite different reasons: (1) they provide simple, "small number" contexts for the development of mathematical ideas pertaining to an operation, and (2) pupils' memorization and recall of such facts are essential to *efficient* computational skill with larger numbers.

Teachers know that children pass through a series of stages from counting to automatic recall in their work with basic number facts. Pupils use various ways to obtain answers to combinations—guessing, counting, and solving from known combinations, as well as immediate recall. Brownell has stated that "children attain 'mastery' only after a period during which they deal with combinations by procedures less advanced (but to them more meaningful) than automatic responses" (1941, p. 96).

We know from the research of some years ago that drill per se is not effective in developing mathematical concepts. Programs stressing relationships and generalizations among the combinations have been found to be preferable for developing understanding and the ability to transfer (see, for example, Thiele [1938]; Anderson [1949]; Swenson [1949]).

Brownell and Chazal (1935) summarized their research work with third graders by concluding that drill *must be preceded by meaningful instruction.* The type of thinking that is developed and the child's facility with the *process* of thinking are of greater importance than mere recall. Drill in itself makes little contribution to growth in quantitative thinking, since it fails to supply more mature ways of dealing with numbers. Pincus (1956) also found that whether drill did or did not incorporate an emphasis on relationships was not significant, *when the drill followed meaningful instruction.*

Another procedure was checked by Fullerton (1955). He compared two methods of teaching "easy" multiplication facts to third graders, (1) an inductive method by which pupils developed multiplication facts from word problems, using a variety of procedures; and (2) a "conventional" method that presented multiplication facts to pupils without involving them in the development of such facts. In this instance a significant difference in favor of the inductive method was found on a measure of immediate recall of taught facts as well as on measures of transfer and retention.

Teachers know that the number of specific basic facts to be memorized is reduced substantially if pupils are able to apply the properties of an operation. This is illustrated by the following examples:

1. $3 \times 5 = 15$; so $5 \times 3 = 15$ (commutative property of multiplication)
2. $8 \times 1 = 8$; so $1 \times 8 = 8$ (identity property of multiplication)

3. $7 \times 0 = 0$, so $0 \times 7 = 0$ (zero property of multiplication)
4. $8 \times 6 = 8 \times (5 + 1) = (8 \times 5) + (8 \times 1)$ (distributive property of multiplication over addition)
5. $8 \times 3 = (2 \times 4) \times 3 = 2 \times (4 \times 3)$ (associative property of multiplication)

Little research attention has been directed explicitly toward the use of properties. Hall's (1967) research on teaching selected multiplication facts to third-grade pupils appears to support an emphasis on the use of the commutative property.

In an investigation with third-grade pupils and their beginning work with multiplication, Gray (1965) found that an emphasis on distributivity led to "superior" results when compared with an approach that did not include work with this property. The superiority was statistically significant on three of four measures: a posttest of transfer ability, a retention test of multiplication achievement, and a retention test of transfer. On the remaining measure, a posttest of multiplication achievement, children who had worked with distributivity scored higher than those who had not, but the difference was not statistically significant.

Gray's findings add further support to the body of evidence on advantages that can be expected from instruction that emphasizes mathematical meaning and understanding. These advantages may not always be particularly evident in terms of skills-achievement immediately following instruction; rather, the payoff is much more clearly evident in relation to factors such as comprehension, transfer, and retention.

what do we know about the relative difficulty of basic facts?

At one time, especially when stimulus-response theories of learning were prevalent, there was great interest in ascertaining the relative difficulty of the basic number facts or combinations. Textbook writers as well as classroom teachers used the results of such research to determine the order in which facts would be presented for mastery. The assumption was that if the combinations were sequenced appropriately, the time needed to memorize them could be reduced.

Although there was variability in these studies on relative difficulty, some generalizations can be drawn:

1. Subtraction combinations are more difficult for children to learn than addition combinations.
2. An addition combination and the combination with its addends reversed tend to be of comparable difficulty.
3. The size of the addends is the principal indicator of difficulty, rather than the size of the sum.
4. Combinations with a common addend appear to be of similar—but not equal—difficulty.
5. The "doubles" and those combinations in which 1 is added to a number appear to be least difficult in addition, whereas those combinations with differences of 1 or 2 are easiest in subtraction.
6. Multiplication combinations involving 0 present unique difficulties.
7. The size of the product is positively correlated with difficulty.

Recently, Suppes (1967) used the data-gathering potential of the computer to explore the relative difficulty of mathematical examples, including the basic facts. A drill-and-practice program served as the vehicle to determine a suggested order of presentation and a suggested amount of practice.

In one aspect of his study on the strategies and processes used by pupils as they learn multiplication, Jerman (1970) reported data on the difficulty level of multiplication combinations. He found support for the findings of Brownell and Carper (1943) that children do use different strategies for different combinations and that the strategy used may be a function of the combination itself. Certain combinations, such as $n \times 0$, $n \times 5$, $n \times n$, $n \times (n - 1)$, and $n \times (n + 1)$, appear to be easier to retain.

A number of years ago, Swenson (1944) questioned whether results on the relative difficulty of facts—results that are obtained under repetitive drill-oriented methods of learning—are valid when applied to learning situations that are not so definitely drill oriented. When second-grade children were taught by drill, by generalization, and by a combined method, it was found that the order of difficulty seemed to be—at least in part—a function of the teaching method. Thus research that aims at establishing the difficulty of arithmetic skills and processes should probably do so by means of a clearly defined teaching and learning method.

what is the role of mathematical sentences?

Within the past few years, much attention has been given to *open sentences*, such as $\square + 7 = 11$, $9 - \square = 3$, $6 \times \square = 24$, $\square \div 2 = 14$. Some research on the relative difficulty of such open sentences is just now beginning to appear. Within the context of basic facts having sums between 10 and 18, sentences of the form $\square - b = c$ or $c = \square - b$ are significantly more difficult than sentences of the form $\square + b = c$ or $c = \square + b$ (Weaver 1971). We may expect more definitive research along these lines. Weaver also found that the position of the placeholder in the sentence affected the difficulty of the sentences. That is, sentences of the form $a + \square = c$ were less difficult than sentences like $\square + b = c$.

Grouws (1972) studied sentences of the form $N + b = c$, $a + N = c$, $N - b = c$, and $a - N = c$. He found that sentences using basic facts with sums between 10 and 18 were significantly easier than similar open sentences using addends and sums between 20 and 100. Thus the difficulty appears to be a function not only of the form of the sentence but also of the magnitude of the numbers.

Engle and Lerch (1971) surveyed a group of first graders who had had no instruction on closed number sentences (that is, sentences of the form $3 + 4 = 7$, to which the children respond "true" or "false"). They found that the pupils could respond at about the same level of correctness to closed number sentences as they could to exercises of the computational type. It seems logical to conclude that a variety of forms of mathematical sentences and inherent interrelationships should be emphasized.

what algorithms are most effective?

In order to compute efficiently with larger numbers, algorithms are essential. When applied to whole numbers, research on these computational forms and procedures has focused almost exclusively on subtraction and division.

Over the years, researchers have been very concerned with procedures for teaching subtraction that involves *renaming* (once commonly called "borrowing"). The question of most concern has been whether to teach subtraction by equal additions or by decomposition. Consider the example

$$\begin{array}{r} 91 \\ -24 \\ \hline 67 \end{array}$$

With the *decomposition* algorithm, the subtraction is done this way:

$11 - 4 = 7$ (ones); $8 - 2 = 6$ (tens)

With the *equal additions* algorithm, it is done as follows:

$11 - 4 = 7$ (ones); $9 - 3 = 6$ (tens)

In a classic study, Brownell and Moser (1949) investigated the comparative merits of these two algorithms—decomposition and equal additions—in combination with two methods of instruction—meaningful (or rational) and mechanical:

	meaningful	mechanical
decomposition	a	b
equal additions	c	d

They found that at the time of initial instruction—

1. meaningful decomposition [a] was better than mechanical decomposition [b] on measures of understanding and accuracy;

2. meaningful equal additions [c] was significantly better than mechanical equal additions [d] on measures of understanding;
3. mechanical decomposition [b] was not as effective as either equal additions procedure [c or d];
4. meaningful decomposition [a] was superior to each equal additions procedure [c and d] on measures of understanding and accuracy.

It was concluded that whether to teach the equal additions or the decomposition algorithm depends on the desired outcome.

In recent years, the decomposition procedure has been used almost exclusively in the United States, since it was considered easier to explain in a meaningful way. However, some question has recently been raised about this method. With the increased emphasis in many programs on properties and on compensation in particular, the equal additions method can also be presented with meaning. For instance, pupils are learning that, in essence,

$$a - b = \square \text{ means that } \square + b = a, \text{ or } b + \square = a.$$

They are also learning that, in essence,

$$a - b = (a + k) - (b + k).$$

The development of such ideas should facilitate the teaching of the equal additions procedure. Whether there will again be a shift toward a wider use of this procedure remains to be seen. Evidence from other studies indicates that its use leads to greater accuracy and speed.

Ekman (1967) reported that when third graders manipulated materials before an addition algorithm was presented, both understanding and ability to transfer increased. Using materials was better than either using pictures alone before introducing the algorithm or developing the algorithm with neither aid.

Trafton (1971) found that third-grade children who had experienced a "general" approach based on the main concepts of subtraction and using the number line as an aid to solution before working with the decomposition algorithm had no greater understanding of, or performance with, the decomposition agorithm than children who had had no such "general" approach but instead had had a prolonged development of the algorithm.

Variants of two algorithms for division have been used in the schools:

"Distributive" algorithm

```
   29
3 |87
   6
   --
   27
   27
   --
```

"Subtractive" algorithm

```
3 |87 |
   30 | 10
   -- |
   57 |
   30 | 10
   -- |
   27 |
   21 |  7
   -- |
    6 |
    6 |  2
   --
        29
```

In one investigation comparing the use of the conventional (or distributive) and the subtractive forms, Van Engen and Gibb (1956) reported that there were some advantages for each. They evaluated pupil achievement in terms of understanding the process of division, transfer of learning, retention, and problem-solving achievement. Among their conclusions were these:

1. Children taught the subtractive method had a better understanding of the process or idea of division than children taught with the conventional method. The use of this algorithm was especially effective for children with low ability—those with high ability used the two methods with equivalent effectiveness.
2. Children taught the conventional (distributive) method achieved higher problem-solving scores (for the type of problem used in the study).
3. The subtractive method was more effective in enabling children to transfer to unfamiliar but similar situations.
4. The two procedures appeared to be equally effective on measures of the retention of skill and understanding. The retention factor seems to be related more to teaching procedures than to the method of division.

5 Children who used the distributive algorithm had greater success with partition situations; those who used the subtractive algorithm had greater success with measurement situations.

Scott (1963) used the subtractive algorithm for measurement situations and the distributive algorithm for partition situations. He suggested that (1) the use of the two algorithms was not too difficult for third-grade children; (2) two algorithms demanded no more teaching time than only one algorithm; and (3) children taught both algorithms had a greater understanding of division.

should children check their computations?

Children are encouraged to check their work because teachers believe that checking contributes to greater accuracy. Although there is some research evidence to support this belief, Grossnickle (1938) reported data that are contrary to this idea. He analyzed the work of 174 third-grade children who used addition to check subtraction answers. He found that pupils frequently "forced the check," that is, made the sums agree without actually adding. In many cases, the checking was perfunctory Generally, only a chance difference existed between the mean accuracy of the group of pupils when they checked and their mean accuracy when they did not check.

What does this indicate to teachers? Obviously, children must understand the *purpose* of checking and must know what to do if the solution in the check does not agree with the original solution.

what can young children learn about fractions?

Surveys of what children know about mathematics on entering school indicate that many of them can recognize halves, fourths, and thirds and have acquired some facility in using these fractions. Gunderson and Gunderson (1957) interviewed twenty-two second-grade children following their initial experience with a lesson on fractional parts of circles. The investigators concluded that fractions could be introduced at this grade level if manipulative materials were used and if the oral work made no use of symbols.

A planned, systematic program for developing fractional ideas seems essential as readiness for work with symbols. The use of manipulative materials is vital in this preparation.

Sension (1971) reported that area, set-subset, and combination representations of introducing rational-number concepts appeared to be equally effective on tests using two types of pictorial models with second-grade children.

what can children be taught about geometry and measurement?

Little research on geometry and measurement has been done with young children. From a set of tests administered after two weeks of teaching, Shah (1969) reported that children aged seven to eleven learned concepts associated with plane figures, symmetry, reflection, rotation, translation, bending and stretching, and networks.

Williford (1971) found that second- and third-grade children can be taught to perform particular transformational geometry skills (such as producing slide images, turn images, etc.) to a greater degree than they can be taught to apply such skills toward the solution of more general problems.

Four- and five-year-olds exhibit wide differences in their familiarity with ideas of time, linear and liquid measures, and money, with little mastery evident. In a survey with first graders, Mascho (1961) reported that as their age, socioeconomic level, or mental ability increased, the children's familiarity with measurement also increased. Familiarity was greater when the terms were used in context. It was suggested that (1) some ideas now considered appropriate for first grade should be considered part of the child's knowledge when he enters school and (2) teachers need to study the composition of their classes in terms of age, socioeconomic level, and mental ability when planning curricular activities with measurement.

miscellaneous content

Certain content areas (for example, probability) are new to mathematics programs in the elementary school, and at present there is very little agreement on how much of such areas can, or should be included. Most of the very limited relevant research has been in the form of status or feasibility studies, which have varied widely in the particular concepts or subtopics investigated within a given domain of content.

Probability. Gipson (1972) developed a set of lessons for teaching children in grades 3 and 6 the concepts of a finite sample space and the probability of a simple event. She found that pupils at each of these two grade levels could "learn" the two concepts that were taught, although some subordinate ideas were easier than others for the children to grasp. McLeod (1971) observed that in grades 2 and 4, children who studied a unit on probability—using either a "laboratory participation" approach or a "teacher demonstration" approach—scored no better on immediate and delayed post-experimental measures than pupils who had not studied the instructional unit. This observation was equally true for both boys and girls.

Logic. If a child is to learn to think critically, it is important that he make logically correct inferences, recognize fallacies, and identify inconsistencies among statements. Hill (1961) concluded that children aged six through eight are able to recognize valid conclusions derived from sets of given premises. There seems to be a "gradual, steady growth which is nearly uniform for all types of formal logic" (p. 3359). Differences in difficulty were associated with the type of inference, but these difficulties were specific to age. Difficulties associated with sex were not significant. Children can learn to recognize identical logical form in different content, although the addition of negation very significantly increased the difficulty in recognizing validity.

O'Brien and Shapiro (1968) confirmed Hill's findings, except that little growth was observed between ages seven and eight. Using a modification of Hill's test, they found that children experienced great difficulty in testing the logical necessity of a conclusion and showed slow growth in this ability. The investigators suggested that Hill's research should be interpreted and applied with caution. Hypothetical-deductive ability cannot be taken for granted in children of this age. This seems consistent with Piaget's theory on the attainment of formal logic.

In a later study, Shapiro and O'Brien (1970) reported that at all age levels children's ability to *recognize* logical necessity was significantly easier than their ability to *test* for it.

McAloon (1969) used units in logic with third- and sixth-grade classes. These units included such topics as sets, the relation of sets, statements; truth-falsity, class logic, such as the meaning of *all*, *some*, and *none*; disjunction and conjunction, negations and implications; and basic ideas of validity and invalidity. The group that was taught logic—either interwoven with mathematics or separate from mathematics—scored significantly higher on the mathematics achievement test and on reasoning tests than the group that was not taught logic. Taking time from mathematics did not decrease mathematical achievement but in fact increased it. Symbolism and the principles of transitivity in both class and conditional reasoning were easy for both grades, although in general class reasoning was easier than conditional reasoning. Fallacy principles and the principles of contraposition and converse were more difficult to teach in grade 3 than in grade 6. Items with a correct answer of "maybe" were answered least well.

The identification of conjunctive concepts involves noting attributes common to positive instances, the identification of disjunctive concepts requires noting attributes *never* contained in negative instances. McCreary (1965), working with third graders, and Snow and Rabinovitch (1969), studying children at ages five through thirteen, found that disjunctive tasks were more difficult than conjunctive tasks. Weeks (1971) reported that the use of attribute blocks had a strong positive effect at grades 2 and 3 on both logical and perceptual reasoning ability.

what procedures are effective in teaching verbal problem solving?

Although problem solving may be viewed broadly, its consideration here is restricted to those situations commonly referred to as "word problems" or "story problems."

Most of the relevant research on verbal problem solving has been done with children above the third-grade level. This is understandable in light of the highly related reading factor, although this factor can be controlled with younger children by giving them word problems orally.

Steffe (1970) investigated the relation between the ability of first-grade pupils to make quantitative comparisons and their ability to solve addition problems. He found that children who failed on a test of quantitative comparisons performed significantly lower on a test of addition problems. LeBlanc (1968) reported a similar finding with quantitative comparisons and subtraction problems.

Somewhat conflicting findings were noted regarding whether or not children were able to solve more easily those problems in which a described action was embedded in the problem statement (Steffe 1970; Steffe and Johnson 1971). Such studies have agreed, however, that the presence of manipulative objects facilitated problem solving. Other studies also agree with this finding (see, for example, Bolduc [1970]).

There are some verbal problem-solving techniques for which there is little explicit research evidence, even with older children. Common sense, however, says that these techniques may be applicable at many points in the problem-solving program. Among the techniques that researchers suggest are the following, whereby the teacher might—

1. provide a differentiated program, with problems at appropriate levels of difficulty;
2. have pupils write the number question or mathematical sentence for a problem;
3. have pupils dramatize problem situations and their solutions;
4. have pupils make drawings and diagrams and use them to solve problems or to verify solutions to problems;
5. have pupils formulate problems for given conditions;
6. use problems without numbers;
7. have pupils designate the process to be used;
8. have pupils test the reasonableness of their answers;
9. have pupils work together to solve problems;
10. encourage pupils to find alternate ways in which to solve problems.

concluding observations

The research cited in this chapter is not as definitive as one would like. Many questions remain to be answered. However, the research does give the teacher some broad guidelines for instructional practices and some broad boundary conditions within which to explore and experiment informally. Teachers are encouraged to use the research to focus exploration and experimentation in directions that have potential for improving classroom instruction and to avoid pathways that have little or no promise.

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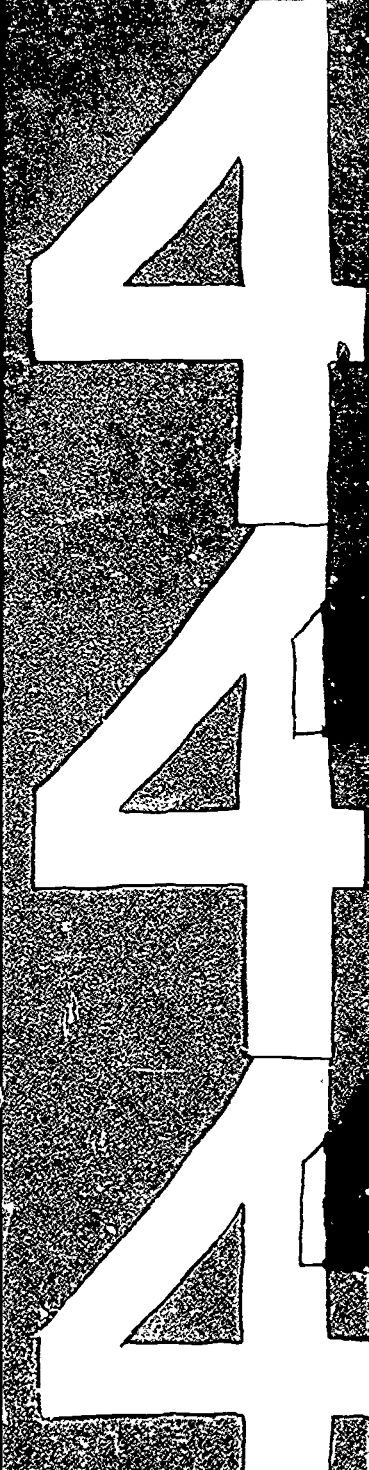
problem solving



doyal nelson

joan kirkpatrick

THE central purpose of any instruction in mathematics at the early childhood level is to help the child see order and meaning in the situations and events that occur in his day-to-day activities. Once the child learns how to relate a situation to its mathematical representation, he can see new meaning in it and begins to exercise some control over it. In so doing, he learns to change or modify it so that his needs and purposes are better served.

Mathematics instruction should emphasize strategies for relating events and situations to their mathematical models. For example, we help a child to estimate numbers, distances, and capacities and to apply these estimating abilities to other situations in his real world. We teach him to develop criteria for comparing two lengths and for ordering a series of objects according to length. We introduce him to certain aspects of the coordinate plane, assist him in moving about on it, and then deliberately relate such moves to the trips he makes through his own neighborhood; the regularities of the coordinate system when applied to his real world make the child's movements in that world more controlled and easier to modify.

In order to help the child see the relationship between day-to-day events and their mathematical structure, we must somehow develop for him problems that are derived from real-world situations and events that make sense to him. In such problems, a real situation occurs (or is created or simulated), and the child is required to make some modifications or impose some order on the situation, which can sooner or later be summarized in mathematical form.

Consider, for example, the following situation An empty egg carton that holds twelve eggs is on a table. A child wants to fill the carton from a basket of eggs on the floor. He can, of course, fill the carton by putting eggs in it one at a time, thereby taking twelve trips to accomplish the task. He could increase his efficiency by taking two eggs at a time. Or he could further increase his efficiency by making what may be thought of as a nonmathematical modification on the situation—he could move the carton down beside the basket. In any event, the problem provides him with the opportunity of seeing that there is a definite relation between the number of eggs the carton will hold, the number of eggs taken in each trip, and the number of trips required. Eventually, of course, the mathematical model for the situation will be fully developed, and the child can use it to predict precisely how many trips are associated with any particular number of eggs per trip for a carton of any reasonable size. However, once he has gained this measure of control over the situation, it is no longer a problem for him.

It is not the purpose here to discuss at any great length the steps the child goes through in solving a problem. It is more important to observe the behavior of the child in problem situations and to make inferences from his actions about how he is thinking. Our task is to create and select significant problems for the child. To do this, we must know the characteristics of a good problem.

some characteristics of good problems

Perhaps some popular misconceptions about the nature of problem solving should first be dispelled It is quite common even today to hear someone claim that a child faced with pages of computation exercises is solving problems. This activity is not problem solving, for practice exercises do not establish any kind of relation between a real situation and its mathematical model.

Another common activity that goes under the name of problem solving is the verbal description of a situation for which the child is expected to write a mathematical model. Consider the following example:

> Jim had some shells. His sister gave him 4 more. Now Jim has 11 shells. How many shells did Jim have at first?

The child's response to this verbal description may be something like this:

$$\square + 4 = 11$$
$$7 + 4 = 11.$$

Jim had 7 shells at first.

In the early stages of thinking about the verbal descriptions and their mathematical structures, the child may indeed be involved in a form of problem-solving activity, particularly if the described event can be simulated in a realistic way. In the later stages, when he is required merely to write down mathematical sentences that fit contrived verbal descriptions of imagined events, his involvement in a problem-solving sense is likely to be minimal. This is not to say that associating the verbal description of such an event, however contrived, with a mathematical description of the event is undesirable—it is an essential form of mathematical activity—but it is better defined as mathematical modeling than as problem solving.

If the purpose of problem solving is to help the child learn to see a relationship between an event and its mathematical model, what then are the characteristics of a good problem designed for young children?

1. *The problem should be of significance mathematically.* The potential of a situation as a vehicle for the development of mathematical ideas determines whether we choose one particular problem situation over another.
2. *The situation in which the problem occurs should involve real objects or obvious simulations of real objects.* The problem must be comprehensible to the child and easily related to his world of reality.

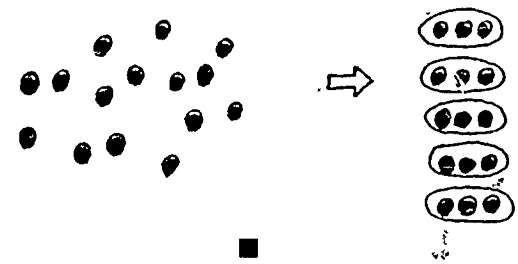
3. *The problem situation should capture the interest of the child.* This is done by the nature of the materials, by the situation itself, by the transformations the child can impose on the materials, or by some combination of these factors.
4. *The problem should require the child himself to move, transform, or modify the materials.* It would be difficult to overemphasize the role of action in early childhood learning. Most of the mathematical models at this level are what might be called "action models."
5. *The problem should offer opportunities for different levels of solution.* If the problem allows the child to move immediately from the problem situation to an expression of its mathematical structure, it is not a problem.
6. *The problem situation should have many physical embodiments.* Whatever situation is chosen as the particular vehicle for the problem, it should be possible to create other situations having the same mathematical structure. It may not be possible for a child to generalize a solution to a certain kind of problem until the problem has come up in a variety of situations.
7. *The child should be convinced that he can solve the problem, and he should know when he has a solution for it.* If the child is somehow required to react with, or transform, materials used in the problem situation, it is usually easy to determine whether the problem meets the criterion or not.

Two examples of good problems for young children follow. They illustrate the characteristics listed above.

sample problem 1

The process of division is usually a source of difficulty for the young child. The difficulty could possibly be eased if at an early age the child could be provided with problems involving actions that could be associated with the division process. For example, the following problem is designed to help the child see the relation between certain actions he must carry out in real-world situations and the process we call division. It is apparent that the first characteristic of a good problem—that it have some mathematical significance—is present in division problems.

Let us examine the basic structure of a situation related to division. Suppose we have fifteen objects and we wish to know how many groups of three objects each can be formed from them. We would regroup them as shown in ■.



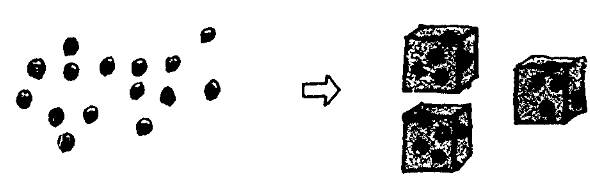
■

The action involved is usually a successive subtraction of groups of three until all the objects in the original group have been used up. The mathematical statement of what has happened is usually written as follows:

$$15 \div 3 = 5$$

The 5 refers to the number of groups of three. Here, division is used in what is called a *measurement* situation.

But another type of situation also uses division. Consider, for example, the same fifteen objects, but suppose this time the objects are to be placed in three boxes so that each box contains the same number of objects, as shown in ●.



●

● shows the situation, but in order to complete the action and answer the question, the objects would have to be partitioned among the

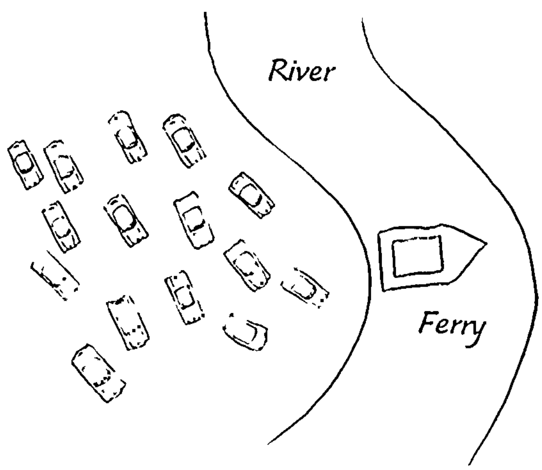
boxes. This partitioning, or sharing, would continue until all the objects were in the boxes. The sharing action would ensure that each box contained the same number of objects. Again we would write

$$15 \div 3 = 5.$$

The 3 in the mathematical statement refers to the number of objects used up in each round of the sharing. The 5 refers to the number of objects in each box. Here, division is used in a *partitive* situation.

The mathematical structure of such situations is clear, and it is also evident that children often get into play situations that would involve these division actions. However, if we design problems for the child around such situations, they must be more interesting than those that have appeared in our discussion so far. Thus it remains for us to devise a situation that would appeal to the child.

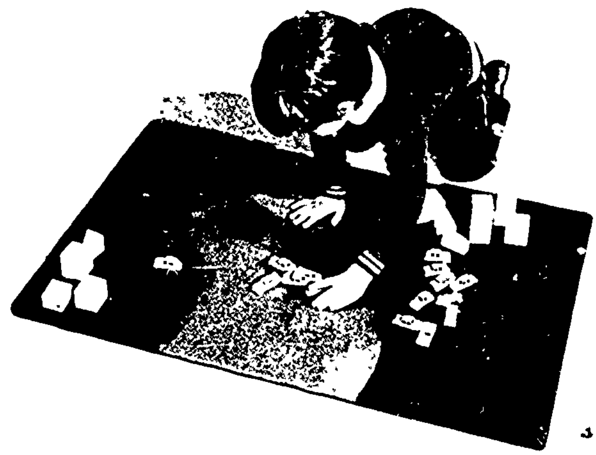
Suppose we collect fifteen toy cars and simulate on a table model a river on which a ferry moves back and forth. (See ■.) The child might be asked a problem like the following:



■

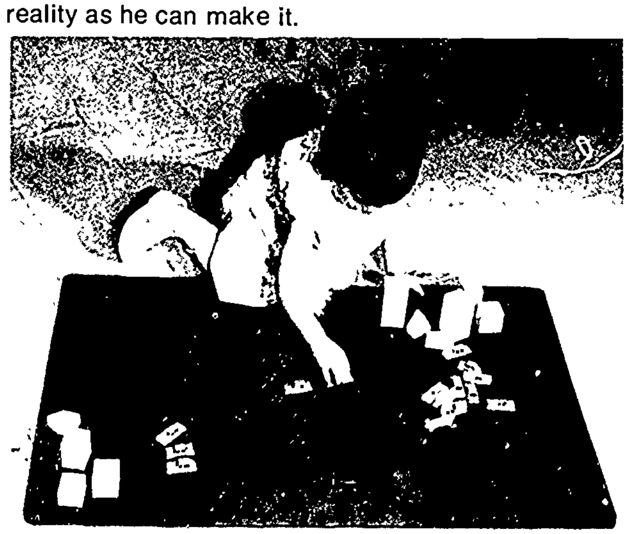
> ferry must be used to get these cars across the river. If it can carry only three cars at a time, how many trips must the ferry make to get the cars across?

Even at the age of three, a child shows some understanding of the problem and is interested in performing the actions that will get the cars across the river. It is common, however, for a child this young to forget the requirement that



the ferry must carry three and only three cars at a time. He may load one car and ferry it across, and for the next trip he may pile on five cars, as the three-year-old in the photograph is doing. Yet some measure of control over the situation is indicated by this kind of solution: the child almost always uses the ferry and somehow gets the cars across. But it is obvious that the three-year-old still has many facets of the problem to learn.

In general, the four-year-old solves the problem at a much more advanced level. He seldom forgets, for example, that the ferry must carry exactly three cars. However, he often becomes so involved in making the action completely authentic that he forgets the question he set out to answer. For example, here the child has carefully turned the ferry around and has driven each car on over the rear of the boat. Likewise, the unloading is just as authentic and as close to reality as he can make it.

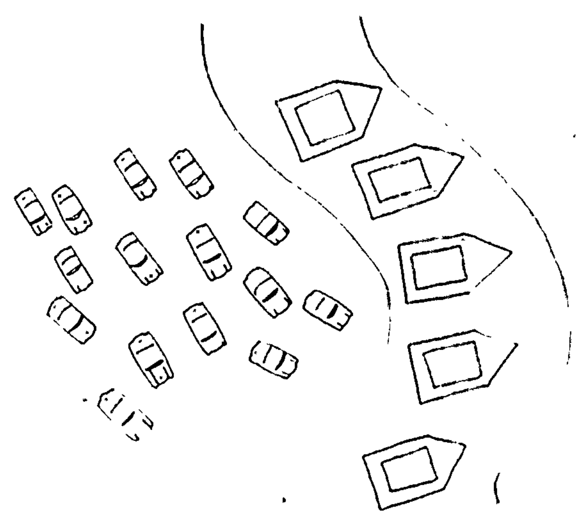


A five-year-old tends to give less emphasis to the action and will generally answer the question about the number of trips. This child had to perform all the actions, but he could remember in the end how many trips were required to get the cars across.

By the time the child is six he may have lost interest in going through all the motions. He tends to give his full attention to what is required in the problem. One six-year-old asked, "May I get the cars ready?" She proceeded to group them in threes, and then said, "I would have to make five trips." Her only action was grouping the cars; not one car was physically put on the ferry. This child had stripped the problem of all actions except those absolutely essential to the solution. When a child has reached this stage, the next step is to help him learn how to write the mathematical model for the situation. The timing of this last step is left to the wisdom of the teacher.

With only slight modification, this problem can be changed into a partitive rather than a measurement situation. Partitive situations are often more difficult for the child to understand. Although he may have had extensive experience in sharing, he has not usually generalized the action.

In ■, the diagram and its problem suggest a partitive situation for division.

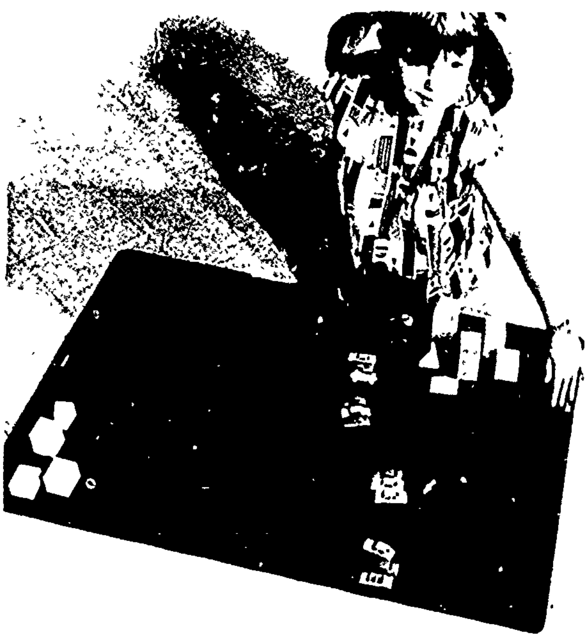


■

These five ferries can all be used at once to move the cars across the river. Each ferry must carry the same number of cars. How many cars would be on each ferry?

The three- or four-year-old child has difficulty seeing how sharing would work in this problem. He can use only trial and error to equalize the number of cars. Even if the number of cars and the number of ferries were reduced, the three- or four-year-old child does not have a strategy for solving this type of problem.

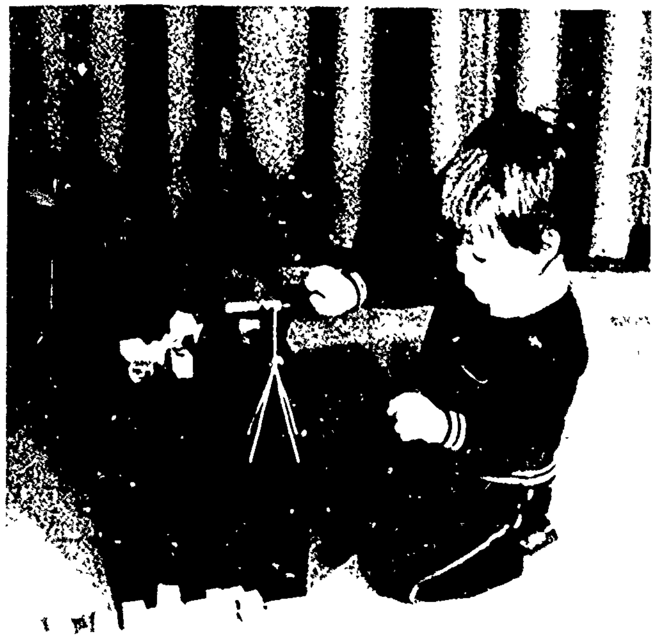
The older child with some instruction begins to see that sharing is not an action restricted to a narrow range of experiences. He usually partitions the set of cars among the ferries if he is given a start. Once he uses sharing in this situation, he seems to be able to take it out of simple sharing situations and use it readily to solve any partitive problem. The girl here is using partition to get the loads ready for the ferries to take across.



It can be seen that the problem presented here meets most of the requirements of a good problem:

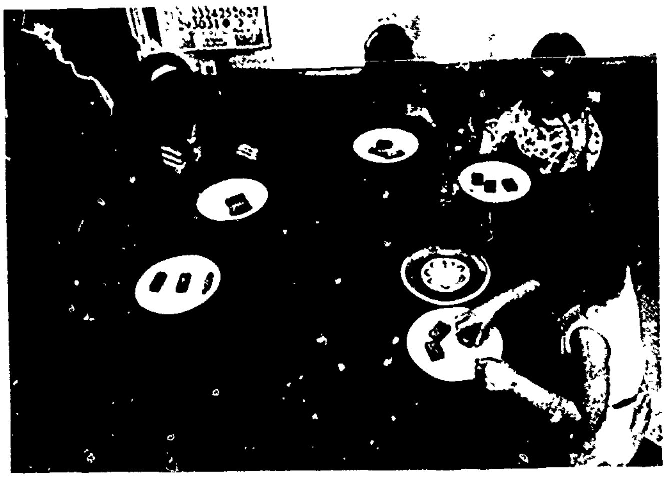
- It is significant mathematically.
- It involves real objects.
- The child is interested in the problem.
- The child must make modifications in the situation.
- Several levels of solution are available.
- The readiness with which the child attempts a solution indicates that he is convinced he can solve the problem.

One important requirement, however, has not yet been satisfied. In order for the child to be able to generalize division solutions, he must be given varying problem situations with the same mathematical structure. The following examples illustrate the ease with which teachers can vary the problem situation:



The builder wants to move all the bricks to the top of the building. The tray on the crane can hold only 5 bricks. How many trips will it take to get all the bricks up?

There are 5 trucks to move the crates from the terminal to the warehouse. Each truck carries the same number of crates. How many crates will be on each truck?



Each girl can take 3 cookies from the tray. How many girls can get 3 cookies?

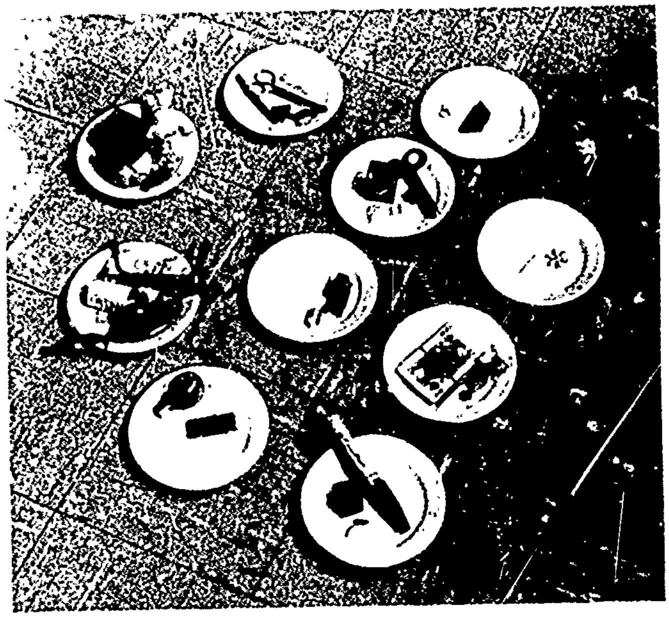
The 30 students in this class are going to a picnic in 6 cars. Each car is taking the same number of students. How many students will go in each car?

It is important that the child be faced with as many variations as he needs to generalize the actions associated with division. The total number of objects and the number in subgroups can be varied as easily as the type of objects.

Situations in which there is a remainder have not been suggested. Problems that help the child think about how to handle remainders could form another kind of variation to be used later.

sample problem 2

It is important that the child become flexible in his ability to classify. Classifying objects on the basis of some physical attribute is a common activity not only for young children but for adults as well. The ability to form classes on the basis of number, however, requires the child to ignore all physical attributes of the objects in the classes. Consider the following photograph and problems.



You must not move anything that is on a plate to another plate. Try to sort the plates so that there are two or more groups of plates. Explain why you grouped the plates together as you did.

If the child has a rudimentary idea of one-to-one correspondence and is flexible in his ability to classify, he will sort out the plates somewhat in the manner shown, which would enable the teacher to determine the child's ability to recognize sets that have the same cardinal number. This is not to say, however, that the exercise is a good problem.

A better problem in classifying could be provided by supplying the child with a fairly large collection of objects that can be sorted in a variety of ways. For example, attribute blocks, which are commercially available, can be sorted by color, shape, size, thickness, texture, and so on. Appropriate problems in classifying do not necessarily depend on such sets; they can be built around common materials like beads, buttons, spools, blocks, containers, and a host of others.

For example, a child is provided with a pile of objects made up of a set of farm animals, a set of plastic cars, and a set of prehistoric animals.

> Sort the things into two piles. Explain how you did it. Next make three piles, then four piles. Now make as many piles as you can There must be at least two things in each pile when you have finished. Can you explain how you sorted them out as you did?

One solution to the problem by a five-year-old child is shown here.

It is not difficult to see that a great variety of problems like this can be designed from readily available materials Such problems should greatly increase the child's flexibility in recognizing and forming classes.

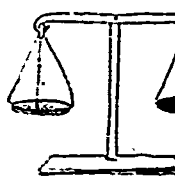
selected problems for early childhood

The characteristics of a good problem have been outlined, and examples have been used to show how problems can be generated with these characteristics as a guide. The next task is to select a sample of good problems for early childhood and describe a setting for them. For convenience, the selection will appear in two parts--those problems appropriate for the beginning stages of early childhood, covering approximately the ages of three to five years, and those problems for the older child, approximately five to eight years of age.

problems for the beginning stages

comparing and ordering weights

Materials: A beam balance
Six or eight clay balls that look identical
A steel ball to conceal in one of the balls of clay



Problem: Use the balance to find the ball that is heavier than the others.

Variations:

► Collect a variety of objects that differ markedly in appearance and size but only slightly in weight, and have the child use the beam balance to find the heaviest object.

► Have the child use the beam balance to order a collection of objects from heaviest to lightest.

► Have the child use the beam balance as an aid in making one ball of clay just as heavy as another one.

► Give the child a ball of clay or other object and a handful of identical steel washers and have him use the beam balance to find how many washers would weigh about the same as the ball of clay. (NOTE: Spring scales should not be used in these problems for two reasons: first, the child must read the number on the scale; second, the spring scale is not a direct indication of weight, whereas the balance is.)

making inferences

Materials: A metal container with a cover that can be sealed
Six beads (four black, two white)
A length of masking tape for sealing the can

Problem: What is in the can?

For the basic problem, the six beads are put in the container, and the cover is affixed but not sealed. The child should be permitted to manipulate the container in any way he pleases, but he should not remove the cover. Then let him open the container and draw out only one bead. He must not see the others, and he must replace the bead after he has looked at it. Allow several draws and get the child to make conjectures about the contents after each draw. In making his responses, the child tends to jump to conclusions. For example, if he draws a black bead on the first draw, he is likely to conclude that all the beads are black. The object is to get him to modify his conclusions as he gains more information. In this problem he should come to the conclusion after repeated drawings that there are some black beads and some white beads and probably more black beads than white ones.

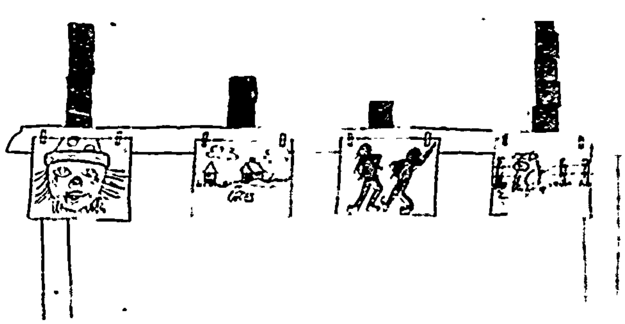
Variations:

► Place a single object in the container and have the child make inferences about its nature after he has manipulated the container.

► Place an object in the container and seal it with masking tape. Leave it in the classroom for a day. Each child should be encouraged to handle the container and to make conjectures about its contents.

rudimentary graphs

Materials: A collection of pictures representing television programs
One 1-inch cube for each child in the classroom
A table to which the pictures may be attached



Problem: What are the favorite television programs of students in this classroom?

To solve the problem, each child places his block above the picture representing his favorite program.

Variations:

▸ Use a piece of graph paper with 1-inch squares and have children take turns coloring a square over a picture of their favorite pet, food, game, color, and so on.

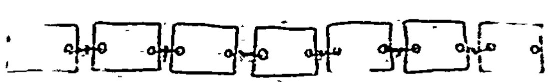
▸ Have the child keep records of weather with the same kinds of charts; for example, he could determine whether the month of March has more cloudy days than sunny days.

▸ Have the children prepare graphs that show the number of people in their classroom with blue eyes and the number with brown eyes.

▸ Draw graphs or charts to record information about children in the classroom: their birthdays, hair color, color of toothbrush, color of socks, size of families, ways of coming to school, distance from school, time taken to get to school, height, reach, and so on.

patterns in number

Materials: A train of cardboard rectangles fastened together with paper clips, string, or some other device



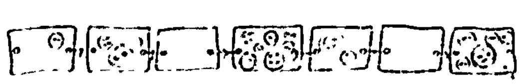
A collection of small counters such as cubes, beans, buttons

Problem: Make a pattern of counters on the train so that all the counters you have are used up.

Suppose there are seven cars and you gave the child eleven counters. One solution might look like this:

Variations:

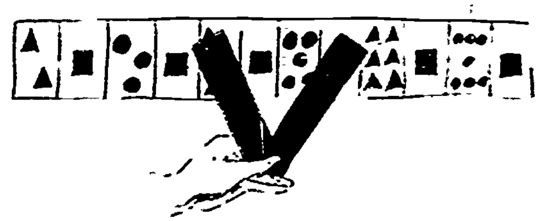
▸ Place counters on the cars as shown:



Give the child additional counters and ask him to put the correct number of them on the empty cars. This problem could be arranged so that many alternate solutions are possible.

▸ Provide colored shapes and set up a pattern that the child must continue. A simple example might look like this:

▸ Rule off a strip of cardboard into a series of squares and put counters or shapes or colored squares or some other objects on all the spaces in some pattern. Then cover some spaces and have the child determine what is on those spaces.



spatial relations

Materials: Several sets of seven 1-inch cubes

Problem: How many different buildings can you make, using seven blocks in each building?

Here are some sample responses:

Variations:

▸ Vary the number of blocks in the sets, or vary the rules: for example, all the buildings must be L-shaped.

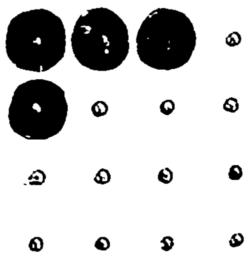
▸ Once the buildings are completed, have the child group them in classes.

▸ Take a set of blocks and construct a building; ask the child to build one just like it.

patterns in the coordinate plane

Materials: A geoboard or spoolboard at least 4 × 4

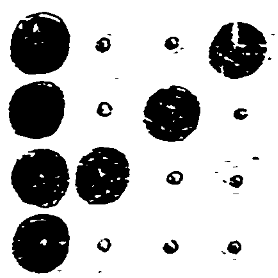
At least sixteen colored beads with large holes or colored spools



Problem: Can you complete the pattern?

Variations:

► Give one child red beads and the other blue ones. Have them move in turn. The first child to get four in a row is the winner. The sketch shows a win for red. (It is important in this problem to develop a strategy for blocking an opponent.)



► Use the geoboard as a parking lot and let the child code the places on the board as if they were places to park. He gives the code (for example, fourth car in red row) to another child who then tries to "park" in the right place.

► Have the child use beads of three colors to construct as many patterns as he can on the board. Require that all places on the board are to be used up.

many-to-one correspondence

Materials: A supply of paper plates or flat boxes.

A supply of sugar cubes, beans, or colored shapes

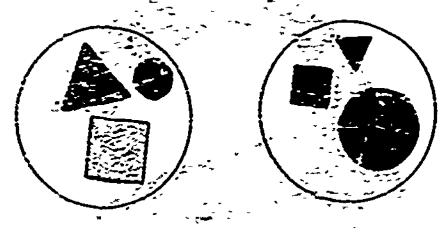
The child is given five paper plates and fifteen sugar cubes.

Problem: Are there enough sugar cubes so that you can put two on a plate? Can you put four on each plate? On how many plates can you put five sugar cubes?

Variations:

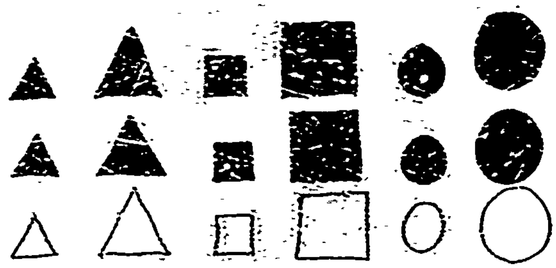
► Give the child five plates and seven each of circles, triangles, and squares. First ask him to find one of each shape for each plate and then to get as many more plates as he needs to use up all the shapes.

► Let each plate contain only one triangular shape, one square shape, and one circle. Ask if there are the same number of shapes on each plate. The shapes should vary in color and size.

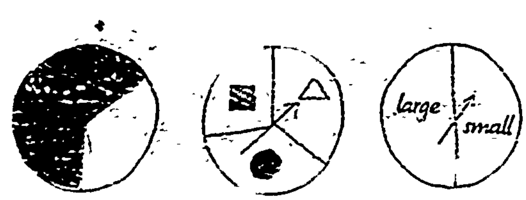


a problem with shapes

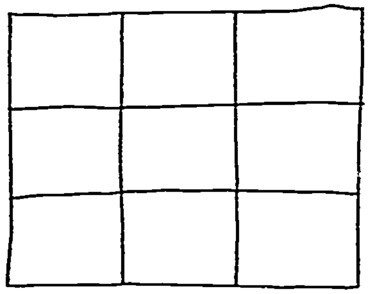
Materials: A set of shapes as follows:



Three spinners like these:



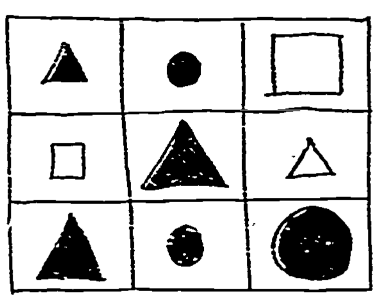
A set of cards ruled to make a playing board:



Have the child flip the spinners in turn. choose the piece indicated, and put it on the board wherever there is an empty space. (The dials on the diagram shown indicate a small red triangle.) Then have him spin again and place that object in an empty space. This goes on until he has objects that are in some way alike in a row, column, or diagonal. He can then record a win.

Problem: Place the pieces on the board so that you can win in the least number of flips on the spinners.

The diagram shows a win on both diagonals. (The pieces on the top right-to-left diagonal are all large. The pieces on the top left-to-right diagonal are all red.) The board does not have to be full to declare a win.

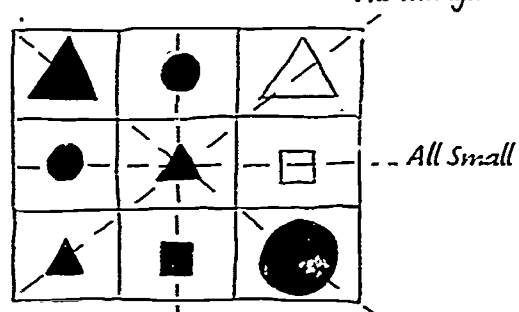


Variations:

► Have the child use only one spinner, for example, the one indicating color. He can choose an object of any shape or size to put on the board, so long as it is the color the spinner indicates. Again, it is up to the child to choose where he will put the piece, but once he has chosen a piece and a position for it, it cannot be changed.

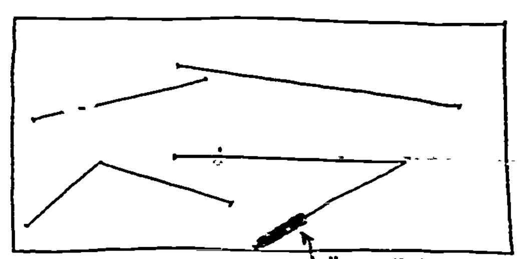
► Have the child try to arrange pieces on the board so that all the spaces are filled and all the objects in the rows and columns are alike in some way.

► Have the child start with a particular piece on the center space of the board, for example, the small red triangle. The problem is to complete the rows and diagonals so that the pieces in each are alike in some way. Here is one possible solution:



rudimentary measurement

Materials: Sheets of paper on which line segments of various lengths have been drawn
A supply of beans, candies, sugar cubes, or blocks to be used as units of measure



Problem: On which line could you put the most beans?

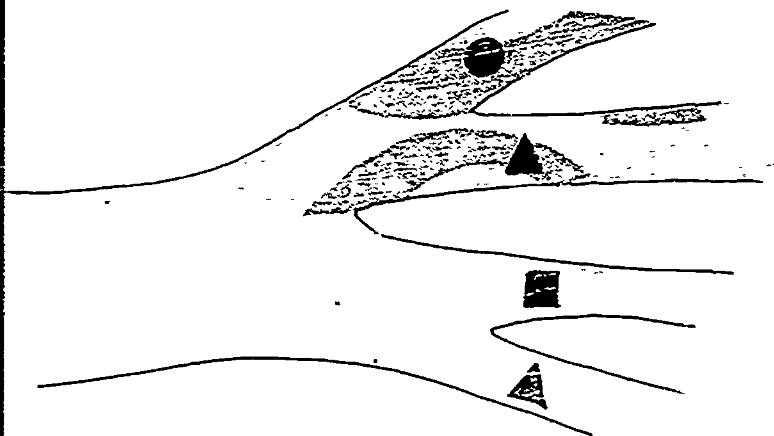
Variations:

► Ask the child to try to draw a line segment that will hold just five candies.

► Ask the child if there is a way of putting more beans on the line. (For example, he can change the unit by laying beans across the line instead of end to end, or he may stand candies upright if he uses candies of the right shape.)

a sorting device

Materials: Various diagrams of branching roads



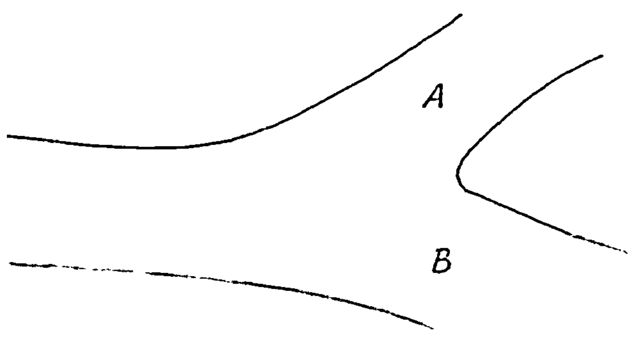
Shapes used in the previous problem

A red sign means only red things can pass, a O means only circles can pass, and so on.

Problem: Can you take each shape down the road and show where it comes out?

Variations:

► Draw a path with chalk on the floor. As they file through, five-year-olds take path A; all others take path B.



► Give these directions. If your name begins with A, B, C, . . ., K, take path A. If it begins with any other letter, take path B.

► Give directions such as this: If you are a boy and blond, take path B, otherwise, take path A.

► Let the child make more complicated sorters: he can make his own paths and the rules for following them.

The problems above and their variations represent only a small sample of all those that might have been selected, and few of them are completely original. It is hoped, however, that these situations will prove helpful to those wanting to provide good problems for young children.

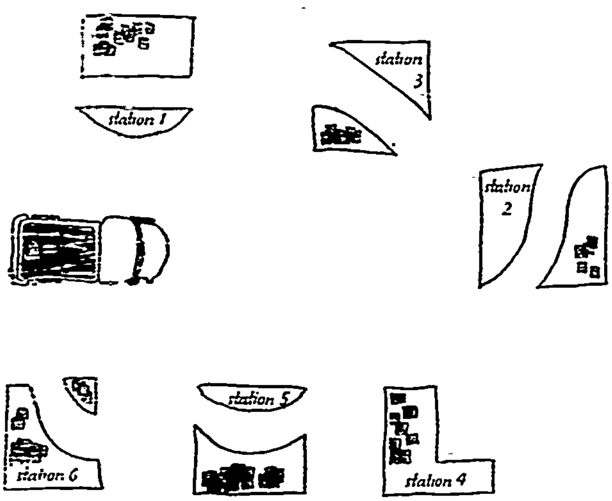
problems for the later stages

In the ages from five to eight, the child shows increasing ability to make estimates, to keep a record of his observations, to make statements of a mathematical nature that fit events or situations, and even to make some decisions on the basis of what may happen. This does not mean that the problems should be more abstract or that their solution should depend less on the actual manipulation of things in the real world. It does mean, however, that more attention should be given to having the child keep written records of the observations and maneuvers that are involved in the solution of problems. A child will learn through appropriate problems how to express as written statements mathematical models related to many aspects of reality. For example, when he sees a group of five birds join a group of three birds on a fence, he may think 3 + 5 = 8. Or he may be

able to calculate the number of 10-centimeter square tiles needed to cover a 60-by-70-centimeter area by thinking or writing $6 \times 7 = 42$. It is with these developments in the older child in mind that the following selection of sample problems is presented.

loading and unloading

Materials: A toy truck
Twelve or fifteen blocks or objects that will all fit in the truck at once
A table top that will accommodate five or six (or more) stations for the truck.

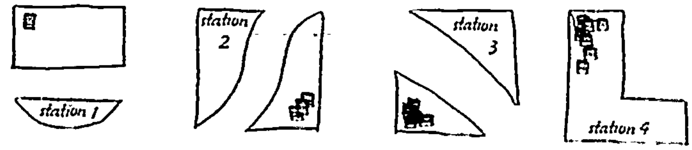


Problem: At how many stations can the truck pick up three blocks if the truck holds twelve blocks?

Variations:

► Set up the problem so that the driver starts with eight blocks and drops off three blocks at every second station. At the first station and every additional one after that he picks up two blocks. The question is how many stations could he visit before the truck is empty. The number of blocks to begin with, the number dropped, and the number picked up can all be varied.

► Have twelve blocks (for example) in the truck at the beginning. The driver drops off one block at the first station and at each successive



station two more blocks than at the station before. Ask the child to determine at which station there will not be enough blocks to make a drop. The number of blocks on the truck can be varied, and the child can keep a record of the number and of which loads have just enough blocks for the last drop.

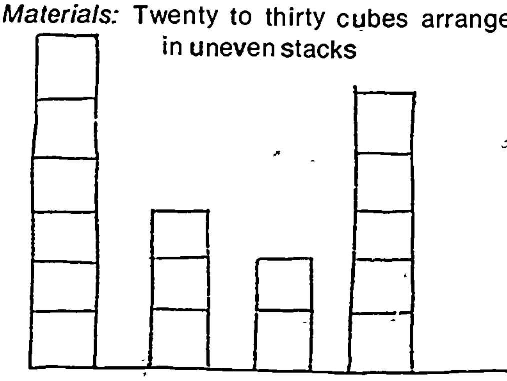
► Have the child pick up two blocks, for example, at each station and predict how many blocks will be on the truck after four stops. He could write down the number of blocks in his truck after two stops, four stops, six stops, and so on.

► Permit the child to devise his own schedules for picking up and dropping off blocks and to keep a record of his findings.

► Using different materials, have the child gather things off plates around a table or serve things in some kind of progression or pattern.

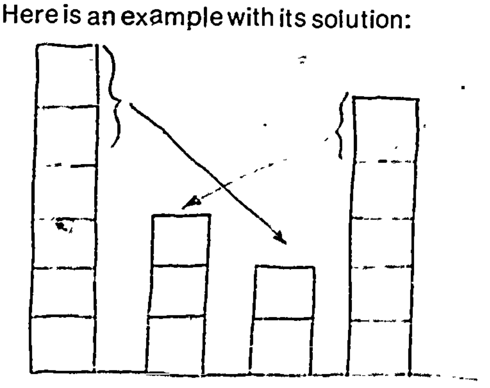
making stacks

Materials: Twenty to thirty cubes arranged in uneven stacks



Problem: Make all the stacks the same height in as few moves as possible.

Here is an example with its solution:



Variations:

► Present a different arrangement of stacked blocks, for example, 7, 3, 3, 3, which must be equalized in the least number of moves.

► Give the child 18 blocks and arrange them with 6, 4, 2, and 6 blocks in the four stacks. Then ask him to make all the stacks the same height in the least number of moves. (Of course, there is no way to solve this problem. The child soon learns that making all stacks the same height requires that the total number of blocks be some multiple of the number of stacks.)

► Vary the number of stacks with which the child works. With proper record keeping and flexibility in stacking, the child can discover the factors of many composite numbers, though the word *composite* need not be introduced. He can discover, too, that if the total number is 7, 11, 13, 17 (prime numbers), there is no way to make equal stacks.

► Start with four stacks and continue to require the child to make equal stacks with the least number of moves, but allow him to change the number of stacks as he thinks necessary. For example, if the stacks number 3, 4, 3, 2, he would keep four stacks because one move would be sufficient to make them all equal. If they number 3, 1, 3, 3, he could change to either two or five stacks because the 10 cubes cannot be made into four equal stacks. It would take the same number of moves to make two stacks of the same height as it would to make five stacks of the same height. Or, when they number 2, 4, 3, 2, the problem cannot be solved if there is to be more than one stack and at least 2 cubes in a stack.

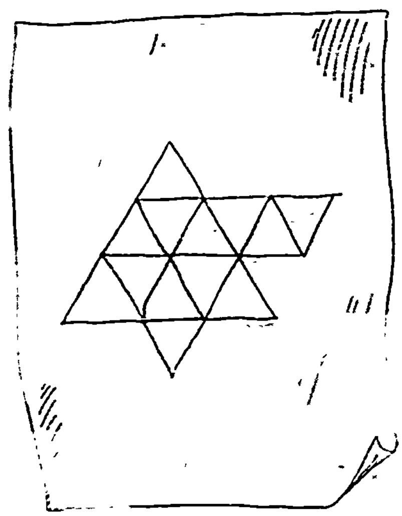
tessellations

Materials: A sheet of colored paper, 8 1/2 inches by 11 inches

A variety of regular and irregular shapes (triangles, quadrilaterals, pentagons, hexagons, rectangles, squares), with enough of each to cover the sheet of paper. The area of each shape should be between one and two square inches.

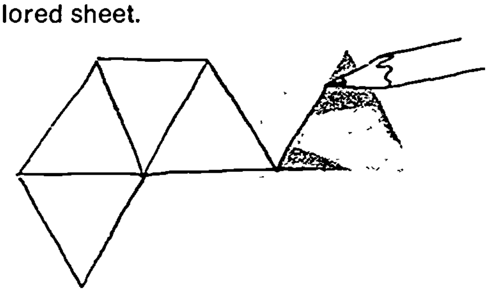
Problem: Pick out all the shapes of one kind. Can you put these shapes together so that you can cover the colored sheet of paper without leaving any gaps or spaces?

The beginning of a solution using equilateral triangles is shown here.



Variations:

► Instead of using a large number of units, prepare a template of the desired shape; the child can then trace around this shape to fill the colored sheet.

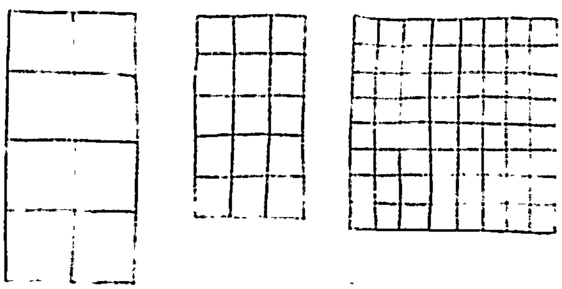


► Permit the child to make his own shape for the tessellating.

► Use the process to compare the area of two dissimilar shapes: each shape is covered with units, and the numbers of units required to cover each one are compared.

estimating

Materials: Several sheets of paper of varying dimensions squared off in equal units



—Problem: **Estimate the number of squares in each sheet. Do not count all the squares—try to find a shortcut that will get you close to the correct number without having to count every one.**

Variations:

► Ask the child to estimate the number of beans in a jar. He should be provided with materials such as measuring cups, scales, rules, other quantities of beans, and so forth. The object is for him to make progressive improvements in the estimate.

► Have the child make estimates of the length of the classroom, the playground, the gymnasium, the number of windows in the school, the number of tiles in the floor or ceiling, his own height and that of other children, weights, what can be done in one minute, how long it takes to tie his shoes, and so on.

► Have the child keep a record of situations in which an exact count or measure is not available or even possible to get. Require only an estimate (attendance at games, population figures, cost estimates, time and movement estimates, crowds, etc.).

► Require estimates in situations like the following:

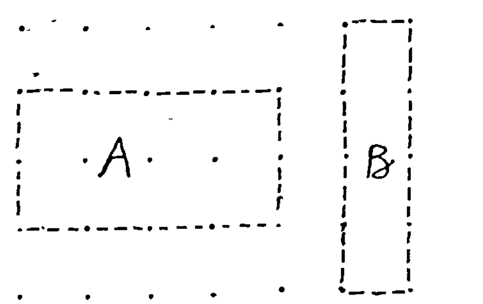
a) Is two hours enough time to drive from New York to Washington?
b) Plastic cars can be bought for 35 cents, 20 cents, and 25 cents. A child is given a dollar and told he can buy four cars. Which can he buy? Or the child can buy as many cars as he wants, but he can spend only 50 cents. Which can he buy?
c) Is this shelf long enough to hold the books piled on the table?
d) How long will the batteries last in this flashlight?

area-perimeter relations

Materials: A geoboard (at least 6 x 6) for each child
A supply of rubber bands
Paper for recording

Problem: How is the distance around a rectangle related to the number of squares inside?

A sample procedure might work like this:



The dotted line around A represents a rubber band. The child counts the units around (perimeter) and the number of squares inside the rectangle (area) and keeps a record of the count. In this case, the perimeter is 10 and the area is 6. The child then tries to change the shape of the rectangle, keeping the same number of units in the perimeter. For example, he might change to a one by four (B), which would have the same perimeter (10) but a different area (4). A variety of situations would have to be investigated before the relation would be apparent.

The child will find, of course, that the greatest area will occur when the shape is nearest to being square. In the example chosen, a square is not possible, but the area is greater when the dimensions are 2 x 3 than when they are 1 x 4. The investigation will undoubtedly lead the child to situations in which the square is a possibility (i.e., the number of units in the perimeter is exactly divisible by 4).

The child may decide to keep the area the same but to change the shape of the rectangle and then see what happens to the perimeter. It

will be found that the perimeter is least when the shape is a square (or nearest to being a square). He may even come to attach some special significance to 4, 9, 16, 25 as areas, since they can be made into squares.

Variations:

► Have the child start with squares (2 × 2, 3 × 3, 4 × 4) and record the perimeter and the area. Then have him change the shape to a rectangle, keeping the perimeter the same and noting any change in area.

► Once the child becomes involved in the investigation, he will think of keeping the area constant and varying the perimeter. The child might require some assistance in keeping a systematic record. Verbal summaries should not be forced.

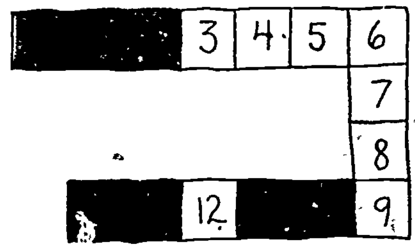
► Let the child make up a problem for the rest of the class, using the data he has gathered.

► Conduct an investigation to find the relationship between people's height, weight, reach, length of shadow, size of neck, waist, foot, ankle, wrist, and so forth. Other relationships could be investigated, such as the weight of materials in air and their weight in water.

investigations in probability

Materials: A pair of dice
Paper for recording
Fifteen 1-inch squares made of cardboard painted black on one side. The first square has a zero written on both sides, the second has a one written on both sides, and so on, until all the squares are numbered on both sides up to fourteen.

The child is taught a game procedure. The squares are laid out in numerical order, some with white sides up and some with black. Before he rolls the dice, the child chooses a color, either black or white. If the dice total a number of the color he chose, then he wins. Suppose he has chosen white and the dice show a 3 and a 2. The total is 5; so the child records a win because 5 is white. Suppose on the next throw he chooses black and the total on the dice is 8. He then records a loss, since 8 is white.



Problem: Try to learn the best way to play this game.

Variations:

► Change the squares so that a different proportion of black and white squares is used.

► Permit the child to arrange the squares as he likes, but there must not be more than eight of one color turned up.

► Allow the child to arrange the squares in any way he pleases to help him win.

(NOTE: The child should discover that numbers like 7, 6, and 5 are much more likely to occur than 2, 3, or 12. He should discover that some numbers—0, 1, 13, 14—can never come up. It is not expected, however, that he make a complete analysis of the probabilities.)

traffic counts

Materials: A timer
A sheet of paper for recording

Problem: **How many cars pass through the traffic lights at the corner?**

It is clear that the problem will have to be restricted in some way. For example, observations must be limited to a certain period of time, and "cars" must be explicitly defined. It is best to have the child impose those restrictions he thinks necessary. As the child gains experience in activities of this kind, he will improve his methods of tallying and recording.

Variations:

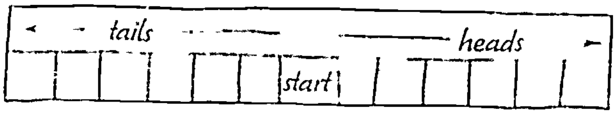
▸ Activities like this one do not have to be conducted entirely during school hours. For example, have the child compare the number of people entering a place of business during the week with the number entering on Saturday.

▸ Show the child how to measure pulse rate; he can compare rates after different forms of exercise and report the results.

▸ If circumstances permit, allow the child to watch as cars come to a gasoline pump; he could record the kinds of cars and the amounts of gasoline bought. The results could form a good basis for a report.

predictions of heads and tails

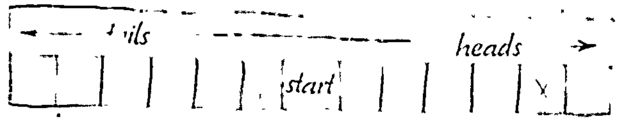
Materials: Three pennies
A tally board, which can be drawn on a sheet of paper and duplicated



Problem: Learn the best way to play this game.

Explain to the child that he will toss the three coins all at once and observe whether they come up heads or tails. Whenever a head comes up, he will move one place to the right on the board; when a tail comes up, he will move one place to the left. He will record his actual position on the board by writing the number of the toss (1 for the first toss, 2 for the second, etc.) in the proper space. Give the child some practice in tossing the pennies and observing the possible outcomes. Then ask him to mark with an X the place on the tally board where he thinks he will be after he has tossed the three coins ten times.

Here is how the first toss would be recorded if two tails and a head come up. (The most likely place for him to be after ten tosses, or any number of tosses, is back at START.)



Variations:

▸ This game can be played competitively between two or more players. Give each player 10 points at the beginning of the game. After each ten tosses, the child finds the number of spaces between where he is on the board and where he thought he would be and subtracts this number from 10. When a player has no points left, he is out of the game. The last player in the game is the winner.

▸ Vary the number of coins used in tossing.

▸ Scoring need not be based on the estimate but on the number of spaces from START. Do not require the child to make an estimate of where he will be after a given number of plays. The score is the actual number of spaces from START.

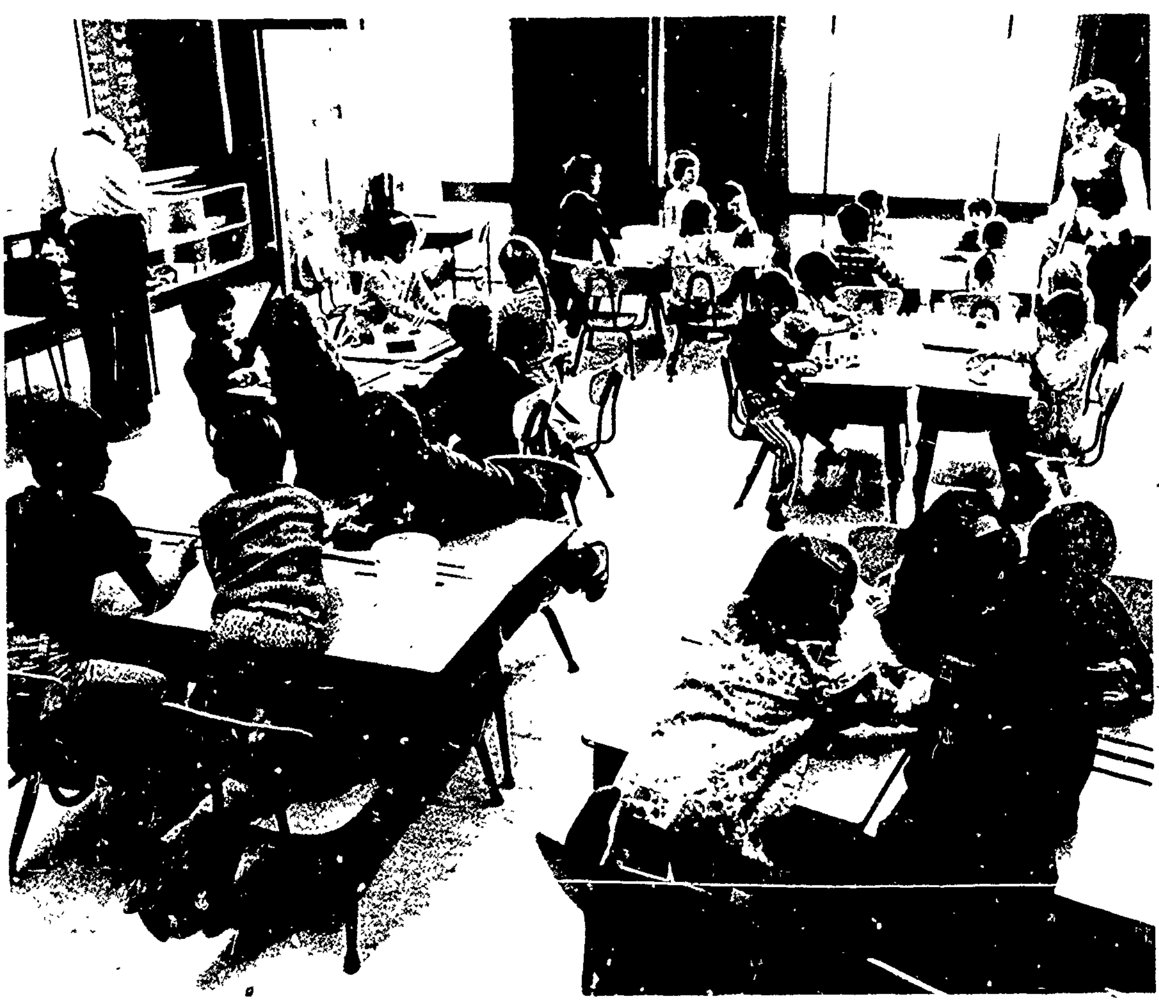
classroom organization for problem solving

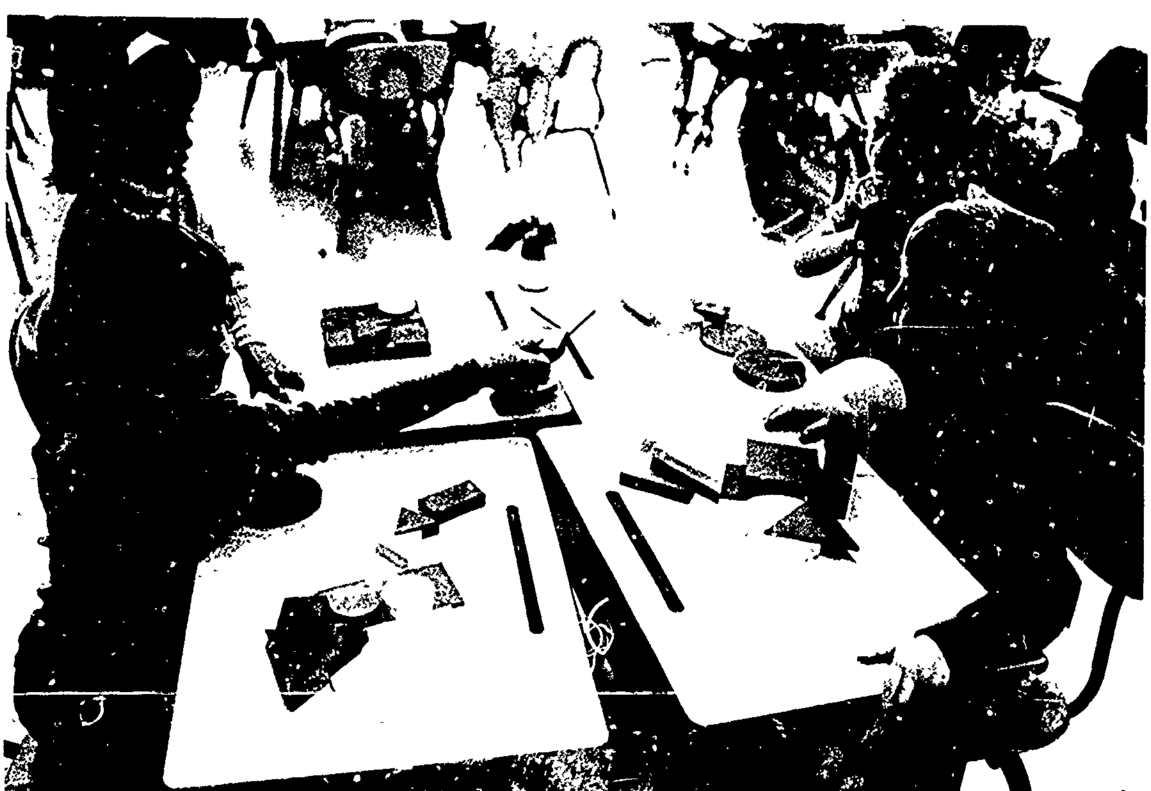
The problem examples introduced here have been described as if only one child were involved. It is highly unlikely that all problem solving could be arranged on a one teacher-one child basis in all the years of early childhood. Indeed, the development of the child's ability to deal with problems probably depends in some measure on his interaction with other children. In any event, it is important that teachers have some ideas on how the classroom can be organized for activities that might be called problem-solving sessions. It is equally important that teachers choose those approaches that are best adapted to their own interests and abilities as well as to those of the children they teach. There are many ways of organizing the classroom for such activities, but only three are presented here. Each involves a child working with other children either in pairs or in small groups. This is not to suggest that the child will always work with someone else, for there are many opportunities for individual work with problems—indeed, it is often needed.

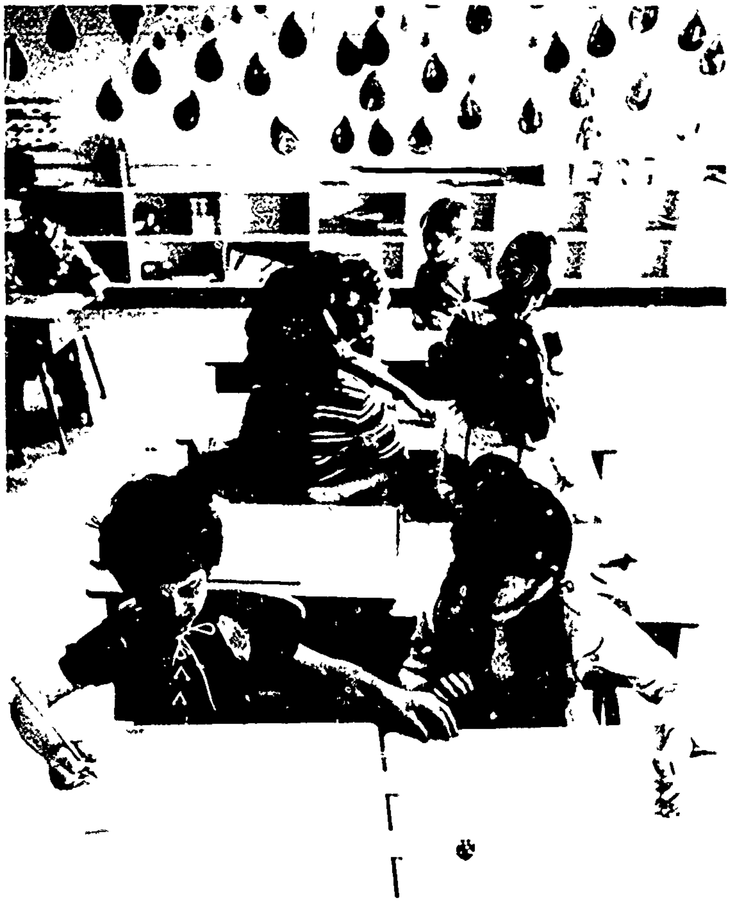
working in groups on the same problem

The photographs in the following series show a first-grade classroom organized into eight groups of about four children in each group. The groups are all working in the general area of classification. Some are working with attribute blocks and some with collections of toys or objects. However, all are working on the same problems, which were posed orally by the teacher as follows:

1. Sort out the objects on your desk so that you have two piles and so that each pile has things that go with one another.
2. Now sort the objects into three piles so that each pile still has things that go together.
3. Now try sorting them into four piles.
4. Everyone take an object by turns until the objects are all used up. Now, play a "one difference" game: the first person to take a turn puts out one of his objects. The next person must put out an object that is different from the first *in any one way*. The next person puts out an object that is different in any one way from the last object put out. Play in turns until one person has used all his objects.

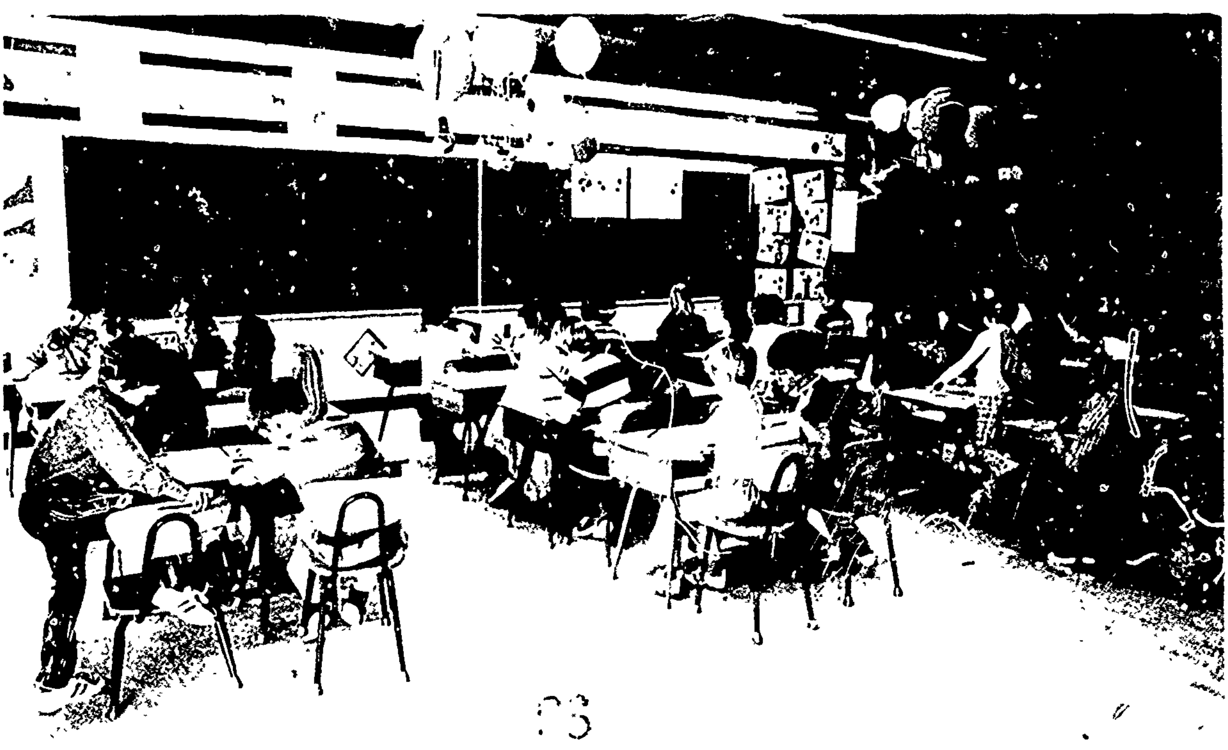






working in pairs on the same problem

The desks in the second-grade classroom pictured here are usually arranged in rows; so it was an easy matter to have adjacent rows moved together to allow children to work in pairs. This is often a convenient arrangement if materials are to be shared or if the children need partners for an activity. In the situation pictured here, the children are working on the "investigations in probability" problem, outlined earlier. The teacher has given the directions to the whole group and is thus able to spend time with pairs of children, watching, listening, and discussing.



working in groups, each group on a different problem

The children in the third-grade class shown here are working at stations on problems that differ greatly from one another. The problems are in the following general areas. tessellations. estimations. stacking. comparing and ordering weights. measuring with different units. traffic counts. and probability.

The problems for each group are on cards and are somewhat open ended. As a general rule. problem cards provide minimal directions. and children must *do* something. then *record* their results for class discussion later. The problem cards being used by this class are pictured below.

Each group works at a station for a specified length of time. then moves to another station in some orderly way. In this class. each group worked on a problem for one class period. This could be done on consecutive days or on one day a week until each group worked each problem card. The conclusion of the activity is in the form of a presentation and discussion by each group about its findings at one station. Some groups or individuals within groups will present fairly sophisticated solutions: other groups will present solutions that are quite rudimentary. The teacher may serve as a discussion leader or may let each group handle the discussion in its own way. The teacher generally helps the children sum up their conclusions and arrive at desired generalizations.

The actual problems this class is working on and the materials needed for each one are listed below.

Problem	*Materials*
1. Arrange the differently shaped tiles on the colored paper so as to leave no empty spaces.	—colored paper. about 8 1/2 inches by 11 inches —sets of geometric shapes—squares. circles. triangles. pentagons. hexagons. parallelograms
2. How many grains of puffed rice are in the	—jar containing puffed rice. sealed tightly

jar? (The actual number of grains of puffed rice is written in a secret place in the room. When all the groups have made estimates, you will be given some clues to help you find the secret place.)

—extra puffed rice
—extra jar
—ruler. length of string. small box. and small cup

3. Make four stacks of blocks with—
   6 blocks in one stack:
   3 blocks in one stack;
   2 blocks in one stack:
   5 blocks in one stack.

   Can you make all the stacks the same height in two moves? (You may move more than one block at a time if you wish.)

   Now make stacks of 7. 3. 3. 3 blocks. Can you make all the stacks the same height in three moves?

   Now make stacks of 6. 6. 1. 3 blocks. What is the smallest number of moves needed to make all the stacks the same height?

   Now make stacks of 6. 4. 2. 6. How many moves will it take to make all the stacks the same height?

   Make up some stacking problems of your own.

—18 wooden 1-inch blocks

4. Look at each of the things in the box, and match each with a name card. Guess what order the objects would be in from lightest to heaviest and put the cards in that order.

   Now use the balance scale to help you put the objects themselves in order. lightest to heaviest. How many did you guess correctly?

—balance scale
—box containing lead weight. crayon box. large piece of Styrofoam. pair of scissors. plastic toy. wooden block
—name card for each item in the box

5. Use the things on the table and see how many ways you can find the following measurements. Make a chart that shows your measurements.
   How tall are you?
   How high can you reach?
   How big are your hands?
   How big are your feet?

—ruler. tape measure. ball of string. jar of pennies. book. shoebox. large-squared paper
—paper for a large chart
—pens

6. List all the things with wheels that you think might pass the school on Fortieth Avenue. Then watch the road for ten minutes and keep a record of the things you see passing.

--paper for a large chart
--pens

7. You are going to toss three coins and keep a record of how many heads and tails you get. For every head. you must move one place to the right: for every tail. one place to the left. To record your position at the end of each toss. write the number of the toss in the proper space on your tally board. To work the problem. mark with an X the space on the tally board where you think you will be after you have tossed the three coins ten times and see how close you are after your ten tosses.

—three coins
—paper cup
—tally sheets

concluding remarks

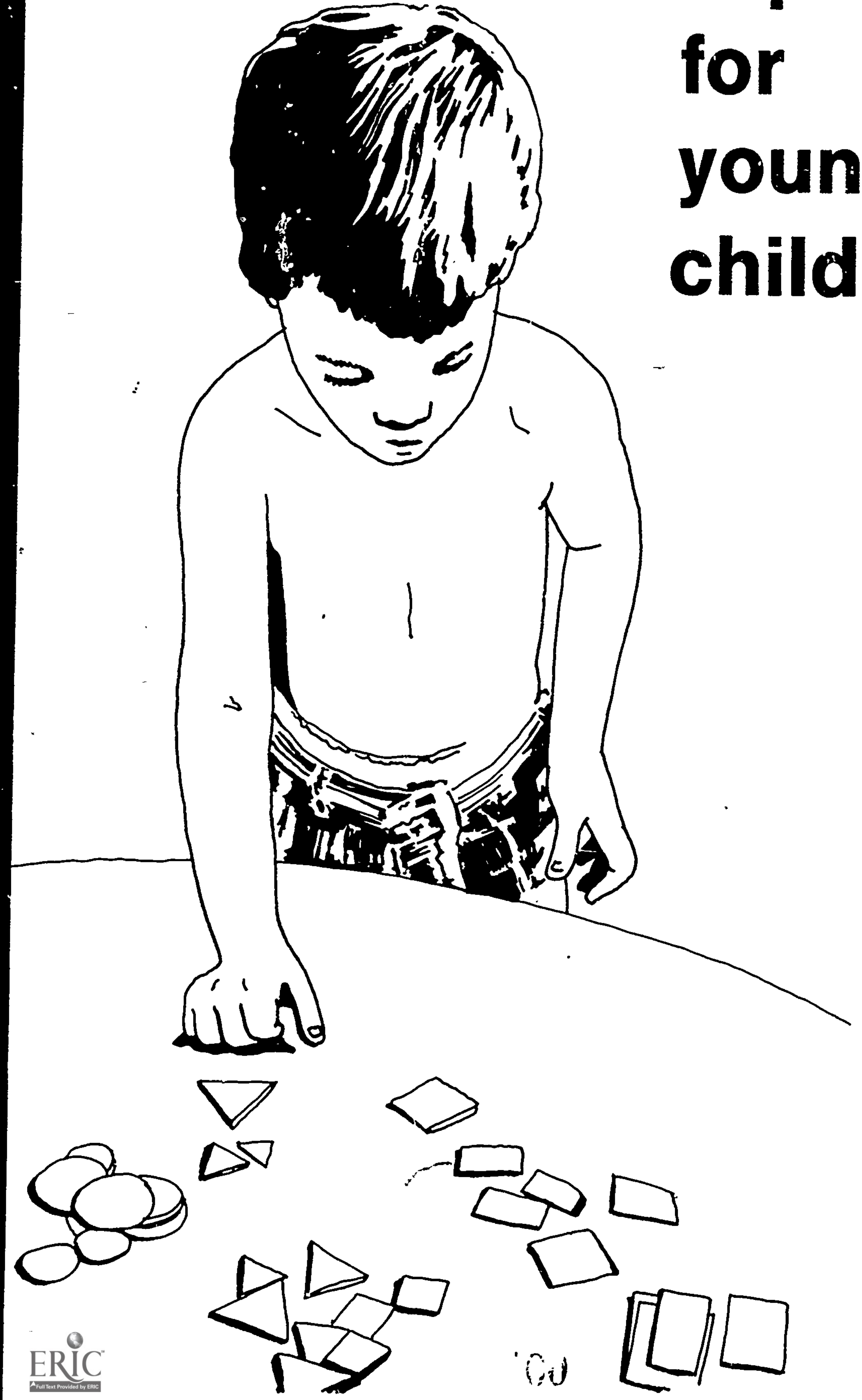
In early childhood, the main purpose of mathematics teaching is to help the child learn how to relate a situation or event that occurs in his daily life to a mathematical representation of that situation or event. Since the most effective way to accomplish this purpose is to provide carefully selected, significant problems, it is more important to consider the basic characteristics of good problems at the early childhood level than to trace the steps in the problem-solving process itself. These characteristics are summarized as follows:

1. The problem should have demonstrable significance mathematically.
2. The situation in which the problem occurs should involve real objects or simulations of real objects.
3. The problem should be interesting to the child.
4. The problem should require the child to make some modifications or transformations in the materials used.
5. The problem should allow for different levels of solution.
6. Many physical embodiments should be possible for the same problem.
7. The child should believe he can solve the problem and should know when he has a solution for it.

The problems included as examples were chosen because they met these criteria and because they have been effective in classroom and individual use.

When selecting or designing appropriate problems, it is very important that one have a particular child with a particular background in mind. After a good problem has been designed in this way, it can easily be modified and adjusted for the whole class—it is much more difficult to create a good problem than it is to modify a good one for classroom use. Whatever the method of design may be, however, problem solving is of such basic importance in early childhood that the selection and design of *good* problems is worth any amount of creative energy that teachers can devote to them.

experiences for young children



glenadine gibb

alberta castaneda

THE search for effective education programs for very young children has been heightened by two developments: (1) the rapidly growing number of preschools, kindergartens, and day-care centers and (2) a growing awareness of the importance of early experiences in the total development of the human learner. Traditionally, preschool programs have concentrated on the development of a healthy, well-adjusted personality. Spurred by a concern for children from low-income homes and from minority ethnic groups, attention has been directed increasingly to the intellectual growth of children. As a result of this emphasis on intellectual growth, attempts have been made to use preschool experience as a guarantee of future school success, which in turn has produced some programs directed so exclusively toward academic achievement that other aspects of the young child's life have been ignored.

When the interrelatedness—one might even say the inseparability—of physical, social, and intellectual development is accepted and acted on, it is possible to devise a balanced program with a broadly intellectual orientation. Jerome Bruner emphasized this in a statement about mathematics for the young child (1965, p. 1009):

> The more "elementary" a course and the younger its students, the more serious must be its pedagogical aim of forming the intellectual power of those whom it serves. It is as important to justify a good mathematics course by the intellectual discipline it provides or the honesty it promotes as by the mathematics it transmits. Indeed, neither can be accomplished without the other.

The work of Piaget has contributed much toward producing programs that emphasize the development of broad intellectual power rather than the attainment of narrow academic skills. Piaget's research has significance for those who guide the mathematical learning experiences of young children. It indicates that one cannot teach children ideas such as number, length, time, and shape in isolation from natural experiences in their environment. This, of course, does not preclude the planned experience of direct instruction.

planning instruction

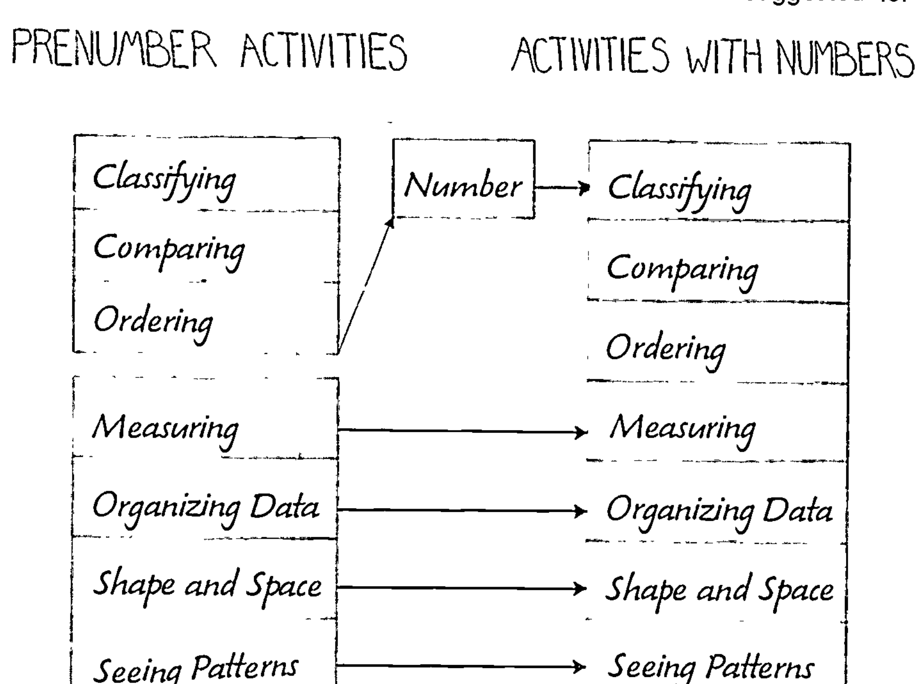
In planning any instruction, one must choose content and instructional strategies that are based on the characteristics of the learners and of the subject matter.

The young child as a learner relies on sensory information of a concrete or pictorial nature. Since mathematical concepts are abstract, however, it is essential that he relate the sensory experiences and the mathematical abstractions For example, a child cannot experience adding numbers in the same sense that he can experience melting things. But he *can* manipulate a set of objects and gradually, through much experience, form in his mind a mathematical idea or model that is abstract. This "abstractness," this quality of being one step removed from the real world, is what is distinctive of even beginning mathematics concepts. In order to communicate these concepts, some language is needed, and this language must be based on sensory data.

It is with these steps—the accumulation of sensory data, its organization into concepts, the acquisition of the appropriate language, and the development of an intuitive background for further symbolization—that a beginning mathematics program must be concerned. In fact, a beginning mathematics program can best be thought of as one that emphasizes the development of concepts and the acquisition of oral language. Great caution must be exercised in moving to new language or to any written symbolization.

The activities suggested in this chapter have been chosen because they take into account the young child's need for perceptual information and yet do not distort the mathematical ideas being generated. It is a basic assumption of the authors that all experiences, including planned activities, must be directed toward building the young child's trust and confidence in his own perception and cognition, his trust in his own ability to learn, and his image of himself as a learner and a valued human being.

The activities, organized around a series of topics and abilities, are listed in the chart below. All the activities suggested for any one



topic need not be exhausted before another topic is begun, a teacher might well consider moving back and forth between topics. However. one obvious break in the sequence must be noted—the point where number is conceptualized and named. The chart shows this point by the arrow running from the first three topics to "Number." Activities in classifying, comparing. and ordering must precede the naming of number. Other activities on the left-hand side of the chart may or may not precede the introduction of number but must precede the corresponding activities on the right-hand side that do use number.

prenumber activities

Classifying, comparing, and ordering are three processes that underlie the concept of number. Once the child is able to classify objects and see the collections thus formed as entities, then he can classify sets of objects as equivalent or nonequivalent and order them on the basis of numerousness. Yet in order to associate number with these processes, the identities and attributes of the objects must be disregarded. Hence experience in classifying. comparing, and ordering provides the necessary background for the higher degree of abstraction required for number.

classifying

Very young children classify. Applying identity labels such as *ball* and *sock* is a form of classification. Whenever a child accurately names an unfamiliar object belonging to a familiar identity class, he is classifying. For example, he may never have had a banana oatmeal cookie, but if given one, he may readily identify it as a cookie. He has a concept and a label for "cookieness," and he has classified this object as being a cookie.

Classifying is a part of everyday life and of many activities of the preschool program. It is often presented to children as a sorting task. These sorting tasks may be informal. "Let's put the blocks in this basket" or "The socks go in this drawer and the shirts in that drawer." Or they may be carefully planned and sequenced Planned sorting tasks may be either independently chosen and carried out by the child, or teacher directed, they may be performed either with a small collection of objects varying in few and obvious attributes, or with a large collection varying in several attributes, they may be planned either on the basis of a criterion of the child's own choice, with or without later description of the criterion, or on the basis of a stated, predetermined criterion.

early activities

Children can classify—that is, put together things that are alike or that belong together—even before they have the precise language with which to label the objects, describe the likeness, or justify the "belongingness." But they cannot do this before they have an understanding of "put together," "alike," and "belong together" when these words are used in instructions. Any classification is dependent on having concepts and labels for identities, attributes, purposes, locations, positions, and so on. The acquisition of such concepts and labels should be an integral part of early mathematics learning.

Discussing the differences and likenesses found in the following activities will usually lead to the discovery of other similarities and will prevent children from becoming tied to the traditional classifications of size, shape, and color.

▸ Have boxes of assorted "junk" freely available for the children to handle, use, or group as they wish. On occasion ask a child about the groupings he has made. Do not try to impose any correct or acceptable criterion; just talk with him to see how he has done the grouping.

▸ Assemble a small "junk pile" and challenge the children to find two objects that are "the same." Ask them to tell how they are the same.

▸ Set one stamp aside from a pile of stamps and ask the children to find another one that is in some way "like" the first stamp.

▸ Have a child select two children who are in

some way "the same." Have other children try to guess the similarity, that is, how the two children are "the same."

later activities

Criteria for sorting may include such attributes as simple identity (blocks), larger identities (toys), properties (blue), utility (things to write with), composition (things made of wood), and combinations of attributes (things that are red and roll). The teacher should be careful to choose materials that the child can talk about freely.

► Assemble a collection of cars, pencils, and blocks—some red, some blue, and some yellow, for instance, and some larger than others. Have the children put them in piles so that the things in each pile are alike in at least one way.

► Display a collection of cutouts with distinctive shapes and ask the children to arrange the cutouts in two piles so that the cutouts in each pile are alike in some way.

► Set up a collection of doll furniture and eating utensils. Tell the children to put all the chairs together in one pile, all the beds in another pile, all the forks in another, and so on. Later, they can sort the furniture into one pile and the eating utensils into another pile.

► Put together a collection of shells, dried legumes, rocks, dry pasta, buttons, beads, samples of fabric, or any other materials amenable to sorting. Have the children put together the things that are alike or the same. Then ask them how they decided to put them together the way they did.

► Take one of the collections that has been partially sorted and ask the children to decide which group the next object will be part of and why, or ask them to describe how the things that are together are alike.

Classifying can be used to develop concepts and language related to sets of objects. Although set language is rarely a part of everyday conversation, children have had experiences with sets of objects. They may not have identified building blocks, their fingers, a pair of socks, their crayons, or their mother's dishes as sets, but they *have* considered them as single entities; when children think of these things as single entities, they are using the concept of set.

early activities

Initial, planned experiences and the accompanying introduction of some set language might include opportunities for the child to conclude (1) that a set can be named by describing or listing its members, (2) that the name of a set determines its membership, (3) that the phrases "a set of" and "the set of all the" can name different sets, and (4) that an object can be a member of more than one set.

In introducing the language "this is a set of," teachers should not use sets with one member—it is awkward and perhaps confusing to speak of "a set of pencils" when only one pencil is in view. Also, the number in the sets should be varied to prevent the child from concluding that all sets have some given number of members.

► Show the children several objects, such as chalkboard erasers, and say, "This is a set of erasers." Show other sets of objects and name them, leaving them in view. Show other familiar objects, such as shells, and ask, "What is this?" The children may reply "a set of shells" or just "shells." In either event, agree that they are right and say, "They are shells. This is a set of shells." After many sets have been named and are still visible, ask again about each set and say, "What is the name of this set?" Then ask them to show you the set of pencils, cups, and so on.

Building on earlier classification skills, the children readily use the words "this is a set of" to describe sets whose members have common qualities with which they are familiar. If the members of those sets are distinguishable, it is possible to name the individual members of the set. Once this is possible, children also can name the members of a set where no common

quality exists. These experiences can lead them to conclude that members of a set need not be related.

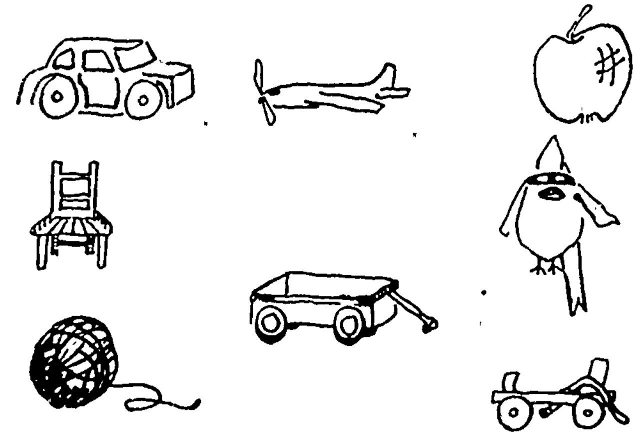
Sometimes a child is inclined to think that a set of objects is inherently what he sees and names it to be. If so, the following activities might be used to show that a set of objects can be named in more than one way and moreover that the name will determine what other objects might be members of the set.

▸ From a collection of objects, select some that have two easily identified common qualities, for example, red airplanes, animals with wings, or toys with wheels. Display the objects—for example, a robin, a chicken, and a duck—and name them by using one of the common qualities: "This is a set of animals." Have the children find other members for the set of animals. Again show the robin, the chicken, and the duck, and name the set in a different way: "This is a set of things with wings." Now have the children find other members of the set of things with wings. Ask them why they choose a dog, cat, and rabbit to go with the robin, chicken, and duck the first time and an airplane and ladybug the second time.

▸ Whisper to one child the name for a set of objects. He then decides which objects suggested by other children are members of the set. See if the children can figure out what the one child knows that allows him to decide which objects can be members of the set.

▸ Using a collection such as the one shown in ■, take a single object such as a red car and start a set with it. Let the children suggest a name for the set and assemble the other members of that set to one side of the original collection. Call the children's attention to the original object again and have another appropriate name suggested. Assemble other members of this newly named set to the other side of the original set. Talk with children about the original object and which set or sets it is a member of.

A further refinement of the naming of sets might include the distinction between "a set of" and "the set of all the." For these activities, the teacher must be sure that the children understand the restrictions on the objects for the activity—for example, only those objects on the rug, or the table, or the display board.



■

▸ Begin any set (for example, a set of scissors) and have a child name it. Ask if there is anything else that could be a member of the set and when there are no more possible members, ask if that is all the scissors. When the children decide that it is, say, "Yes, it is not just a set of scissors, it is the set of all scissors in the room."

▸ Have children name a given set and then make it "the set of all the ___s." Or ask them to name a set and when all possible members have been included, ask them to rename the set.

later activities

Activities with sets of objects offer an opportunity for perceptual experiences that include some ideas of logic, proof, and Venn diagramming. The following activities combine the use of negation and the idea of the complement of a set.

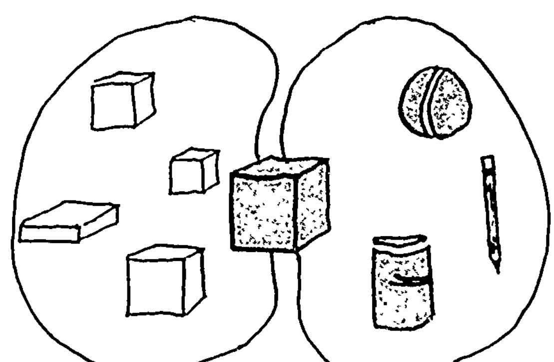
▸ Have the children work with a sorting device. (See page 82 of chapter 4.)

▸ Bring out a variety of objects. Begin to put the objects into two sets, perhaps on a piece of tagboard marked into two sections. Into one set put things with a common property or identity—animals, perhaps—and into the other set objects that are not animals. As soon as they think they know it, have the children tell the name of either set. They will easily name the set of animals; the difficulty comes in finding a name for the other set. They may suggest names like "things," "toys," or "objects," but if any of these were the name of the set, then all the animals would be members of that set, too, and they are not. If no satisfactory name can be suggested, put that display aside, still in view, and begin another partitioning, such as "yellow" and "not yellow." Again seek names for the two sets. If several such examples do not elicit the desired response from the children, return to the set of animals and say, "Listen while I think how I'll decide where this one goes. Is it an animal? Yes; so I'll put it here with the set of animals. Is this an animal? No; so I'll put it with that set. This is the set of animals. What could be a name for that set?"

The following activities approach Venn diagramming of the intersection of sets by showing with circular enclosures that an object can be a member of two sets.

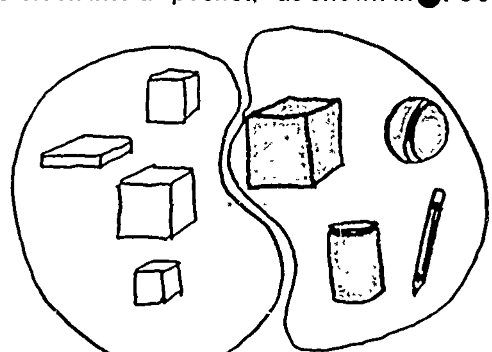
▸ Position two pieces of yarn or string in nonintersecting, circular shapes on the floor or table so that all the children can see. From a collection of objects—a universe—begin to form two sets of objects (blocks and red things), building each set within one of the circular areas. (Perhaps labels reading "red" and "blocks" could be used.) As each object is taken from the universe, have the children assign it to one of the sets. Be sure the universe contains some objects that could be members of both sets. When the children clearly understand this procedure, produce a red block and ask them where it should go. They will agree both that it is red and therefore belongs in the set of red things and that it is a block and therefore belongs with the set of blocks. The problem is how to get it physically into both sets, that is, within both circular boundaries.

Solutions that 4 1/2-to-5-year-old children have suggested include putting the block equidistant between the two yarn circles, cutting the block in half and putting a piece in each set; getting another red block so that there will be one for each set; putting the red block aside and going on to the next object; moving the pieces of yarn together and balancing the block on them (■); and putting the two pieces of yarn



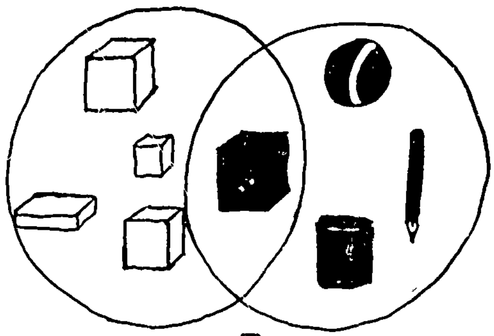
■

close together in a zigzag fashion and inserting the block into a "pocket," as shown in ●. Occa-



●

sionally, a rare five-year-old will arrange the pieces of yarn so that they overlap, thus forming an area that is enclosed by the boundaries of *both* set areas as shown in □



□

If no child solves the problem, have the materials available at free-activity time so that those interested can continue to explore and discuss the possibilities. While this activity provides an opportunity for children to use the idea that an object can be a member of more than one set, it tests for other cognitive behaviors, including a certain flexibility of action and thought. It is quite possible for a child to accept, and even be able to state, that an object can have membership in two sets and yet not be able to solve the preceding problem.

► Set off two circular areas with string. Overlap the strings to indicate an intersection and begin two sets—blue things and cars. When several objects have been placed in each set and in the intersection, have the children describe the objects that are in the overlapping area (blue cars). Point out that they are things that share the identifying quality of both sets.

After children are able to place objects correctly within the boundaries of the intersection, start two new sets and give them an opportunity to speculate about what would go into the intersection of the two sets. Later, provide an opportunity for them to speculate about what would go into the intersection of two sets even before the sets have been started: "If we made this a set of oranges and that a set of plastic things, what would go here? Why?" Also include some instances for which the intersection is the empty set.

comparing

Comparing is the process by which the child establishes a relation between two objects on the basis of some specific attribute. In such simple activities as comparing the length of two sticks and stating, "This stick is longer than that stick," the child establishes the relation "is longer than." Also, comparing is the forerunner of ordering and measuring.

Just as a child can classify only in terms of the attributes and identities that are known to him, so can he make comparisons only on the basis of those attributes that have already been clearly discerned by him. Thus, the development of concepts for attributes of objects contributes to the ability to compare. It is essential also to develop (1) the techniques that allow the child to make perceptual comparisons and (2) the specific language that is required for describing particular comparisons. Comparing heights, for example, requires techniques and language different from those used in comparing sizes. The technique of attempting to establish a one-to-one correspondence allows the child to compare visually the numerousness of two sets. Juxtaposing, hefting, and using a common baseline are activities that allow the child to compare area, weight, and height perceptually.

early activities

Because children are so intimately concerned with their own height, the activities in this section will involve comparisons of height. (Height and length are both forms of line measure; it is only physical position that makes height measurement appropriate in one situation and length measurement appropriate in another.)

► Ask two children who distinctly differ in height to stand. Ask the other children which child is taller and how they know.

► Give the children two objects that differ only in height. Have them decide which is taller, or shorter. Ask them what they had to do to be sure.

► Give children two objects of different heights and in some instances of contrasting bulk, that is, tall and narrow, short and broad. Be sure they compare height and not bulk. Ask what they had to do to compare the two objects.

► Have the children attempt to compare the heights of two objects placed some distance apart. Ask them what they must do to confirm their judgment.

► Have the children attempt to compare the heights of objects that are on different baselines—for example, one child standing on a chair and another standing on the floor. Ask them what they must do to be sure which is taller or shorter or if they are of the same height.

later activities

Moving from comparisons of amount to comparisons of numerousness requires the child to pair the members of two sets, attempting to set up a one-to-one correspondence between the two sets.

► Give three red spoons to one child and two blue spoons to another. Ask who has more spoons. Have each child put a spoon on the table, thus establishing a pair of spoons, a red one and a blue one. With each pairing, comment on what can be seen. "There is a blue spoon for that red spoon. There is a blue spoon for *that* red spoon. But there is no blue spoon for this red spoon. There is a red spoon left over. There are more red spoons than blue ones."

► Give one child three jars of paste and another child three paste brushes. "Is there a brush for each jar of paste? A jar for each paste brush? How can we find out?"

► Using sets of more than five members, have the children compare the number of dolls and hats, cowboys and horses, and so on. Set up situations and ask questions similar to those described above.

► Show a set of objects in a linear arrangement. With other sets in random arrangement, ask the children to find which sets have more members and which sets have fewer members than the set being used as a model.

► Show a model set of objects. Have the children sort other sets into those that are equivalent to the model set and those that are not.

For some kindergarten children, later activities can include the use of the words "fewer than" and "equivalent." It is acceptable, however, to continue to use only the terminology "more than" and "as many as."

Daily classroom routines provide many opportunities for children to use pairing to find if there are "as many as" or "enough" or "more than." These opportunities can be used to practice the actual pairing, the use of the information thus gained, and the language to report that information.

ordering

Ordering builds on comparing. For the young child, ordering involves a physical arrangement of objects or sets of objects. The arrangement must have an origin and a direction and must reflect some rule. For example, things can be ordered from shortest to longest, the rule being that each thing after the first must be longer than the one it follows; sets of objects can be ordered from those with the most members to those with the fewest members, or vice versa. Obviously, children can order only on the basis of attributes that are known to them and that they can compare.

early activities

Materials that lend themselves to ordering on some clearly perceptible property should be available to children for their independent use. Examples of such materials include Cuisenaire

rods, the Great Wall, blocks of progressive size, unit blocks of graduated length, stacking barrels, a collection of dolls or pencils or Tinker Toy sticks of graduated lengths, objects of varying weight, color chips in varying shades of one color, tone blocks of varying pitch, and so on. Beginning activities in ordering should use no more than five objects, and their differences should be clearly discernible.

▸ Show the children three objects differing in height. Locate the one of medium height. Have a child choose one of the remaining objects, compare it to the medium one, describe that comparison, and place it to one side of the medium one. Have another child compare the other remaining object to the medium one, describe that comparison, and place the object on the other side of the medium one. Have the children describe the resulting arrangement.

▸ Show three objects that have been arranged according to height. Indicate one end as a starting point and ask the children to describe how each object is different from the one it follows. Indicate the other end as the beginning and have the children describe how each object is different from the one it follows.

▸ Show three objects ordered by height. Indicate and identify the shortest of three other objects. Have the children arrange the other two so that this set will look just like the first set. Tell them that the objects are ordered so that each object after the first is taller than the one it follows.

▸ Show the children five objects that have been ordered according to height. Then show them five more objects. Arrange the first, third, and fifth of these objects in order of height. Have the children insert the other two objects so that the set is ordered like the first set.

▸ Give the children sets of five objects and have the children order the sets according to some physical property with which they are familiar. Have them describe how they arranged their objects.

▸ Show five objects that have been ordered by height and have the children describe the arrangement. Ask them to tell where a particular object is and why it belongs there: "The blue one is between the red and green ones. The orange one is last. It comes after the green one because it is the tallest of all. . . ."

later activities

After comparing and ordering objects and comparing sets of objects for numerousness, children can use numerousness as a basis for ordering sets of objects. Their first orderings should involve no more than three sets, with the numerousness clearly disparate. When children are able to order three sets so that each set after the first has more members than the set it follows, they can order as many as five sets. In a similar way, sets can be ordered so that each one has fewer members than the one it follows. As with the preceding sequence of activities, these activities should move, not just toward more sets to be ordered, but also toward the acquisition of the language, both receptive and productive, with which to describe the sets and the arrangements.

The preceding activities on classifying, comparing, and ordering are a prelude to the concept of number. The following activities need not necessarily precede the naming of number, but neither do they require the use of number names. Many of them build on ideas already begun in the activities using classifying, comparing, and ordering.

measuring

Children's initial experiences with measurement should be leisurely enough to allow them to build concepts both of the properties to be measured and of the appropriate units. Later, there should be kinesthetic and physical experiences with standard units before any standard measuring device is used. The teacher should be in no hurry to use standard units; rather, she should make sure that each experience related to measurement contributes to a general concept of the process of measuring and to the concept of measurement.

Activities related to measurement begin with tasks of comparing and ordering. The earlier sections on comparing and ordering concentrated on height as a variable dimension. This

section on measuring concentrates on weight and capacity ways for developing these concepts, and the language and techniques needed.

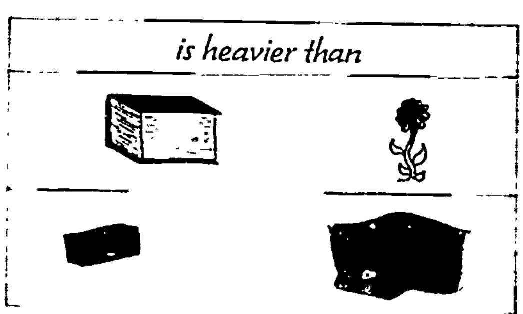
Weight. Obviously, it is not possible to measure a property unless the property itself has consistent meaning. The child must be able to distinguish weight from size, texture, or other elements of appearance before he can compare and measure it.

early activities

It is easy to plan experiences that will lead the child to realize "I cannot say which is heavier by looking—I must lift the objects to be sure." In this way he realizes the power of, and the limitations on, his perception.

▸ Using a collection of objects including ball bearings, Ping-Pong balls, golf balls, solid rubber and foam rubber balls, balsam wood, boxes of cotton, and other objects whose weight and size seem disproportionate, show the children two objects and ask them to find out which is heavier. The first discriminations should be obvious and not necessarily offer any contradiction in size and weight. Ask the children to describe the different feelings of lifting very heavy and very light objects. Activities with finer discriminations and with pairs that include large, light objects and small, heavy objects can follow.

▸ Have collections of objects available for a self-chosen activity. Let the children collect ordered pairs of objects and arrange them so that the relation "is heavier than" or "is lighter than" exists between the two objects. A large piece of tagboard placed on a table or the floor with the relation inscribed at the top and spaces ruled off can help in the physical arrangement of the pairs of objects (■).



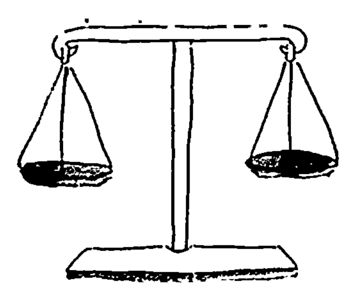
■

▸ Expand the preceding activity, both as a directed and as a spontaneous experience, to include ordering three or more objects. The rule might be that each object after the first is heavier than the object it follows. Thus, the first object is the lightest and the last is the heaviest.

The preceding activities have used physical sensation in comparing the weight of one object to the weight of another object. Building on the knowledge of muscular comparison, the following activities use a balance beam in comparing weights.

▸ Have children lift heavy things that require grasping by two arms and have them describe what it feels like to lift them. Then have them lift buckets of sand or rocks by using only one hand to grasp the handle. Ask them to describe how *that* feels, how it makes their arm feel, and what they had to do with their lifting arm. Have other children observe and describe the lifting posture.

Show the children a balance beam like the one in ● and ask them to predict what would happen if they put something heavy in one pan. Then have them do it to check what does happen.



●

► With a balance beam, have the children compare the weight of two objects of grossly different weights, establish which is heavier and which is lighter, and predict what would happen if one object were put in one pan and one object in the other. Have them check their predictions. When the children agree that the heavier object causes the arm from which it is suspended to come down, let them compare several different pairs of objects, this time reading the balance beam rather than relying on their physical, kinesthetic sensations to say that one object is "heavier than," "almost as heavy as," or "lighter than" another object.

► Provide a balance beam and from two to six look-alike Plasticine balls, with one ball heavier than the others (a metal ball can be placed inside). Have the children use the balance beam to find the heavier ball.

Capacity. Although capacity and volume are related properties, a visual comparison between capacities can easily be made, whereas a visual or perceptible comparison between volumes, in the sense of space occupied, cannot be so clearly made. For example, we can pour from one container to another and have evidence that the first container holds more because material is left in it when the second container is full (or that it holds less because we have emptied its contents into the other container and the other container is not full).

Before introducing standard units of capacity—liters, cups, pints, quarts—have the children compare the capacities of containers of different size and shape.

► Provide liquids and other "pourables," such as rice, sand, lentils, or beans, and a variety of containers, some identical in shape and size, some varying in shape but the same in capacity, and some differing in capacity. Let the children pour from one container to another. They can see that containers of quite different dimensions can hold the same amount.

Careful attention should be given, in guided activities, to the acquisition of the language needed to describe the result of pouring the contents of one container into another.

► Have the children compare the capacities of two containers: "This holds more than (less than, almost as much as) that." Have them order three or more containers by capacity. Let them label the ones that hold the most and the least.

organizing and presenting data

Presenting information in graphic form and reading the graphs are interesting activities for young children. Real objects can be arranged in three-dimensional bar graphs. With just the ability to compare height and use one-to-one correspondence, the young child can "read" and interpret such an arrangement, provided the objects are stackable and of the same size.

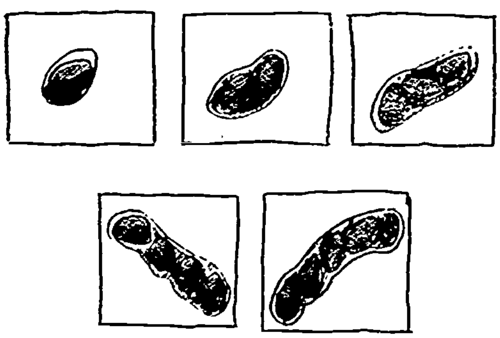
► Provide a shoe box full of colored cubic blocks and ask the children to name the colors of the blocks in the box. Ask them to find out which colored blocks are more numerous. Let the children sort the blocks, stack them, and arrange the stacks in a side-by-side arrangement. They can tell perceptually which stack is taller, establish the one-to-one relationship across the columns, and then tell which stack has more blocks.

► Translate the preceding activity to pictorial form by using drawings or pictures of blocks, colored like the real blocks. Have the children find an appropriately colored drawing for each colored block and place the drawings by color on a paper marked off in columns. Again, without naming the number in any column, they can tell which color of block is most (or least) numerous.

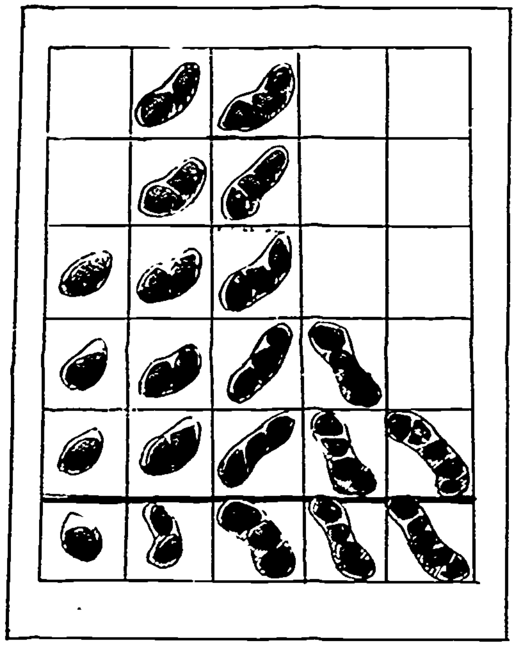
▸ Have the children choose pictures to represent each one of their brothers and sisters—a picture of a boy represents a brother; a picture of a girl represents a sister. Have each child arrange his pictures in a column above his name. The tallest columns show the children with the most brothers and sisters. Comparing the heights of the columns allows the children to say that this child has as many brothers and sisters as that child, these children have more brothers and sisters, and so on.

▸ Children can organize into a graphic presentation pictorial data related to attendance, shoe styles, pets, birthday months, dessert preferences, and so on.

Many science and social studies projects offer the possibility of a pictorial or symbolic presentation of data. One simple example developed from a study of peanuts. Each child was given a peanut to open and was asked to note how many kernels were inside. The teacher previously had prepared cards with drawings showing peanuts containing from one to five kernels and a chart on which these cards could be displayed. The cards looked like this:



Each child found a card showing a peanut shell with as many kernels as his peanut. He placed the card on the chart in the column marked with a similar picture. The completed chart is shown in ■. Again, the data were organized and presented with no need for counting.



■

shape and space

The geometric idea of shape has been used, overused, and misused. Children are asked frequently to handle, name, and describe cutouts shaped like triangles, circles, squares, and so on. From some such sequences of instruction, children could conclude logically—but incorrectly—(1) that a circle can be held in the hand, (2) that a shape is made of posterboard, (3) that a shape is red, green, blue, or yellow, (4) that "square" and "rectangle" are mutually exclusive categories, and (5) that triangles have at least two equivalent sides and their bases are firmly horizontal. To prevent such incorrect conclusions, activities with objects should be planned to develop a general concept of shape, to give the children the opportunity to observe and describe likenesses and differences in the shapes of objects, and, finally, to enable the children to learn the names for the most common shapes.

Explorations of shape begin in infancy. Babies and small children feel shape as they grasp rattles, clutch bottles, and roll balls, as they crawl around chairs and through cartons, climb up and down stairs, and finger the edge of a coffee table while walking around it; as they play with blocks, dishes, stuffed animals, wheel toys, and dolls; and as they learn to feed themselves by holding their glasses of juice or milk. Although small children may not consciously

communicate what they see and feel, these experiences during the early years of their lives develop in them an awareness of the likenesses and differences in the shapes of a variety of objects. With guided learning experiences, children learn to describe what they see and feel.

► Gather a collection of objects, such as checkers, buttons, bottle caps, small wheels, sheets of paper, corks, crayon boxes, stamps, pennants, toy irons, cards or sheets of paper cut diagonally, or other objects that can be classified by an attribute of shape. Select one object, such as the cork, and place it on one table and select another object with distinctly different shape, such as the crayon box, and place it on another table. Place a stamp with the crayon box and a checker with the cork. Continue to select objects or ask children to select objects, one at a time, and have the children indicate the table on which the object should be placed. Verify that their suggestion is correct, or if it is not, say, "No, let's put the ______ on the other table." After all the objects have been classified into one or the other of the two sets, ask what is alike about the objects in each set. Check each suggestion to see if it applies to all the objects in that set. Although you have imposed a means of classification, the children may not perceive it. Try to look through their eyes, not yours.

Further activities can be designed to use children's sight, touch, and kinesthetic senses.

► Blindfold a child and have him trace with his fingers the outline of a cutout (circular, rectangular, or triangular) and then give the cutout to you to conceal before taking off his blindfold. Have him rely on his kinesthetic memory to trace the shape in the air with his eyes open and describe it as being round or as having sides or corners.

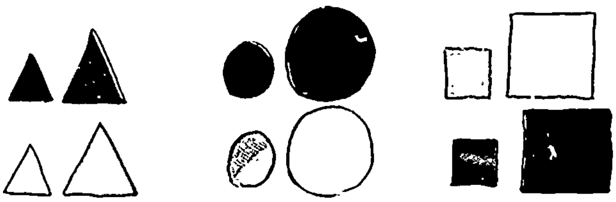
► Seat the children in a circle facing a pile of blocks of various shapes and ask them to close their eyes and put their hands behind their backs. Pick up one object from the pile and place it in a child's hands. Ask him to find, without seeing the one he has handled, one exactly like it or similar to it.

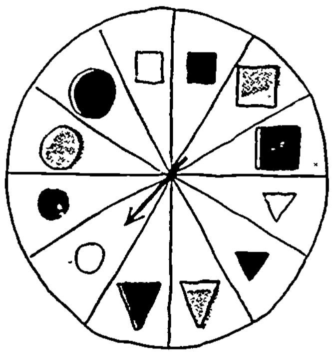
► Vary the preceding activity, using blocks that vary in only one dimension with their shape staying the same; for example, use dowels of varying diameter or sticks of varying length (Cuisenaire rods).

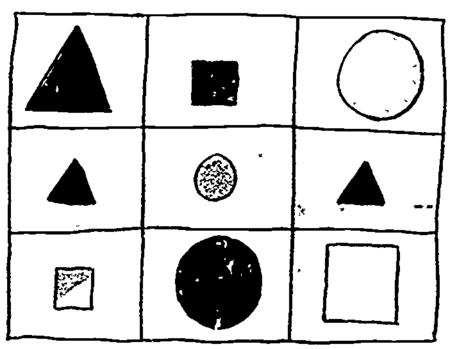
► Provide a collection of felt or cardboard pieces. Include some with sides and corners and some with no corners, some that are examples of common shapes and some that are not, some with the same shape but of varying sizes, and some with the same shape but with different proportions. Let the children sort the pieces according to whatever likenesses or differences they can see. Successive sortings can help the children to see and describe the characteristics of the pieces that have, for example, a rectangular shape. It is, however, quite possible to know and state that two things have a common shape or common elements in their shape without being able to name the shape. Do not be in a hurry to use the words *circle*, *square*, *triangle*, and the like.

► From a collection of three-dimensional objects, select one object that can be a model of a cylinder, another that can be a model of a rectangular box, and another that can be a model of a sphere. Choose other objects that will go with each. The first grouping is called *cans*, the second *boxes*, and the third *balls*. Have the children continue the sorting. Include some objects whose names offer competing ideas—for example, a sardine can, which is shaped like a box, or a salt box, which is shaped like a can. Invite the children to bring in other objects that would fit into one of the groupings.

► Provide a collection of cutouts that vary in shape, color, or size; a spinner showing the same assortment of figures; and, for each child, a Bingo type of card with nine such figures (■). For his turn a child spins and if he finds a figure on his card like the one indicated by the spinner, he takes such a figure from the collection and covers the appropriate space on his card. The winner must have three objects in a row or a full card.



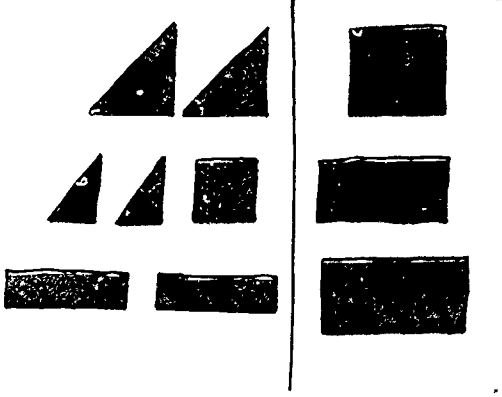




▸ Provide each child with the needed paper cutouts. (See table 1 for suggestions.) Hold up a square and say, "Find two of these and fit them together to make this." (Hold up a rectangle.) After the children have become familiar with the task, the instructions may be given as shown in the table.

TABLE 1

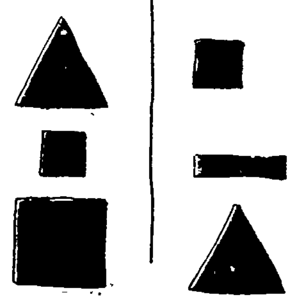
Find | *Make*



▸ Vary the preceding activity by having the children use scissors to convert one shape into another. Some suggestions are shown in table 2.

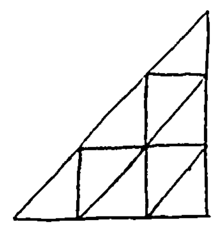
TABLE 2

Find | *Make*

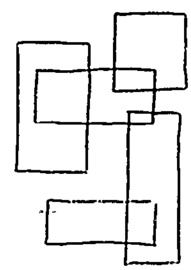


▸ When the names for some common shapes have been learned, give the children such instructions as these:

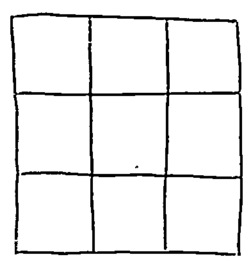
"Point out some of the triangles you can see in this picture"



"Point out some of the rectangles you can see in this picture"

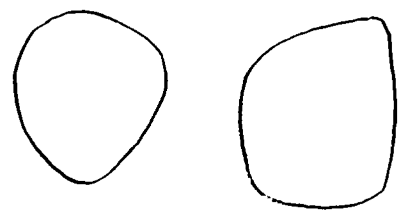


"Point out some of the squares you can see in this picture"

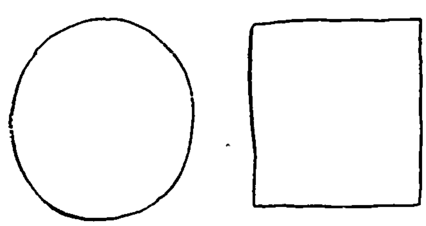


"Draw a picture using just triangles and another using just rectangles. Which picture do you like better? Why?"

Studies by Piaget have suggested that the child's first geometric discoveries do not necessarily involve shape but are concerned with such ideas as separation, proximity, closure, and order. Children have been observed to respond to the topological aspects of figures before perceiving sides and corners. For example, ■ shows how one child reproduced the shapes shown in ●. Responses like this imply that children see and represent "closedness" rather than configuration. However, one also might speculate that because of their age, their responses are governed by a lack of muscle control rather than by an error in perception.



■



●

Other ideas that are geometric in nature and that depend on these intuitive experiences are closed and not-closed paths. The idea of a closed path is related to the various geometric figures that children have used in observing and comparing shapes. Consider the path that they take in walking around the room from one corner along all four walls and back to the starting place. This is a closed path. Tracing with their fingers the closed path made by the edge of a table, the rim of a wastepaper basket, or the edge of a book provides other sensory experiences with which they associate the words *closed path*. By contrast, let them trace with their fingers the not-closed path made by a piece of string or a stick lying on the table. This is different from a closed path, for they cannot get back to the starting point unless they retrace their steps.

▸ Have the children use ropes or cords to form paths on the floor so that they can walk on them from one point back to that same point without having to turn around. Have them form other paths that cannot be so traced.

▸ Have the children tell if the path formed by a rope is closed or not closed and then let them test their conclusion by walking on the rope path or tracing it with toy cars.

▸ Give the children pipe cleaners formed into closed and not-closed curves and have them sort them into two piles so that those in each pile are somehow alike.

▸ Have the children use pipe cleaners to form a variety of closed and not-closed curves.

It can be expected that children have acquired definitions for "inside" and "outside" from hearing such everyday statements as "It is time to come inside" or "You may go outside to play." Learning to associate these ideas with closed and not-closed paths can bring another dimension to their learning. For example:

▸ Have the children make closed paths on the floor with pieces of string. Have them place objects or other children (or even themselves) inside the path, outside the path, and on the path. Show them that they must cross a closed path to go from the inside to the outside but that they do not have to cross a not-closed path to go from one side to the other.

These activities all help children to observe and classify objects according to shape, to compare objects for similarities and differences in shape, and to perceive differences between closed and not-closed paths. Their purpose is to illustrate the spirit of involvement necessary for the young child to develop an awareness of geometric ideas and to acquire the appropriate language for communicating these ideas. In this spirit of involvement, other activities can be devised.

patterns

Basic to mathematical insight is the ability to recognize pattern. With the use of real objects and pictures and drawings, children can be made aware of patterns and can practice seeing, describing, extending and completing, or repeating patterns. They can be required merely to perceive and repeat or extend a pattern; they can be asked to describe a pattern or to explain why they think a particular thing comes next; or they can be asked to make a pattern themselves and describe it. The teacher must be aware of the burden of language that a given task places on the child and must make a considered judgment about the appropriateness of that burden for each group of pupils.

Patterns may be made up of concrete objects, pictures, designs in weaving and stitchery, symbols, and so on. The first patterns should be strong and uncomplicated.

► Using blocks, shells, dry pasta, or anything of distinctive shape, color, or other attribute, begin a pattern, say, of two red blocks followed by two green blocks. Have the children make a pattern directly below the one shown and just like it, and another directly below that. They have *replicated* the pattern.

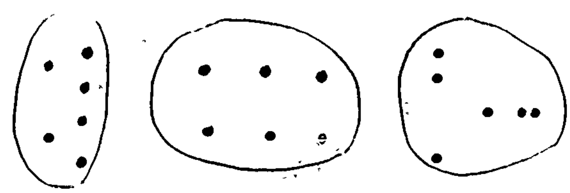
► Show another pattern, such as two oyster shells and one snail shell. Have the children repeat that pattern in a linear arrangement following the original. They have *continued*, or *extended*, the pattern.

► Show another arrangement—for example, two large blocks, two small blocks, two large blocks. Have a child choose the next block and then the next one. Ask why each of those things comes next. (Note that what he chooses next depends on the pattern he sees.)

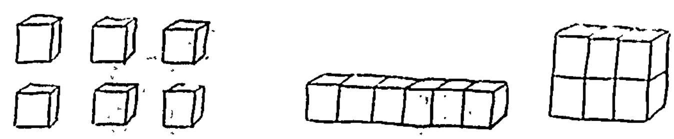
► Show another pattern (a piece of bowknot pasta, a piece of elbow macaroni, a piece of rigatoni, a piece of bowknot pasta, a piece of elbow macaroni). Ask the child to show or tell what the pattern is and to choose the next item in the pattern.

Patterns need not be in a strictly linear arrangement. The following activities will allow the child to practice seeing nonlinear patterns.

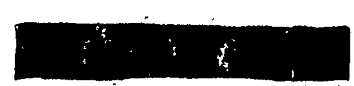
► Show the child a row of evenly spaced dots and have him put a counter on each of the dots. Ask him to take these counters and make a pattern or arrangement. Then have him match another set of counters to the dots so that he now has another set with just as many counters in it. Say, "Try to make these look different." Here are three possible patterns using six counters:



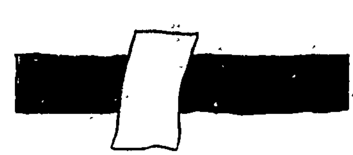
► Show the child different arrangements of six or more blocks. Ask how the different arrangements are alike or different. Challenge the child to build a different arrangement with the same number of blocks.



► Show a linear pattern of red and blue cutouts such as this one:

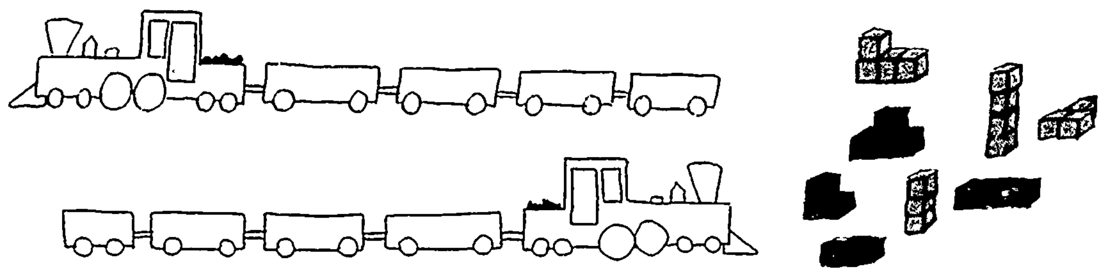


Cover one of the cutouts like this:

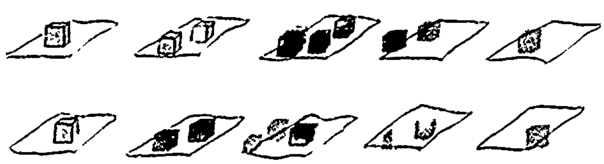


Ask, "What is covered? How do you know?"

Cover more than one cutout. "What color are the ones that are covered? How do you know?"



▸ Show pieces of paper in a linear arrangement and begin a pattern on the papers with blocks:



Ask a child to put blocks into the next "car," or onto the next paper. "How do you know how many to put there? Why did you use that many blocks?"

▸ Show the children eight block "buildings" and two trains like the ones shown in ■. Then give them the following problem: "All the houses have to be moved at the same time. The houses on each train must be the same in some way. How will you load the trains?" The children can choose to sort on the basis of color, shape, or numerousness.

▸ Begin making an arrangement on a geoboard with differently shaped colored beads or spools (●). "Which bead comes next? Where do we put it? How do you know?" The sequence used can involve spatial arrangement only or color and shape.

summary

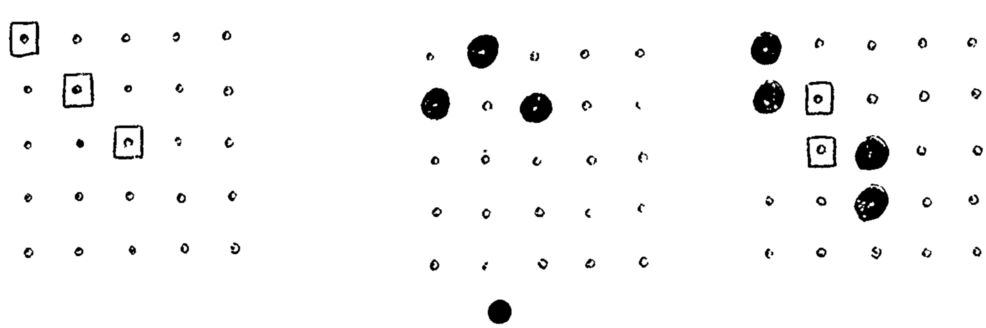
Considerable attention has been given in this section to activities in which children can engage before they are able to recognize and name number. None of these ideas or activities is trivial. They all contribute to the child's first-hand experience and allow him to store images, meanings, and language that he can bring to later, more symbolic learning experiences.

Teachers of young children should be able to defend a mathematics program that does not begin with the naming of number, counting, and the recognition of numerals. Work with numbers and number names can be meaningful only to the extent that the child is allowed to accumulate meanings for the symbols used.

developing the concept of number

number awareness

The young child's awareness of number seems to parallel the historical development of



mankind's concept of number. In response to the question of "how many," primitive man counted only to two—any set beyond this level was dismissed as "many." In a spurt of exploration, a two-year-old seeks out a balloon and then scurries off to find another balloon. He proudly reports "two balloons" as he exhibits his collection. He announces that there is "one car" as he peers out the window and sees a car in the driveway. If he is playing surrounded by toys and the toys are removed one at a time, he seems to show no concern until only two or three remain. At that point, however, any change in the location of one of the remaining toys causes him to investigate what has happened to it. A year later he continues to report his age by holding up his fingers, changing from two fingers to three. Also, he may report that there are two toys in his tent without having the set in sight as he relates the information. Throughout these early years, the child associates number names with collections of objects just as he associates the word *milk* with a glass of milk, *cookie* with a certain food object, *red* with a certain item of clothing, and so on.

A premature effort to teach the child to count may have the undesired effect of teaching him to associate number names with single objects rather than with sets of a certain number. For example, in response to the question, How many cookies? a child may count "one, two" and report two cookies. (We are pleased with his response.) Later, when he is directed to put two toys into his toy box, he counts "one, two" but carries *only the second toy* to the box. Although the first experience does not clearly indicate what he is thinking, the second indicates that he is associating a number name with each object much as he associates a name with his mother, pet, toy, or food. If early experiences have caused the child to attend to the numbers one, two, and three superficially at the level of language and counting, then there is need for activities to associate meaningfully and conceptually the language with perceived sets for the purpose of naming the numbers of those sets.

Readiness for the systematic development of the concept of number is provided by developing the child's ability to perceive a collection of objects as an entity, that is, as a set of objects. This is followed by experiences to ascertain the number relation—as many as, more than, fewer than—between two or more sets. These relationships, as described in the previous section, are established by pairing members in the sets.

The numbers 0–4. Sets with zero through four members are called "elemental" sets because their number can be seen, and named, without counting or partitioning. This tie with perception makes them special sets and the numbers zero through four special numbers in the young child's learning. The imagery and knowledge accumulated for these numbers contribute to the concepts that the child is able to build for numbers associated with sets of five and more members. It is important that the child have a variety of experiences with sets with zero to four members.

A concept of cardinal number evolves from a knowledge of specific numbers. Because cardinal number is the property of equivalent sets and because the body provides so many model sets for two—two hands, two eyes, two ears, two feet, two knees, two elbows—it is suggested that the first number considered be two.

► Arrange several sets of two objects each on a display board (magnetic board or flannel board), table, or floor. Ask the children to describe each set. Then ask what is alike about all the sets. (There are just as many in one set as in the others.) "How many?" The children must learn to name the "manyness" two. The number is not identified, however, by counting "one, two." The number of each set is two.

► Have the children identify or collect sets of two objects in their classroom or pictured in story books or magazines.

► Have the children draw pictures of sets of two objects.

► Have the children paste on sheets of paper their collections of pictures showing two objects.

► From a collection of objects, have the children form sets of two. Ask them to describe each set formed and tell the number of the set.

Children learn to recognize the spoken name "two" and to associate it with sets of two just as they learn to recognize other spoken words and to associate these words with certain objects and actions. However, written symbols, either numerals or words, should be used cautiously with the preschool child. Recognition and naming numerals should be restricted to those numerals for which meaning specifically has been built. When children do have the imagery and language for twoness, the numeral "2" can be introduced. Children learn to associate meaning with the symbol. Certainly they should not be expected to write numerals themselves during these early years.

When the children can identify the number of sets of two objects, sets of three can be considered.

▸ Show two sets of two objects. Have the number of each set named. Change one set by putting in another object and ask, "Is the number of this set (the changed set) two?" By comparing this set with the other set they can see that the new set has more members and thus has a different number. Form sets of three.

▸ Extend the activities for twoness to develop the meaning of threeness. Also, since number is independent of arrangement, be sure to vary the arrangement of the objects.

The knowledge of twoness and threeness can be used to introduce one and four. Although work with sets of one member may seem trivial, it is important that children think of a single object as being a set whose number is one and that they relate the cardinal characteristic (numerousness) of a set of one to those sets whose numbers are two, three, and so on.

▸ Display a set of three objects and ask the children to name the number of the set. Then change the set by removing an object and ask the children to report the number of the set. Change the set again by removing another object.

▸ Show a set with a single object. Name the set and identify the number of the set as one. Have the children find other sets with one member from a collection of sets having one to four members. Have them name each set and its number. "This is a set of apples. The number of the set of apples is one."

▸ Ask, "What is the number of the set of pianos in our room? What is the number of sinks in our room?" Have children look for another set with one member, such as the set of clocks, paper cutters, soap dispensers.

Changing sets from one to two to three to four objects by putting in another object leads to other activities related to fourness.

▸ Give each child three objects. Have several children tell how many they have. Give each child one more object. Establish that each child now has four objects. Have the children pick up the four objects in their hands and tell how many they have in each hand. Have them put the objects back together and tell how many there are in all.

▸ Give each child two small containers and four small objects. Have them put their objects into the two containers and report how many there are in each container.

▸ Give each child four objects and have them make as many arrangements as possible. Have children describe their arrangements as, for example, "I put one and then two more and then another one" or "I put three down and one on top."

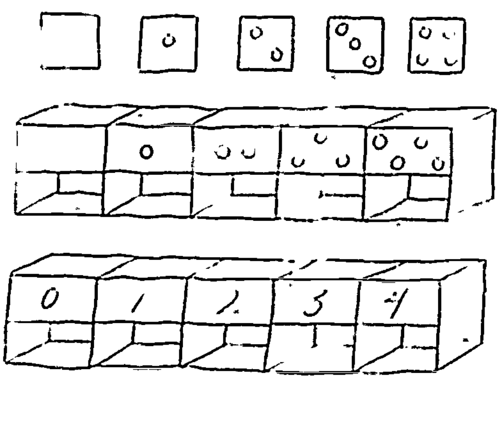
The idea of a set having no members is not foreign to the young child. He has seen and accepted the significance of an empty plate when someone has said regretfully, "No more cookies," and he has seen his empty cereal bowl as his mother has approvingly said, "All gone." What is not familiar to him is the use of the word *zero* to tell how many cookies there

are when there are "no more" or to tell how many cereal flakes are in the bowl when he has eaten them all—that is, to name the number of the empty set.

► Begin with a set whose number the child can name—two, three, or four cutouts of a rabbit on a flannel or magnetic board. Follow the same procedure that was used when focusing on the set with a single member. Each time, after removing a rabbit from the set, ask the children to tell the number of the new set. Repeat until all the rabbits have been removed. "How many rabbits are in the set on the board now? Yes, you are right, there are no rabbits left. There are no members in the set. The number of rabbits on the board is zero. There are zero rabbits on the board." Repeat with other sets. Then ask, "What is the number of the set of real, live elephants in our room?" Ask children to name other sets whose number is zero.

After developing the names for the numbers zero, one, two, three, and four, continue with activities that reinforce these ideas and enable the children to establish firmly the concepts of these numbers and the recognition of numerals if the symbols have been provided. Little is to be gained from teaching numbers greater than four too soon, before meaning and language for zero through four are firmly established. Informal activities like the following can be used to attain this goal:

► Have the children sort picture cards into appropriate boxes that show model sets. (Later, the boxes can be labeled with numerals.)



► Attach model sets or numeral cards to pockets or boxes and have the children deposit the number of cards (or beans, sticks, etc.) indicated by the numeral.

► Use interlocking blocks to provide practice with matching model sets and their number names. Ask questions about the numerousness of the sets and the numerals fitted together to direct the children's attention to the purpose of the task.

The number 5. Some children may recognize sets of five as elemental sets, but others must build the meaning of five from their perception of the subsets of two and three or one and four as they partition a set of five. A model set can be introduced by starting with a set of four, as illustrated in the following activity:

► Display a set with four members. Have the children name the set and its number. Then change the set by putting another object in it. Ask if the number of the new set is four or if anyone knows the number of this set. Their responses to such questions show how familiar they may be with the number five. Reaffirm or tell them that the number of this set is five. Have them use it as a model set to find other sets whose number is five.

► Give each child a set of five objects and identify the number of the set. Ask the children to pick up a subset of a given number and to observe how many are in the remaining set. Do this until they are able to state from some given display that a set of five can be a set of three objects and a set of two objects, and so on. After the objects they picked up are returned to the original set and the number of the total group is again identified, continue until the children, using the imagery built earlier, can correctly respond to the question, "If you picked up three blocks, how many would be left?"

► Give each child a collection of six to eight objects and ask the children to form a set of

two objects. Then ask them to put more objects with that set to make a set whose number is five. Repeat the activity, starting with a set of three, a set of four, and a set of one.

If the children have shown any confusion or hesitancy about the numbers zero through five and their symbols, the teacher would be well advised to go no further than five in the development of numbers at this time. Using additional activities can provide greater experience with sets having zero to five members, the associations of numbers with those sets, and the recognition of the numerals if they have been introduced. It should be remembered that a child's *perception* is a potent factor in the naming of number in sets with five or fewer members. For sets having more than five members, the child must rely heavily on his *cognition*, that is, on what he has learned about the elemental sets as he learns to associate numbers with sets of more than five members. Many five-year-olds need to engage in a variety of activities involving zero through five before being introduced to numbers greater than five.

The following activity not only will allow the child to use the numbers he already knows but also will contribute to an experiential background on which he can build the idea of base, a concept fundamental to our numeration system.

▸ If the child can name the number for sets having at least three objects, show a set having from three to eleven members—for instance, a set of ten pencils. "Let's find out how many pencils we have. Can you show me a set of three pencils? Put it here. Is there another set of three pencils? Put it here. Can we make another set of three pencils? Put it here. Look, we have three sets of three pencils and one pencil left over—three sets of three, and one left over. That's how many pencils we have."

▸ Repeat with other sets of three to eleven members until the child can perform the grouping and can describe it. Give each child a set of eight objects. "Show me how many things are here by making sets of three. Tell me how many there are. Good, there are two sets of three, and two left over."

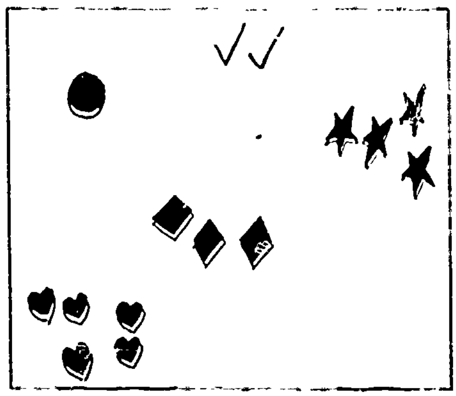
▸ Some children may need more guidance. "Do you have enough to make one set of three? Do it. Do you have enough to make another set of three? Do it. Do you have enough to make another set of three? How many do you have left over? How many sets of three do you have? How many left over? Yes, you have two sets of three, and two left over."

For example, a child's arrangement of eight dolls might look like this:

Using equivalent sets and "leftovers," a child who has number concepts only to five can name as many as twenty-nine objects as five sets of five, and four more. Even if he can recognize the number of only those sets having no more than three members, he can name eleven as three sets of three, and two more. The use of equivalent sets and leftovers is analogous to the use of base in our numeration system. When the child can communicate number in such terms, he will be prepared for communicating number in terms of tens and ones. "Place value" in two-digit numerals would then be a way of representing something the child already sees as meaningful.

The numbers 6-10. Sets associated with each of the numbers six through ten should be manipulated and partitioned in ways similar to those suggested for the number five. The

teacher should move slowly in introducing these numbers. It is difficult for five-year-olds to establish strong images of these numbers because they cannot name the number without counting or partitioning. Then, too, the numbers zero through ten are the building blocks of our numeration system, and only as *they* are meaningful can numbers greater than ten be made meaningful. The children should, therefore, be given ample opportunity to establish meaning for each number before being introduced to the next number.



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counting

In order to develop counting as a meaningful process, the child must be asked to count only with number names that are meaningful to him, the order of the counting numbers must be seen as having some logical basis, and the process of counting must be introduced in such a way that it is seen as naming the number of successive sets.

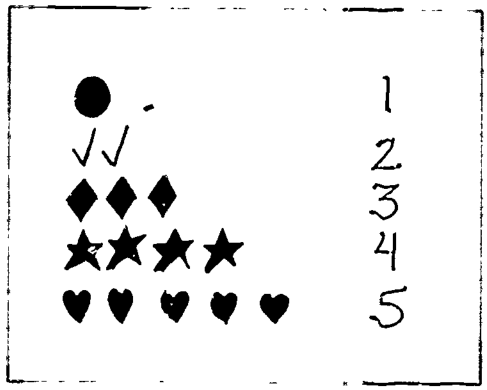
Because sets with five or more members require counting or partitioning for their number to be named, the process of counting should follow the establishment of meaning for one, two, three, four, and five. It is assumed that ordering as a process has been previously developed, and that the children have had experience with ordering sets so that each succeeding set has "more members" and also "one member more" than the set it follows.

The following activities give the child an opportunity to conclude that numbers appear in the counting order in such a way that each successive number is one more than the preceding one.

▸ Display in random order sets with one, two, three, four, and five members, as shown in ■

Have the children begin with the set of one and order the sets so that each set has one member more than the one it follows. Have the children tell the number of each set. If the numerals have been introduced earlier, have them displayed by each set (●).

Ask the children to look at the display and tell how many are in the set having one member more than the set of three. Continue with other questions that call attention to the "one more than" and "one fewer than" relationships between two adjoining sets.



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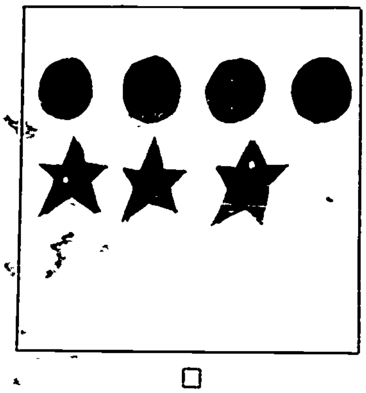
▸ Give children boxes containing pairs of sets with no more than five in any set, with one set having one member more than the other set. Have each child arrange his sets to see if they have the same number of members, or if one set has more members than the other. A child's arrangement might look like □.

Ask, "How many are in each of your sets? Which set has more members? How many more?" After one child has reported that he has sets with four members and three members and that the set with four has one member more than the set with three, ask each of several other children who have sets of four and



three if his set of four also has one member more than his set of three. Perhaps find other sets with four members and three members in the room and verify that all sets with four members have one member more than the sets with three members.

Repeated orderings of sets with one through five members so that each set has one member more than the set it follows will always produce the same order of sets: a set of one is always followed by a set of two, which is always followed by a set of three, and so on. Since the number of each set is named in succession, the word order is always "one, two, three, four, five."

▸ Again display, in a random arrangement, sets with one, two, three, four, and five members. Have the children order the sets so that each set has one member more than the set it follows. Then have the children name the number of the sets in order. Repeat with other similar orderings, each time asking the children to listen to the order of the words.

Such activities provide the child with the ordered set of number names. The next activities (1) apply the ordered set of number names to the members of one set, (2) emphasize that counting names the number for the sets so far counted, and (3) establish that the order in which the objects are counted does not change the accuracy of the count.

▸ Conceal in your hand or in a container a set with five members. "I have a set of tops. Let's find out how many there are." Take one top out and ask, "How many? Good, there is one." Take out another. "How many are there now? Good, there are two." Continue until the container is empty. "There are five. We counted and found that there are five."

Repeat with other sets of four or five, having the children take out the objects one by one. Each time, after a set has been counted, describe what has happened, noting the last number name used and its significance.

There is a reason to begin counting with a set whose members are not visible to the children. When the objects are taken out one by one, the child each time is naming the number of the set counted. When he says "one," he sees one. When he says "two," he sees two. By contrast, when the first counting experiences are with a set whose entire membership is fully visible and the child points to the objects as the counting progresses, it may seem to him that the number names are applied to individual objects and that a particular object is, for instance, named three.

The child's first experience with counting fully visible sets should be with sets whose number he already knows, and it should assure him that if he uses one number name in the counting order for one object, he can count the objects in any order.

▸ Give the child a set whose number he can see and name and whose members are distinguishable (perhaps four blocks—a red one, a blue one, a green one, and a yellow one). Let him experiment freely with counting his blocks several times—he can put them in different arrangements as he counts, he can begin with different blocks, he can proceed with a different order of blocks. He knows there are four, so

this experience is not counting to find out "how many" but rather to experiment with different ways of counting the same set.

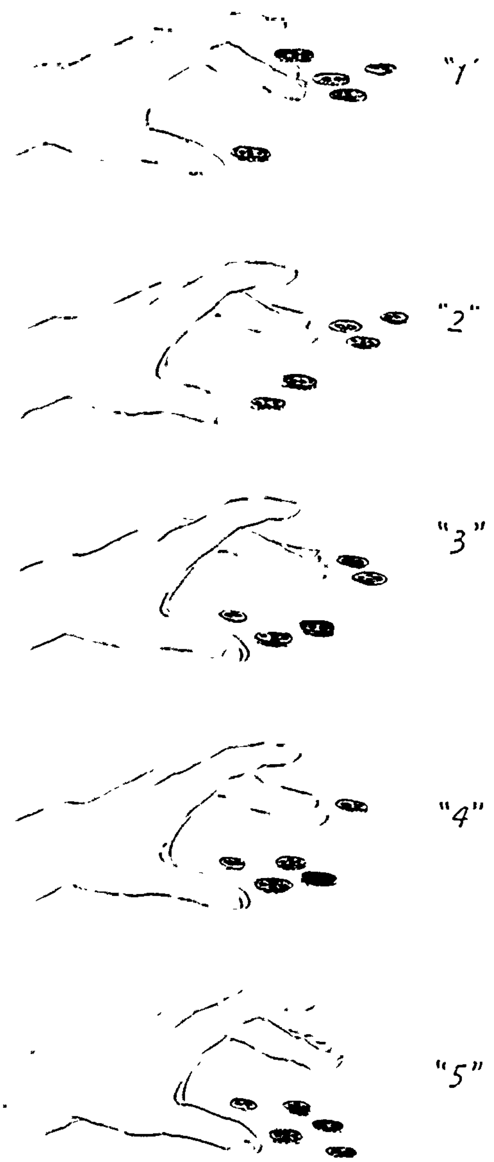
▸ Let the children see the teacher "make a mistake"—either failing to count objects or using more than one number name for an object so that the last number does not name the number in the set. When the number in the set is readily perceptible, they can confidently tell her she is wrong and how and where she went wrong.

After experiencing a variety of activities that use counting to name the number for sets whose number is already known, children develop confidence in using the process of counting to answer the question, How many?

As counting is used with sets with more members, the teacher can reinforce that a number name is associated with successive sets by moving the objects apart as they are counted or indicating the sets with her cupped hand, as in ■

It is easy enough to find opportunities for counting, but they should be restricted to those that require using only the number names for which meanings have been built. Counting to find how many children are in attendance in a class of twenty would require most pre-first-grade children to use words for which they have no meanings.

A child should be encouraged to vary his counting procedure by naming the number of an elemental set or some set whose number is known to him and then counting on to find the number of the set.



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▸ Show a child a set of nine members arranged in groups. Seeing, perhaps, a set of three members together, the child could name the number of that set, "three," and count on. "four, five, six, seven, eight, nine." If he knows that a set of three and a set of two is one arrangement of a set of five, he could start by naming the number of the set of five and then count on: "six, seven, eight, nine." Children should note that they can name the number in the set, nine, by using either procedure.

Many older children join sets and count from one to find the solution to any addition equation. With practice in "counting on," these children could name the number of the starting set and count on through the set that "came to join" to find the number of the union set. This procedure would help them retain the number of the starting set, attend to the whole equation, and, perhaps, remember the addition fact, rather than simply stating the number they found from a process of counting.

summary

The focus in this section has been (1) on developing meaning and names for some whole numbers and (2) on counting. These developments should come only after much experience with other mathematical situations and ideas.

The suggestion was made that many five-year-olds should be introduced only to the numbers zero through five. Because the numbers zero through ten are basic elements of the whole numbers, it is important that children have special and individual concepts for each of these numbers. Additional suggestions for activities with zero through ten and for beginning concepts related to the numeration system can be found in later chapters.

activities with numbers

When children have acquired stable concepts and language for some numbers, such activities as classifying, ordering, and measuring can include the use of number. Using number in these activities not only allows for their extension but also provides practice with those numbers with which the children are familiar.

classifying, comparing, and ordering

Because activities with classifying, comparing, and ordering are not changed qualitatively with the introduction of number, these three areas are considered together in this section. When children can recognize and name number, they can use it to describe objects being classified, compared, or ordered. They also can use number in the description of the rules for these activities. (Only numbers with which the children are familiar should be used.)

► Have the children put together all the sets that have the same number.

► Display pictures and have the children put together those pictures that show sets with the same number.

► Have the children sort sets of objects so that in one location are sets with, for example, eight members, in another location are sets with fewer than eight members, and in another location are sets with more than eight members.

► Have the children put together objects that have the same number of sides (corners, faces, edges).

► Show the children a set and have them find another set with one (two, three) member(s) more. "How do you know it has one member more? Show me."

► Show sets of different number and ask, "Which set has more members? How many more are in this set than in that one?"

► Again show sets of different number and have the children order them so that each set after the first has three (two, one) members more than the set it follows.

► Show the child two numeral cards at a time. Have him read them and arrange them in order with the one with the greater number on the right. Be sure to use only numerals that the child knows.

► Repeat the preceding activity using more than two cards or varying the order.

Teachers should be aware of the numbers with which the children have had experience and take every opportunity to use those numbers in the everyday classroom routines.

measuring

Being able to use numbers makes a significant difference in what children can do with measurement. When children can recognize and name number, they can use, and describe the use of, a repeated unit. They are no longer limited to saying, "This thing is longer than that one." With number, they can say, "This thing is almost as long as three of those." The use of a

repeated unit is basic to the process of measuring, and young children can be ingenious in devising such units. (Exploring what constitutes an appropriate unit might help prevent the confusion frequently seen in older children when they must choose appropriate units for area and volume and when they try to understand our standard units. However, neither the use of standard units nor formal information about equivalencies among those units seems appropriate for the preschool child.)

Linear measurement. In the sections on comparing and ordering, only height was considered. As children move into activities related to measuring length, they should be introduced to the more general process of measuring a line segment. Both height and length involve linear measurements. One speaks of how long a block is when it is placed in a horizontal position; it is measured for height when it is in a vertical position.

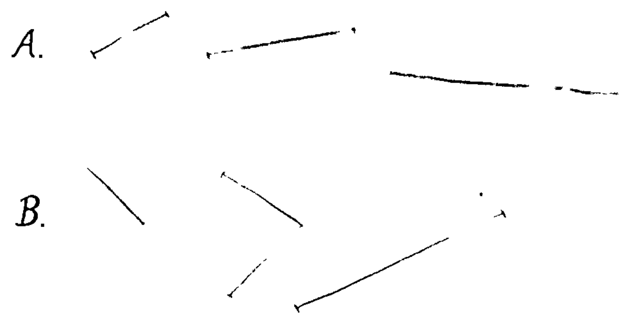
► As children build with cubical blocks, stand another, longer block by the stack and help them make the statement that is appropriate: "This block is not as long as the six you have stacked together," or "It is a little longer than four of them."

► Have one child stack blocks to see approximately how tall another child is. Then have that child lie down and the same blocks laid end to end to find out how "long" he is. Introduce the language for describing that he is longer than, about as long as, or shorter than, a specified number of blocks.

► Have the children place erasers, blocks, or any other objects of reasonable size along the edge of a table to see if it is longer than some other piece of furniture in the room.

► Show a picture such as the segments in A (■). Ask, "On which 'line' could you put the most candies (blocks, beans)?" Repeat when one "line" is broken, as in B.

► Ask the children to draw a line that is two blocks long (or as long as three of these candies, etc.) and to check to see how close they came.



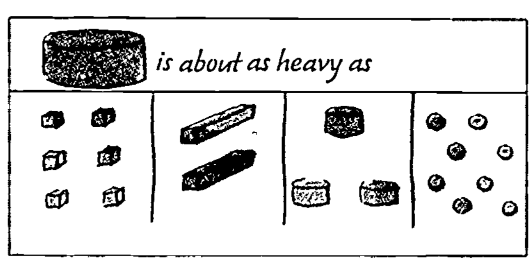
■

Weight

► Supply a balance beam. Have the children put an object in one pan and balance it as closely as possible with some other, lighter objects—perhaps one-centimeter cubes, metal nuts and bolts, marbles, or ball bearings—and describe what they see: "The ______ is almost as heavy as five of these."

► If possible, have available a spring scale with no markings on it and with the spring encased in a clear plastic cylinder so that the children can see how far down an object pulls the marker. Have them decide how many lighter objects it takes to pull the marker that far. The heavier object is "almost as heavy as" or "a little heavier than" or "just about as heavy as" some number of the lighter objects.

► Have the children record weight comparisons using repeated units as shown in ●. Let them place actual objects on the chart. Individual children or teams of children can work independently, or the work can be done as a directed activity.



●

Capacity

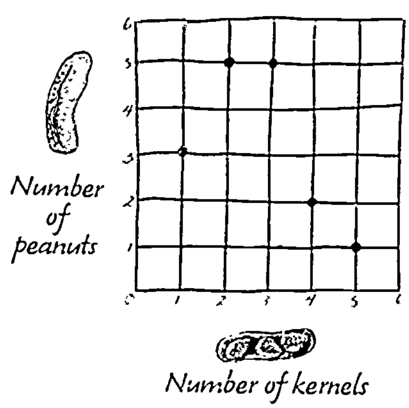
▸ With some pourable material, such as liquid or sand, and a collection of containers, including several of like size and shape, give children opportunities to pour from a larger container into several smaller, similar containers and describe what they have discovered. The big jar holds "as much as," "a little more than," or "a little less than" some number of the smaller jars.

▸ Have the children order objects on the basis of their capacity after they have filled them with repeated use of a smaller container. "The tall jar holds as much as two cups. The blue jar holds as much as three cups. The can holds as much as four cups. The tall jar holds the least, the blue jar holds a little more, and the can holds the most."

▸ Have the children record comparisons of capacity in the same way they did comparisons of weight, using real objects or drawings, or let the children express the comparisons in narrative form.

organizing and presenting data

With the ability to recognize and name numbers, the child no longer needs to rely on one-to-one correspondence to organize and present data. Quantities can now be represented numerically rather than pictorially. In contrast to the earlier suggestions for showing the number of kernels in a peanut shell, consider how that data might be organized if the children can count to five or six and can read those numerals.



□

▸ Have each child carefully open a peanut shell and count out how many kernels are in it. Have them assemble all the peanuts with the same number of kernels and count to see how many are in each group. Record the data on a chart in narrative form, as shown here.

3 peanuts had 1 kernel.
5 peanuts had 2 kernels.
5 peanuts had 3 kernels.
2 peanuts had 4 kernels.
1 peanut had 5 kernels.
Most peanuts had 2 or 3 kernels.
Fewest peanuts had 5 kernels.
No peanut had more than 5 kernels.

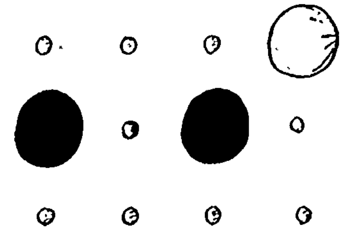
A graphic representation of the same information is shown in □

shape and pattern

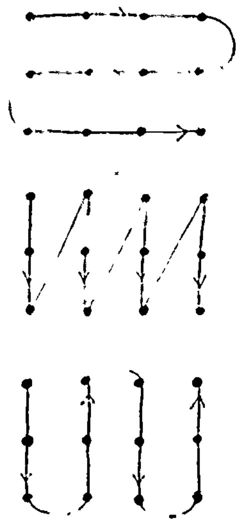
The use of number allows for different descriptions of shapes and patterns. For instance, the teacher and children can now talk about "the objects with three sides" or "the pattern with two blue blocks and three green blocks." Children can be asked about shapes and patterns with questions that require them to recognize and name number.

▸ Show the child a rectangular array of dots, staying within the limits of numbers that are meaningful to him and that he can count. Have him put a counter on each dot. "How many counters are there? How do you know?"

▸ Using the same arrangement of dots, put a colored rubber band around some counters or used colored counters. Give a number name to the red counter and have the children guess from which corner you started to count. Continue to ask questions such as, "If the red one is 5, what should we call the black one? If the pink one is 3, what should we call the black one now?"



"What would the names for the black one be if we use each of the following patterns?"

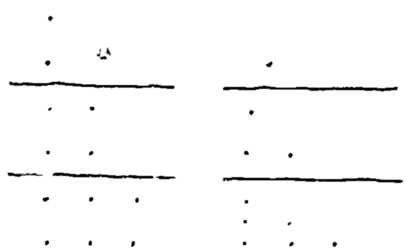


▸ Show the child a pattern of numerals and have him decide what comes next. (Use only those numerals that are meaningful to him.)

1	2	3	1	2	

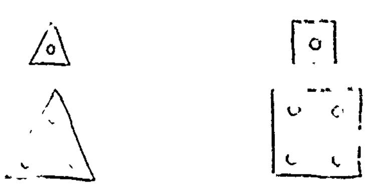
5	4	3	2	

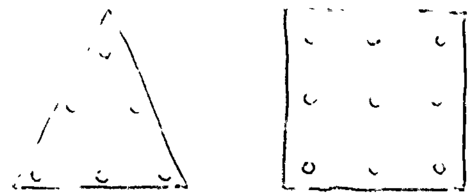
▸ Show one of the following patterns, or a similar one:



Have the children put a counter or block on each dot. "How are the three parts alike or the same? How are they different? Can you make another part that will look like these but be different? Another?"

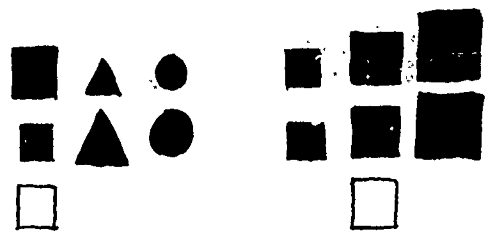
▸ Show the child one of the following sequences:





"Can you make the next one?"

▸ Begin a pattern with blocks so that the rows and columns vary in size, shape, or color: Have the child find the appropriate blocks for the empty places and tell how he knew what should be used. After the pattern is completed, cover or remove one object and ask the child what has been removed from view.



summary

The activities suggested in this section are only a few of the many kinds of things that can be done with young children who have acquired concepts and language for some whole numbers and who can count meaningfully. In later chapters, the reader will find extensions of the areas considered here and additional activities for children who are ready for them.

concluding remarks

The activities in this chapter are designed for children who are ready for directed study. Although this directed study need not take place in a group setting, most children work well in groups. For children who are at ease in the school situation, who are self-directed, who trust the adults in charge, and who have sufficient language to carry on dialogues and conversations about the planned experiences,

large-group instruction has some advantages. The social setting of large groups can be used (1) to foster group consciousness, social skills, and communications skills, such as those of repartee and argument; (2) to increase pupil exchange and minimize teacher domination; and (3) to build a sense of responsibility and appreciation for each other. If a teacher plans to work with groups as large as twelve to twenty children, she must develop her skills of differential questioning and task assignment to allow each child to participate successfully at his own level; learn to take full advantage of the interaction between children having different abilities, experiential backgrounds, and motivation; and make a careful selection of activities and materials to avoid having many children look on while only one or two perform a task.

For younger children, and for five-year-olds who need more language practice and who are less secure in a group and with the adults in the school, small-group instruction can be more profitable. Small groups allow more interaction with the adult, thus providing a more dependable language model; offer each child more opportunity for language practice; and offer the teacher an opportunity to observe more carefully each child's response to the tasks given him.

Since it is possible for a child to manipulate and observe materials without perceiving the mathematical significance of his actions, it is essential to have in any program verbal interchange between the learner and some knowing, verbal person. Language can be acquired only in dialogue with someone who possesses language.

When directed instruction in mathematics is a part of the school program, it needs to be balanced with self-initiated activity, that is, times when children may independently and individually manipulate and observe materials and perhaps record their observations. Even when a carefully planned, sequential program involving directed group instruction is used, the teacher should not overlook the tremendous value of the child's play with materials in self-chosen activities.

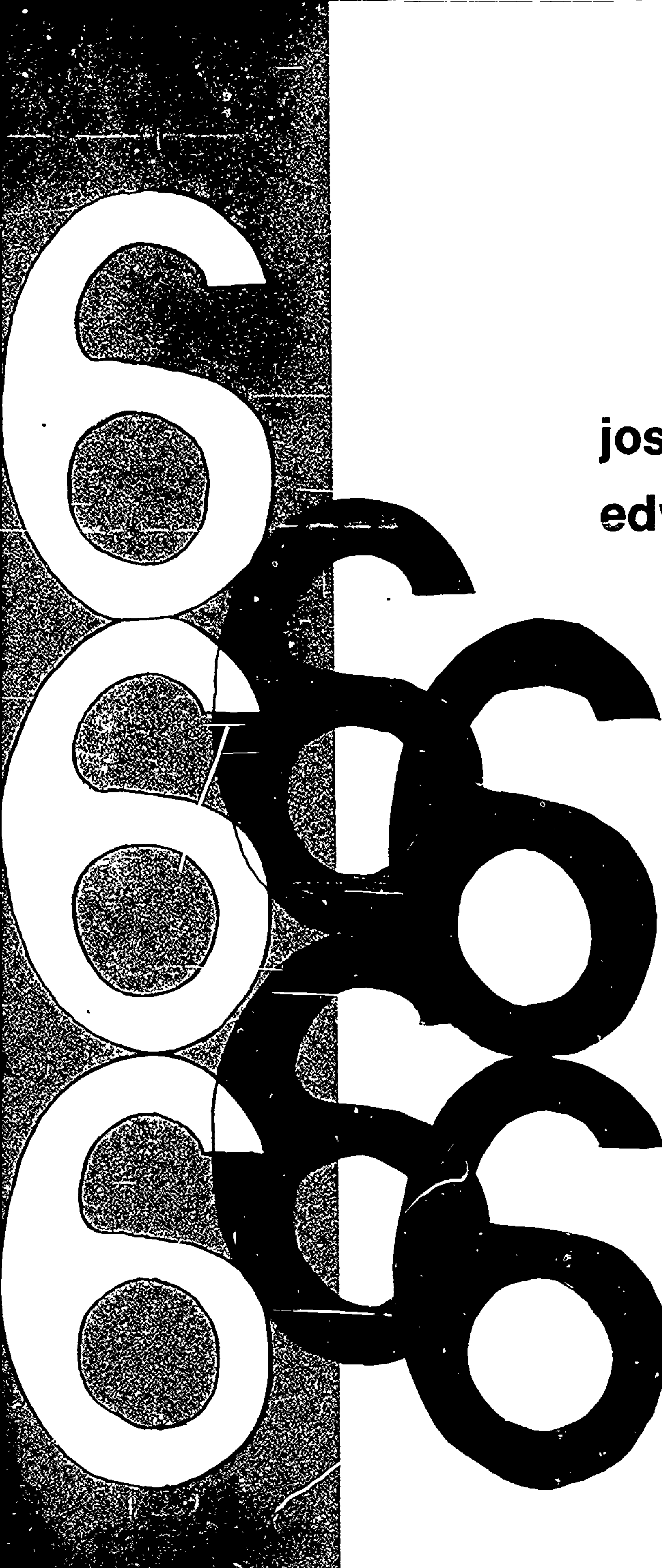
No suggestion has been made for using behaviorally defined objectives or for developing a sequence of carefully determined and minutely ordered abilities. It is hoped that when provision is made for learning that is insightful rather than totally incremental, at least some children can experience the thrill of a "flight of intuition." By not employing mastery tasks with the very young, teachers can prolong their opportunity to learn from their involvement and for the fun of it. When the ability to do and to say is considered an integral part of each lesson or teaching episode, the teacher has ample opportunity to make judgments about each child's understanding and his acquisition of new knowledge and language. Progress between children and even within each child will be uneven, but this will not prevent children of varying abilities and backgrounds from responding satisfactorily to a common experience.

It is suggested (1) that mathematics instruction for young children emphasize concept development and language acquisition, (2) that instruction be planned to allow the child maximum use of oral language, (3) that a primary goal of instruction be to build the child's confidence in his perception and cognition and in his powers to seek out information and draw conclusions from it, (4) that movement to new language, and especially to written symbols, be unhurried, and (5) that instruction be planned to allow and encourage the perception, description, and extension of patterns, for "the mathematician, like the poet and painter, is a maker of patterns."

reference

Bruner, Jerome S. "The Growth of Mind." *American Psychologist* 20 (1965):1007-17.

number & numeration



joseph payne
edward rathmell

How old am I? When is my next birthday?
How much do I need to grow to be as big as you are?
How many are coming to the party?
Who has the most—you or me?
What time is my TV program? What channel?

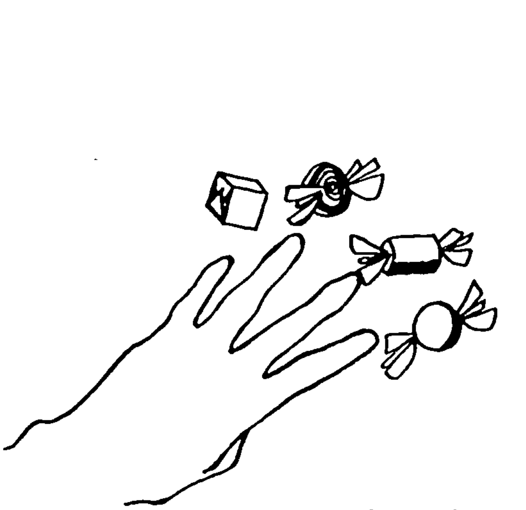
These questions reveal personal needs children have for number ideas, for number language, and for number symbolism. Children encounter problem situations that require quantitative solutions and number responses. Parents ask children quantitative questions. Number ideas and language permeate children's television programs.

At times the use of number ideas and language is indirect and unplanned. At other times the content is systematically planned and taught to children. In either event, the need for numbers and the language and symbolism of numbers is universal.

early stages with number

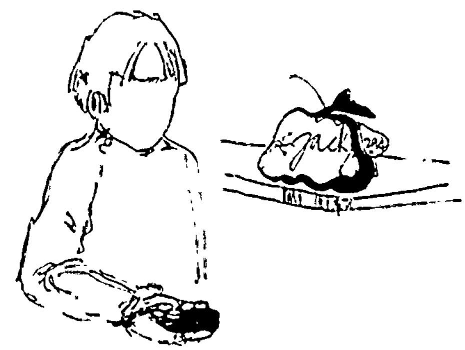
Children very probably begin their number and quantitative notions with comparisons of size. Early in life the child is aware that he is not as big as his mother or his older brother. He knows he cannot reach the top of the cabinet to get a drink of water. These size comparisons are not very precisely delineated for the child. At times the child gives attention—

- to length—"You are higher up";
- to area—"My hand can't cover yours";



How many pieces of candy?

How old are you?



Do I have enough to buy the toy?

- to weight—"I can lift the cat but not the dog";
- to volume—"That pile of dirt is bigger."

These early perceptual comparisons help children develop the vocabulary needed for expressing quantitative ideas. For example, words such as *same, more, not as many, just as many, higher, lower, bigger,* and *smaller* arise naturally in such situations.

Whatever the features chosen, some of them involve comparisons between amounts that are about the same. "We have the same amount of cookies," a child might say, thinking either of the quantity or of two sets that match one to one. It is from the idea of matching one to one that number idea emerges.

The main component of the concept of whole number is the classification of sets that are in one-to-one correspondence. If the elements of two sets match one to one, then the two sets have the *same* number. One set has *just as many* as the other. When a child holds up the correct number of fingers in response to the question, "How many pieces of candy are there?" he is showing *the* essential feature of the attribute called number.

The concept of one-to-one correspondence requires that other features, or attributes, be excluded. The texture, the feel, the softness or hardness, the color, the length, and the position must receive no attention. Consequently, initial developmental activities should aim to help children exclude such irrelevant features. Chapter 5 provides many suggestions for this initial experience.

If two sets match one to one, they do have the same number of members. The *specific* number of members, however, is of use most often, and it needs a name, both oral and written.

The oral name "two" or the written symbol "2" may get attached simply by association to any set that corresponds to the eyes of a normal person. Enough examples may have occurred for the name to come to tell how many are in a set. Likewise, the name "three" and the symbol "3" may get attached to the wheels of a tricycle. With "two" and "three," counting may not have been necessary for the child. Counting will, however, likely accompany the learning of the names.

Counting is itself an example of one-to-one correspondence, though more abstract than matching two sets of objects. In counting, a correspondence is made between objects and the number names. Wilder, a mathematician well recognized for his work in foundations of mathematics, states this idea clearly (1968, p. 34):

> True counting is a process whereby a correspondence is set up between objects of the collection to be counted and certain symbols, verbal or written. As practiced today, the symbols used are the natural number symbols 1, 2, 3, and so on. But any other symbols will do, tally marks on a stick, knots on a string, or marks on paper such as ~~////~~ // are all sufficient for elementary counting purposes. Counting is, then, a *symbolic process* employed only by man, the sole symbol-creating animal.

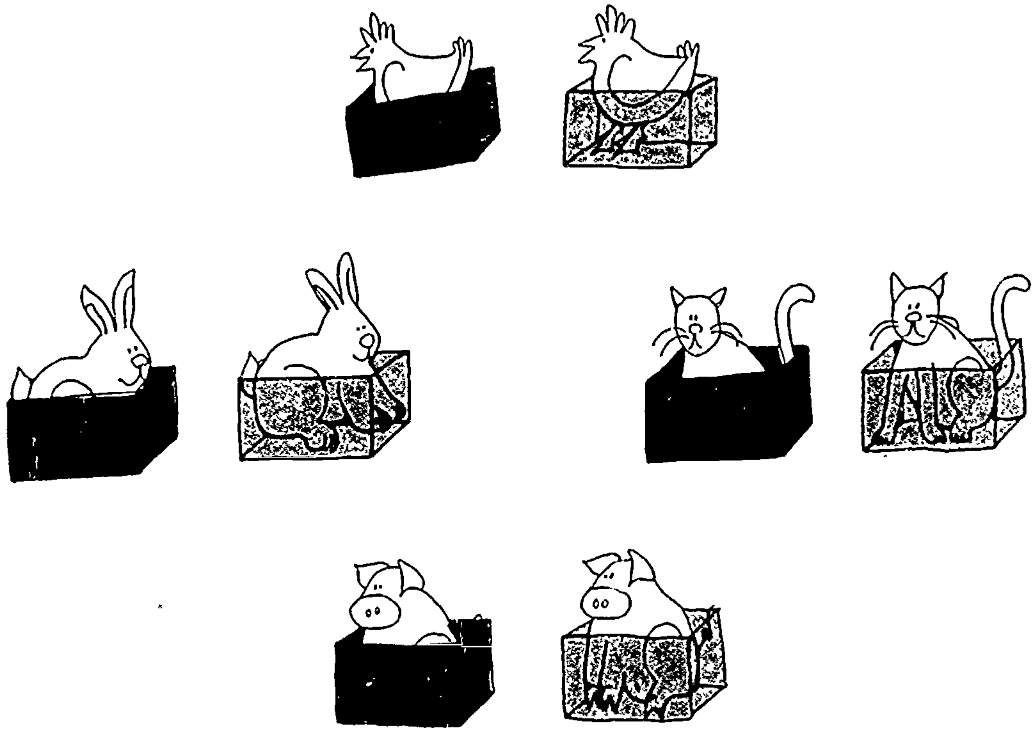
In counting a set of five members, one object is matched as each name is said. one, two, three, four, five.' The last name said tells the number of members in the set counted. Thus 'five' identifies the last object counted as well as the total number in the set. More is said on this later.

Learning to distinguish those transformations of a set that leave the number of its members unchanged from those that change the number of members is of major importance in the learning of number in the early stages. The transformation most often reported is a change in position of the objects in a set. When a child views a change of position as not affecting the number, the child is *conserving* that number.

conservation

More has been written on the topic of conservation than on any other aspect of the young child's mathematical development. Piaget and his associates have written extensively about the topic. In turn, their initial findings and conjectures have led to extensive research and voluminous writing by psychologists and educators in the United States and other countries. It is difficult to interpret the total effect that all this work has had on educational practice. It does seem certain, however, that one result is better understanding of what children know at various ages.

The age at which conservation occurs, the reasons for lack of conservation, and the effects of training have been studied extensively. The classic conservation task for number begins with two sets of objects, the same number in each set. Through questioning, it is determined that the child recognizes that the sets match one to one. In the initial setting, language such as "same," "just as many," "one here for each one there" is used. Then the objects in one of the sets are rearranged, and the child is asked, "Do both sets have the same number of things, or does one have more than the other?" When the child recognizes that the rearrangement of objects does not change the number, the child is said to conserve. Piaget has said that conservation occurs at about age six or seven.

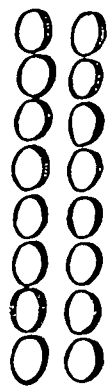


Task 1. Before transformation

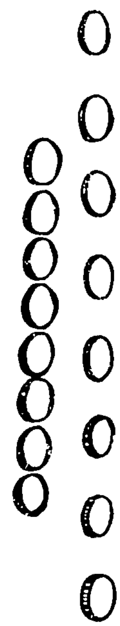
In a study of conservation, Miller, Heldmeyer, and Miller (1973) found that perceptual clues affected conservation, and they pointed to the developmental nature of conservation. They studied 64 children ranging in age from three years and one month to five years and ten months, with a mean age of four years and four months. (Sixteen children were excluded from the study because they did not have facility with the language of "same number," "more," "not as many," and "just as many.") On task 1, the experimenter conversed with the child to emphasize that the animals were the same kind and that a cage of one color was for the child and a cage of another color was for the experimenter. The purpose was to provide the child with perceptual clues that emphasized one-to-one correspondence. The questions asked were, "Do we both have the same number of animals or does one of us have more animals?" After the child replied, he was asked, "How did you figure that out?"

On task 1, 77 percent of the children answered the conservation question correctly, and 50 percent of the children could give an adequate explanation of conservation.

In task 2, flat beads were arranged in a vertical line before and after the transformation. No perceptual clues were provided. Only 41 percent of the children were correct on this typical conservation task, and only 22 percent were able to give an adequate explanation of conservation. It seems clear that perceptual clues helped more children perform the conservation task correctly.



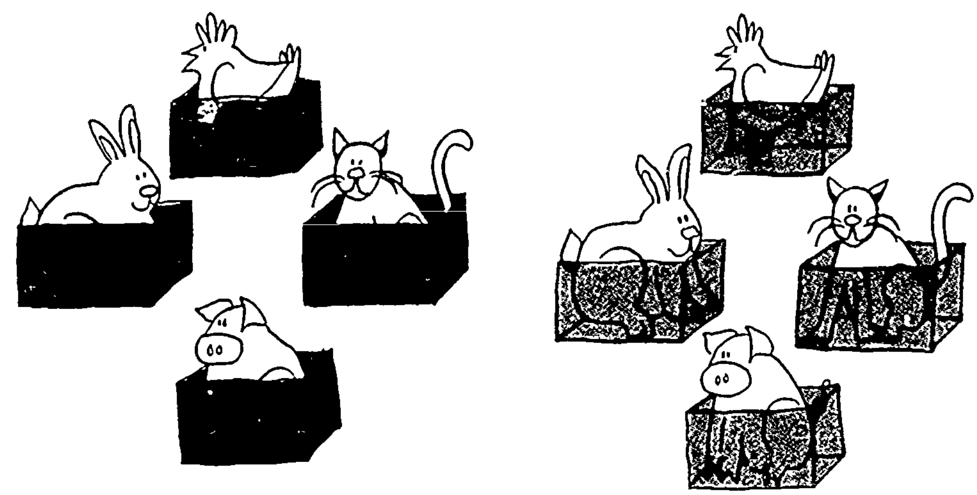
Task 2 Before transformation



Task 2 After transformation

Six other conservation tasks, having difficulty levels between the two tasks described, were included in this study. The tasks varied the use of color, the kind of objects, the number of objects, and the linear or nonlinear placement of the objects.

The experimenters found that any response made after the arrangement of objects was transformed by length substantially reduced the number of children who conserved. Length was a major perceptual distraction. The *kind* of objects, however, seemed to have no effect,



Task 1. After transformation

and the effect of the number of objects was not conclusive, although for objects that seemed to go together, more children conserved with four objects than with eight objects. When objects were arranged linearly, about the same percent conserved using four objects as eight objects, which suggests the overriding influence of the linear arrangement.

> The children in this study demonstrated a beginning understanding of conservation of number at an earlier age than that generally suggested by the literature. For example, 17 of the 20 three-year-olds made at least one conservation judgment, and 15 of these children could also supply at least one adequate explanation. [Miller, Heldmeyer, and Miller 1973, p. 9]

Gelman, in an earlier study, found that "young children fail to conserve because of inattention to relevant quantitative relationships and attention to irrelevant features in classical conservation tests" (1969, p. 167). She also found that discrimination training with feedback helped children with a median age of five years and four and a half months who did not conserve. She provided tasks with objects varying in color, size and shape (large or small rectangular or circular chips), starting arrangements (horizontal, vertical, and geometrically arranged rows of chips or sticks), and number of objects. When a task was presented to the child, he was provided immediate feedback on whether or not he was correct. The training and testing took place in a three-day period.

At the end of the training, the children had attained a performance level of approximately 95 percent correct on number problems and on problems involving length. On a test two to three weeks later, the same results were found.

The importance of feedback was evident in her study. Children who were given the tasks with no feedback performed only slightly better than children in a control group that used "junk" stimuli, essentially a placebo treatment.

It seems clear that a key component in attaining conservation of number is the attention the learner gives to one-to-one correspondence under different arrangements and the extent to which he can ignore irrelevant attributes.

knowledge of entering kindergartners

Any planned mathematics program must take into account the knowledge already possessed by the children. Helpful data have been obtained by Rea and Reys. They took a cross section of 727 urban children entering kindergarten and assessed their knowledge of mathematics. Some excerpts from their results related to number follow (1971, pp. 391-92):

Oral Names for Numerals

Numeral	*% of Correct Responses*
1	83
3	71
4	71
5	71
10	38
16	17
21	7

Sequence	
What comes next? (Count orally: "one, two, three")	90
What comes next? (Count orally: "five, six, seven")	88
What comes after three?	75
What comes before three?	43
Cardinal Number	
Give me three disks.	82
Give me seven disks.	55
Give me thirteen disks.	34
How many pins are here? (Show card with three pins)	92
How many stars are here? (Show card with eight stars)	72
Ordinal Number	
Point to the second ball.	48
Point to the first tree.	70
Point to the third ball.	25
Group Comparison	
Which group has more (Show four spades and three buckets)	72
Is there a disk for each child? (Show four disks and four children)	88

Among the conclusions drawn by Rea and Reys are these (pp. 393-94):

> Skills in counting and recognition of small groups were well developed by most of the entering kindergarteners.

> Development in ordinal relationships has not reached the level of competence displayed in cardinality.
>
> Most children were able to make accurate small group comparisons.

Another part of the study was reported in the *Arithmetic Teacher*. There Rea and Reys reported that almost 40 percent of entering kindergarten children could count by rote to twenty and about the same percent could count rationally to twenty. more than 75 percent could count by rote beyond ten and about the same percent could count rationally beyond ten (1970. p. 68).

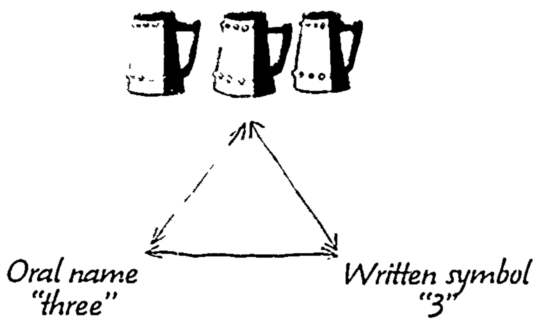
activities for numbers 0 to 10

Most children have some well-developed ideas about number at the time they enter kindergarten. The kind of meaningful activities for these children will depend on the background of particular groups of children and particular children in a group. Overall, the activities should be directed at the following five major objectives:

1 *Matching sets.* This includes matching objects of two sets one to one to determine if the two sets have the same number. If the two sets do not match one to one, then one set is identified as having "fewer" and the other "more."

Rearranging objects should be perceived as a transformation that does not change the number of members. Removing or adding an object should be recognized as a transformation that does change the number of members.

2. *Relating a set of objects, the oral name, and the written symbol to show "how many"* (cardinal number). These relations can be viewed in this triangular arrangement:



Each of the three paired relations—set ↔ oral name, set ↔ written symbol, and oral name ↔ written symbol—requires two questions to assess the objective: (1) for example, given a set, the child is asked to give the oral name; (2) given the oral name, the child is asked to choose the set. The child's production of the written symbol, "3," should come substantially after the recognition stage for the numeral.

3. *Ordering numbers 0 to 10.* This includes arranging sets of from zero to ten members in order, counting in order to ten, and giving the names "one, two, three, . . ," and "first, second, third, . . ." to positions of objects in an ordered set (ordinal number).

4. *Recognizing sets of from two to five members without counting.* This is facilitated by patterned arrangements of objects.

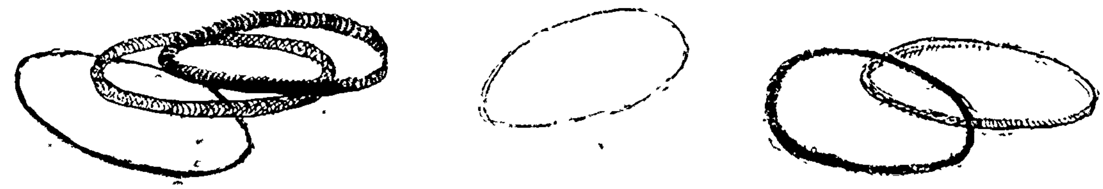
5. *Separating sets of up to ten members into subsets and naming the numbers of members in each.* This is in preparation for addition and subtraction and also for place value.

matching sets and the numbers 0 to 10

Activities for matching should include the physical pairing of real objects and subsequent questions about the number of members. At the same time, the objects can be counted to find out what the number is. This gives the child two dependable and varied ways to work with number. By making this variety available, the teacher provides the child with the flexibility of thought needed in his subsequent use of number.

Activities involving matching should begin with natural situations where there is some need for the matching. The language should suggest the reversibility of one-to-one correspondence.

► Match parts of the body. "Look at your shoes and think about your feet. Is there a shoe for each foot? A foot for each shoe?" Then ask about the numbers of the two sets. "How many shoes do you have? How many feet?"



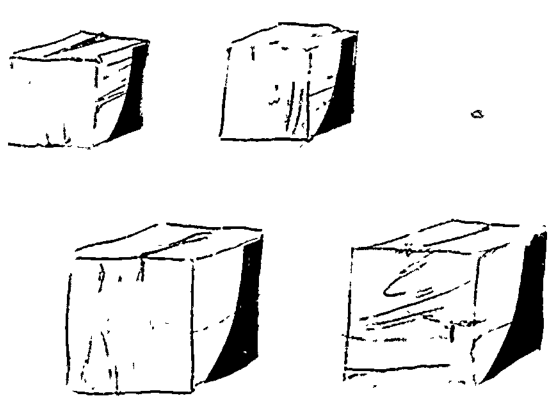
■

Repeat with parts that do not match one to one. "Do you have a nose for each hand? Do you have the same number of noses as hands? How many hands do you have? How many noses?"

► Use situations where matching seems natural in the classroom setting. "From this box of pencils, take one for each hand. Now is a pencil in this hand? In the other hand? Is a hand with each pencil? How many hands do you have? How many pencils do you have in your hands?"

► Vary the procedure and the way questions are asked. "Which set of bracelets gives you one for each arm? How many arms do you have? How many bracelets do you need?" (■.)

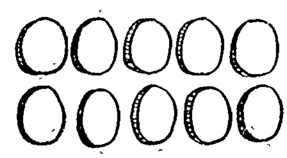
► Vary the size of the objects used. Place a small block of wood in each hand of a child. "Is there a block of wood for each hand? Does each hand have a block? How many hands have you? How many blocks of wood are in your hands?"



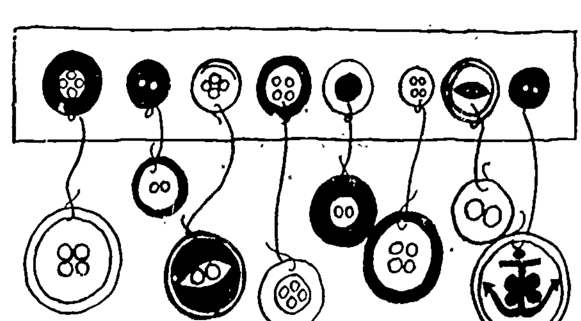
► Use the larger blocks of wood and repeat the questions. Then ask further questions. "Which blocks are bigger? Longer? Do the blocks match one to one? How many small blocks are there? How many large blocks?"

Activities like these help a child distinguish number from size, and they provide experience with transitivity (hands have the same number as small blocks, and hands have the same number as large blocks; so small blocks have the same number as large blocks).

► Use rows of equal-sized chips (poker chips are excellent). "Look at the two rows of chips. What do you see that is the same?" Children may say something about color, about the size of the chips, and almost certainly about the fact that they stretch out the same, that is, that they are the same length. Pursue the matching and number questions. "Is there a chip in the top row for each chip in the bottom row? Is there a chip in the bottom row for each chip in the top row? Count the row of chips on top. Now without counting, can you tell me how many chips are in the bottom row? Do I still have five chips in this row? How can you find out?" To the last question, the child may match to find out or he may count.



► To help cue for number instead of length, link each member of one set to each member of another set with a piece of masking tape or string (buttons are durable and easy to use).



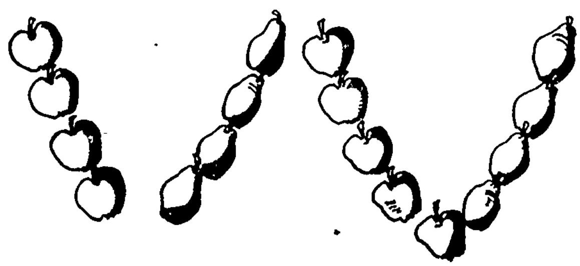
"Look at the top row of buttons. Is a string in each button? Does each string go to another button? Is there a button on top for each button

on the bottom? Is there a button on the bottom for each button on top? How many buttons are on top? Can you tell without counting how many buttons are on the bottom?"

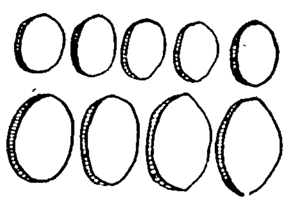
Spread out the buttons on the bottom and ask similar questions to bring out the idea that the number does not change with different arrangements.

As an extension of this activity, hold in your hand three of the buttons in the bottom row. "Can you begin here—with three—and count the rest of the buttons?" Repeat with a different number of buttons in your hand. This "counting on" strategy is a valuable one for children to develop.

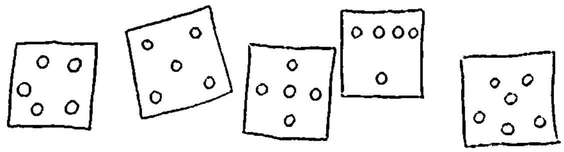
▸ Contrast sets that match one to one and those that do not. Develop the language of "more" and "fewer." "Look at the two sets at the left Is there a pear for each apple? An apple for each pear? How many apples? How many pears?" Then point to the sets on the right. "Do the sets match one to one? Is there the same number of pears as apples? Which set has more members? Which set has fewer members?"



Ask similar questions using two rows of chips with the chips in one row larger than those in the other row. This again brings out the need to focus on matching instead of on the size of the objects.



▸ Make "Match Me" cards. For each number one through ten, make a set of five cards with five different arrangements of dots representing that number. Substitute a card from one set for a card in another set. Ask the child, "Which card does not belong?"



A child can answer the question by matching the dots visually or by counting the dots. Mark the back of the odd card with colored tape to allow for individualized work.

▸ Use rhythmic activities for matching. "Let's clap our hands together. Each time we do it, say, 'Clap.' Do we say something each time we clap our hands?" Variations include beats on a drum or other musical instrument or taps of the finger on a desk. The sounds are matched with an oral response.

Matching members of one set—whether claps of the hands, sounds on a musical instrument, or objects themselves—with verbal sounds is one essential part of oral counting. Oral counting can also be done as hands are clapped or as beats are made with fingers or a musical instrument. As an extension of this activity, tap your fingers three times and have the children count silently. Then tell the number. Repeat with other numbers and have the children say the number. Challenge the children to go as far as they can.

▸ Use matching activities for larger numbers for which the children do not have the names. "Look at the chairs in the room. Is there a person on each chair? Is there a chair left over? Are there more people, more chairs, or the same number of each?" Pose similar questions as cups are passed out for juice. Stress matching as a way to tell if two sets do or do not have the same number and which set has more or fewer members.

The previous activities emphasize matching and oral language. In such activities, quantitative language—"same," "more," "fewer," "just as many," "one here for each one there," "match," "big," "small," and so on—should be deliberately used.

▸ Make a set of "Show Me" numeral cards for each child. Use cards about seven centimeters by ten centimeters. Put one of the numerals 0 through 9 on each card. Tell the children to hold up the correct numeral card when a set is displayed or when the oral name is given. "I am holding up some fingers. You pick the card that shows how many I am holding up. When I say 'Show me,' hold up your card." It is easy to check the entire group this way, and each child is responding to the question.

Repeat with questions that use the oral name. "I am going to say the name of a number. You hold up the card that goes with the name when I say 'Show me.' "

▸ Make a set of "Show Me" dot cards for each child. Use cards about seven centimeters by ten centimeters. Put a different set of dots on each card to represent the numbers 0 through 9. Arrange the dots in some regular pattern for ease in determining the number and for practice in recognizing the number without counting. Dots can be drawn on the cards or gummed stars may be affixed. Have the children respond to the written symbol for numbers and to the oral names of numbers by holding up a card. "I am going to say a number name When I do, you pick the correct dot card. When I say 'Show me,' hold it up," Repeat the questions using a numeral written on the chalkboard.

▸ Recognition of the numerals is helped by talking about the essential features of the symbols. Establish the meaning of "straight" and "curved." Then write the digits on the chalkboard. "Which numerals have straight marks?" (1, 2, 4, 5, 6, 7, 9) "Which ones have curves?" (2, 3, 5, 6, 8, 9, 0) "Hold your finger to show the way the straight marks go in each." (Have them show the finger in vertical and horizontal positions, depending on the numeral.) "Which ones have loops that are closed up?" (6, 8, 9, 0) "Which ones have both curves and straight marks?" (2, 5, 6, 9)

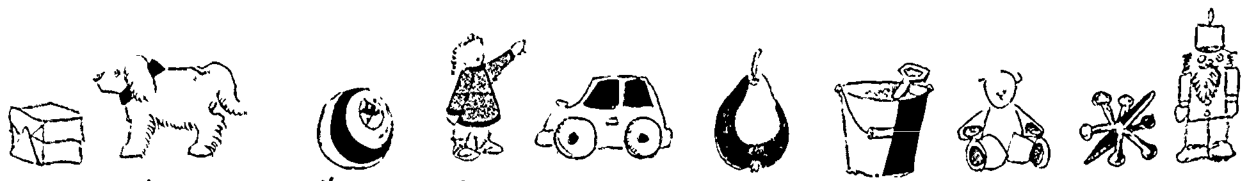
▸ Make sets of "Match Me" numeral cards for the numerals 0 through 9, with five cards in each set. Then rearrange so that four cards have the same numeral on them and the fifth card has a different, but similar, numeral. For example, four cards might have a "9" and the other card a "6." Have the child choose the one that is different. Mark the different card on the back so that the children can check themselves.

comparing, ordering, and ordinal numbers

Comparing and ordering using sets of objects, oral names, and written symbols are both implicit and explicit in many of the prior activities. Children should be able (1) to compare two numbers and tell which is greater and which is less, (2) to arrange any three numbers so that the first one is least and the last one is greatest, (3) to give the number "before" and the number "after" a given number, (4) to arrange the numbers 0 to 10 in order, and (5) to give the ordinal names "one, two, three, . . ." and "first, second, third, . . ." As the numbers 0 to 10 are learned, comparing and ordering questions arise naturally. Some other activities follow.

▸ Use the "Show Me" numeral cards and the "Show Me" dot cards. Choose two cards from one of the sets. Put one on the chalkboard tray. "Does the other card go 'before' or 'after' the card in the tray?" Now choose three cards. Put two of the cards in the tray. "Does the other card go 'before,' 'in between,' or 'after' the two on the tray? Can you arrange all the numeral cards in order? Can you arrange the dot cards in order?"

Counting itself gives one ordinal name for each object. When counting children in line, you say "three" as you point to a given child.



one *two* *three* *four* *five* *six* *seven* *eight* *nine* *ten*

first *second* *third* *fourth* *fifth* *sixth* *seventh* *eighth* *ninth* *tenth*

■

That child is "number three"—an ordinal use of numbers. It would seem reasonable that children do indeed use ordinal numbers just as well as cardinal numbers. The difficulty is more likely to be with the special ordinal names of "first, second, third, . . ." and not with ordinal ideas themselves. The names for "third" through "tenth" are easily related to the names "three" through "ten." The special name "first" is learned early. The name "second" must be learned separately. (See ■.)

► Have children count off as they line up to go to lunch or recess. Count "one, two, three, . . ." Then count off "first, second, third, . . ." "Will the 'number five' hold up his hand? Will the fifth person hold up his hand?"

► Show how the ordinal name tells not only "which one" but "how many" as well. "Will the fourth person hold up his hand? Are you 'number four' also?" (Yes) "How many children are in line, counting you?" (Four) Ask similar questions using pages in a book. "Turn to page 10 [an ordinal idea]. How many pages are there to here, counting the tenth one?" (Ten)

Knowing how many numbers are in a set tells nothing about how the members are ordered. However, knowing the ordinal name of an object does tell the cardinal number of the set containing that member and all before it.

reproducing symbols

Although it is not unusual for children of ages three through five to begin trying to write their names and certain numerals, most educators prefer to delay formal writing until age five or six. The main reason for this delay is the lack of development of the small muscles needed to write. However, it is doubtful that any harm is done if children, on their own initiative, begin early to write the numerals; indeed, many children get much satisfaction from it.

The following activities are suggested for the time when a child is learning to write the numerals.

► Trace the numerals in the air, using the large muscles. Then have the child trace the same motion, making sure enough space is between successive children to avoid collision. As the tracing is done in the air, talk about the shapes used to make the numerals. Short descriptions help: 1—"straight down"; 2—"around and back"; 3—"around and around again"; 4—"down and out, then cross it down"; 5—"straight down and around, put the hat on top", 6—"down and around"; 7—"out and down", 8—"down around one way and back up the other"; 9—"around and down"; 0—"all the way around."

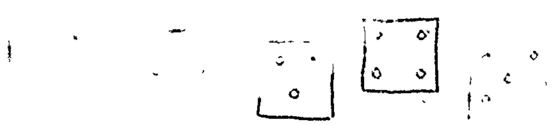
► Make the numerals from sandpaper or felt and paste them on cards. Put a red mark to show where the child is to begin the numeral. Allow the child to trace the numeral many times, using the rough texture for touch. Another child, a teacher aide, or the teacher can check to see if the child can make the numeral in the air first by looking at what he traced and then without looking. It is difficult for a child to check himself because he can be reversing and not be aware of it.

Dots can be used to show patterns for the children to follow in writing numerals. A cross mark might show where to begin.

recognizing sets of two through five without counting

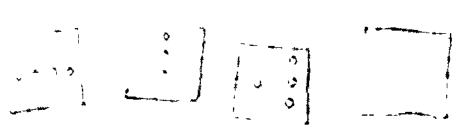
A major goal of the extensive activities and experiences with numbers in the age range three through six is to enable children to recognize without counting the number of members in sets containing zero through five members. This ability is especially important as the numbers six through ten are studied, as grouping is done to teach the base ten of our numeration system, and as addition and subtraction are learned. For numbers beyond five, it is difficult to recognize the number without counting or without making subsets whose numbers are easily recognized.

Initial instruction should involve arrangements of objects and dots that are easy to see, such as the ones shown in this diagram.



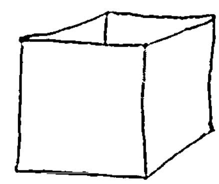
► Hold up a dot card, allow no more than two seconds for the child to see it, and then ask, "How many dots?" After the children see what they are to do, let them practice in pairs. (This activity could easily be carried out by an older child or by a teacher aide.)

As the child gains proficiency in recognizing the number in easy arrangements, use a wider variety of arrangements. Several arrangements of four are suggested in this diagram:



The number zero. Usually the number zero is not introduced until some other numbers are studied, the concepts well established, the numbers ordered, and the written and oral symbols learned. Frequently the study of zero, then, comes after the numbers one, two, three, and four have been studied. It should be treated as any other number and should be given its correct name, "zero," from the start.

The number property of the empty set is zero. The word name "zero" and the written name "0" show the number. The number zero answers the question, *How many members in the box?*



► Using a paper bag or metal can, place objects inside the container and move it back and forth. "Do I have objects inside? Can you hear the objects move?" Then remove the objects and ask similar questions to illustrate the empty set. Introduce the oral and written names as the way to answer the question, "How many objects are there?"

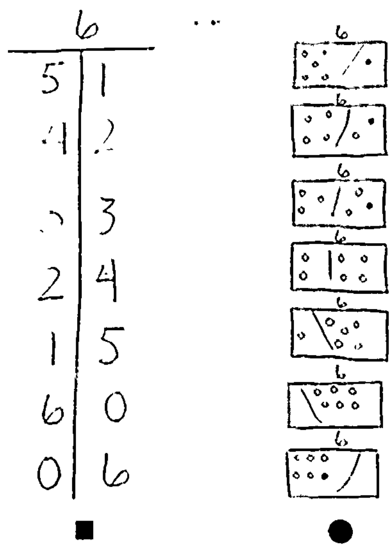
Another effective activity is to have all pupils extend their five fingers. "How many fingers?" Then turn one down and ask, "How many fingers now?" As each finger is turned down, ask the pupils to tell how many. When all the fingers are turned down, it will then seem natural to answer the question "How many?" by replying, "Zero."

separating sets into subsets and naming the numbers

As numbers beyond two are studied, sets should be partitioned and the numbers named. For example, the chart shows the numbers of the two subsets of a set of six partitioned just once (■).

The partition can be shown on the chalkboard or on a felt board using a piece of yarn. For an individual activity, a child can make several sets of six and do his own partitions (●).

Words that children could use to describe this action include *split, part, partition,* or *separate*. Using the words *whole* and *parts* emphasizes the essential relation to be learned. Sets should be partitioned more than once. Six, for



example, might be partitioned as 2, 3, and 1; equal partitions, such as 2, 2, and 2, should also be made.

Activities that focus on partitioning a set and naming the numbers of the subsets have two important goals. (1) the recognition of the subset or inclusion relation, and (2) preparation for addition and subtraction.

To focus on goal 1, such questions as these can be asked: "Where is the whole set? Where are the parts? Can you cover the whole set with one hand? Can you use one hand to cover one part? The other hand to cover the other part?"

The preparation for addition and subtraction is very direct. The concept of addition involves naming the number of the "whole" when two "parts" that do not overlap are joined. For subtraction, the concept is giving the number of one "part" when the numbers for the "whole" and the other "part" are known.

Partitions also provide a good opportunity for "counting on." Cover four objects—perhaps just recognizing them as "four" rather than counting them—and then "count on" to get six.

Competence with the numbers zero to ten is essential for the meaningful learning of larger numbers. It is essential that the relation of the sets of objects, the written symbols, and the numerals be well understood, as well as the order relations. The progress chart at the end of the chapter suggests a way that information on these topics may be collected by the teacher.

numeration

The term *numeration* is used here to denote those concepts, skills, and understandings necessary for naming and processing numbers ten or greater. The purposes of this section are to describe the components of numeration and to present a sequence of learning experiences that will help children deal meaningfully with number.

Most children are introduced to numbers greater than ten by counting. Hence, it is only natural for them to associate cardinality with these numbers. They think of eleven simply as one more than ten and twelve as one more than eleven.

Although counting is essential, children have much greater use for thinking of numbers between 10 and 100 as tens and ones. Thinking using tens and ones, called *base representation,* gives the child flexibility and facility in dealing with a great variety of tasks. For example, base representation helps with oral naming and counting, with writing numerals, and with comparing numbers; moreover, it is essential for all algorithmic work with whole numbers.

Learning about numeration is not a singular task. Children need to develop many concepts and skills before they can deal with numbers flexibly. The major components of this complex of ideas called numeration are listed below. A brief description is included here, with greater elaboration in the following sections.

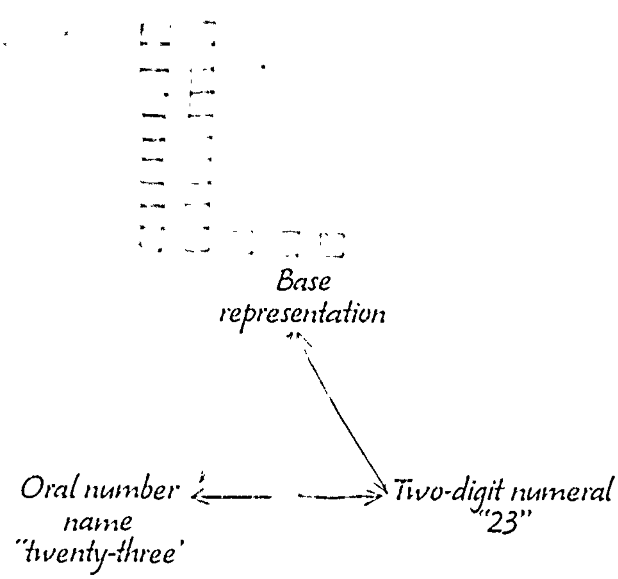
1. *Grouping once and naming the number of groups.* The essential notion is grouping objects into equivalent sets. The number of groups and the number of ones left over are then recorded orally and in written form. For example, a set of sticks may be grouped by tens. If there are two groups of ten and three single sticks left over, the number is noted as "2 tens 3 ones."

2. *Scheme for grouping more than once.* Not only are ten ones grouped to form one ten, but also ten tens to form one hundred, ten hundreds to form one thousand, and so on. A generalized scheme for grouping needs to be considered prior to the introduction of hundreds.

3. *Scheme for recording groups.* After grouping and naming the number of groups, a positional scheme is developed for writing the numerals. This is often referred to simply as "place value." It should be noted that place value requires grouping as a prerequisite.

4. *Representing numbers three ways.* The number of objects in a set can be represented by (*a*) the oral number name, (*b*) the written numeral, and (*c*) a base representation. Base representation means some form that directly indicates the number of groups, for example, tens and ones.

5. *Translating from one representation to another.* Students should learn to translate from any representation of a number to the other two forms of representation, as indicated by the three pairs of translations shown.



If the child can easily shift from one representation to another, he will be able to deal with tasks more flexibly. For example, to add 32 and 45, a child thinks "3 tens 2 ones" and "4 tens 5 ones." Then he adds the tens and the ones, getting 7 tens 7 ones, and converts back to 77. Similarly, ease in translation is of great use in ordering, in approximating and estimating, and in regrouping and renaming.

a sequence for learning two-digit numerals

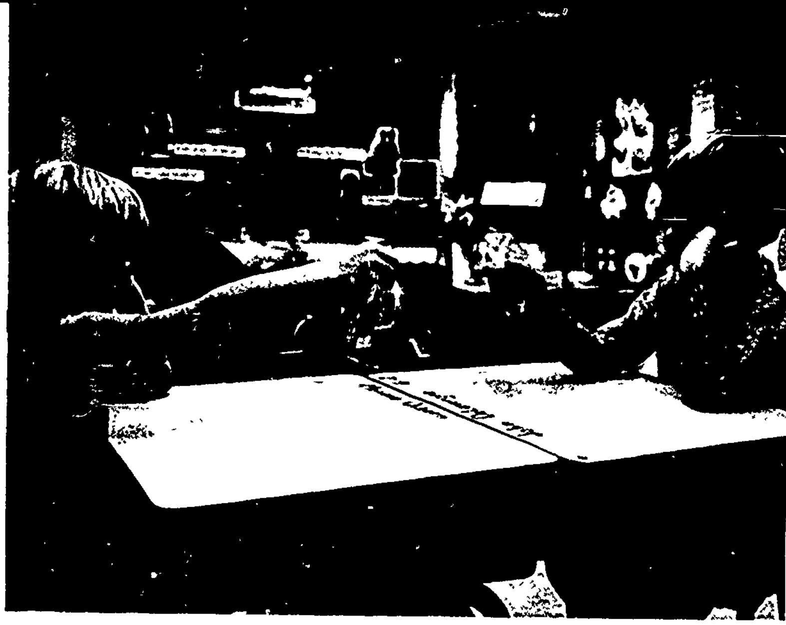
Four major units on numeration are suggested in the following sequence. Suggestions for activities useful in teaching the units are included.

At each stage, the main concern has been the development of a *meaningful* relation between the various numeration components. Each stage begins with few prerequisites and builds competence in such a way that rote learning is minimized.

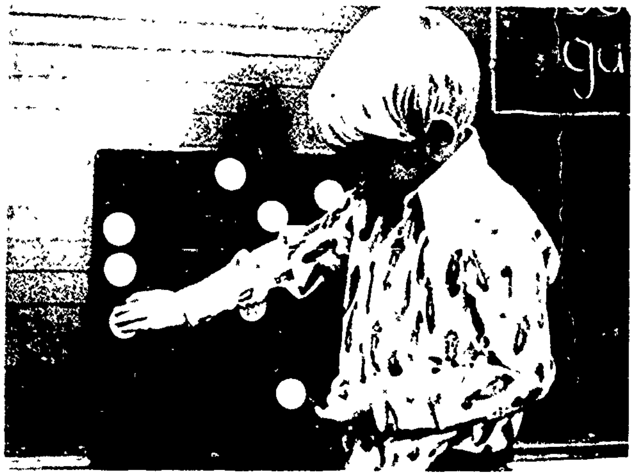
unit 1: grouping once and naming the groups

The primary objective is to teach children to group objects into equivalent sets and name orally the number of sets and single objects left over. Preliminary work on grouping objects has been suggested for inclusion in the learning of the numbers zero to ten. This unit builds on these experiences and extends through a mastery of grouping by tens and ones.

▸ Group objects by four, using plastic bags. "Here are some blocks and some plastic bags. Put four blocks in a bag. Are there enough blocks left over to put another set of four in a

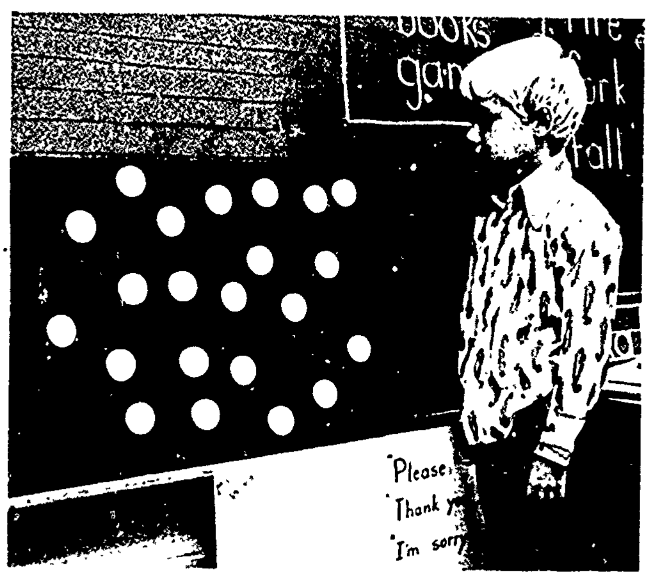


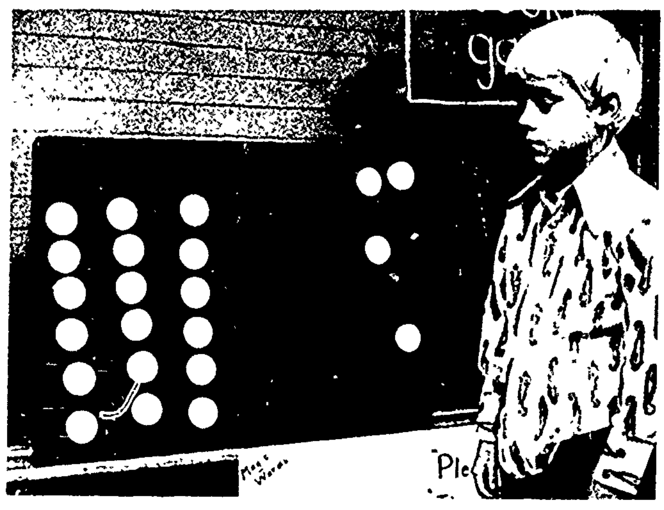
bag?" (Yes) "How many bags of blocks are there?" (Two) "Are there any blocks left over?" (No) Use the bags and group the blocks by fives and by sixes. Have the children tell the number of groups orally.



► Use different kinds of objects, such as sticks and counters. Display sixteen sticks. "Make a group of five sticks. Are there enough sticks left over to make another group of five?" (Yes) "Make as many groups of five as you can. How many groups of five are there?" (Three) "How many sticks are left over?" (One) "Are there enough sticks left over to make another group of five?" (No)

"Group these counters by sixes. How many sixes are there?" (Three) "How many counters left over?" (Four) "Are there enough counters left over to make another group of six?" (No)





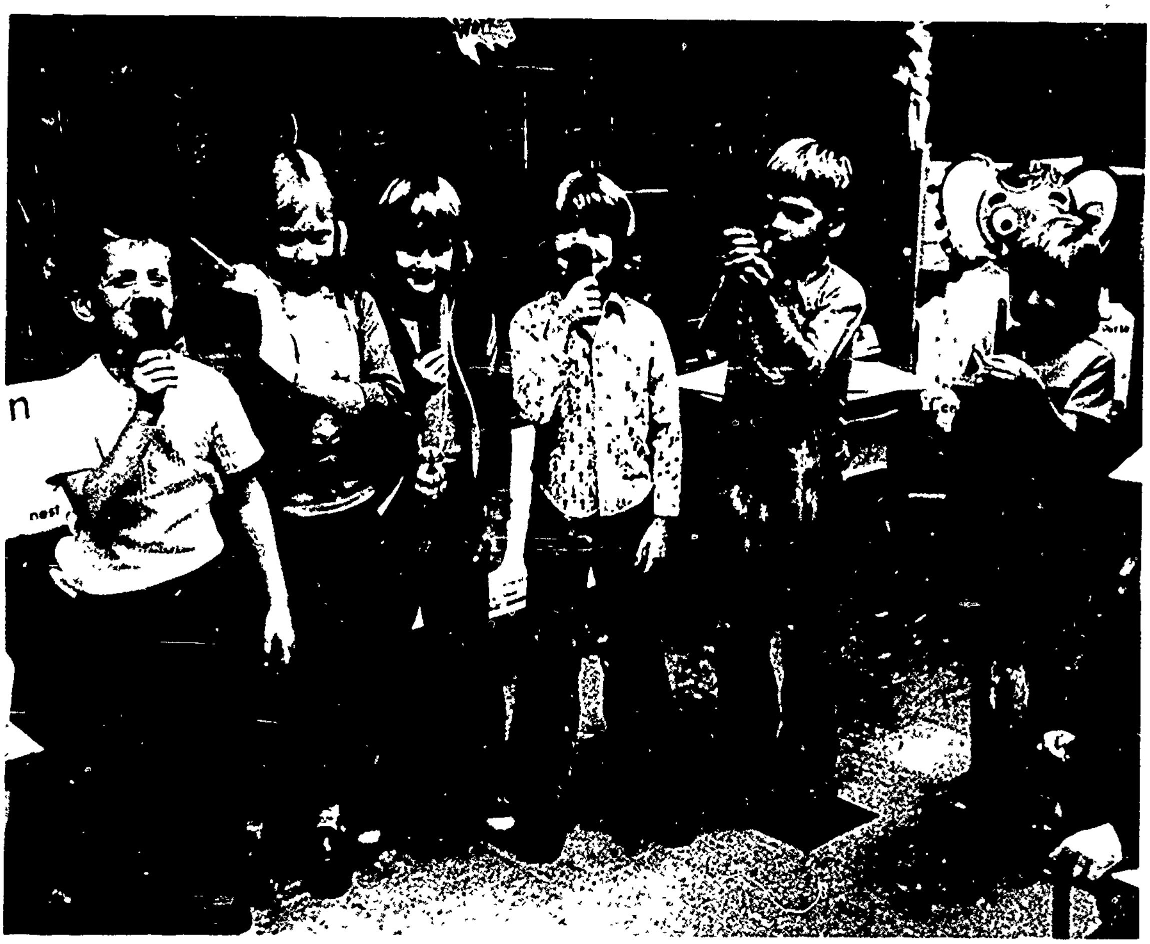
Not only are situations like the ones above preparation for grouping objects by tens, but they also help children to consider a group as a unit just as individual objects are considered units. They help children avoid saying "thirty" in

response to the question (when three tens are shown), "How many tens?"

Next, grouping by tens can be emphasized.

▶ Display thirty-four counters in a random arrangement on a flannel or magnetic board. "Let's group the counters by tens. Are there enough counters to make a set of ten?" (Yes) Have a child line up a set of ten counters on the left side of the board. "Are there enough counters left over to make another set of ten?" (Yes) Let another child show a second set of ten from the remaining counters. Continue this process until the grouping is completed. "Are there enough counters left over to make another set of ten?" (No) "How many sets of ten are there?" (Three) "How many ones are left over?" (Four) "That's correct, there are three tens and four ones."

▶ For a similar class activity, group sticks that have been placed in a box. "Ed, come and take ten sticks from the box. Now, Julie, will you come and take ten." Repeat until all tens have been taken. When fewer than ten sticks remain, have the child taking those sticks stand slightly apart from the others. "Are there enough sticks left over to make another set of ten?" (No) "How many sets of ten are there?" (Five) "How many sticks are left over?" (Two) "Yes, there are five tens and two ones."



With all experiences in grouping, the essential questions are these: "Are there enough objects left over to make another set of ten? How many sets of ten are there? How many objects are left over?" The number of tens and ones is then reported.

The reverse task is also important. When asked, children should be able to show the appropriate number of tens and ones. After developing competence in constructing five tens, two tens, four tens, and so on, the children could construct representations of tens and ones. The girl in the photograph has shown "three tens two ones."

For individual work, each child can be given a set of materials. Coffee stirrers, tongue depressors, Popsicle sticks, or squares cut from heavy paper or cardboard work well. The squares, shown in the photograph above, can easily be drawn by the children. Tall, skinny rectangles represent tens, and little squares represent ones.

Evaluation should be based on the ability to group by tens, to describe groupings of tens and ones orally, and to construct a concrete representation from an oral description of a grouping. Repeated use of the question "Are there enough objects left over to make another set of ten?" should help children focus on the idea of forming as many sets of ten as possible when they are grouping by tens.

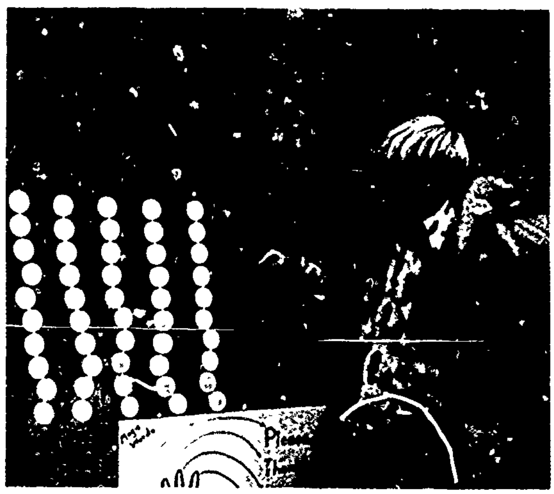
unit 2: oral number names for tens and ones

The major objective of unit 2 is to help children relate groups of ten to the usual oral names. It is here that the child relates the numerosity of a set to some base representation of the set.

The first task of the child is to learn the names for multiples of ten—"ten, twenty, thirty, . . ." One reason for beginning here is that these names are prerequisite to naming numbers such as "3 tens 2 ones" as "thirty-two." Another reason is related to the counting skills that children need to acquire. By learning the names "ten, twenty, thirty, . . ., ninety," children have an effective but simple scheme for naming the number or counting to one hundred.

The knowledge that some children already have in counting by ones can be used to show two ways to count to fifty: counting by ones and counting by tens.

▸ Display fifty counters in a random order. "How many counters are there?" The teacher and children count them together by ones. As they are counted, the teacher groups them by tens. "Yes, there are fifty counters. How many counters are in each group?" (Ten) "When objects are grouped by tens, there is a different way to count them. We say, 'Ten [point to the first group], twenty,' and so on. This is called counting by tens."

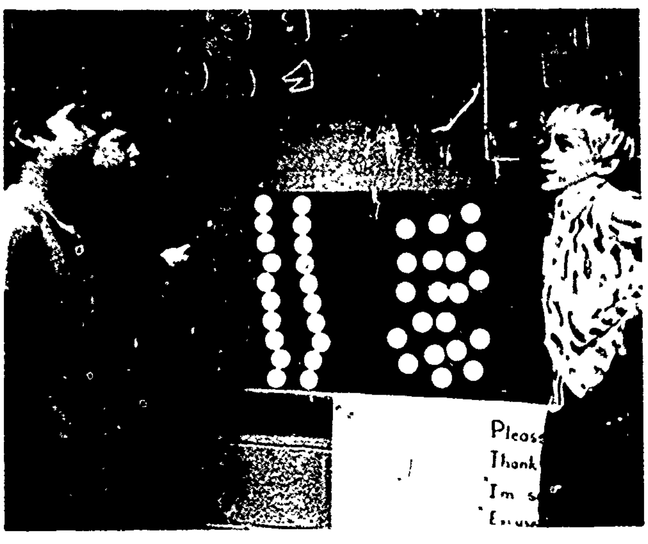


Remove a set of ten. "How many counters are there now? Count them by ones. What do you get?" (Forty) Now count them by tens. What do you get? (Forty) "Do you get the same name when you count both ways?" (Yes)

Remove another set of ten. "Count them by tens. How many are there now?" (Thirty) "What number name would you get if you counted by ones?" (Thirty)

Most children have considerable confidence in their ability to count by ones. The purpose of these activities is to develop a similar confidence in naming numbers when counting by tens. Children need to be thoroughly convinced that counting by tens and counting by ones yield the same result.

► Show a given number of objects grouped by ten and the same number of objects arranged in a random order. Have a student place two sets of ten on one side. Have another student count out twenty counters by ones and place them at the other end. Point to the two tens. "How many counters are at this end?" (Twenty) "How many counters are at the other end?" (Twenty) "Are more counters at one end than the other end, or is the same number at both ends?"



An interesting discussion should follow. There will probably be some disagreement. Guidance should be provided to help compare the sets. Previous experience with comparison by constructing a one-to-one correspondence can be used. The counters could be lined up in a one-to-one correspondence, or yarn used to match counters, or perhaps some children could pick up one counter from each end until all the counters have been removed to see if it "comes out even." Some children may suggest counting the two sets of ten by ones. This procedure should be supplemented with a matching comparison.

With the understandings and skills described above, children are ready to relate the number of tens meaningfully with the usual number names. This task is not difficult. They quickly learn to name concrete representations two different ways.

The photograph on the next page shows physical representations for one ten, for two tens, and for three tens. By using such representations, count two ways—"one ten, two tens, three tens" and "ten, twenty, thirty."

Learning the names can be facilitated by comparing the names of the number of tens with the names of the numbers one through nine.

one ten	*two* tens	*three* tens	*four* tens
ten	*twen*ty	*thir*ty	*for*ty

five tens	*six* tens	*seven* tens
*fif*ty	*six*ty	*seven*ty

eight tens	*nine* tens
*eigh*ty	*nine*ty

Note that all names beyond ten end in *-ty*. The *ty* means *ten*. Discuss the similarity and differences in names:

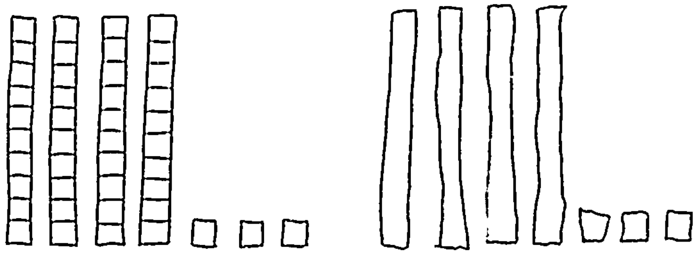
- "twen-"—sounds like "twins", twins mean *two*
- "thir-"—sounds like "third"
- "for-"—sounds just like "four"
- "fif-"—sounds like "fifth"
- "sixty through ninety"—just like the names you know

The child should be able to move easily from the name using "tens" to the usual name, and conversely. Varied experiences are needed in counting objects by ones, by tens using the name "tens," and by using the usual names for tens. Since the same concrete representation is used for all the ways to name, the child has a simple but powerful way to think about numbers, regardless of the form in which a task is posed.

Next, numbers other than multiples of ten can be considered. The usual oral number names should be developed for groupings of tens and ones.

Because the students already know that two tens is twenty, that three tens is thirty, and so on, they should have little difficulty developing the name "forty-three" from a grouping of four tens and three ones. The translation is nearly direct. The reverse task of constructing a concrete representation that corresponds to an oral number name is also relatively easy.

If ten-unit strips and unit squares are used as materials, tens and ones can be easily represented by children pictorially. Since they enjoy drawing, this variation of the task is a pleasant change for them. An example of the materials and a first grader's drawing of them is shown.



In counting the number shown, children should learn to count by tens as far as they can and then count the ones. They would count, "Ten, twenty, thirty, forty, forty-one, forty-two, forty-three."

Next the teens can be considered. It should be pointed out that "eleven" and "twelve" are special names for "1 ten 1 one" and "1 ten 2 ones." The other names are all backwards. For example, in "fourteen," the part referring to four is named first and indicates ones. It should be made explicit that the names are backwards and that *teen* indicates "one ten."

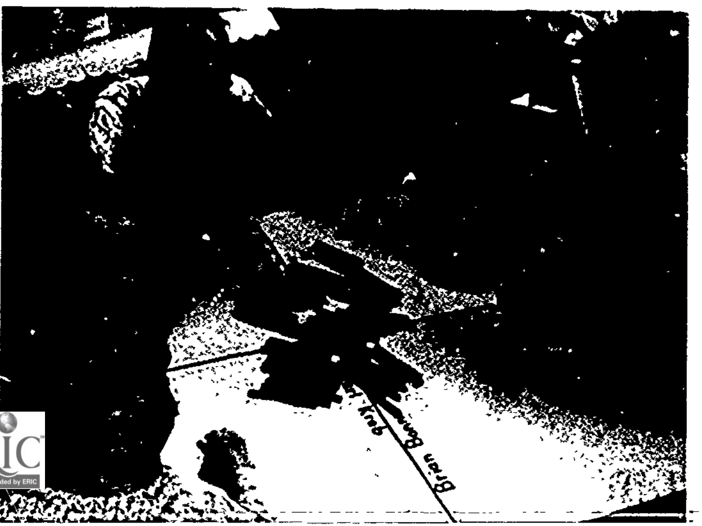
The naming irregularities cause many errors in naming the teens. Frequent reminders to stop and think about the naming for the teens will help. Many experiences discriminating between numbers such as sixteen and sixty-one, eighteen and eighty-one, and so on are also necessary.

Evaluation should be based on the children's ability both to group objects by ten and give the correct name and to construct groups of tens and ones when they are given the name.

The children's depth of understanding can be assessed with tasks such as these:

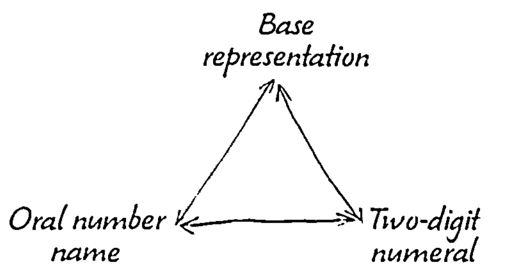
▸ Show some bundles of ten sticks and several single sticks.

1. Say, "Pick up forty sticks." See if the child needs to count by ones or if he chooses four bundles immediately. Repeat with other numbers, such as thirty-two.
2. Point to the tens and then to a large group of single sticks. Ask, "Which set has more, or do they have the same?" See if the child arranges the single sticks in tens for ease in comparison.
3. "Choose as many sticks as there are fingers on all our hands." See if the child chooses by groups or by single sticks. "Give two names for the number of fingers." See if the child can give the two names.



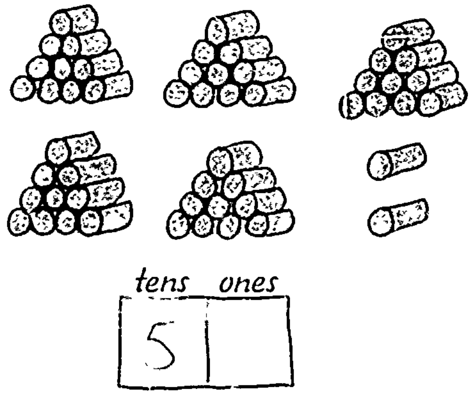
unit 3: two-digit numerals

The major objective is to enable the child to write and interpret two-digit numerals and to relate two-digit numerals to the usual oral names This completes the initial instruction for the translation tasks suggested in component 5.

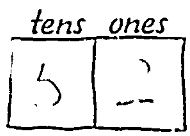


The prerequisites for the instruction up to this point in the sequence are minimal; however, before moving into unit 3, children should be able to read and write the numerals 0 through 9. They will also need to recognize the written words *tens* and *ones*.

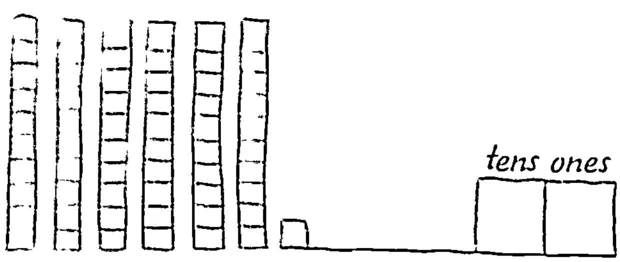
▸ Show a grouping of tens and ones. "How many tens and ones are there?" (Five tens and two ones) "We can keep a record of how many tens and ones there are by writing numerals under the words *tens* and *ones*. How many tens are there?" (Five) Write the numeral 5 under the word *tens* to show five tens. "How many ones are left over?" (Two) Write the numeral 2



under the word *ones* to show two ones. "This chart tells us that there are five tens and two ones."

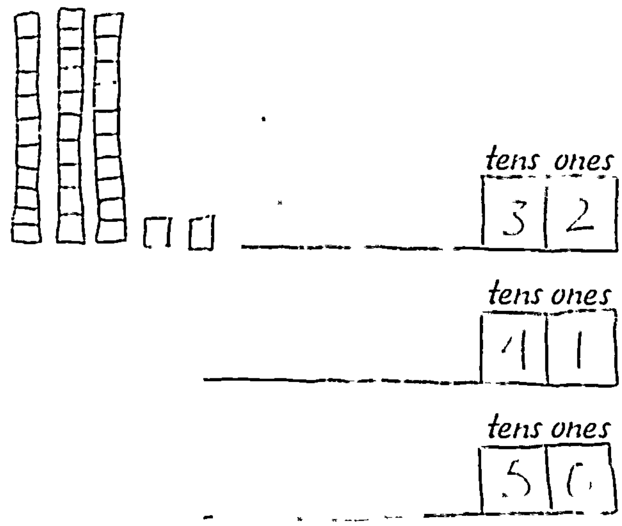


After discussing several examples, distribute worksheets on which the students can record the number of tens and ones. Drawings of tens and ones could be used when the children are working individually. Concrete objects could be used when the class is recording the numerals together.



Include a review of grouping by tens by varying the task as shown at the top of page 146.

Next have the children draw pictures of tens and ones for the numerals provided on a chart.

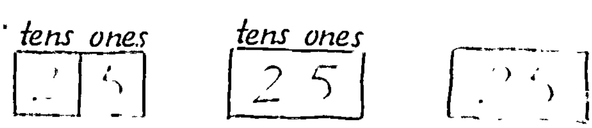


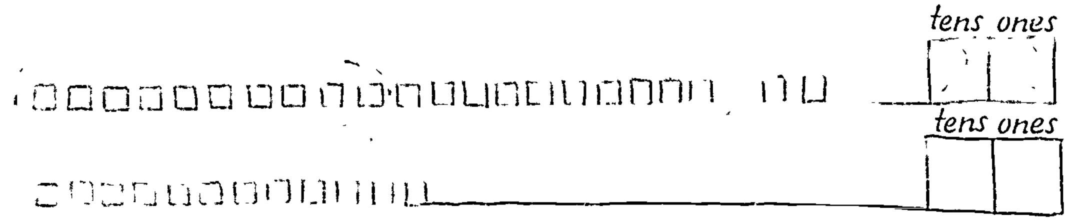
Finally, all the missing parts could be filled in, as shown in the chart at the bottom of page 146.

Children will generally attack these worksheets with great enthusiasm. After the task is clear, the only serious problem they have is dealing with zero. Special attention must be given to the fact that "no ones" is recorded by writing the numeral 0 under the word *ones*.

Similar tasks could be performed by letting children form their own sets of tens and ones from graph paper. They can cut strips ten units long for tens and single squares for ones. These cutouts can then be matched to the appropriate tens and ones charts.

Soon after beginning this unit, the students can learn to write two-digit numerals. First have them record the numerals in charts without a separating line. Then after a few experiences like this, they will be able to write two-digit numerals without reference to the words *tens* and *ones*.

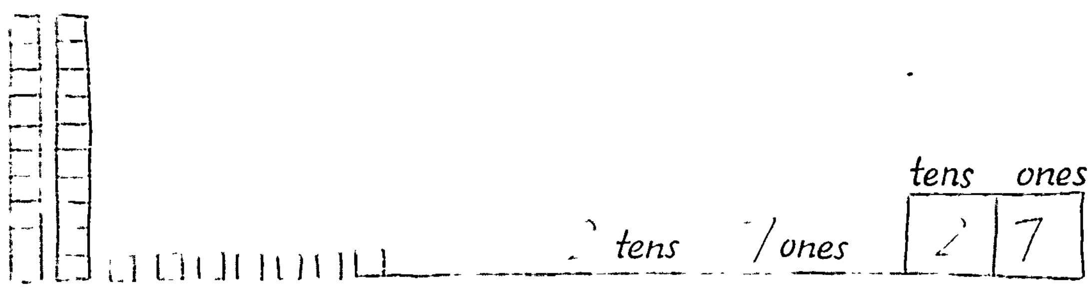


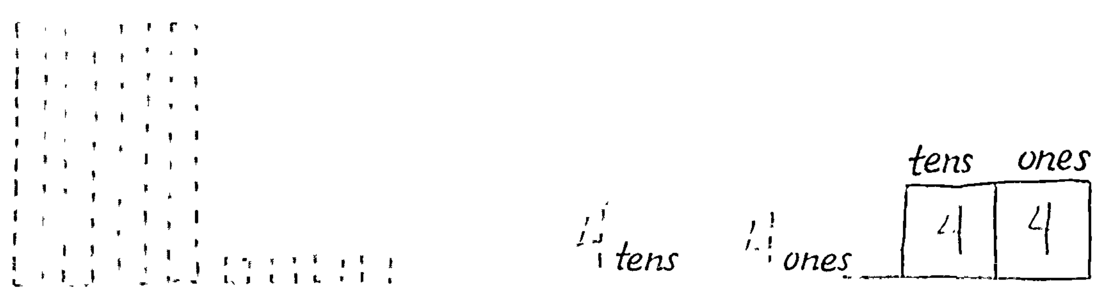


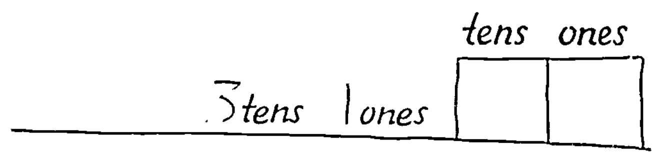
Worksheets like the one on the following page provide opportunities for children to relate two-digit numerals to groupings of tens and ones. Similar problems can be used to assess their ability to relate base representations to two-digit numerals.

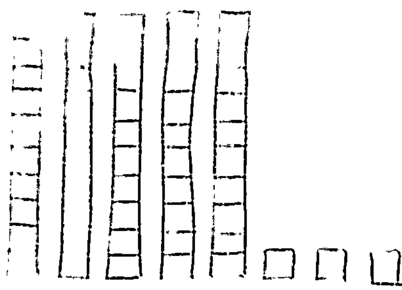
Reading and writing numerals is generally taught earlier in a numeration sequence than indicated here. The reason for the postponement in this sequence is to insure that the students have a sound basis on which to relate the number names with the two-digit numerals. Since groupings have been closely tied both to the numerals and to the usual oral number names, reading and writing numerals can now be made more meaningful by reference to the base ideas. Without this reference to base representation, connections between the oral names and the two-digit numerals must necessarily border on rote learning.

The students are able to produce number names for symbols such as 68 by thinking. "68 is 6 tens 8 ones. Since 6 tens is sixty, 68 is sixty-eight." By reversing the procedure, the children



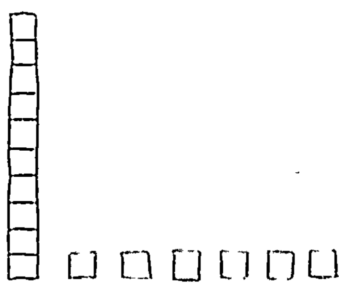






5 tens 3 ones 53

3 tens 0 ones

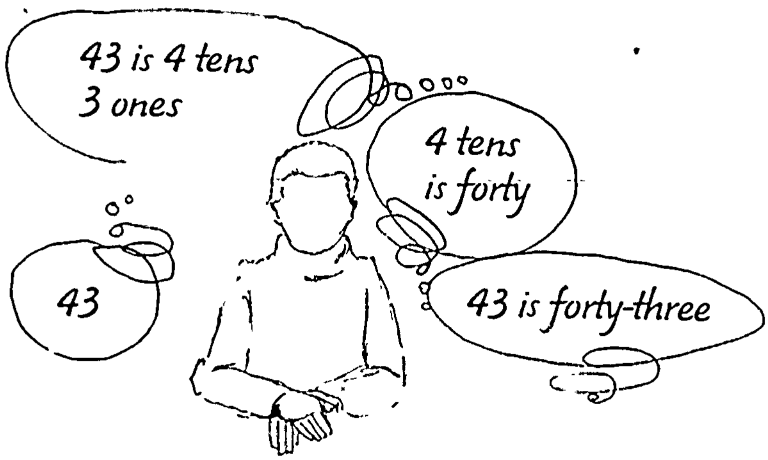


tens ones

tens ones 40

1 tens 2 ones

can develop two-digit symbols just as easily. All the prerequisites for making this a meaningful process are present.



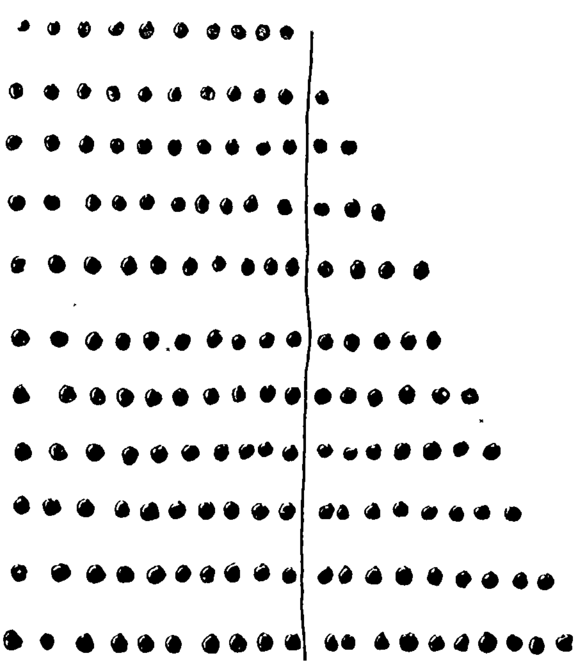
unit 4: sequence of numbers and numerals 10 through 100

The major objective of this unit is to teach the child to count and write the numerals in order up to 100. The sequence of the numerals was not presented in earlier units so that the central notion of base representation could receive greater emphasis. Since counting is such a prominent component of the cognitive framework that children have developed for number, emphasis on sequence too early may distract attention from groupings of tens and ones and lead children to make responses based on patterns they have learned for counting.

The numbers zero to ten are ordered using the "one more" idea. This idea is also used in writing the numerals in order and in reading them as they are written. The chart at the bottom of the page shows how this idea can be used in ordering the two-digit numbers.

Many other activities can help students master the two-digit number sequence. For example, they could fill in the missing numerals in a hundreds chart. Some sample lines are shown.

1			4			7			10
11		13			16			19	20
	22			25			28		
	72			75		77			80
91			94			97			100



1 tens 0 ones 10
1 tens 1 ones 11
1 tens 2 ones 12
1 tens 3 ones 13
1 tens 4 ones 14
1 tens 5 ones 15
1 tens 6 ones 16
1 tens 7 ones 17
1 tens 8 ones 18
1 tens 9 ones 19
2 tens 0 ones 20

A slightly more difficult task is to have the children fill in missing numerals in a portion of a sequence.

____ ____ 41 ____ ____ 44

68 ____ ____ ____ 78 ____

Another related task is to ask the children to write numerals for the numbers one less and one more than a given numeral.

____ 38 ____

____ 79 ____

____ 60 ____

A hundred board with circular tags containing the numerals 1 through 100 is useful for counting in order and recognizing the numerals in order. By turning all the tags over, a child can choose one and give the number before he turns it over to check the numeral. For example, if a child turns over the fourth tag in the third row and says, "Twenty-four," he is right.

comments on the sequence for two-digit numerals

That children have difficulty learning about numeration has been well documented by informal observations of teachers as well as by more careful evaluations of researchers. Three possible reasons for this difficulty, along with precautions that should be taken to minimize them in this sequence, follow.

1 Children often have difficulty with numeration because too many different but closely related tasks are presented at nearly the same time It is not uncommon for children to be expected to count by ones, count by tens, group by tens, make tens and ones charts, name numbers orally, read and write two-digit numerals, recognize the significance of ten in our base-ten system of numeration, recognize the equivalence of numbers named orally and named by their respective base representations, and perhaps grasp several other ideas—all during their initial exposure to tens and ones. Children need time and assistance to sort out these base and place-value ideas and to relate all the components of numeration. In this sequence, all the major components are related so that children will be able to deal with numbers flexibly.

2. Another problem that children have with numeration involves the use of oral number names. Besides the normal confusion due to the irregular number names, such as the teens and some multiples of ten, there is a more subtle and more significant difficulty. Children generally do not have a way to think about how numbers are named. Initially, they do not learn to relate the names to groupings of tens and ones. For example, they do not recognize that the name "sixty-four" is derived from, or is even related to, six tens and four ones. Rather, they learn the number names in sequence from counting patterns. "Sixty-four" is simply the number after "sixty-three." This type of thinking severely restricts the use of number. It accounts for some of the difficulties that children have with numbers taken out of the context of sequence. It also accounts for the slow development of meanings and skills related to oral number names, particularly the tasks of reading and writing numerals (Rathmell 1972).

The sequence presented here has been carefully structured to insure that naming numbers can be meaningful. Simple counting skills are used to relate the oral names to the base representations from which they are derived. This provides students with a thinking pattern for naming numbers. Base representations can then be used as a mediator between naming and symboling numbers. Names or numerals that have been forgotten can easily be reconstructed by the patterns. Since the teens are the exception, they are presented after the general scheme is well established. Reversal errors are less likely because the patterns stem from base ideas. Unit 2, which is devoted to naming numbers, also provides the opportunity for children to have many experiences with grouping and naming before symboling is necessary.

After these developmental experiences, writing numerals is a natural way to record the ideas already established.

3 Many children do not recognize the significance of grouping by tens. One contributing factor to this lack of understanding is that grouping activities are often performed in connection with counting by ones. When children count, they may not recognize that each new decade is really a set of ten.

The sequence presented here features a special emphasis on grouping objects to stress the base idea. Base representation is central to the entire development. Besides the concentration on grouping and the link between grouping and the oral number names, tens and ones are used as the initial model for two-digit numerals. Since references to counting may distract from the significance of grouping and base ten, the usual number sequence is not emphasized until base ideas are thoroughly developed. Children are also given the opportunity to compare base representations of a number to the idea of numerosity.

other related topics

Related to numeration is a need to order numbers, to use estimation and approximation, and to begin regrouping and renaming. Activities for these topics are suggested in this section. Many of the activities could be intertwined with those for two-digit numerals, but mastery should probably be expected later.

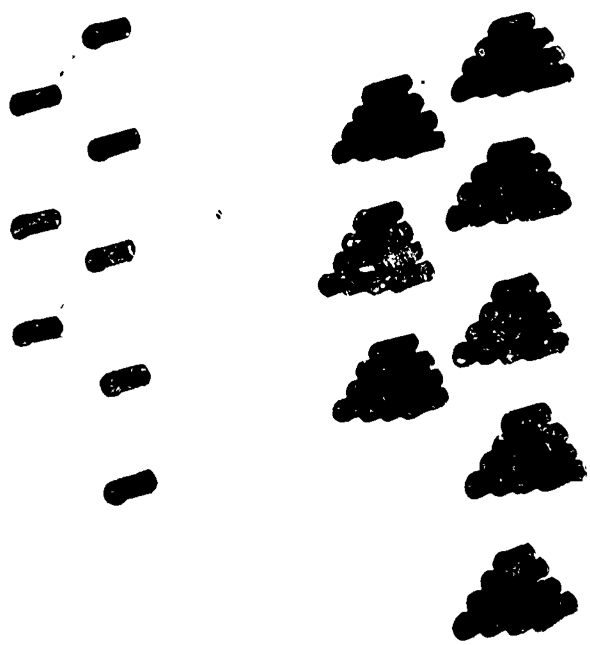
order

Initially, children probably learn to order numbers greater than ten by using the number sequence they have learned from counting. For example, a child might say, "Twenty five comes before twenty-eight, so it is less than twenty-eight."

Although such thinking is not incorrect, ordering from a number sequence is inadequate. If two numbers being compared are not close in the sequence, counting is not practical, and if counting is not done, a child may focus on the size of the greatest digit. For example, students often say that 39 is greater than 51 because 9 is greater than either 5 or 1.

A more effective method is to use base representations. Children who can compare 3 and 5 have little trouble comparing three tens and five tens.

► Relate initial comparisons of tens to comparisons of ones. "Which is greater, three or five?" (Five) "Which is less?" (Three) "How can you tell? What would happen if you tried to match the sets?" (There would be some left over.) "Which is greater, three tens or five tens?" (Five tens) "Which is less?" (Three tens) "How can you tell? What would happen if you tried to match the tens?" (There would be some tens left over.)



The students will soon learn that numbers of tens are ordered the same way as numbers of ones. After the oral number names have been learned, they may also be used in comparing numbers.

"Which is greater, thirty or fifty? Which is

less? How can you tell? Yes, thirty is only three tens, and fifty is five tens."

Sets of tens and ones can then be compared with or without the oral names. Teach the children to look at the "big things"—the tens. Only if the tens are the same will they need to look at the ones. Later, after the two-digit numerals have been learned, the children might respond as follows:

▸ "How many tens are in thirty-nine?" (Three) "How many tens are in fifty-one?" (Five) "Which is greater, thirty-nine or fifty-one?" (Fifty-one) "Which is less?" (Thirty-nine) "How can you tell? Yes, thirty-nine has only three tens, but fifty-one has five tens." (See ■.)

Besides comparing two given numbers, many other appropriate order experiences with numbers are possible. Concrete representation will probably be necessary during the first encounters with such tasks as these:

Name a number greater than forty-nine.

Name a number less than thirty-five.

Name a number greater than twenty-three, but less than forty-one. Can you find all the possibilities?

Order forty-three, seventy-one, and seventeen from smallest to largest.

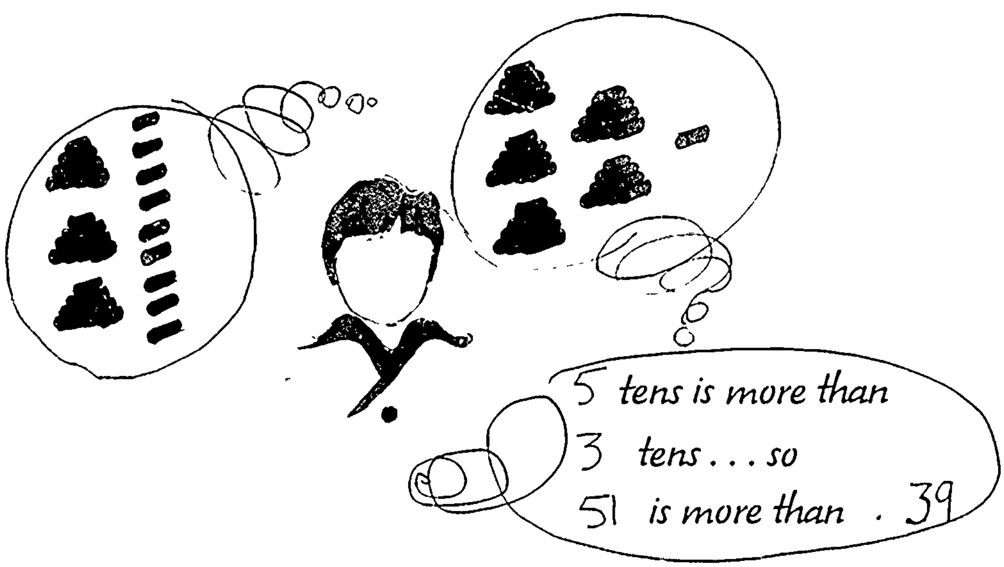
Here are two boxes, A and B. Box A has more than forty beads. Box B has less than forty beads. Which box has more beads? How can you tell?

Here are two boxes, A and B. Box A has more than thirty beads. Box B has more than twenty-five beads. Can you tell which box has more beads? (No)

estimation and approximation

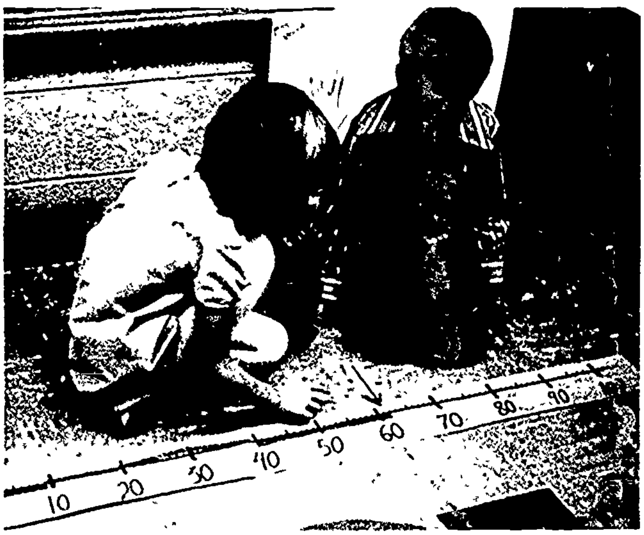
Skills in both estimation and approximation depend heavily on the ability to deal with tens and multiples of ten. Developmental experiences for these estimation and approximation tasks must be provided in the primary grades.

Initially, an intuition should be developed for the approximate location of points representing given numbers on the number line. Students should not have to scan the teens to find 83.



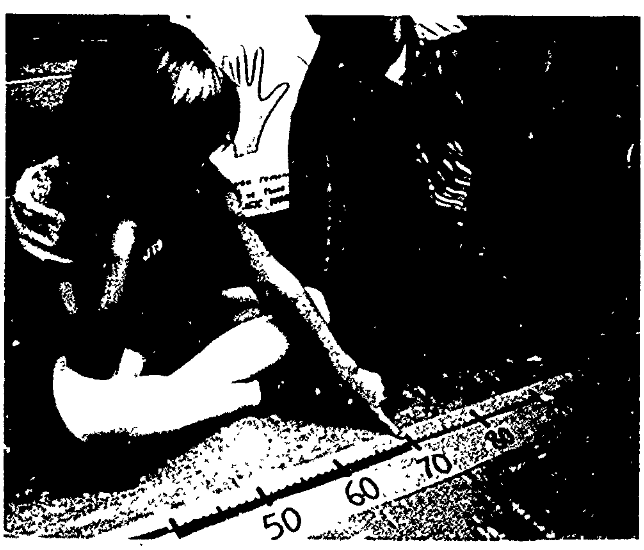
■

Locating points on the number line can be done in conjunction with ten-unit strips and unit squares matched to the points on the number line. This activity has a built-in self-check.



▸ Predict where six tens and two ones will be on the number line.

The number line is also useful for rounding to the nearest ten. The difference between two numbers is represented visually by a line segment on the number line, and so rounding to the nearest ten involves the comparison of two of these line segments. For example:



▸ "Find 69 on the number line. Is it closer to 60 or 70?" (70) "Which is the nearest ten, six tens or seven tens?" (seven tens)

▸ "Find 32 on the number line. Is it closer to 30 or 40?" (30) "Which is the nearest ten, three tens or four tens?" (three tens)

▸ Now show only a portion of the number line with cards covering up both ends. "Here is 48. Is it closer to 40 or 50?" (50) "Which is the nearest ten, four tens or five tens?" (five tens)

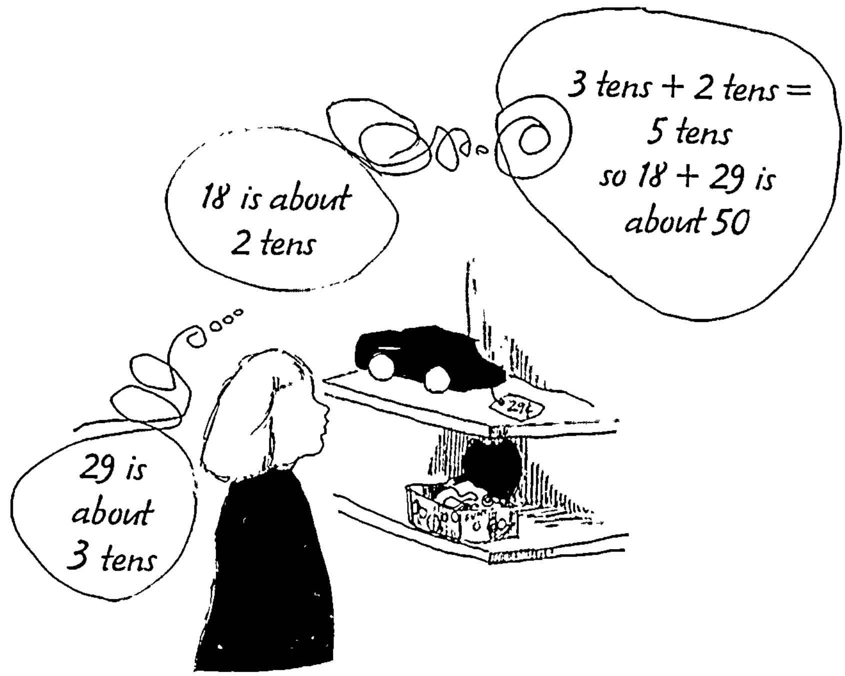
▸ "Think of 42. What is the nearest ten?" (40) Show this on the number line. "Think of 72. What is the nearest ten?" (70) Show this on the number line. Do the same for 12, 32, 92, 52, 22, 82, and 62. Do the same for other numbers in the ones place.

These questions should lead to a generalization about rounding to the nearest ten. If the ones digit is 0, 1, 2, 3, or 4, the nearest ten is the number of tens indicated. If the ones digit is 6, 7, 8, or 9, the nearest ten is one more than the number of tens indicated. An opportunity for creative expression comes when dealing with a 5 in the ones place. Usually 5 is rounded up. Later, this generalization is an aid in estimation for computation. (See ■.)

regrouping and renaming

Although regrouping objects and renaming numbers are ideas that are not generally thought to be of major concern for the early primary grades, it is essential for students to understand these ideas if they are to understand the computational algorithms. Unfortunately, the ideas of renaming cause a great deal of difficulty. More developmental activities dealing with regrouping and renaming may prove to be quite beneficial.

Several renaming ideas have already been presented. They include the naming of ten ones as one ten and the tasks comparing the results of counting by ones with counting by tens. The possibility of a number's having more than one name should be made explicit during this instruction.

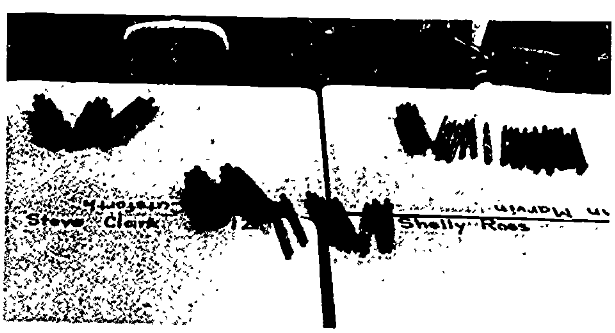


What is the approximate cost of the car and the balloon together?

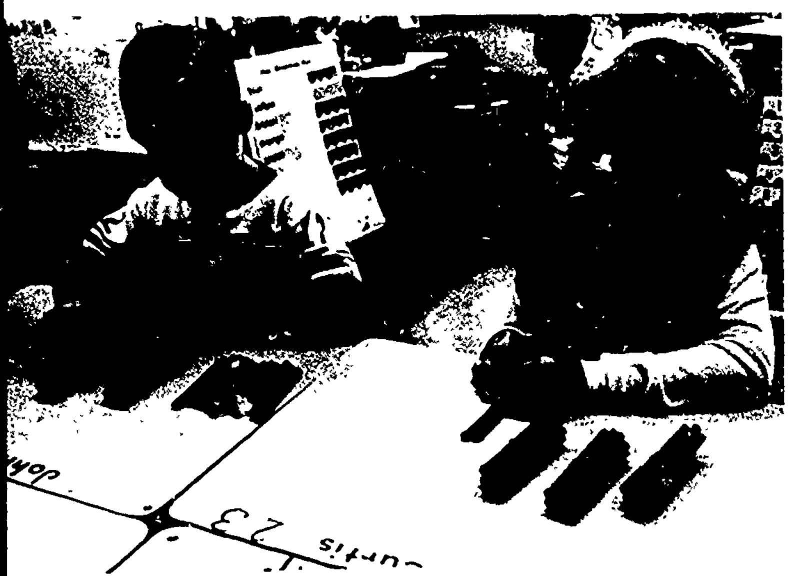
Other developmental tasks could also be presented. After the students can represent numbers concretely by tens and ones and relate this to counting objects by ones, they could be asked to find different ways to show the number concretely. For example, after showing thirty both as thirty ones and as three tens, the teacher might ask, "Can you show thirty in a still different way?" Some direction will have to be provided at first.

The preceding task is no doubt easier if the materials used do not require physical exchanges. Bundling sticks by tens is quite appropriate, since the removal of a rubber band regroups the sticks.

The comparison of different representations of numbers is also an effective way to provide developmental experiences for renaming. Consider the following:



▸ Give one student three bundles of ten sticks and two single sticks. "How many tens and ones do you have?" (Three tens and two ones) Give another student two bundles of ten sticks and twelve single sticks. "How many tens and ones do you have?" (Two tens and twelve ones) "Who has more sticks, or do you have the same?" (Same)



By establishing a bank, where exchanges of one dime for ten pennies can be made, small groups of students could compare different representations of numbers. The group first might decide what exchanges they would need to make, and then one student could go to the bank to make these exchanges. On his return, a direct comparison of the representations could be made.

Other situations that force physical exchanges could also be presented. For example, three students could be given four dimes and two pennies to share equally. A bank could again be used for making the appropriate exchanges.

Nearly all the renaming experiences in the primary grades should include concrete referents. Money, bundles of sticks, unit squares and strips, and many other materials can be used to provide excellent problem-solving activities that promote the ideas of regrouping and renaming.

These developmental experiences should provide an intuitive understanding of renaming before it is required for the computational algorithms. This should facilitate both an understanding of the algorithms and computational ability.

a sequence for learning three-digit numerals

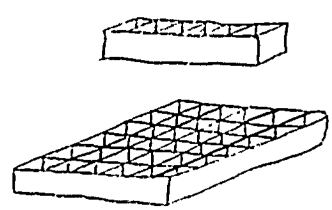
Base ideas, or groupings, must continue to be the central theme in the teaching of three-digit numerals. This not only helps promote the basic concepts of our numeration system but also helps students maintain a concrete referent for the symbolism.

The major steps in the sequence suggested here include (1) using a general scheme for grouping from which 100 emerges as ten sets of ten, (2) naming the number by using a place-value numeral, (3) giving the usual oral name for groupings and the three-digit numerals, and (4) sequencing and ordering the numbers.

general scheme for grouping

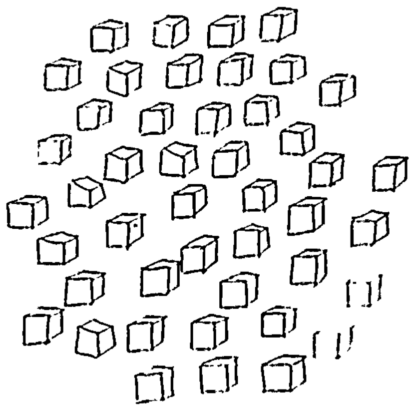
The generalized scheme for grouping can be developed by grouping more than once using numbers less than ten. In this way, pupils can experience grouping activities without having to work with so many objects.

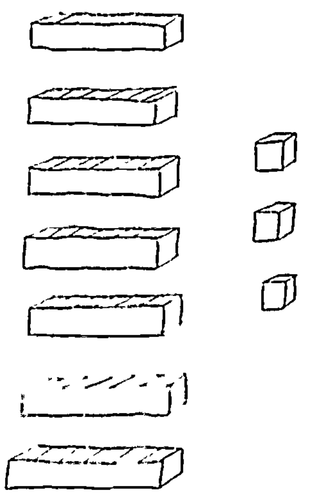
▸ "Today we are going to work in Sixland. In Sixland, things are grouped by sixes: six objects to a box, six boxes to a carton, and so on.

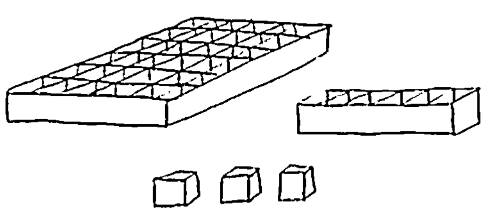


Here are some blocks. Let's group them as they would in Sixland. How many blocks to a box?" (Six) "Now let's make as many boxes as we can." Have the children help complete this first grouping. "How many blocks are left over?" (Three) "Are there enough left over to make another box?" (No) "How many boxes to a carton?" (Six) "Are there enough boxes to make a carton?" (Yes) "Let's make a carton Are there enough boxes left over to make an-

other carton?" (No) "What did we end up with? Yes, one carton, one box, and three single blocks."







cartons	boxes	singles
1	1	3

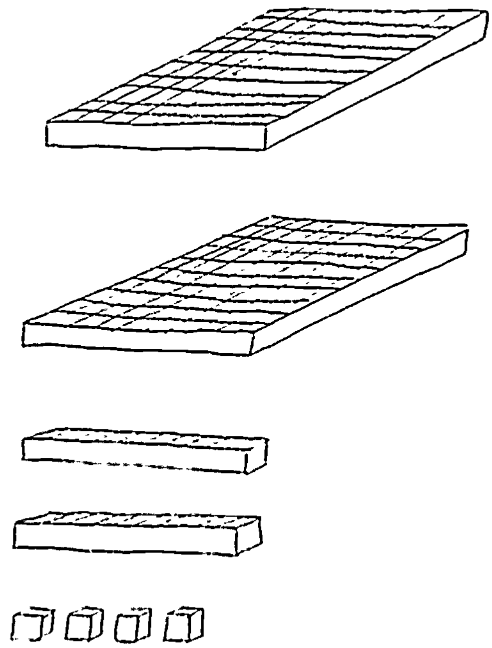
Grouping activities like the preceding one can be completed by using strips of paper six units long and separate square units. Children can line up six squares to form a "box" and six boxes to form a "carton."

Similar grouping activities could be provided in other bases; for example, in Threeland there would be three objects to a box and three boxes to a carton. In Fourland, four objects make a box and four boxes make a carton. Poker chips may also be used to establish the grouping. In Fourland, four white chips can be traded for one red chip; four red chips for one blue chip.

A few activities like the preceding ones will help children anticipate the possibility of grouping groups of objects. Now special attention can be given to grouping ten tens to form one hundred.

► "Here is a container with lots of sticks. Let's group them by tens, ten sticks to a box, ten boxes to a carton." Let several children in turn form sets of ten. Have them remove each set of ten from the container and place it in an appropriate box or bundle it with a rubber band to represent a box. Continue this process until there are no longer enough sticks left over to form another set of ten. "How many sticks are left over?" (Four) "Are enough left over to make another box?" (No) Place the sticks that are left over on a desk top or table. "How many boxes are needed to make a carton?" (Ten) "Are there enough boxes to make a carton?" (Yes) Have the students form as many cartons as possible. "Are enough boxes left over to make another carton?" (No) "What did we end up with? Yes, two cartons, two boxes, and four single sticks. How could we record this in the chart?"

Since there are ten sticks in a box, count by tens to see how many sticks are in a carton. "The name for ten tens is one hundred. A carton in Tenland is one hundred." Record the answers now in a hundreds-tens-ones box (■).

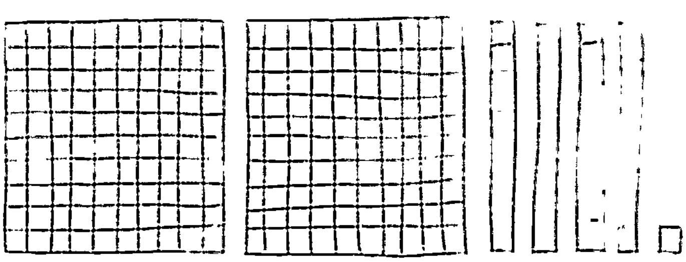


cartons	boxes	singles
2	2	4

hundreds	tens	ones
2	2	4

three-digit numerals

It is important that all the components of numeration be taught and related well. Familiarity with the different forms of representing numbers will grow as groupings of hundreds, tens, and ones are recorded. Recording physical and pictorial groupings in a chart should require little practice. The reverse task—constructing a grouping from a chart—should also be completed. Worksheets such as the one shown provide opportunities for both tasks.



____ hundreds ____ tens ____ ones   hundreds $\square$ tens $\square$ ones $\square$

1 hundreds 4 tens 3 ones   hundreds $\square$ tens $\square$ ones $\square$

____ hundreds ____ tens ____ ones   hundreds $\boxed{2}$ tens $\boxed{1}$ ones $\boxed{5}$

As with two-digit numerals, three-digit numerals may be constructed from the charts in two steps. First, the separation lines can be removed; then later, the words can be removed.

hundreds	tens	ones
3	7	8

hundreds tens ones

3 7 8

3 7 8

reading three-digit numerals

Some guidance for reading three-digit numerals can be quite helpful. Children need to be given a way to decide which numerals involve hundreds and also a process that will enable them to be successful at reading the numerals.

Discrimination tasks can be used to focus attention on the number of digits in the numeral. For example:

Circle the numerals that involve hundreds

42 375 506 98 111 694 63

After the child realizes that a numeral involves hundreds, he needs to remember only that the first digit (on the left) indicates the number of hundreds and that the other two digits are read just as other two-digit numerals are read. Consider 632. A student should think, "There are three digits; so the numeral involves hundreds. The first digit indicates hundreds; so the number name is six hundred thirty-two." It is sometimes helpful to separate the numeral with a long slash. Writing 6/32 emphasizes the distinction that needs to be made among the digits when numerals are read. When a student is unable to name a number correctly, this long slash is quite often enough of a cue for the student to reconstruct it without calling it "sixty-three-two."

number sequence

After becoming fairly proficient at the six basic translation tasks, children need to consider the number sequence for three-digit numerals. Activities like those for the number sequence with two-digit numerals are quite appropriate. The children could fill in the missing numerals in a chart:

241			244		246			249	
		253				257			

Special attention will need to be given to those numbers between 100 and 110, 200 and 210, and so on.

Writing numerals for numbers one more or one less than the number represented by a given numeral is also appropriate:

___ 299 ___

___ 410 ___

.

___ 735 ___

___ 560 ___

In preparation for the computational algorithms, base representations can be used to name the numbers ten more or less and one hundred more or less than a given number. For 347, the students might think, "347 is 3 hundreds 4 tens 7 ones, and so ten more would be 3 hundreds 5 tens 7 ones, or 357." Similar thought patterns could be used for the other problems.

oth otational schemes

In the past few years, nondecimal numeration has become a popular topic in the elementary mathematics curriculum. Originally, the rationale for including this topic was that it would improve our understanding of our decimal numeration system. Research studies, primarily at the middle- and upper-grade levels, have generally supported the hypothesis that an equivalent amount of instruction in base ten is more effective.

Although a study of nondecimal systems of numeration has not proved to be particularly beneficial, a study of the Egyptian numeration system by third graders has been shown to be

effective in promoting the concepts and usage of zero (Scrivens 1968). Evidently the lack of a symbol for zero, as in the Egyptian system, draws attention to the way in which zero is used in our own system.

Roman numerals should also be taught. Although they are not used excessively, they do arise often enough to be included in the curriculum. However, addition and subtraction are prerequisite for a comprehension of this system.

record of progress

The key to a successful mathematics program is its ability to provide experiences to a child when he is most likely to benefit from them. Regardless of whether the program is geared toward the whole group, is flexibly grouped, or is designed for individualized instruction, the teacher must know what a child understands to be able to provide these experiences.

One of the best ways to obtain the information on which curriculum decisions can be based is to conduct short interviews to supplement paper-and-pencil assessments. Records of these interviews can be kept on charts such as the ones that follow. Each teacher can devise his own scheme, but by using checks or marking the date of achievement, he will always have available a current profile of student achievement.

RECORD OF PROGRESS

One-digit

	Jeff	Kara	Lon
[illegible] sets	OK	OK	OK
[illegible], to order sets	OK	OK	OK
recognizing that matched sets have the same number	counts to check	only for small numbers	OK
using cardinal number	OK	OK for small numbers	OK
using ordinal number	needs a little more work on "third" and "fifth"	---	OK
counting to ten	OK	only to five	OK
recognizing numerals	OK	---	OK
writing numerals	needs more work	---	reverses some
recognizing dot patterns	OK to four	only two	OK
using "same," "more," "less," "fewer"	needs more work with "less" and "fewer"	---	needs more work with "less" and "fewer"
identifying "parts" and "whole"	---	---	---
recognizing that rearrangement doesn't change numbers (conservation)	transition stage for larger numbers	---	usually OK

Two-digit	Jody	Neal	Patrick
grouping	needs more work with grouping once	OK	OK
representing numbers	needs more work with oral names	a few reversals, needs more work with base	OK
counting	OK for ones; needs some help for tens	OK	OK
ordering numbers	uses misleading cues sometimes	OK	OK
rounding to nearest ten	guesses	depends on number line	OK
regrouping tens and ones	not ready	ready for initial exposure	depends on concrete materials
Three-digit			
grouping	---	ready for grouping more than once	needs review for grouping more than once
representing numbers	---	beginning to try	oral names cause some problem, also zero-tens
counting	---	trying, but has trouble from 100 to 110	OK
ordering numbers	---	---	has some trouble when hundreds are the same

conclusion

Throughout the learning of the numbers 0 to 10, 10 to 100, and 100 to 1,000, concrete referents for thinking are critical. The concepts that grow from the concrete referents with such heavy emphasis on base representation must be related carefully and thoroughly to the oral names and written symbols. If a child learns all three of these and can move easily from any one to the other two, we shall have provided him with the best foundation possible for all subsequent work with whole numbers.

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operations on whole numbers

mary folsom

OPERATIONS on whole numbers represent a major area of learning for the primary school child. Understanding and competency in this area are important both in the everyday world of the child and for his future mathematical learning. The four operations on whole numbers are frequently equated with techniques for computing answers to examples; yet learning about operations on numbers encompasses far more than this. In the early stages of learning, a primary goal is to help the child move from physical-world situations to the abstract ideas of operating on numbers. Consequently, it is important to provide the child with many experiences involving both physical representations of the operations and the properties of a given operation so that he has the basis needed for abstracting the mathematical ideas. For example, from a multitude of situations involving two sets being joined, the idea of finding the sum of a pair of whole numbers may be slowly abstracted. Although the child does determine sums, proficiency in computing is not the goal of early instruction.

the meaning of operations and their properties

An integral part of the work with operations is guiding children in their investigation of the properties associated with each operation. This examination helps the child focus on the idea of

operations rather than on the computational techniques alone. Because the emphasis shifts later to techniques for computing numbers, an early emphasis on conceptualizing the operations and investigating their properties has significant value. Indeed, it complements and facilitates the learning of basic facts and the development of algorithms.

After having many experiences with combining sets, the child can learn, for example, to associate the number five with the number pair (3, 2) under the operation of addition. From carefully planned lessons, the following properties can be developed:

1. The order in which the numbers are added does not affect the sum; that is, $a + b = b + a$. (Commutative property of addition)
2. If one of the addends is zero, the sum is the other addend; that is, $a + 0 = 0 + a = a$. (Identity element of addition)
3. Given three numbers, the way in which they are grouped does not affect the sum; that is, $a + (b + c) = (a + b) + c$. (Associative property of addition)
4. If two whole numbers are added, the sum is a whole number; that is, $a + b$ is a whole number if a and b are whole numbers. (Closure property of the whole numbers under addition)

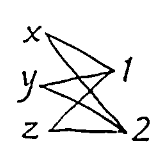
Since none of the properties of addition holds under the operation of subtraction, care should be taken to prevent children from making such assumptions. Because the identity element for addition is zero, any number minus itself is zero, and any number minus zero is itself; that is, $n - n = 0$ and $n - 0 = n$. Both these generalizations can be developed through the use of set removal and set partitions.

The idea of subtraction as the inverse operation of addition is difficult for a primary-grade child to grasp. Although it is used when he is ready to master basic subtraction facts, his understanding of it is probably only intuitive. To be precise, the inverse operation for the equation $3 + 2 = 5$ must be either $5 - 2 = 3$ if 2 was added to 3 or $5 - 3 = 2$ if 3 was added to 2. When a child knows that $4 + 5 = 9$ and sees $9 - 5 = \underline{?}$, he can think, "What number plus 5 equals 9?" This does not necessarily mean that he grasps the full significance of an inverse operation. Teachers might well question the advisability of asking young children to write an equation to show the inverse operations for given addition equations. Time might be used to better advantage by using a concrete or pictorial situation in structuring the addition and subtraction equations associated with a given set partition.

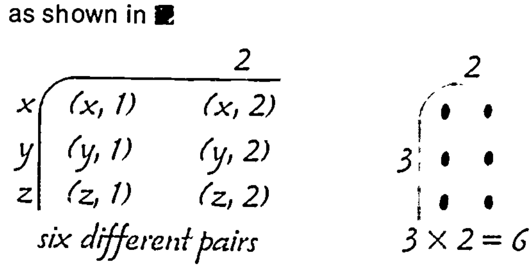
Multiplication of whole numbers, like addition, is a binary operation that associates with a given pair of numbers a unique third number. For example, (3, 2) and 6 are associated under the operation of multiplication. The numbers in the pair are called *factors*, and the third number is called the *product*. Multiplication is related to the product set of two sets just as addition is related to the union of two disjoint sets. The product set of two sets, A and B, results from the pairing of each element of set A with each element of set B. If A and B are two sets,

$$A = \{x, y, z\} \text{ and } B = \{1, 2\},$$

we may find the product set of $A \times B$ by pairing the elements as shown:



Thus the product set is the set of ordered pairs, $A \times B = \{(x, 1), (x, 2), (y, 1), (y, 2), (z, 1), (z, 2)\}$. These six pairs can be arranged into three rows and two columns, called a three-by-two *array*, as shown in ■



■

Since primary-grade children tend to see, not six pairings as shown in the array on the left in ■, but rather twelve members in the array, only arrays such as those shown on the right will be used with children of this age. Every

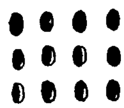
product involving counting numbers may be represented by an array. The product $a \times b$ can be visualized as a rows with b objects in each row. Consequently, it is possible to guide the child to abstract from his experiences involving arrays the properties of the operation of multiplication:

1. The order in which the numbers are multiplied does not affect the product; that is, $a \times b = b \times a$. (Commutative property of multiplication)
2. If one of the factors is one, the product is the other factor; that is, $a \times 1 = 1 \times a = a$. (Identity element of multiplication)
3. Given three numbers, the way in which they are grouped does not affect the product; that is, $a \times (b \times c) = (a \times b) \times c$. (Associative property of multiplication)
4. If two whole numbers are multiplied, the product is a whole number; that is, $a \times b$ is a whole number if a and b are whole numbers. (Closure property of the whole numbers under multiplication)
5. Given two factors a and $(b + c)$, their product is the sum of the two products $a \times b$ and $a \times c$; that is, $a \times (b + c) = (a \times b) + (a \times c)$. (Distributive property of multiplication)

Trying to make an array that is 3 by 0 or 0 by 3 helps the child develop the generalization that if one of the factors is zero, the product is zero.

Since a similarity exists between the operations of addition and multiplication (both operations are commutative, associative, and closed in the set of whole numbers), it seems natural to ask if there is an operation that bears the same relation to multiplication that subtraction bears to addition. Of course, the answer is yes. The inverse operation for multiplication is division.

To find the product 3×4, we count the members in a 3-by-4 array, that is, in three disjoint sets with four members each. An associated question is to start with twelve objects and ask how many disjoint subsets are in this set if each subset has four members. In terms of arrays, the question is, If a set of twelve is arranged in three rows, how many columns will there be? The answer is four:



Twelve objects arranged four to a row.

Often there is no whole-number answer to the question, How many rows? For example, eighteen objects arranged in seven rows does not yield a whole number of equivalent columns. Although we do carry out such a division process as eighteen divided by seven, obtaining a quotient and a remainder, we cannot speak of this process as the operation of division, since the expression is meaningless in the set of whole numbers—there is no whole-number quotient. The process, however, can be developed.

Division by zero must be excluded because there are many answers in one case and no answers in all other cases. For example, there are many answers for $0 \div 0$, and no answers for $6 \div 0$. Both kinds of examples are explained using multiplication.

$$0 \div 0 = \square \text{ means } \square \times 0 = 0$$

Since any number times 0 is 0, any number solves the equation.

$$6 \div 0 = \square \text{ means } \square \times 0 = 6$$

Since no number times 0 is 6, there is no solution. Consequently, any multiplication chart that children use for practicing basic division facts should eliminate zero as a divisor.

The role of division as the inverse operation for multiplication suffers from the same lack of understanding as that of subtraction. But with division another hazard is involved. Whereas it is true that $15 \div 5 = 3$ and the statement for the inverse operation is $5 \times 3 = 15$, $8 \div 3$ is not defined in the set of whole numbers, since 8 is not a whole-number multiple of 3. Even if the child obtained the true quotient 2 2/3 from the set of rational numbers, it is doubtful that he would check his answer by using the inverse operation $3 \times 2\,2/3$. The usual procedure would be $(3 \times 2) + 2 = 8$.

Because the identity element for multiplication is one, the following generalizations in

division may be developed: $n \div 1 = n$ and $n \div n = 1$, provided n does not equal zero. Division distributes over addition when division is on the right. For example, $91 \div 7 = (70 + 21) \div 7 = (70 \div 7) + (21 \div 7)$. Of course $7 \div 91 \neq (7 \div 70) + (7 \div 21)$, but this is not usually attempted by primary school children, although many of them, unless carefully instructed, are tempted to subtract 9 from 2! The point is that properties of addition and multiplication must not be thoughtlessly carried over to subtraction and division.

addition and subtraction with one-digit addends

the role of counting

Since meaningful counting is the main strategy used by children to find the sum in addition and the difference in subtraction, counting should play an important role in the number experiences of primary school children. Fingers are often used in counting because they are convenient. The teacher can develop alternatives to finger counting by providing more counting experiences and by using concrete objects in counting.

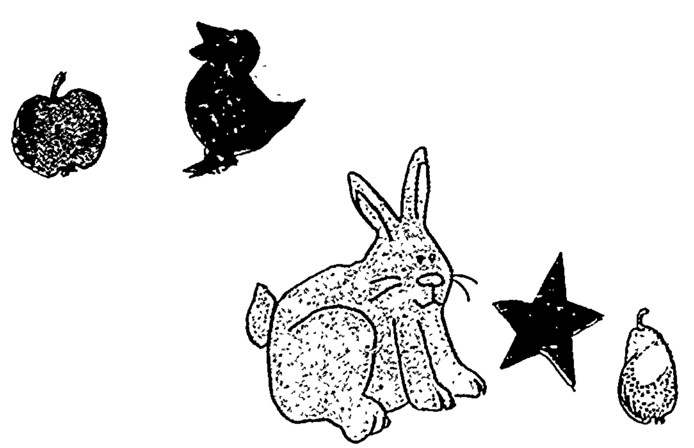
By the end of the first grade, the majority of children should have mastered counting to 100 by ones, tens, and fives and to 20 by twos. At the next grade level these counts should be reviewed so that those who have not mastered them will be able to do so, and counting by threes and fours should be introduced, using the number line. The number line should be placed so that children can touch the symbols. A mastery of counting by fives, twos, threes, and fours is a prerequisite for learning multiplication facts once an understanding of the concept of multiplication has been developed. Hence, counting activities should occur frequently in the early grades.

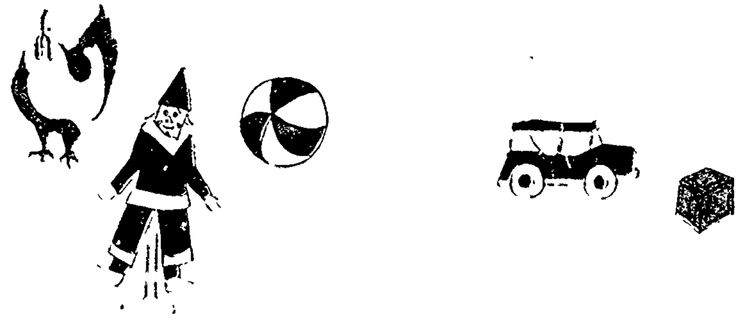
procedure and sequence for addition

The physical representation for addition is the union of two disjoint sets. The words *union* and *disjoint* need not be used with children. The meaning of *set* should be developed by usage rather than by definition.

► Place a number of dissimilar objects on a table (dissimilar so that children will not develop the idea that the members of a set must have some characteristic in common other than that they are members of the same set). Ask the children to name the members of the set (a block, a pencil, a key) as each is pointed to. The set will be well defined if the children can answer such questions as these: "Is the block a member of the set?" (Yes) "Why?" (It's there.) "Is the elephant a member of the set?" (No) "Why?" (It's not there.)

► Use the flannel board to develop the idea of joining sets to find the number of a new set and begin work on the commutative property of addition. Ask the children, "What things do you see in each set? How many members are in the first set? How many in the other set?" Move the set with the apple and the duck to join the other set and ask, "If you join a set of two with a set of three, how many will be in the new set?" Separate the sets and then join the set of three with the set of two and ask the same question. Present several different joinings, using numbers other than zero. The children may need to count to determine the number in the new set.





3 2 5

■

Every child should be given a box with at least ten objects, such as blocks, disks, and other materials. Much practice can be given by asking children to show, for example, a set of four and a set of two and then to join them and name the number of the new set. There are several advantages to such a procedure: all the children are busy working; they have many experiences in seeing sets of three to five members, which they ordinarily cannot recognize without practice, and after the novelty of the objects has worn off, the children will regard them simply as things used to determine number, in the same way that an adult regards paper and pencil.

Partial counting, that is, counting on from a given number, is a difficult task for almost all children. Much practice is needed before children master the task.

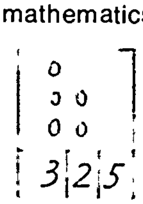
► Place several flannel cutouts on the flannel board. Say to one child, "Come up and place your hands around the number you can see without counting and tell me how many that is." Then have the class help the child count the rest of the members in the set. The ability to do partial counting will help increase the children's sophistication as they learn the basic addition facts. This will be shown later.

Slow Child	Average Child	Very Competent
(1), 2, 3, 4, 5, 6, 7	(2), 3, 4, 5, 6, 7	(5), 6, 7

Building equations should be a slow process. Symbolism should be introduced only after the concept being taught is well developed. The following sequence is suggested for addition.

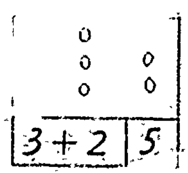
► Use objects and numerals. (Cutouts on a flannel board are easy to manipulate.) Point to the first set and ask, "How many are there in this set? Put the numeral under it. How many in the other set? Put the numeral under it. If we join the two sets, how many will be in the new set?" Place a 5 to the right of the joined sets. Do this with several different sets, but have no more than nine objects in the joined sets. (■.)

► Transfer to the chalkboard and represent the objects there. Explain, "We bought three cans of orange juice." Draw rings to represent the cans of orange juice. "Where should the 3 be written?" Put the 3 in the proper space. "Then we bought two cans of applesauce." Make the rings to represent the two cans. "Where should the 2 be written? How many things did we buy? Where should the 5 be written?" (It is not necessary to explain the use of rings for representing the cans. If the rings are drawn as the words are spoken, the children will understand. The reasoning behind the use of rings is simple. It prevents wasting time to draw objects and enables the teacher to devote more time to the mathematics.)



► Transfer to a worksheet that will hold about six boxes like those shown in the preceding activity. Make up word problems similar to those in the previous activity and have the children draw the rings and write the numerals as the questions are posed. Thus you can walk about observing while telling the story and make certain that everyone understands what is to be done. Six such exercises may take the entire mathematics period.

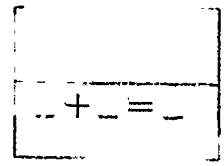
► On another day, after it is clear that everyone has understood, return to the flannel board and use cutouts to represent the sets. After the numbers of the sets and the number of the joined set have been identified and the numerals placed under the objects, say, "We join sets, but we add the numbers. We have a special sign that says 'add the numbers.' It is called the plus sign." Show the children the flannel plus sign and place it between the numbers of the sets. "We can read this sentence like this: Three plus two is the same as five." Repeat with other sentences, and then transfer the activity to the chalkboard, and finally to the worksheet, using story problems. The figures on the worksheet will now look like this:



(It is not necessary to explain what a sentence is. Simply read it for the children as you point to it. Ask them to read the sentence after each story is completed. To reinforce the language, ask where each numeral should be written, that is, before or after the plus sign.

► Return to the flannel board and use the set objects, flannel numerals, and plus sign. Introduce the equals sign. Say, "We have a sign that means 'the same as.'" Show the children the equals sign and place it in the proper position on the flannel board. "We can now read the equation as three plus two equals five." Note that the word *equation* has now been introduced. Again, there is no need to give the children an explanation. Once 3 + 2 = 5 is written, an equation is represented; in the previous activity, the children were just supplying an answer.

► Follow the same sequence from flannel board to chalkboard to worksheets. The worksheet now looks like this:



► When the children are ready, teach them to find the sum for simple equations, such as 4 + 2 = ____, without using story problems. Have them represent the number with concrete objects. Let them show four blocks, join a set of two, and count to find the number of the new set.

Children will not use the blocks as a crutch unless they really need them. When some of them ask if they can use marks instead of the blocks, encourage them to do so. Usually they will follow a pattern like this:

(a) ///// // to (b) // to (c)
5 + 2 = 7   5 + 2 = 7   5 + 2 = 7

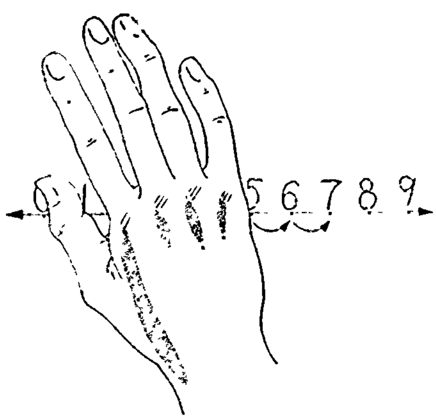
Advancing from (a) to (b) indicates that the child has mastered partial counting. Going from (b) to (c) tells us that the child "just knows."

Worksheets like the following may be used to determine (by watching the children at work) which children have generalized the commutative property of addition and the idea that adding one produces the next whole number after the first addend.

2 + 3 =	1 + 1 =
3 + 2 =	2 + 1 =
4 + 1 =	3 + 1 =
1 + 4 =	4 + 1 =

If children are still using blocks or marks to find the sum of 3 + 2 after using them to find the sum of 2 + 3 or are still using material to find the sum when one is added, it is evident that they have not generalized these concepts. Further attention will be given to the commutative property when set partitions are used, but more counting experience is needed to generalize the addition of one.

▸ Use the number line to show addition. (This is a rather sophisticated idea, which may be postponed until later in the year, depending on the ability level of the class. To be successful, the child must be able to move his finger from 0 to 1 to 2, and so on, which he should have mastered in counting exercises. He must also be able to do partial counting.) Show the child how to find, for example, 5 + 2 = 7. It is often necessary to teach him to cover the first five spaces with his hand as he counts two more to reach the 7.

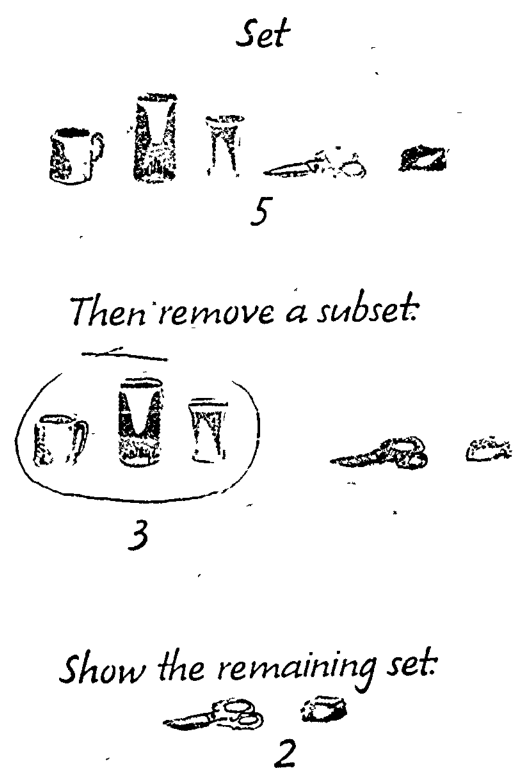


procedure and sequence for subtraction

The physical representation for subtraction is the removal of a subset. It is not necessary to define *subset*, nor is it necessary to dwell on the empty set's being a subset of every set, on every set's being a subset of itself, or on the distinction between a single element and the subset consisting of that one element.

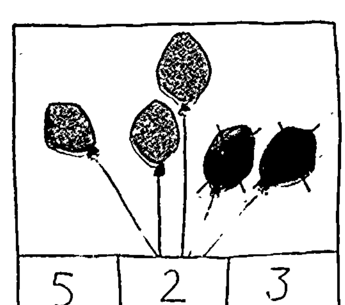
▸ Place five objects on the flannel board. Ask a child to remove a subset. "How many are now in the remaining set?" Eventually someone will be brave enough to remove the entire set so that the number of the remaining set is zero. Someone, perhaps the teacher, might even remove the subset with no elements so that the number of the remaining set is five.

▸ Place objects and numerals on a flannel board like this:

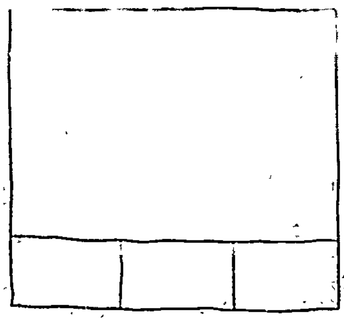


Many examples similar to this are needed.

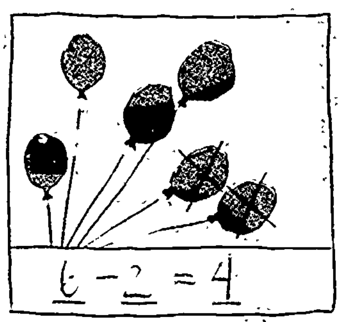
▸ Move to a chalkboard and represent the objects on it. "John had five balloons. His brother broke two. How many were left?"



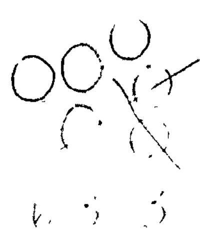
▸ Transfer to a worksheet like the one shown. Tell a story and have the children work together using rings to represent the objects.



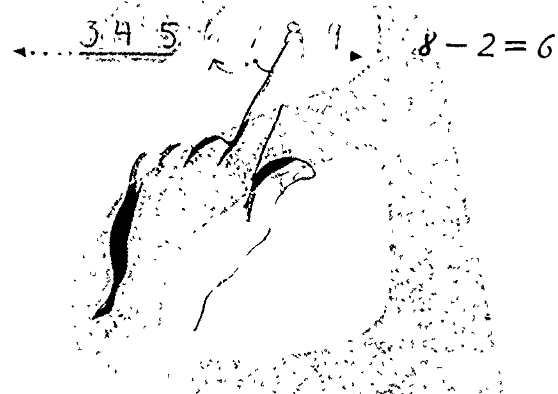
▸ Introduce the minus sign. Explain that it is related to the idea of removing a subset but that it means to subtract the number. Read the equation as "six minus two equals four." Stress the fact that in subtraction the numeral for the number of the entire set is always written first. Do this at every opportunity. Since subtraction is not commutative, every effort should be made to see that no one writes 2 – 6.



▸ Have the children complete subtraction equations such as 6 - 3 = ___. Use concrete objects first. Ask the children to show six blocks and remove three to find the number of the remaining set. On paper, their work will look like this:



Next, show subtraction on the number line.

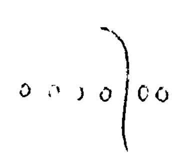


Teach the child to put his finger on the numeral for the number of the entire set. Use a pencil to make the swing back to 7 and then to 6 as he counts, "One, two."

teaching the related facts

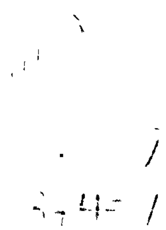
Partitioning a set into two subsets will be used to teach the commutative property of addition and later the connection between the inverse operations of addition and subtraction. In attempting to relate addition and subtraction facts, the teacher should work on the addition facts for a given partition first. Again, there is no need to define partition. Simply do it so that the children understand what is meant.

▸ Partition a set and give equations orally. Place a number of objects on the flannel board and say, "I am going to partition the set into two subsets." Use colored yarn to make the partition.



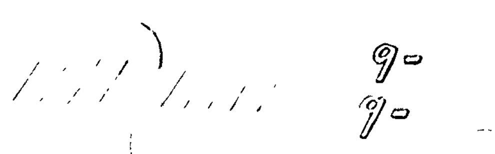
Stand on the side of the flannel board next to the subset of four. "If I stand on this side of the flannel board, how many are in the first subset I see?" (Four) "How many are in the other subset?" (Two) "What addition equation can I write?" (Four plus two equals six.) Move to the other side of the board and ask the same question, which will result in the equation two plus four equals six. This activity illustrates the commutative property of addition and should help most, if not all, of those children who have not made the generalization.

▸ Move to the chalkboard and do several other partitions so that all the children understand. Ask the same questions as before and write the equations suggested by the children.



▸ Provide worksheets with enough partitions to fill the page but not crowd it and have the children work on their own. It may be necessary to use two different worksheets for children having different levels of knowledge—one might be with partitions not greater than five and another with partitions as great as ten. It is better to divide the group and have the children working on their own level than to give them work that is too difficult, which will develop feelings of frustration and failure.

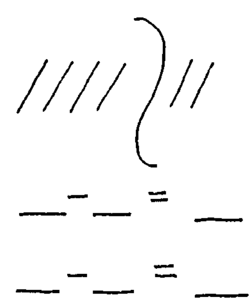
▸ Once the children are able to do the preceding activity, move to subtraction facts for the partition Say, "Today we are going to write subtraction equations for a partition. What is the first thing we do? That's right; we write the numeral for the nur - of the entire set first." (They may say, "Wri. le number of the set.") Give the children flannel numerals and minus signs to show 9 and - twice.



Cover the subset of four and ask a child to finish the first equation. (9 - 4 - 5) Uncover the subset of four and cover the subset of five. "Finish the other equation." (9 - 5 = 4)

▸ Do several activities on the chalkboard, but instead of covering a subset, just point to it with your finger.

▸ Do similar activities with worksheets.



While the children are working, the teacher should walk about, observing (1) those children who must count to find the number of the set, whether or not they write that numeral for both equations before completing the first, (2) those who cover the subset to be removed for each equation, and (3) those who go through some of these steps to obtain the first equation and then automatically write the second. (Of course, the same types of observations should have been made while the children were doing the addition equations for a given partition.) Obviously, the child who must count the number in the first subset and the number in the second subset and then count the members of the partition by ones is far below the child who can recognize a subset of four and a subset of three, count "five, six, seven" to find the sum, and then simply write the second addition equation. It is possible to discern various levels of maturity by watching what the children are doing and how they are doing it.

▸ Use the partitions to write all four addition and subtraction facts suggested for

. For the partition , where

the subsets are equivalent, only two facts are possible. When this occurs, it is not necessary to explain it to the children—provide only the model for one addition and one subtraction equation. If a child asks why this is so, simply ask him to write the equations; he will discover that he has written the same thing twice. A child bright enough to ask the question can understand if you explain that this occurs when the subsets are equivalent.

No one should take it for granted that writing the addition and subtraction equations for a given partition is an easy thing to teach all children. It is difficult and requires great patience and a slow pace from one idea to the next. At any given moment it is possible to observe children doing the task eight or nine different ways.

The child who looks at and

writes $4 + 5 = 9$ is indicating that he knows that fact, when he automatically writes the other three, he is showing that he has the structure well in hand. He has grasped four basic facts at one time. Although others may labor at lower levels of sophistication, they will move from one level to the next if they have the freedom to move at their own speed.

It may be useful at this point to consider how the physical representation of subtraction as the removal of a subset has been expanded by partitions so that it encompasses far more than this simple idea. The word problems and equations with pictures, such as , allowed the subset that was to be removed to remain in full view. The use of the partitions took the child even further. After the initial covering of the subset to be removed, he not only does not cover, remove, or mark it out but visualizes the separation of the set into two subsets without even moving the subset. Relating the appropriate addition and subtraction equations to a given partition is a flowering of the ideas of part-part-whole and whole-part-part. If the child can write the equations for a given partition, he has attained not only conservation of number but also the concept of reversibility.

Frames and partitions. Once the children understand the work with partitions, it is possible to use the partitions to solve other equations. Some teachers use frames. Instead of writing $6 + 4 = \_\_\_$, they write $6 + 4 = \square$. Ask the children if it would make any difference if we wrote $\square = 6 + 4$. Let the children discuss the matter until they agree that it makes no difference, that the sum can be written on either side. Moving the position of the frame to an addend in addition or to a sum in subtraction can cause difficulty for both teachers and pupils. In the equation $\square + 2 = 7$, for example, children do not understand that they are being asked, "What number plus two equals seven?" They do not know that this can be determined by subtracting 2 from 7. Use partitions to help children develop these concepts.

► Give the children the following problem: $\square + 2 = 7$. "Do we know how many are in the entire set?" (Yes, seven) "Make seven marks." (/////) "Do we know how many are in one subset?" (Yes, two) "Mark off two." ( ) "The other subset, then, must have five members." Use the same procedure for $3 + \square = 8$ and for $7 - \square = 2$. For $\square - 3 = 4$, say, "Do we know how many are in the entire set?" (No) "Do we know how many are in each subset?" (Yes, three and four) "Make a drawing and count to find the number of the entire set." ( ) A child might recognize three and count on to seven, or recognize four and count on to seven, or simply recognize groups of three and four as seven.

Practice can be given in addition by using the partition to give different whole-number name combinations for a given number. The following example shows names for 4 corresponding to the partitions.

4

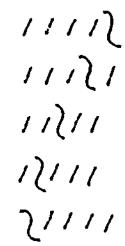
4 + 0

3 + 1

2 + 2

1 + 3

0 + 4



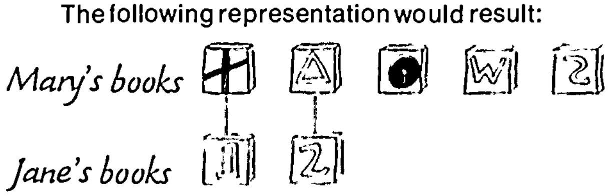
Later this can be written as □ + = 4. Point out that the two frames are of a different shape, making it possible to use either the same number or different numbers in both frames. If the frames are the same shape, as in □ + □ = 4, the same number must be used in both. Inserting 2 makes this sentence true. (Using the same frame is equivalent to $x + x = 4$, and x must be the same number both places. Using two different frames is the equivalent of the algebraic equation $x + y = 4$, and there is no reason why x and y cannot be either the same or different numbers.)

Subtraction word problems. In the primary grades children encounter three types of subtraction word problems:

1. How many are left?
2. How many more?
3. How many more are needed?

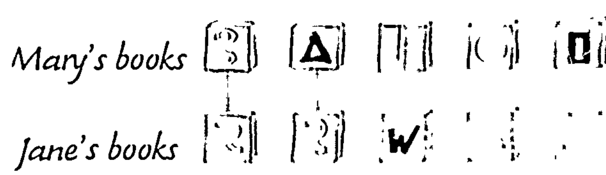
The first type does not prove difficult, since it is common in life situations and is used extensively in developing the subtraction equation. Looking at the drawing 0 0 0 Ø Ø, children easily recognize that the operation of subtraction is suggested and that the question is of this type: "Mary had 5 pieces of candy. She ate 2 pieces. How many pieces does she have left?"

The second type, however, involves a comparison, and many children do not recognize the question as one solved by subtraction. It may help to ask the children to represent the word problem with blocks or other concrete materials. For example: "Mary has 5 books. Jane has 2. How many more does Mary have?" The following representation would result:

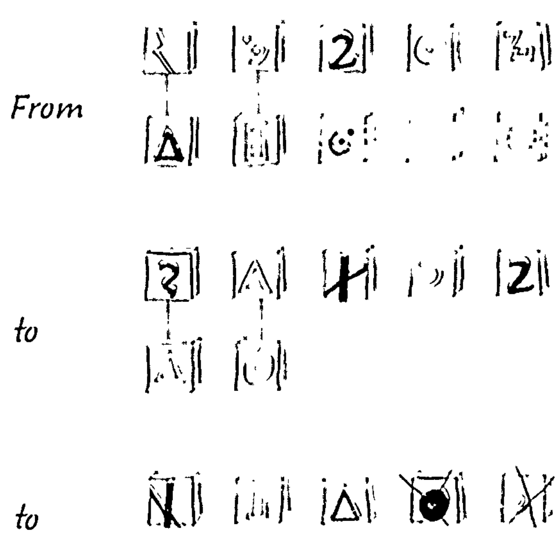


From this arrangement, it now becomes apparent that Mary has three more than Jane. It is not equally apparent that the equation suggested is 5 - 2 = 3. Many experiences with blocks or pictures are needed before the teacher can suggest that it is not necessary to represent the number of books Jane has. "Can't we remove the number of Jane's books from Mary's?" The picture will look like this: Ø Ø 0 0 0. The children will then recognize the operation of subtraction.

Many adults tend to consider the third type of problem the same as the second type. Many children, however, do not agree. Using blocks to represent the problem should convince any teacher that this type of problem needs attention. When asked to represent the number of Mary's books and the number of Jane's books, some children will move in three more blocks to make the sets equivalent.



The teacher must first convince them that it is not necessary to move in the three blocks to make the sets equivalent and then suggest that they remove the number of Jane's books from Mary's. In other words, it may be necessary to move through three different physical representations before the children see that the question suggests subtraction.



using the associative property of addition

If children know the addition facts up to sums of ten, the associative property of addition can be used to obtain facts involving sums of eleven to eighteen. They must learn to think of 8 + 4 as 10 + 2, for example. The following sequence is suggested:

1. Present the problem at the flannel board, using objects.

▸ Place a set of eight objects and a set of four on the flannel board. "Do I have enough to make a set of ten?" (Yes) "How many must we move from the set of four to make a set of ten?" (Two) Move the two objects accordingly. "How many do we have now?" (Ten plus two, or twelve)

This must, of course, be done with many other combinations that will result in a sum greater than ten.

2. Present the problem at the chalkboard, but let the children use the objects.

▸ Give the children paste sticks, lollipop sticks, tongue depressors, or blocks. "Show me a set of seven and a set of four. Are there enough to make a set of ten?" (Yes) "How many must you move from the set of four to make ten?" (Three) After they have done this, ask, "What numbers are represented in the two sets now?" (Ten and one) "What is their sum?" (Eleven)

After many such examples have been done, it may be helpful to write on the chalkboard what has taken place. For the preceding example, write

$$\begin{aligned} 7 + 4 &= 10 + 1 \\ &= 11 \end{aligned}$$

3. Transfer to a worksheet, using marks for the objects in both sets.

▸ Distribute worksheets to the children with problems like this:

|||||||| ||||

8+4=10+___

= ___

4. After the children are successful at working problems like those on the preceding worksheets, change the worksheets to show marks for the second addend only.

▸ Distribute worksheets like this:

||||

8+4=10+___

= ___

If a child does not make all the marks, as on the first worksheets, but makes marks as shown above, he either knows the sum of 8 + 2 or he is doing partial counting: "eight, nine, ten."

5. Provide no marks at all on the worksheets.

▸ Give only combinations such as 8 + 5, 9 + 3, 7 + 5, and so on. At this point, children either make all the marks, as in the third step, make only enough marks for the second addend, as in the fourth step, or write only the sum, since they have done enough of these to "just know." Permitting these different levels of solutions on the same worksheet is one way of providing for individual differences.

using partitions for sums of 11 to 18

Partitions can be used to practice the related facts with sums greater than ten. The usual sequence from flannel board to chalkboard to worksheet is used. Any of the three forms indicated here may be used on the worksheet.

1.

||||||| \ |||||||

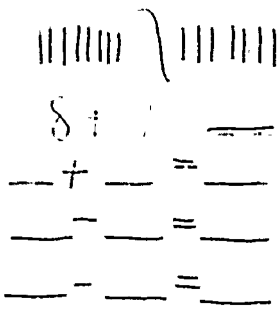
___ + ___ = ___

___ + ___ = ___

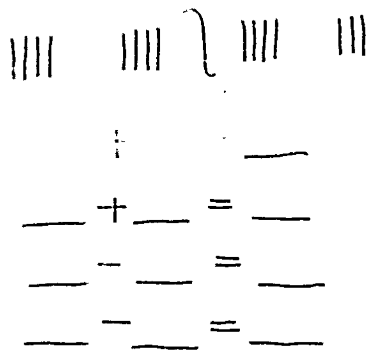
___ − ___ = ___

___ − ___ = ___

2.



3.



If form 1 is used, the child must count to find the number of each subset in order to get the combination. Then he either knows the sum and writes it, counts on from eight to fifteen, or counts the members of the entire set by ones. Once he has the first equation finished, he may write the others, indicating that he knows the commutative property of addition and the inverse operations. Others may combine in various ways counting and covering subsets, as suggested for sums of ten or less. Form 2 allows the child to complete the equations if he knows them without having to count to determine the number of each subset. If he does not know the structure, he will use all the ways indicated in form 1. Form 3 allows the child who recognizes two sets of four as 8 and a set of four and a set of three as 7 to obtain the numbers of the subsets without counting.

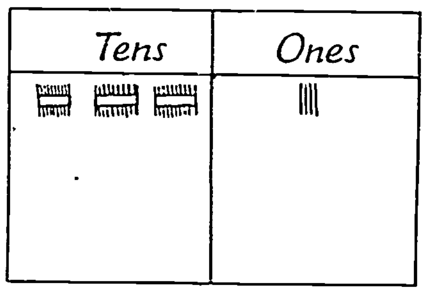
lesson plans

The ideal lesson plan should be a olend of several ingredients. At least ten minutes of the period might well be devoted to group activities in which everyone is involved and the feeling of a class working together is maintained. The group lesson provides an opportunity for questions and discussion. Everyone can participate at his own level—the fact that a student does not know all his basic facts does not preclude his learning other things. The development of concepts should continue while the students are learning the facts and ways of thinking about the facts. The combination of adequate time and good teaching results in a greater range of ability and skill in solving problems and handling computation. Therefore, the teacher must base the small-group assignments for a particular day on a careful evaluation of the work done by the children the previous day and allow enough time for every child to learn. The following lesson plan might be feasible by midyear in grade 1.

All pupils: Count by tens and fives to 100 on the number line. Any child who volunteers may be the leader and move his finger along a number line as the class counts aloud. (When a child first starts counting, the teacher should be sure that he puts his finger on 0.) No child will ask to "do the tens" or "do the fives" if he is not confident that he has mastered the count.

Use a word problem for group solution. Example: There are 16 boys and 15 girls present today. How many children are here? Use the number line and have a child find 16 on the line. Let the class count 15 more to arrive at 31. (Note that this word problem can be solved even though the children have not had experience in adding 16 and 15. Every class period in the primary grades should include a word problem so that children have experience in solving word problems and build confidence in their ability to do so.)

Play a game, such as identifying a number by its representation on charts showing tens and ones, for examnle, 34:



New work Introduce subtraction equations for partitions (sums 6-10).

Grouping: Eddie. Rosa. Leroy. and Susan—worksheet with partitions involving sums not greater than five. They will write subtraction equations for these partitions.

Joe. Bobby. and Leon—worksheet with subtraction equations (sums to ten). still using blocks.

Ethel and Jim—worksheet with addition equations (sums to ten). using blocks or marks.

Rest of class—worksheet with subtraction equations involving partitions with sets of five to ten.

addition and subtraction with two-digit addends

Before adding and subtracting with two-digit addends. the teacher must change the equation from the horizontal form to the vertical form. Tell the children a story problem: "Bill has five marbles. His brother gave him four more. How many marbles does he have now?" Ask the children what addition equation can be written for this problem. When the children say. "Five plus four equals nine." write it on the chalkboard like this:

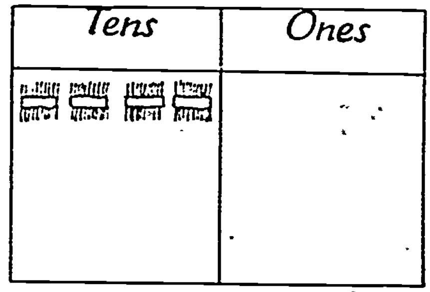
$$\begin{array}{r} 5 \\ +4 \\ \hline 9 \end{array}$$

Continue with other. similar stories and write the suggested equations in the vertical form. It is not unusual for children to accept the form without question. If no questions are asked, they have probably assumed that the teacher has simply decided to write the equation up and down rather than across. Subtraction equations from story problems can then be written in the vertical form. From this point on, practice worksheets should take this form. If previous equations have been read as "five plus four equals nine" and "seven minus two equals five," then the reading is the same from the top down.

The following sequence of steps is suggested for adding and subtracting with two-digit addends.

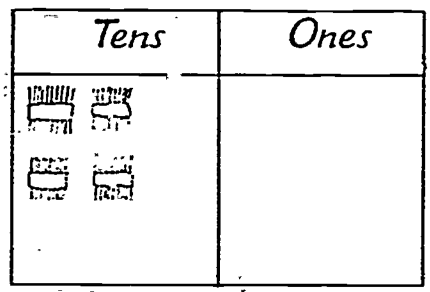
1. Add multiples of ten less than 100 by using the tens and ones charts.

► Show 20 + 20 = 40 this way first:



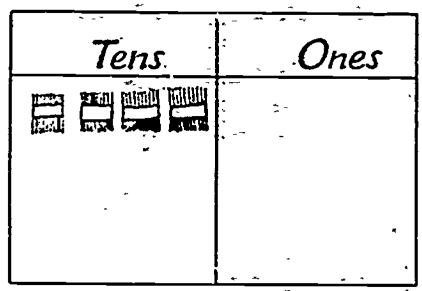
Then like this:

$$\begin{array}{lr} 2 \text{ tens} & 20 \\ +2 \text{ tens} & +20 \\ \hline 4 \text{ tens} & 40 \end{array}$$



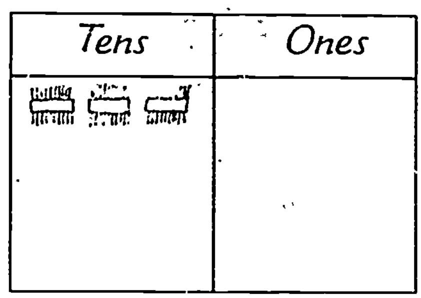
2. Subtract multiples of ten less than 100 by using the tens and ones charts.

► Show 40 - 10 = 30. Begin by showing the four tens:



Then show the charts after one ten has been removed:

$$\begin{array}{lr} 4 \text{ tens} & 40 \\ -1 \text{ ten} & -10 \\ \hline 3 \text{ tens} & 30 \end{array}$$



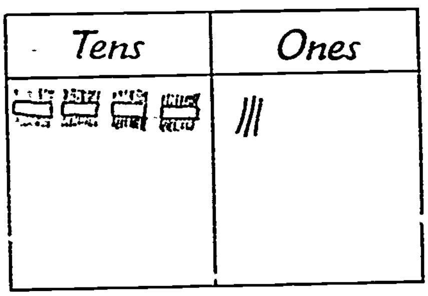
3. Practice facts orally while the children are waiting in line to go to lunch or to physical education.

► Ask what is 20 + 30? 40 + 10? 70 + 20? 80 - 40? 70 - 20? 40 - 20?

The preceding steps are designed to help children think in tens. For example, "Two tens plus two tens equals four tens, and twenty plus twenty equals forty." They should not think mechanically, "0 + 0 = 0 and 2 + 2 = 4."

4. Develop the expanded form for two-digit numerals from the tens and ones charts.

► Show the charts and ask, "What is the number?" (Forty-three) Stand between the charts. "Today we are going to write forty-three in a new way. How many are here?" Indicate the tens chart. (Four tens, or forty) "Plus how many more?" Indicate the ones chart. (Three)



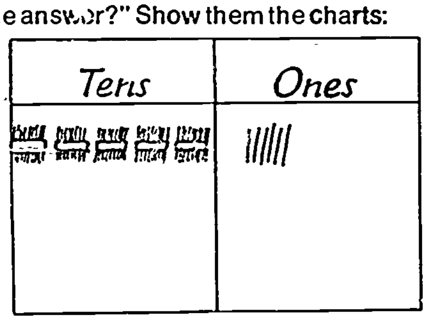
Write on the chalkboard:

$$43 = 4 \text{ tens} + 3$$
$$43 = 40 + 3$$

Do several other examples. Tell the children that 40 + 3 is called the *expanded form*. Give a worksheet that provides practice in writing the expanded form of numerals.

5. Start with the expanded form and use the place-value chart to get the simplest name for the number.

► Present the problem 50 + 6 = ____ and ask the children, "What is the simplest name for the answer?" Show them the charts:



The simplest name is 56.

After doing several examples, distribute worksheets to give the children practice in going from the expanded form to the simplest name for the number.

6. Put together the four skills previously taught.

► Add:

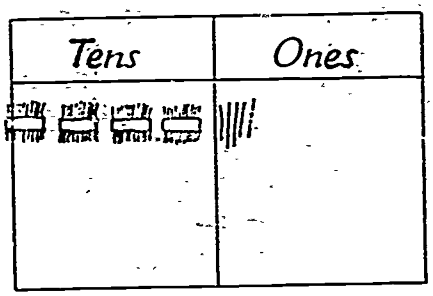
$$\begin{array}{r} 43 \\ +24 \\ \hline 67 \end{array} \qquad \begin{array}{r} 40 + 3 \\ 20 + 4 \\ \hline 60 + 7 \end{array}$$

The children have had practice with the basic addition facts (3 + 4 = 7), adding multiples of ten (40 + 20 = 60), writing the expanded form (40 + 3 and 20 + 4), and using the expanded form to go to the simplest name for the number (60 + 7 = 67).

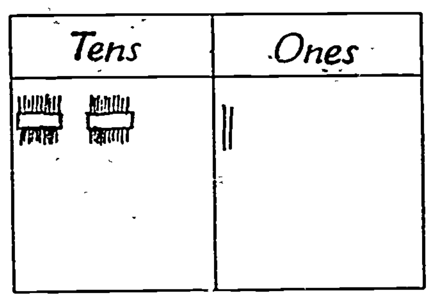
7. For subtraction, represent the number on the place-value chart.

► Subtract:

$$\begin{array}{r} 45 \\ -23 \\ \hline 22 \end{array} \qquad \begin{array}{r} 40 + 5 \\ 20 + 3 \\ \hline 20 + 2 \end{array}$$



Remove three from the ones chart and twenty from the tens chart and show the charts now:



renaming in addition and subtraction

Each step involving renaming in addition and subtraction should be represented on the tens and ones charts.

► Add 46 and 35.

$$\begin{array}{r} 46 \\ \underline{+35} \end{array}$$

Expand and show on the charts:

$$\begin{array}{rcr} 40 & + & 6 \\ \underline{30} & \underline{+} & \underline{5} \end{array}$$

Tens	Ones

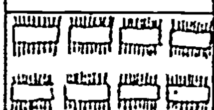
Then show the addition on the charts:

$$70 + 11$$

Tens	Ones

"Do we have enough to make a set of ten on the ones chart?" (Yes) Remove ten ones and place a bundle of ten ones on the tens chart and show the charts:

$$80 + 1$$

Tens	Ones
	

"What is the number?" (Eighty-one) The problem can now be written vertically:

$$\begin{array}{r} 46 \\ \underline{+35} \\ 11 \\ \underline{70} \\ 81 \end{array}$$

► Present 73 − 29 to the children.

$$\begin{array}{r} 73 \\ \underline{-29} \\ 44 \end{array} \qquad \begin{array}{rcr} 60 & & 13 \\ \cancel{70} & + & \cancel{3} \\ \underline{20} & \underline{+} & \underline{9} \\ 40 & + & 4 \end{array}$$

Tens	Ones

Remove a ten from the tens chart and put ten ones on the ones chart:

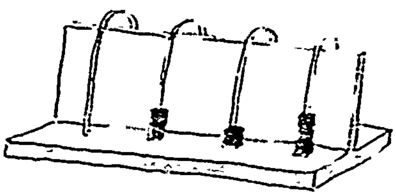
Tens	Ones

Then remove nine from the ones chart and twenty from the tens chart and show the answer, 44:

Tens	Ones

Replacing the charts with a homemade abacus can save time. A useful abacus should have four wires with eighteen disks in the ones place,

nineteen in the tens place, nineteen in the hundreds place, and nine in the thousands place. This type can be used throughout the primary grades and is easy for both teachers and pupils to manipulate.



The children should discuss the difference between how a number is represented on the charts and on the abacus. They will say many things, but the significant difference is that ten is represented on the chart by a bundle of ten ones but on the abacus by one disk. Any true answers the children give, such as difference in color or that one is horizontal and the other vertical, should, of course, be accepted until some child suggests the significant difference.

A few minutes a day should be taken so that eventually each child will have had the opportunity to manipulate the abacus as he renames in addition and subtraction. When they understand the concept, introduce the short form, which can also be adequately represented by the abacus.

Each variation in the algorithms for addition and subtraction will not be discussed here. Techniques similar to those used in the preceding activities can be used in renaming as needed, such as ten ones as 10, ten tens as 100, or 100 as nine tens and ten ones. The usual sequence is available in commercial textbooks.

learning addition and subtraction facts

There are several properties and generalizations that children should know in order to decrease the number of facts in their memory load. Although it is not expected that first-grade children will know all the basic addition and subtraction facts at the end of the year, they will have made several generalizations from their experiences:

1. The children will realize that when zero is an addend, the sum is the other addend. (Zero is the identity element for addition.) This eliminates nineteen facts from the memory load; these are indicated by color A on the addition chart.

2. The children will have generalized that when one is an addend, the sum is one more than the other addend. This eliminates another seventeen facts from the memory load; these are colored B on the chart.

3. The children will have learned that the order in which the addition is performed does not change the sum (commutative property of addition). This eliminates twenty-eight more facts, colored C on the chart.

4. The children will know the seven doubles, which are colored D on the chart, thus eliminating seven more facts.

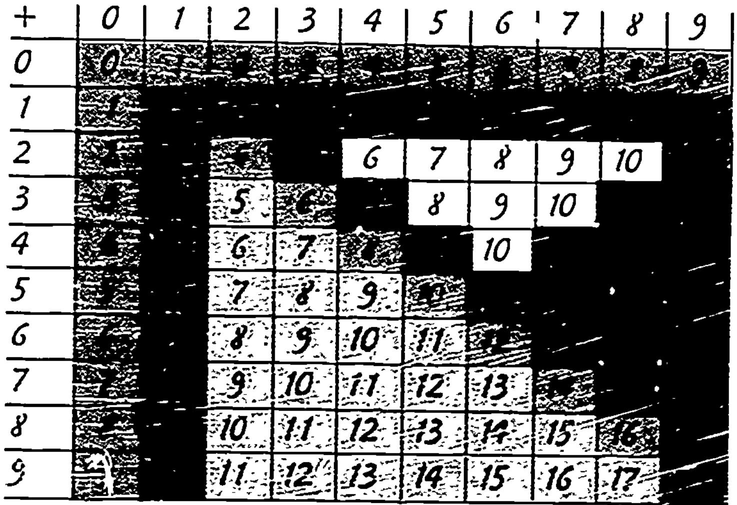
At the next grade level, an effort should be made to master other facts so that they too can be eliminated from the child's memory load. Three facts with sums less than ten, $2 + 3$, $3 + 4$, and $4 + 5$, can be worked around the doubles: if $3 + 3 = 6$, then $3 + 4 = 7$ and $3 + 2 = 5$. These are the facts colored E on the chart. The nines are easy once the children can see that when nine is an addend, the sum is ten plus the other addend less one:

$$\begin{array}{r} 9 \\ +4 \\ \hline 13 \end{array} \qquad \begin{array}{r} 6 \\ +9 \\ \hline 15 \end{array} \qquad \begin{array}{r} 8 \\ +9 \\ \hline 17 \end{array}$$

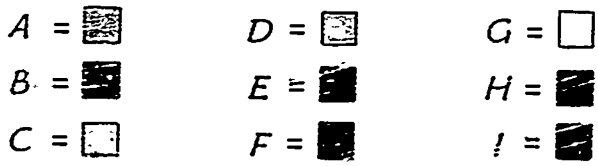
The nines are colored F on the chart. All this leaves nine facts with sums of ten or less (those colored G) to practice and to master.

Six facts with sums greater than ten must be mastered using the associative property of addition. These are colored H on the chart. Three more, those colored I, can be worked around the doubles as before—that is, if $6 + 6 = 12$, then $6 + 5 = 11$ and $6 + 7 = 13$; if $7 + 7 = 14$, then $7 + 8 = 15$.

An understanding of inverse operations helps children relate subtraction to addition. If a child adds 8 to 6 and gets 14, then he should see that under the inverse operation, 14 minus



Addition chart



8 is 6. In the first procedure he adds the 8, and in the second he must subtract the 8. Commercial texts sometimes label these processes "doing" and "undoing." This is not an easy concept for children, who often do not grasp that the "doing" was adding the 8 and that therefore, in order for the result to be the number with which they started (6), the "undoing" must be subtracting the 8.

Thinking "What number added to six equals fourteen?" is a good way of using language and a knowledge of addition facts to eliminate subtraction facts, for if this interpretation is used and if the children know the addition facts, then there are no subtraction facts to be learned.

When correctly administered, timed tests can be used to ascertain what facts need further work. Each problem should be timed, not merely the test itself. If a timed test is given, the teacher must be sure that the children do not have enough time to use immature ways of arriving at the sum or difference. The best way is to use a tape recorder that voices the combinations with just sufficient time (about three seconds) between combinations to allow the children to write the answers if they know them automatically. It is then possible to know exactly which facts are giving which child trouble. The results also permit the teacher to group children so that they work only on the facts they do not know.

Holding up flash cards with combinations such as 4 + 5 or 9 - 2 is not a very productive use of time. Those children who know the answer will shout it out. Those who do not know will relax, since they know that the others will satisfy the teacher by giving the correct response. Flash cards can be effectively used after a timed test like the one described above. When the children have been grouped to work on those combinations they have not mastered—perhaps, for example, into four groups, those working on addition sums of ten or less, those working on sums from ten to eighteen, and those working in similar groups on subtraction facts—one child in each group can be appointed leader for the day and hold the flash cards. As he holds up a card, the others show him the answer by raising their appropriate numeral card. Then the leader turns over his combination card to show the answer. Anyone who has the wrong answer is required to write down that fact for future study. The leaders should be changed every day so that they too have the opportunity to work intensely on those facts they do not know.

games that provide practice

▸ *Chalkboard Race.* Organize two (or more) teams and put two sets of numbers, one for each team, on the chalkboard.

add 6   2 7 5 1 9 4 8 3 0 6
8 13 11 7 15 10 14 9 6 12

add 6   8 3 0 6 2 7 5 1 4 9
14 9 6 12 8 13 11 7 10 15

Have individual members of the teams come to the board. After giving such instructions as "Add six, add nine, add four," have the children write the answer below each number. The first one finished with all correct sums wins a point for his team. This game can also be used for subtraction by putting the numbers 10 through 18 on the board.

▸ *Hidden Numbers.* Think of some combination of numbers, and provide clues to enable the children to guess the numbers. "We shall use the set of numbers zero through nine. I am thinking of two numbers. Their sum is nine." Let the children give all the possibilities: (0, 9), (1, 8), (2, 7), (3, 6), (4, 5). Since it is still not possible for the children to ascertain the correct combination, give another clue: "One number is greater than seven." This leaves (0, 9) and (1, 8). When the children determine that they still do not have enough information, they will ask for another clue. Say, "The difference is seven." Now only one possibility fits the information: (1, 8).

▸ *Show Me.* Provide each child with cards having a zero and two sets of the digits one through nine. Hold up a combination such as 4 + 8. The children show the sum by holding up the cards [1] [2]. Subtraction combinations can also be used.

▸ *Maneuvers on a Lattice.* Make a lattice so that the needed facts are used. For the lattice shown here, the children might generalize as shown by the arrows; that is,

.	.	.	.	.	.
.	.	.	.	.	.
13	14	15	16	17	18
7	8	9	10	11	12
1	2	3	4	5	6

2 → = 9 (add 7)   15↓ = 9 (subtract 6)
3 ↗ = 9 (add 6)   15 ↙ = 5 (subtract 10)
8 ↖ = 10 (add 2)   18 ↙ = 11 (subtract 7)

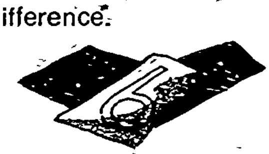
After some practice, give only the bottom row of the lattice so that the children must either visualize the rest of it or make the generalizations.

▸ *Patterns.* Ask the children to find the missing numerals in patterns like these:

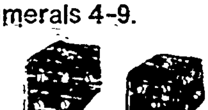
___, ___, 7, 10, 13, 16, ___, ___
3, 7, ___, ___, 19, 23, ___, ___
1, 3, 6, ___, ___, ___, 28, ___

▸ *Card and Dice Games.* Most of these games are played by pairs of children.

Give each child numeral cards that include a zero and two sets of the digits one through nine. Have one child declare the play—for example, to play nines. He then puts out one of his cards. If the other child has a card that he can play that will result in a sum of nine, he plays it, takes the pair, and leads another card. The child with the greater number of cards at the end of the game wins. For subtraction, increase the digits on the cards to eighteen and ask for a difference.



Have one child roll a pair of dice and have the other child name the sum. The child gets one point for each correct answer. The one with the most points wins. For sums greater than twelve, use two plain wooden blocks and number the sides 4–9. For subtraction, ask for the difference between the two numbers. One block could have the numerals 8–13 and the other the numerals 4–9.



problems

As discussed in chapter 4, it is important that children be asked to solve *real* problems. Story problems such as those used in commercial texts and the problems used in the physical representations of addition and subtraction indicated earlier in this chapter are not really problems. They are situations from which the modeling, such as 2 + 3 = 5 and 7 - 2 = 5, can be derived. The following are some problems that may be found suitable for certain children or groups of children.

1. John had a box of chocolates. His mother said that he had more than 12 but fewer than 32. When he counted his chocolates by fours, he had two left over. When he counted them by fives, he had one left over. How many chocolates did he have? (26)

2. The first picture shows two rings. The left-hand ring contains 6 counters, and the right-hand ring contains 4 counters.

If we move a counter from the right-hand ring to the left-hand ring, there will be 3 counters in the right-hand ring and 7 counters in the left-hand ring.

In how many different ways can you arrange 10 counters in the two rings? (10)

3. Complete the following addition table:

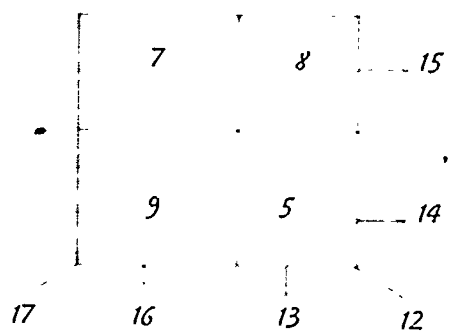
+			3			
	12		11	15		
6		6			7	
2			5	9		
						13
5						14

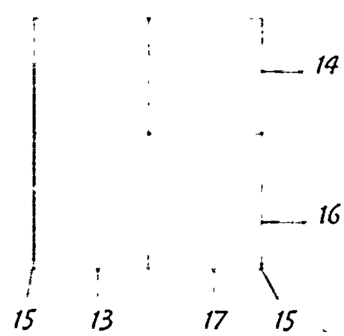
4. Ask a friend to think of a number between 1 and 10. Find out what the answer is by asking him no more than five questions that may be answered only by yes or no. How many questions would you need to ask in order to find a number he has chosen between 1 and 20? Between 1 and 50? Between 1 and 100?

5. Fill in the missing digits in these examples:

3*	*3	*7	*4	63
+*7	+4*	+**	-2*	-*7
62	92	90	35	3*

6. The first box shows how the numbers inside the box were added across, down, and diagonally. Find the missing numbers inside the second box.





multiplication

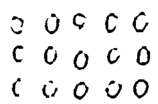
There are two approaches to multiplication, neither of which has been universally accepted: repeated addition and arrays. Both should be

used when appropriate Repeated addition is an effective strategy when used for multiplication with two, since children can be convinced that they know this table if they know the doubles in addition. But is is not very effective to think of 8 × 7 as 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7, since children are not competent in counting by sevens or in column addition by the time multiplication is introduced.

The rectangular array is a physical representation of multiplication. It can be used effectively with the distributive property of multiplication to provide a method for discovering and learning multiplication facts and to discourage the making of marks and other time-consuming counting procedures.

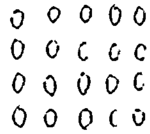
Before beginning multiplication, the teacher should review counting by threes, fours, and fives to enable the children to determine the number of members in an array with a minimum of difficulty. The following activity will serve as a good introduction to arrays.

► Arrange a number of cans in an array like this:

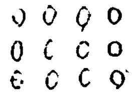


"Mary's mother arranged cans of soup on the kitchen shelf in this way. How many cans of soup does she have?" (15) "How did you find the number of cans?" (Counting by ones to 15. Counting by fives—5, 10, 15. Counting by threes—3, 6, 9, 12, 15.) "When we arrange objects in rows [indicate that the rows are horizontal] and columns [vertical] with the same number of objects in a row, we call it an *array*." Present various arrays on the chalkboard (using no more than five rows and five columns) and ask the children how many rows, how many columns, and how many members are in the array. Explain that the cans shown above are arranged in a 3-by-5 array, which has fifteen members.

After several such activities, the children can be given worksheets containing arrays and problems like the following:

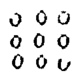


This ___ by ___ array has ___ members.

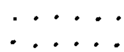


This ___ by ___ array has ___ members.

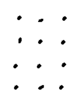
When the children feel comfortable with this idea, change the questions: "How many rows? How many in each row?" These questions lead naturally to the idea that, as expressed for the arrays shown above, four 5s = 20 and three 4s = 12. Worksheets using these questions may be given next:



3s =



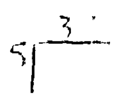
6s = ___



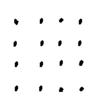
3s = ___

When the children are competent in naming the number of rows, the number of members in each row, and the number of members in an array, review the addition and subtraction signs, "the signs that tell us to add and subtract." Indicate that to find the number of members in an array, there is a special sign that tells them to multiply the number of members in each row by the number of rows. Then introduce the times sign. "For a 3-by-4 array with 12 members, we have been writing three 4s = 12. We can also write the equation as 3 × 4 = 12."

Show the children various arrays (none greater than 5 × 5 or less than 2 × 2) and ask them to give the multiplication equation for the array. Then use arrays like these:



5 x 3 - ___ -



___ x ___ = ___

These types of arrays may be used on worksheets to help children master the following six basic facts:

$3 \times 3 = 9 \quad 4 \times 4 = 16 \quad 5 \times 5 = 25$

$3 \times 4 = 12 \quad 4 \times 5 = 20$

$3 \times 5 = 15$

A mastery of these facts is needed before the distributive property is used to discover the basic multiplication facts greater than $5 \times 5 = 25$. Word problems similar to the following may be used to reinforce the modeling.

1. Mary's teacher brought 5 books to school each day for 3 days. How many books did she bring to school?

2. John broke 2 balloons each day of a school week. How many balloons did he break?

3. Jane put 3 books on each of 4 shelves. How many books did she put on the shelves?

While working on the mastery of these facts, the children can be taught generalizations such as the commutative property of multiplication and the generalizations involving 0, 1, and 2 as factors.

► Prepare cards with various arrays. To develop the commutative property, display a 4-by-5 array. "How many in the array?" Then turn it so that it becomes a 5-by-4 array. "How many in the array now? Does it make any difference whether we multiply 4×5 or 5×4?" (No)

The array is an excellent way to show children that the operation of multiplication is commutative, since they can physically see that the number of members in the array is the same even if the number of rows has been interchanged with the number of columns.

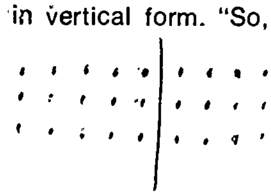
In most programs the language of multiplication is "factor times factor equals product," with the first factor being the number of rows. This terminology is used with the children. Note that the physical representation of $3 \times 5 = 15$ is different from $5 \times 3 = 15$, although the product remains the same.

► Use various array cards having only one row to help the children generalize that if one is a factor, the product is the other factor. Show $1 \times 4 = 4$ and $4 \times 1 = 4$, $1 \times 3 = 3$ and $3 \times 1 = 3$, and so on, as examples of both the identity element for multiplication (one) and the commutative property of multiplication. Write all the equations on the chalkboard. "If one is a factor, what is the product?"

► Use arrays with two rows or two columns and ask the children if they can write an addition equation for the array. For example, for $\begin{matrix} o & o & o & o \\ o & o & o & o \end{matrix}$ the equations $2 \times 4 = 8$ and $4 + 4 = 8$ can both be written. It is necessary to convince the children that since they already know the addition doubles, they know the product when one of the factors is two.

Once the children have learned the six basic facts mentioned previously, an array card can be folded to find the products of facts greater than 5×5.

► Present $3 \times 9 =$ ____, for example, and ask a child to fold the card in two pieces so that he can tell how many members in the array are on each side. "What multiplication equation is suggested by the first side?" ($3 \times 5 = 15$) "By the second side?" ($3 \times 4 = 12$) Write these on the chalkboard and then add $15 + 12 = 27$ in vertical form. "So, $3 \times 9 = 27$."



After doing several of these, give the children their own cards with 3-by-9, 4-by-9, and 5-by-9 arrays. Given such products as 4×8, 3×7, and 5×8, the children can fold the cards, write the multiplication equations, and add the products.

When the children have demonstrated the ability to use this procedure, hold up a 3-by-9 card and fold it so that the array pictured in the preceding activity is evident. "How many columns are on the first side?" (5) "How many columns are on the second side?" (4) "What name have you used for 9?" (5 + 4) Then write on the chalkboard:

$$\begin{array}{r} 5 + 4 \\ \times\ 3 \\ \hline 15 + 12 \end{array} \qquad \begin{array}{r} 15 \\ +12 \\ \hline 27 \end{array} \qquad 3 \times 9 = 27$$

Once the children understand that the folding of the card indicates the renaming of the number of columns, they can dispense with the card and select the name. for the number of columns. This enables them to obtain the product if they do not know it. Note that this procedure introduces the distributive property of multiplication at the simplest level possible.

It is not possible to show the zero facts, other than $0 \times 0 = 0$, on an array card. Asking them to make the array is an effective way to convince children that when zero is a factor, the product is zero. Given 3×0, for example, the child may begin to write in the three rows, but when you ask him how many columns there will be, he will have to erase the three rows. If given 0×3, however, the child cannot start his picture, since there are no rows. Any child making a mistake on a zero fact should be asked to make the array on his paper. By the time factors of zero occur, children will be quite familiar with the idea that 7×6 means seven sets of 6; then 3×0 can be thought of as three sets of 0. Since three sets of $0 = 0$, $3 \times 0 = 0$.

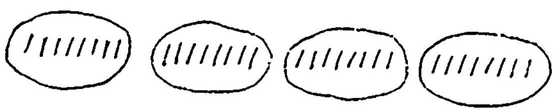
It is worthwhile to help children generalize nine as a factor. Present the products of the nine table—9, 18, 27, 36, 45, 54, 63, 72, 81—in vertical form and ask the children what they can discover about these products. They will probably mention many things, but the important discovery is that the sum of the digits is 9. If no one discovers this, ask what is the sum of 2 + 7, 3 + 6, 4 + 5, and so forth. Then present the following examples:

$$\begin{array}{r} 9 \\ \times 7 \\ \hline 63 \end{array} \qquad \begin{array}{r} 6 \\ \times 9 \\ \hline 54 \end{array} \qquad \begin{array}{r} 9 \\ \times 3 \\ \hline 27 \end{array} \qquad \begin{array}{r} 4 \\ \times 9 \\ \hline 36 \end{array} \qquad \begin{array}{r} 9 \\ \times 9 \\ \hline 81 \end{array}$$

See if they can discover that if nine is a factor, the tens digit is one less than the other factor. If no one is able to see this, cover the 9 with your hand and ask how the tens digit in the product compares with the other factor. Combining the two discoveries, we see that for 7×9, for example, the tens digit is one less than 7, that is, 6, and the ones digit must be 3, since the sum of the digits must be 9.

Also, since children can count by fives so easily, it seems a shame not to allow them to count by fives (using their fingers if necessary) to determine such facts as 5×7. Surely they will know the fact if they count by fives for a sufficient length of time.

The teacher should strongly discourage any child from finding the product of 4×8 by drawing



$3 \times \quad =$

and counting by ones from the first mark to find 32. Instead, start him counting by twos, threes, fours, and fives as suggested previously and follow the procedures as outlined earlier in the chapter.

drill on multiplication facts

By allowing the children to count by fives and by teaching them all the generalizations and properties of multiplication, the teacher can eliminate many of the multiplication facts from the memory load of the children. Drill should be concentrated primarily on those products outlined in black on the multiplication chart (the five perfect squares), those products left without color, and at least two facts from each of the colored products to make certain that the

×	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3				9	12		18	21	24	
4				12	16		24	28	32	
5										
6				18	24		36	42	48	
7				21	28		42	49	56	
8				24	32		48	56	64	
9										

Multiplication chart

A = ■ B = ■ C = ■ D = ■ E = ■

generalization or understanding of a property is still well in hand. Of the one hundred basic multiplication facts, the generalization on zero eliminates nineteen (those indicated by color A), understanding the identity element of multiplication eliminates seventeen more (those colored B), and the generalization on two as a factor (those colored C) cuts fifteen more, leaving forty-nine. The generalization on nine (those products colored D) eliminates thirteen facts. Of the thirty-six remaining, counting by fives eliminates eleven more (those colored E). Of the twenty-five facts remaining, five perfect squares must be learned. Only half of the other twenty facts (those products not colored) must be learned, since the other ten are not needed because of the commutative property of multiplication.

Using a tape recorder as suggested in the section on addition can enable the teacher to determine exactly which facts a child needs to practice. Children can then be grouped to work on the facts they need to learn, and a leader with flash cards can be used as suggested previously in the section on addition. The teacher could then write on the back of a child's report card that he needs to practice 7 × 8, 6 × 6, and 8 × 8, for example, instead of writing that he needs to practice his multiplication facts. If a parent knows exactly what must be done, he is more likely to see that his child learns three, four, or five specific facts than if he is confronted with the problem of practice on a hundred facts. It is important to include facts that involve a generalization or a property, since if the child misses two of these, it is likely that his concept of the whole generalization or property has fallen apart and must be retaught.

games

The following games may be used for practicing multiplication.

► *Chalkboard Race.* Organize two teams and write two sets of numbers on the board.

X by 6	3	8	0	2	9	1	6	4	7	5
	18	48	0	12	54	6	36	24	42	30

X by 6	9	1	3	5	7	4	6	2	8	0
	54	6	18	30	42	24	36	12	48	0

Have two players, one from each team, go to the chalkboard. Say, "Multiply by six," and have the two children record the products beneath the numbers. The first one finished with all correct products wins a point for his team. Use a different factor with the next pair of players. (Record for later drill any specific fact not known by a player.)

▸ *Beanbag Throw.* Organize teams and make a design on the floor (or on paper taped to the floor) consisting of nine squares with a numeral in each square. Provide beanbags numbered 0-9 and have the players stand behind a mark and throw a beanbag on the squares. The player must give the product of the number on the beanbag and the number of the square into which he has thrown the beanbag. One point is given for each correct product. The team having the most points wins.

▸ *Show Me.* Provide cards like those used with this game in the section on addition. Use multiplication combinations rather than addition.

▸ *Hidden Numbers.* Make up the game so that the basic facts to be practiced are needed. (See the section on addition.) For example, the following clues might be given:

I am thinking of four numbers from one to nine.

1. The product of two of the numbers is 24. (Requires either 3 and 8 or 4 and 6)
2. The product of two of the numbers is 32. (Requires 4 and 8)
3. The product of two of the numbers is 48. (Requires 6 and 8)
4. The product of two of the numbers is 64. (Requires 8 and 8)

Answer: 4, 6, 8, 8

It is important to note that most games do not actually provide the practice the teacher wishes. A game like Bingo does not insure that those needing to know $6 \times 7 = 42$ will learn anything when the teacher calls out 6×7. The child who does not know the product will not place a counter on 42, but those who do know the product will. There is no way that the teacher can watch over thirty children and note those who do not know that particular fact. Games like this provide a change of pace and fun for the children, but no adult should assume that they do anything else.

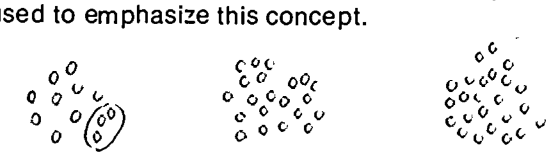
division facts

The inverse operation of the multiplication fact $4 \times 8 = 32$ is shown by $32 \div 4 = 8$. Practice on finding the missing factor can be given as the children work with arrays. If the array is presented as

$$5 \overline{\smash{)}\,25}^{\;N},$$

the teacher can ask, "What is the missing factor?" In this interpretation of division, the question is, How many 5s equal 25? However, it is not sensible to ask such a question when the child is confronted with a problem like $1{,}875 \div 25$.

Children find division difficult not only because they are required to use the other operations of addition, subtraction, and multiplication but because they do not really understand what the question is. When the idea is to develop not only the division facts but the algorithm as well, it is profitable to provide children with concrete materials. Then they may be asked to show a set of eighteen objects, told to make sets of three, and answer the question, "How many sets can you make?" After doing several of these examples, the equation $42 \div 6 = 7$ may be shown. This is now interpreted to mean having forty-two objects, making sets of six, and finding that there are seven sets. A worksheet such as the following may be used to emphasize this concept.



$12 \div 3 =$	$21 \div 7 =$	$24 \div 4 =$
	$7 \times \quad =$	$4 \times \quad =$

Explain that the symbol ÷ is read "divided by" and that in the examples above, ÷ means to make sets of three, seven, and four. Later, when the standard vertical form

$$\begin{array}{r} 6 \\ 3\overline{)\,18} \\ 18 \\ \hline \end{array}$$

is introduced, the same interpretation is used. "There are eighteen objects; we make sets of three and find that there are six sets." When confronted with an example such as 56 ÷ 7, the children are instructed to make as many sets as they are certain they can make in a "sure guess." The results might look like the following:

$$\begin{array}{r} 8 \\ 1 \\ 2 \\ 5 \\ 7\overline{)\,56} \\ \underline{35} \\ 21 \\ \underline{14} \\ 7 \\ \underline{7} \end{array} \qquad \begin{array}{r} 8 \\ 3 \\ 5 \\ 7\overline{)\,56} \\ \underline{35} \\ 21 \\ \underline{21} \end{array} \qquad \begin{array}{r} 8 \\ 2 \\ 2 \\ 2 \\ 2 \\ 7\overline{)\,56} \\ \underline{14} \\ 42 \\ \underline{14} \\ 28 \\ \underline{14} \\ 14 \\ \underline{14} \end{array} \qquad \begin{array}{r} 8 \\ 7\overline{)\,56} \\ \underline{56} \end{array}$$

Such a procedure has many advantages: (1) much practice is given on the multiplication facts; (2) the long-division algorithm is introduced at the lowest level possible; (3) the children understand what the question is; (4) practice is given on making a "sure guess," which will later be called an estimation; (5) the concept of division as a guessing game where the goal is to make the best guesses possible is introduced at a level where children can be successful; (6) by examining the various ways in which children arrive at an answer, the teacher can determine the level of efficiency at which a child is operating; and (7) as the advantages of "just knowing" the multiplication fact become evident, the children become convinced that it is easier for them to "know" than to do all the work required when they do not know.

Once children know the multiplication facts, there are no division facts to be learned. Children should be taught with timed tests to think that 72 ÷ 8, for example, means "8 times what number equals 72."

It is possible to use the multiplication chart (p. 165) for division facts if the chart is used without zero as a factor. Since division by zero is not allowed (see p. 164), only ninety division facts are possible. In the primary grades it would be better to avoid an explanation by simply using a chart that does not have zero as a divisor.

Word problems, similar to those below, may be used to provide practice in thinking about division and to reinforce the modeling.

1. Mr. Jones has 12 cars at his automobile repair shop. If he puts 3 cars in each garage, how many garages does he need?

2. David has 21 balloons. If he decides to give each of his friends 3 balloons, how many friends does he have?

3. There are 30 children in Charlie's classroom. They are going on a field trip and will ride 5 in a car. How many cars will be needed?

multiplication and division algorithms

Teaching division with dividends beyond the basic multiplication facts cannot be done effectively or efficiently if children are not competent in multiplying by tens, by hundreds, and later by thousands. Much practice should be given both orally and in written form to ascertain the competence required before different types of examples are presented.

To teach the multiplication of those multiples of ten that are less than 100, the place-value chart may be used so that children can see that $2 \times 10 = 10 + 10 = 20$, $4 \times 20 = 20 + 20 + 20 + 20 = 80$, and so on. The algorithm should be written on the chalkboard like this:

$$\begin{array}{r} 2 \text{ tens} \\ \underline{\times 2} \\ 4 \text{ tens} \end{array} \quad \begin{array}{r} 20 \\ \underline{\times 2} \\ 40 \end{array} \qquad \begin{array}{r} 4 \text{ tens} \\ \underline{\times 2} \\ 8 \text{ tens} \end{array} \quad \begin{array}{r} 40 \\ \underline{\times 2} \\ 80 \end{array}$$

After both oral and written practice has been

given on this type of multiplication, the following sequence should be developed:

1.
$$\begin{array}{r} 22 \\ \times 4 \\ \hline \end{array} \qquad \begin{array}{r} 2 \text{ tens} + 2 \text{ ones} \\ \times\ 4 \\ \hline 8 \text{ tens} + 8 \text{ ones} \end{array} \qquad \begin{array}{r} 20 + 2 \\ \times\ 4 \\ \hline 80 + 8 = 88 \end{array}$$

or

$$\begin{array}{r} 20 \\ \times 4 \\ \hline 80 \end{array} \qquad \begin{array}{r} 2 \\ \times 4 \\ \hline 8 \end{array} \qquad \begin{array}{r} 80 \\ +8 \\ \hline 88 \end{array} \qquad \begin{array}{r} 22 \\ \times 4 \\ \hline 88 \end{array}$$

2.
$$\begin{array}{r} 22 \\ \times 4 \\ \hline 8 \\ 80 \\ \hline 88 \end{array}$$

and eventually

$$\begin{array}{r} 22 \\ \times 4 \\ \hline 88 \end{array}$$

3.
$$\begin{array}{r} 20 + 8 \\ \times\ 3 \\ \hline 60 + 24 \\ 80 + 4 = 84 \end{array}$$

or

$$\begin{array}{r} 28 \\ \times 3 \\ \hline 24 \\ 60 \\ \hline 84 \end{array}$$

and eventually

$$\begin{array}{r} 28 \\ \times 3 \\ \hline 84 \end{array}$$

4. Before the children can proceed with this sequence, they must be able to multiply hundreds.

$$\begin{array}{r} 200 + 40 + 3 \\ \times\ 4 \\ \hline 800 + 160 + 12 \\ 800 + 170 + 2 \\ 900 + 70 + 2 = 972 \end{array} \qquad \text{or} \qquad \begin{array}{rl} 243 & \\ \times 4 & \\ \hline 12 & (4 \times 3) \\ 160 & (4 \times 40) \\ 800 & (4 \times 200) \\ \hline 972 & \end{array}$$

and eventually

$$\begin{array}{r} 243 \\ \times 4 \\ \hline 972 \end{array}$$

The following sequence for division is suggested:

1.
$$\begin{array}{r} 10 \\ 7 \overline{)\ 70} \\ \underline{70} \end{array}$$

2.
$$\begin{array}{r} 13 \\ \hline 3 \\ 10 \\ 7 \overline{)\ 91} \\ \underline{70} \\ 21 \\ \underline{21} \end{array}$$

3.
$$\begin{array}{r} 24 \\ \hline 4 \\ 10 \\ 10 \\ 6 \overline{)\ 144} \\ \underline{60} \\ 84 \\ \underline{60} \\ 24 \\ \underline{24} \end{array}$$

or

$$
\begin{array}{r}
24 \\
4 \\
20 \\
6\,\overline{)\,144} \\
\underline{120} \\
24 \\
\underline{24}
\end{array}
$$

A good commercial textbook will carefully sequence the work in multiplication and division so that the children are not confronted with several types of difficulty at one time. It is important for the teacher to permit the children to "guess" in division, as shown in the first example of step 3 before moving them on to the more efficient type of algorithm, shown in the second example of step 3. If the children write their examples on the chalkboard and discuss what was done, they will conclude that in the second example given above in step 3, the guess was based on the basic fact $6 \times 2 = 12$, and so $6 \times 20 = 120$. It is recommended that children estimate, or guess, using the basic fact and then use scratch paper to do the second multiplication. When nonapparent quotients are introduced, an incorrect guess on the basic fact is always too great, and so the child will try a lesser number. Fancy estimations, such as rounding up and rounding down, are confusing to many children; children of average or better intelligence will learn how to do this for themselves with experience.

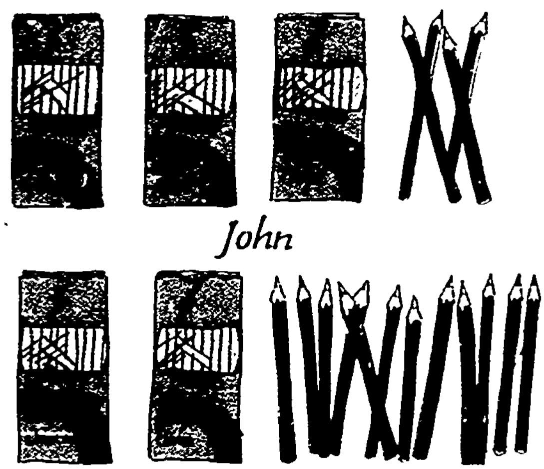
problems

As implied in chapter 4, problems that are real should be included whenever possible. The following may be useful.

1. Use twenty cubes to make four piles so that the first pile contains four more cubes than the second pile, the second pile contains one cube less than the third pile, and the fourth pile contains twice as many cubes as the second pile. (7, 3, 4, 6)

2. Four soldiers came to a wide river and found that the only means they had of crossing it was rowing a boat owned by two small boys. The boat would hold both boys; but it would not hold a boy and a soldier or two soldiers. How many trips were required for the soldiers to cross the river using only the boat and its oars? (17)

3. John and Rose were each given a number of colored pencils for their birthday. In the diagram pictured, each has laid out the same number of pencils, either loose or in boxes. John has three full boxes, and Rose has two full boxes. Each full box contains the same number of pencils. How many pencils are there in a box? (8)



The following problems have more than one solution:

1. Mary has 25¢. How many items costing 2¢, 3¢, and 5¢ can she buy?

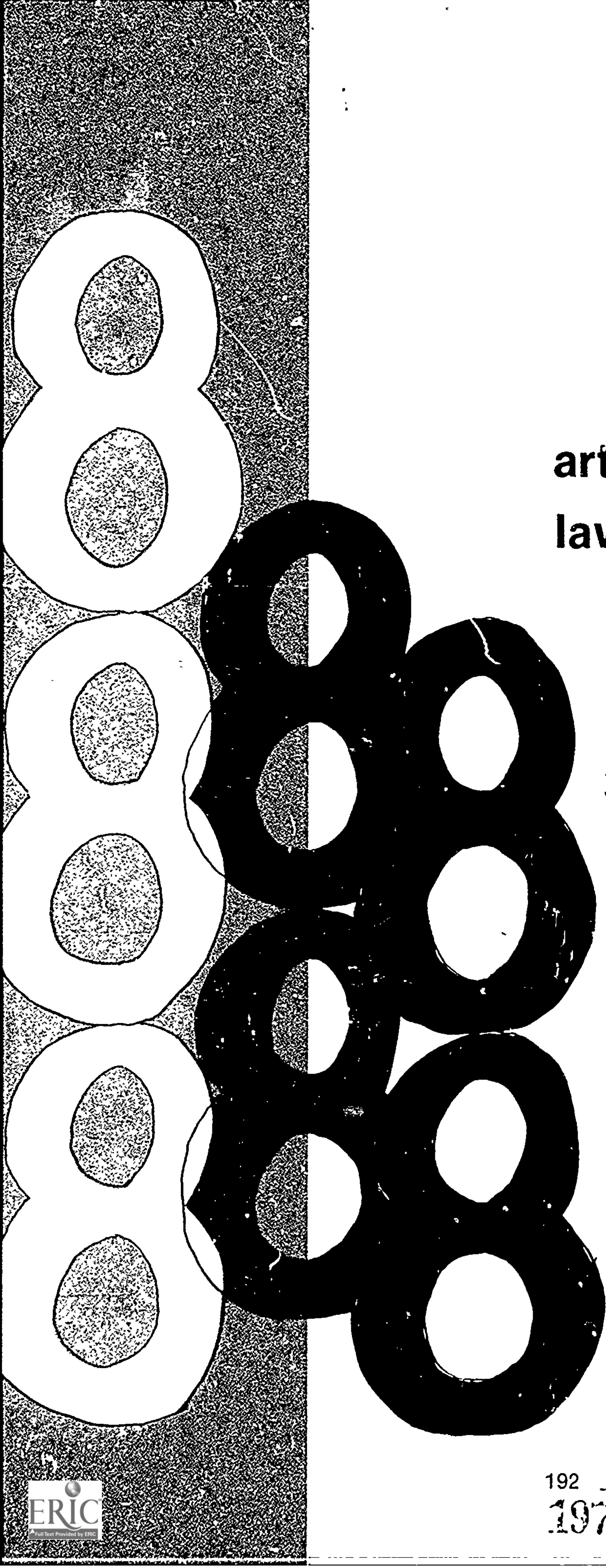
2. John has $1.00. How many items costing 10¢, 20¢, and 25¢ can he buy?

summary

The number of mathematical concepts, operations, symbols, and algorithms introduced, developed, and supposedly mastered by primary school children is of a large magnitude. Without the development of physical represen-

tations from which children can abstract ideas of number and operation, we have probably doomed many to a below-average achievement in mathematics as well as built up in them a dislike for the subject. The idea that we can *start* with symbols such as 2 + 3 5 and then try to make some sense out of them for children by mouthing endless words of explanation is an outmoded, even cruel, thing to do. Children are deserving of our best knowledge, experience, thinking, and observation on what kind of mathematics is suitable, how it can be presented clearly, and what procedures increase logical thinking, sophistication, and mastery in it. It is hoped that teachers will find the suggestions given in this chapter helpful as they try them, change their approaches, and improve on what has been developed here.

fractional numbers



arthur coxford
lawrence ellerbruch

THE word *fraction* has as one of its dictionary meanings a "fragment," a "bit," "a part as distinct from a whole." In common language, *fraction* is used to designate some unspecified part of a whole. For example:

> Only a fraction of the seeds germinated.
>
> A small fraction of the children chose white milk.
>
> We bought it on sale at a fraction of the cost.

The idea of *part of something* is the key idea here. Children acquire this concept early. The youngster who has a piece of toast covered with jam, which becomes part of a piece after several bites, has the idea. It is not unreasonable, then, to suggest that the first idea about the fractional number concept that a youngster may develop is that of a part of a whole, something less than a whole, or a bit of something. It is likely that when the youngster comes to school, the "part of something" idea is his concept of fraction, if he associates anything at all with the term. It is the purpose of the school program to take this idea as a starting point, to expand it, to develop it, and to make it more precise. The result at the end of the total school experience, K-12, should be a relatively sophisticated and useful concept of fractional numbers and of rational numbers as well.

Making precise the idea of "part of something" leads to discovering and to telling *how much* the part is. You may ask the child with part of a candy bar "how much" there is. The typical five-year-old will say "half," regardless of the amount.

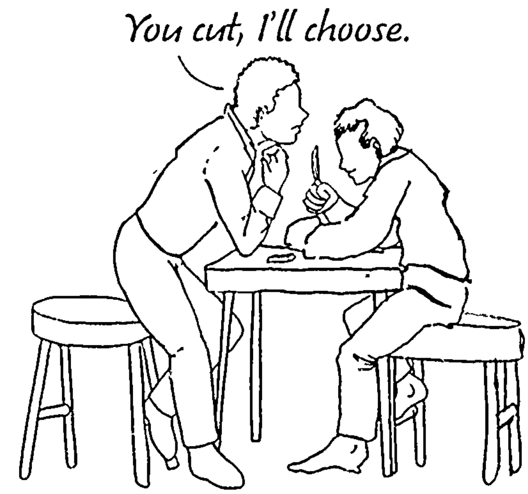
In order to develop the correct fraction concept, a child needs to think of these three questions:

1. What is the unit?
2. How many pieces are in the unit?
3. Are the pieces the same size?

A very young child will see a candy bar cut into three pieces and will believe that there is more candy *after* cutting it than *before* cutting it. The child must learn to think of the whole object when talking about fractions. Practice in identifying the unit and the whole object is extremely important. It is wise to ask often, "What's the unit?" and to have a copy of the unit available so that the children can refer to it as they look at part of the unit.

The number of pieces in the unit gives us information for part of the fraction name. For example, having three equal size pieces in a unit leads to the name *thirds*, four pieces to *fourths*, five pieces to *fifths*, and so on. Thus, the ordinal names help in learning the fraction names. Although ordinal names are not helpful for two equal size pieces, fortunately most children learn the word *half* at an early age, and this special name is not difficult to learn.

To have any consistent meaning for amounts other than a whole number of units, it is necessary to have "equal size" pieces in the unit. Children will see the importance of this if it is related to their experience in receiving a "fair share." For example, if a candy bar is cut into two unequal size pieces, it makes a difference which piece you choose. The "you cut and I choose" method will help reinforce the need for cutting into equal size pieces to get a fair share.



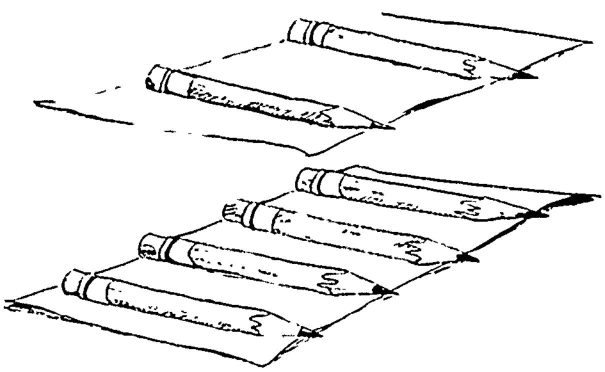
Much emphasis must be placed on equal size pieces. When using fractions to answer the "how much" question, you must not depend on which pieces are chosen.

Children need to be able to recognize equal size pieces and also to do the cutting into equal size pieces. They need a lot of help in learning

how to cut a unit into equal size pieces. Most often, the equal size pieces will be congruent, though this is not necessary.

One method for learning to "cut" a rectangular region into equal size pieces is to fold a sheet of paper to get halves, fourths, eights, and so on. You can also fold the paper to get thirds and ninths. Fifths are somewhat more difficult, but still within reason if marks are placed on the edge of the paper to aid judgment.

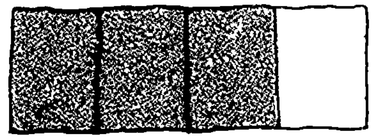
Segments and rectangular regions may be "cut" with a pencil by a halving process. Move the pencil until it appears that you have half on each side, then mark. By halving first the unit, then the resulting pieces, you can get halves, fourths, and eights fairly well. Thirds will require two pencils. Place one pencil on each side of the unit and slowly move them toward each other until it appears that you have equal amounts in the three places. Then mark the positions.



By combining the halving and "cutting into threes" methods, you can cut a unit into the number of pieces most important for children to use initially in working with fractions.

When a child can identify the unit, can tell the number of pieces in a unit, and can decide whether those pieces are equal size pieces, he is ready for naming fractions. The fraction names should be related to ordinal names for ease in remembering, noting the exceptions for one and two pieces.

Fractions are used to answer the question "how much?"



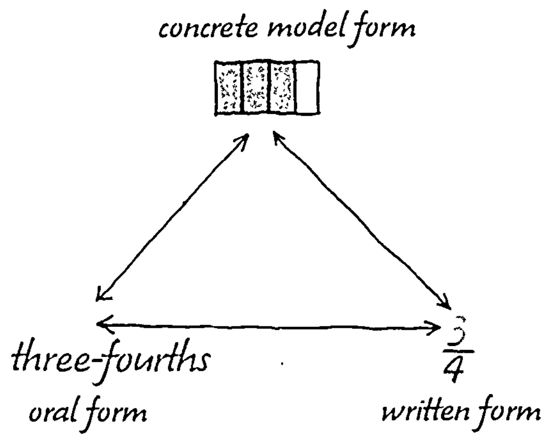
How much is shaded?

The question is answered by working through questions such as these:

What is the unit? (Whole thing)
How many pieces are in the unit? (Four)
Are the pieces all the same size? (Yes)
How much is each piece? (A fourth)
How many pieces are shaded? (Three)
How much is shaded? (Three-fourths)

When a child can successfully use fractions to answer orally the question, "How much?" he can be shown how to use the written form. A special effort is needed to teach the child the connection between the oral form "three-fourths" and the written form "$\frac{3}{4}$." Similarly, the written form must be connected with the concrete model.

Make sure the child can make all six connections between each of the following forms: oral, written, and concrete model.



Number of equal size pieces	1 piece	2 pieces	3 pieces	4 pieces	5 pieces
Ordinal names	first	second	third	fourth	fifth
Fraction names	unit	half	third	fourth	fifth

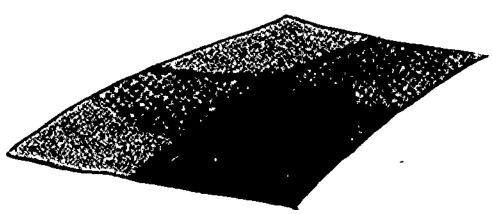
The following are descriptions of sample activities and questions to use to help the child learn the connections between the concrete model, oral, and written forms. A sheet of paper is used as the unit throughout.

▸ Concrete ⟷ Oral

1 Show the child a sheet of paper folded in thirds, with two-thirds marked in red and one-third marked in black. Ask, "How much is marked in red?" (Two-thirds) "How much is marked in black?" (One-third)
2. Say, "Use a sheet of paper to show me one-third." Have the child mark a sheet of paper or point to the part from a sheet already folded or marked.

▸ Concrete ⟷ Written

1 Show the child a sheet of paper folded in fourths, with three-fourths marked in red and one-fourth marked in black. Say, "Write the fraction that tells how much is black." ($\frac{1}{4}$) "Write the fraction that tells how much is red." ($\frac{3}{4}$)



2. Write $\frac{1}{2}$ on the chalkboard. Point to the written fraction and say, "Use a sheet of paper to show this much." Have the child fold a piece of paper or point to a part of a sheet already folded.

▸ Oral ⟷ Written

1. Say, "Write three-fourths." ($\frac{3}{4}$)
2. Write fractions on paper, such as $\frac{2}{3}$, $\frac{3}{5}$, and $\frac{5}{6}$. Say, "Read these fractions aloud." (Two-thirds, three-fifths, five-sixths)

Children should have a great deal of experience working with concrete materials. Sheets of paper can be used to represent a region. Sticks, toothpicks, or coffee stirrers can be used to represent length. These concrete objects should be used to establish all six connections before using diagrams to represent regions or lengths.

It is better to delay the use of sets of objects until the children's understanding of fractions used with regions and segments is very firm.

The following sequence is one that emphasizes the most important parts of the fraction concept:

1. Units
   Identifying the number of units
   Identifying more or less than a unit
2. Parts of a Unit Using Concrete Materials
   Identifying the number of pieces in a unit
   Identifying equal size pieces
   Making a unit into equal size pieces
3. Oral Names for Parts of Units
   Establishing fraction names
   Using fractions to answer "how much"
   Identifying fractions equal to 1
4. Written Fractions for Parts of Units
   Oral to written
   Concrete to written
   Written to oral
   Written to concrete
5. Representing Fractions with Drawings
   Transition from objects to diagrams
   Repetition of parts 1-4 of the sequence, using diagrams
6. Extending Notions of Fractions
   Fractions greater than 1
   Mixed forms
   Set model—using sets to answer questions such as, "What is $\frac{1}{3}$ of 6?"
   Comparing fractions
   Equivalent fractions

learning activities for fractional numbers

This section is organized on the basis of the developmental sequence given in the preceding section. Each part of the sequence is given, followed by a short description of the major learning goals for that heading. This material is followed by one or more activities or situations that can be used to help youngsters develop fractional number ideas.

The teacher should look for opportunities to help children develop fractional number ideas in natural settings. If such settings do not present themselves, then the teacher should plan "lessons" in which appropriate settings do arise. The following illustrations are to be thought of as examples that provide the basis for further teacher-designed activities that have similar purposes.

units

Identifying the number of units. The idea of "fractional number" grows out of the idea of "part of something." The purpose here is to develop or reinforce the idea of "part" as distinct from "the whole" or "all of something."

▸ Assemble a supply of bulbs to be planted in paper cups of various colors. Say, "Plant part of the bulbs in the pink cups. Did you put all the bulbs in pink cups?"

▸ Place some red beads and some blue beads on the table. Ask, "Are all the beads red?" (No) "Give me the part of the beads that are red. Did you give me all the beads?"

▸ Give the children drawings of flowers. Say, "Color part of the flowers red; color another part blue."

▸ Ask, "Is all the class here today, or is only a part here?"

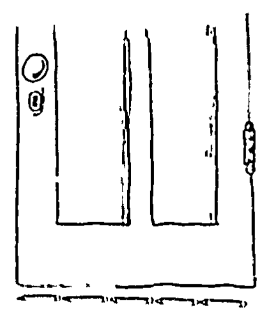
Identifying more or less than a unit. Since fractions depend on the particular unit being used, children should be given experience working with different units and parts of units.

▸ Place several sheets of paper on the desk. Hold up one sheet of the same size. Say, "The piece of paper in my hand is a whole unit. How many whole units are on the desk?" Repeat with different-sized units.

▸ Hold up a unit. Place a unit and another part of a unit on the desk. Say, "I have one whole unit in my hand. Is there more than a unit or less than a unit on the desk?" Repeat with different-sized units.

▸ Have children use sheets of paper to find how many units will cover their desk tops.

▸ Have children use pencils as units of length to find the width of the door.



parts of a unit using concrete materials

Identifying the number of pieces in a unit. Children need experience in identifying the number of pieces relative to a given unit. This reinforces the notion of unit and its role in fractions.

▸ Fold a sheet of paper into two parts. Ask, "If the whole sheet of paper is a unit, how many pieces are in the unit?" (2) Repeat, using different numbers of parts and different units. Have children "cut" a unit into various numbers of pieces.

Identifying and making equal size pieces. Equal size pieces are fundamental to the fraction concept. Practice in both recognizing and making equal size pieces is important. The activities should use objects to represent both area and length. They can be named "cover up" units and "how long" units.

► Fold sheets of paper into equal size pieces. Fold others into unequal size pieces. Have students pick the sheets folded into the equal size pieces.

► Have each child mark and fold separate sheets of paper into two, three, four, five, six, eight, and ten parts. Be sure the paper is an easy size to manage.

► Show students how to fold a sheet of paper into two equal size pieces, four equal size pieces, and eight equal size pieces by matching the corners and edges.

► Use pencils, as shown previously, to mark a sheet into three equal pieces. Mark another sheet with six pieces by folding.

► Use pencils, as shown before, to make five equal pieces. Then make another sheet with ten pieces by folding. Say, "How many equal size pieces are in this sheet?" Hold up the sheet folded into ten equal pieces. "Which sheet shows how you share a cake with ten people?" (See ■.)

► Count ordinally: "First, second, third, fourth, ..." Pair the fraction names with the ordinals. Note the different name for "half" and "halves."

Using fractions to answer "how much."

► Discuss the names *half, third, fourth*, ... Ask the children to tell how much each piece of a unit is when they have the corresponding number of equal size pieces in the unit.

► Discuss the fact that fractions tell us how much something is when compared to a unit. Say, "You know how much by telling how many pieces are used and what part of the unit the equal size pieces are."

► Show the children a sheet of paper folded in six equal size pieces, with four marked in red and two in black. Say, "If this whole sheet of paper is a unit, how many pieces are red?" (Four) "How many equal size pieces are in the unit?" (Six) "What part of the unit is each piece?" (One-sixth) "Say the number of red pieces and the name for each piece together." (Four-sixths) "This is the fraction that tells how much of the unit is red." Repeat with other units and markings.

► Use the sheets of paper each child has and ask similar questions: "Place your finger on two-sixths; hold it up so that I can see. Pick the sheet with ten equal pieces. Put your fingers on three of them. What part do you have your fingers on?" (Three-tenths)

► Repeat the previous activities with units such as straws, sticks, or thin strips of construction paper.

oral names for parts of units

Establishing fraction names. As soon as children can recognize equal size pieces in a given unit, they can start using fractions orally to talk about how much of a unit is being considered. To do this, they first need the fraction names *half, third, fourth*, and so on.

Identifying fractions equal to 1. A special situation occurs when *all* the equal size pieces in the unit are marked, that is, all the pieces make one whole. Both the fraction name and the number one need to be related. For example, "six-sixths" and "one" are different ways to name the same amount.

■

▸ Show a unit cut into fifths. As you mark the pieces to show $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{5}{5}$, have the children say the fractions. After $\frac{5}{5}$, ask, "Is there another name for this much?" (1)

▸ Have the children mark their own sheets of paper to show $\frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \ldots$ and ask, "How many sixths are there in a unit?" (Six)

written fractions for parts of units

After the children are successful at saying fractions when shown a marked unit, they are ready to start writing the fractions. The major problem in writing a fraction is to get the order of the numerals correct. Care must be used to make sure that the children do not reverse the two numerals. Some sensible scheme is valuable. Following is a possibility:

$$\frac{3}{4} \quad \begin{array}{l} \rightarrow \textit{How many pieces are marked?} \\ \rightarrow \textit{What size is each piece?} \end{array}$$

Concrete to written. Oral naming seems to be helpful in moving from the concrete object form to the written form, with fewer reversals. "We say 'three-fourths,' and so we write 3 with a bar underneath it. 'Fourths' means four equal size pieces, and so we write 4." Reinforce the meaning by discussing what the 3 in $\frac{3}{4}$ means and what the 4 in $\frac{3}{4}$ means.

▸ Use sheets of paper cut into eight equal size parts. Point to four of the eight parts. "How many parts are marked?" (Four) How many parts are there in all?" (Eight) "Say the fraction name." (Four-eighths) "Write 4 first. Then write 8." Show $\frac{4}{8}$ beside the sheet of paper. Write the fraction on the chalkboard. Have the children show the amount by holding up the sheet of paper with the amount marked.

▸ Have the children write fractions. First point to parts of a sheet of paper you have marked. "Write the fraction of the unit that is red (or blue, etc.)." Check to see that the children are writing the correct fraction for the right reasons.

Written to oral. When the children can write fractions correctly, they should practice giving the oral names for the fractions.

▸ Make a pack of fraction cards. Use them as flash cards to have children read fractions aloud.

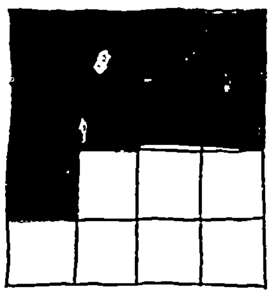
Written to concrete. The most difficult connection to make is the one between the "written" and "concrete model" forms. Later work with fractions will be more meaningful if the children can represent fractions in a correct concrete manner. Some practice activities follow:

▸ Give the child a unit cut into three equal size pieces. Write $\frac{2}{3}$. Say, "Show me how much of this unit this fraction is." Repeat with different units and fractions.

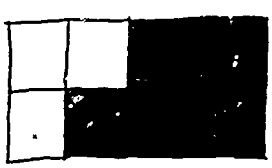
▸ Give the child a sheet of paper to represent a unit. Write $\frac{3}{4}$. Say, "Use this unit to show this fraction." Repeat with different units and fractions.

▸ Write $\frac{1}{2}$ and without providing a unit, ask, "How would you show what this fraction means?" Have the child then follow his answer with a demonstration. Repeat with different fractions.

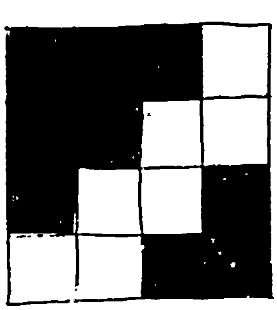
► Give the children paper units "cut" into sixteen equal size pieces, as in the diagram. Say, "Color three-sixteenths of the square red. Color seven-sixteenths yellow. Color six-sixteenths blue." Say, "Draw a square region showing each fraction (for example, one-half, two-sixths, one-third, five-eighths, etc.).



► Give the students paper units "cut" into eight equal size pieces, as in the diagram. Say, "Frost the cake so that one-eighth has white frosting, five-eighths has blue frosting, and two-eighths has red frosting Ask, "Does the whole cake have frosting? Does eight-eighths of the cake have frosting? How do you know?"



► Give the students paper units "cut" into sixteen equal size pieces, as in the diagram. Say, Color three-sixteenths of the square red. Color nine-sixteenths of the square blue. Color the rest yellow." Ask, "What part of the square is yellow?" (Four-sixteenths) "How do you know?"



representing fractions with drawings

Transition from objects to diagrams. Up to this point, the work with fractions has been based on representing fractions with concrete objects. A too-early transition to drawings or diagrams has been intentionally avoided. At this point in the sequence, however, it is appropriate to move to a diagrammatic representation of fractions.

Activities designed to make the transition from objects to diagrams follow.

► Have the children trace around a concrete unit (sheet of paper). Similarly, have them draw a segment along a stick unit to show a length unit. Encourage the children to think of the diagram as a "picture" of the concrete unit. Throughout this process, refer to the diagrams by the object names (paper unit, candy bar unit, stick unit, etc.) to reinforce the connection between the objects and the diagrams.

Repetition of parts 1-4, using diagrams. At this point it is appropriate to repeat the first four parts of the fraction-concept sequence, using diagrams in place of concrete objects.

Establish the roles of the unit, the number of pieces, and the equal size pieces for diagrams. Make the connections between diagrams, oral forms, and written fractions. Use the activities in parts 1 through 4 of the fraction-concept sequence, with diagrams in place of objects.

extending notions of fractions

The topics in this section are included to suggest directions for extending the fraction concept. They are not included as mastery items for young children. They serve as a step between initial work with fractions and computation with fractions. The topics can easily be treated in a problem-solving setting to reinforce fraction knowledge as well as to generalize it.

Fractions greater than 1. A common misconception among children is that a fraction must be less than a unit. Activities that treat a fraction as a certain number of pieces, each of which is a specific part of a given unit, will allow children to generalize about fractions greater than 1. Words such as *proper* and *improper* should be avoided, since they suggest treating fractions greater than 1 as being different from fractions less than 1.

▸ Give the children paper units. Say, "Show me one-third. Show me two-thirds. Show me three-thirds. Show me four-thirds." Ask, "What does four-thirds mean? Is four-thirds greater than the unit, the same as the unit, or less than the unit?" Repeat with other fractions.

▸ Show the children a sheet of paper marked in six equal size pieces. Say, "This shows six-fourths; can you show me the unit?"

Mixed forms. Children use mixed forms even at a young age. A child will say that he is three and one-half years old when he is over three but not yet four. These experiences with mixed forms can be made more precise with the following types of activities.

▸ Place three paper units and one-half of a paper unit on a desk. Ask, "How many units are on the desk?" Move the complete units into a pile and ask, "How many units are in the pile? How much of a unit is not counted? How much paper do I have in all?" (Three and one-half) Write $3\frac{1}{2}$, and give practice in reading it.

▸ Use mixed forms to find the number of sheets to cover the desk.

▸ Use stick units and mixed forms to find the length of a desk, a table, or the chalk tray.

Set model. Children have many opportunities to share sets of objects. Problem solving using sets often involves division, but it can be related to fractions. This means that there is a need for some method of determining (1) what the unit is, (2) how many pieces (or subsets) there will be, and (3) what a fair share is (equal size subsets). That is the purpose of these examples.

▸ Place four red blocks and two pink ones on a table. Say, "Here is a set of blocks. Some are red and some pink. How many are red? How many blocks are there altogether? What part of the whole set is red? What fraction represents the part of the set that is red?" Ask the same questions for pink blocks and for other sets.

▸ Say, "Here is a set of five candies. Three are caramels and two are creams. Three of the five candies are caramels. What fraction of the candies are caramels?" ($\frac{3}{5}$) "How many of the five candies are creams?" (2) "What is the fraction?" ($\frac{2}{5}$) "What does the 5 tell you? The 2? How much of the set is made up of caramels?" ($\frac{3}{5}$) "How much of the set is made up of creams?" ($\frac{2}{5}$)

▸ Say, "Here are seven buttons. Some are red; the others are blue. Write the fraction that tells what part of the set is red." ($\frac{2}{7}$) "Blue?" ($\frac{5}{7}$) "What does the 5 tell you? The 7?"

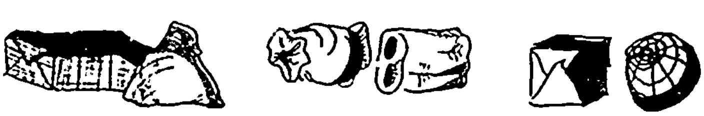
▸ Say, "A candy man has eight licorice sticks—six black and two red. He ties them in bundles of two as shown. What fraction of the licorice is black?" ($\frac{6}{8}$) "Red?" ($\frac{2}{8}$) "What fraction of the bundles of licorice is black?" ($\frac{3}{4}$) "Red?" ($\frac{1}{4}$)

▸ To illustrate the fractions $\frac{2}{3}$ and $\frac{1}{3}$, draw a set of balls that shows two-thirds of the balls black and one-third of the balls red. Repeat with other fractions.

▸ Say, "Draw a set of apples for which two-fifths are red and the rest are yellow." Repeat with similar instructions.

▸ Say, "Here are six candies. If I were to give you one-third of them, how many would you get? Let's draw a picture.

Now let's separate the set of six candies into three sets with the same number in each set.



Why did we make three sets?" (Because the fraction has a 3 below the line, meaning three parts) "How many of these sets should we take to have one-third?" (1) "How many candies is one-third of six candies?" (2) Do the same type of activity with other unit fractions, using different sets of objects.

▸ Say, "Here is a six-foot ribbon. I promised Mary she could have one-third of it. How many feet of ribbon should she get?

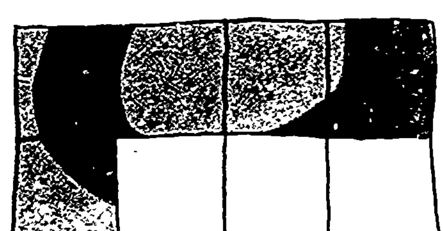
Cut the ribbon into thirds—Mary gets two feet. So one-third of six is two."

▸ Say, "A cake is cut into eight pieces. John is to get one-fourth of the cake. How many pieces should he get?"

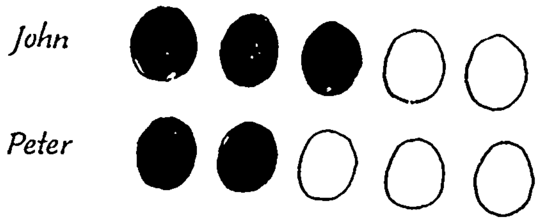
Comparing fractions. Fractions tell "how much." It is natural to ask whether one quantity is more than, less than, or the same as another. The purpose here is to use the "how much" idea to introduce the comparison of fractions.

▸ Say, "Look at these two candy bars. If you get the shaded part of one of them, which one would you want? Why? In which would you get more?" (The first one) "What fractions tell how much you would get in each?" ($\frac{2}{3}, \frac{1}{3}$) "Is two-thirds of a candy bar more than one-third of another candy bar of the same size?" Give other, similar examples.

▸ Say, "Write the fraction for the shaded part." ($\frac{5}{8}$) "For the unshaded part." ($\frac{3}{8}$) "Which is the greater part of the region—the shaded or the unshaded part? How can you tell? We say that five-eighths is greater than three-eighths because there are more shaded parts (five) than unshaded parts (three) in the eight equal size parts." Repeat with similar examples and discuss.

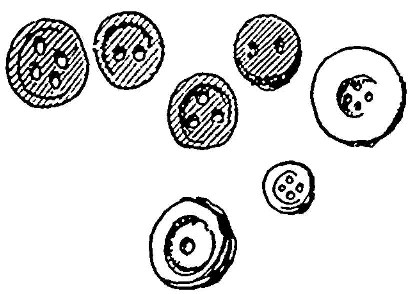


▸ Say, "Three-fifths of John's marbles are red, and two-fifths of Peter's are red."



Ask, "Which has the greater fraction of red marbles? How can you show that you are right by looking at the two sets?" Encourage one-to-one correspondence between Peter's red marbles and a subset of John's red marbles. Ask, "Who has the greater fraction of nonred marbles? Why?"

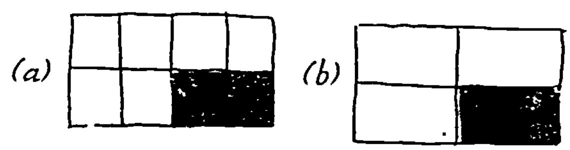
▸ Say, "Here are seven buttons. Some are pink. What fraction is pink?" ($\frac{4}{7}$) "What fraction is not pink?" ($\frac{3}{7}$) "Is the fraction of pink ones greater than, less than, or equal to the fraction of nonpink ones?"



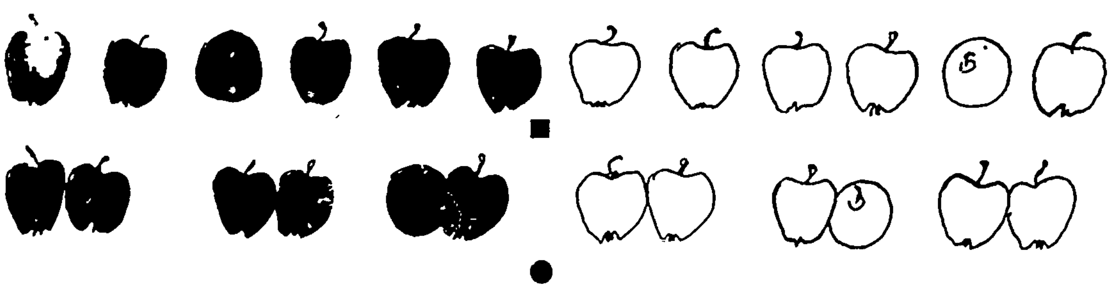
▸ Have the children draw three sets of red and blue balls. In the first, see that the fraction of blue balls is greater than the fraction of red balls. In the second, the fractions of blue and red balls should be equal. In the third, the fraction of blue balls must be less than the fraction of red balls. Ask questions so that the children can see the relationships between sets.

Equivalent fractions. Much of the later work with fractions requires the ability to determine when two fractions name the same amount or the ability to produce two fractions that name the same amount. It may be helpful to show children at this age that two different fractions may represent the same amount.

▸ Ask, "How much of the region in (a) is shaded?" ($\frac{2}{8}$) "How much of the region in (b) is shaded?" ($\frac{1}{4}$) "Is the shaded part in (a) the same amount as the shaded part in (b)? Why? Do you think $\frac{1}{4}$ and $\frac{2}{8}$ name the same amount? Why?"



▸ See ■. Ask, "What part of the apples is red?" ($\frac{6}{12}$) "Can you group by twos?" (Yes) "How many groups of two apples are there?" (6) "Show it." See ●. Ask, "What part of the apples is each set of two?" ($\frac{1}{6}$) "How many sixths are red?" (3) "What part of the apples is red?" ($\frac{3}{6}$) "Do you think three-sixths of the



apples and one-half of the apples are the same? Why?" Continue similarly to get other names for one-half.

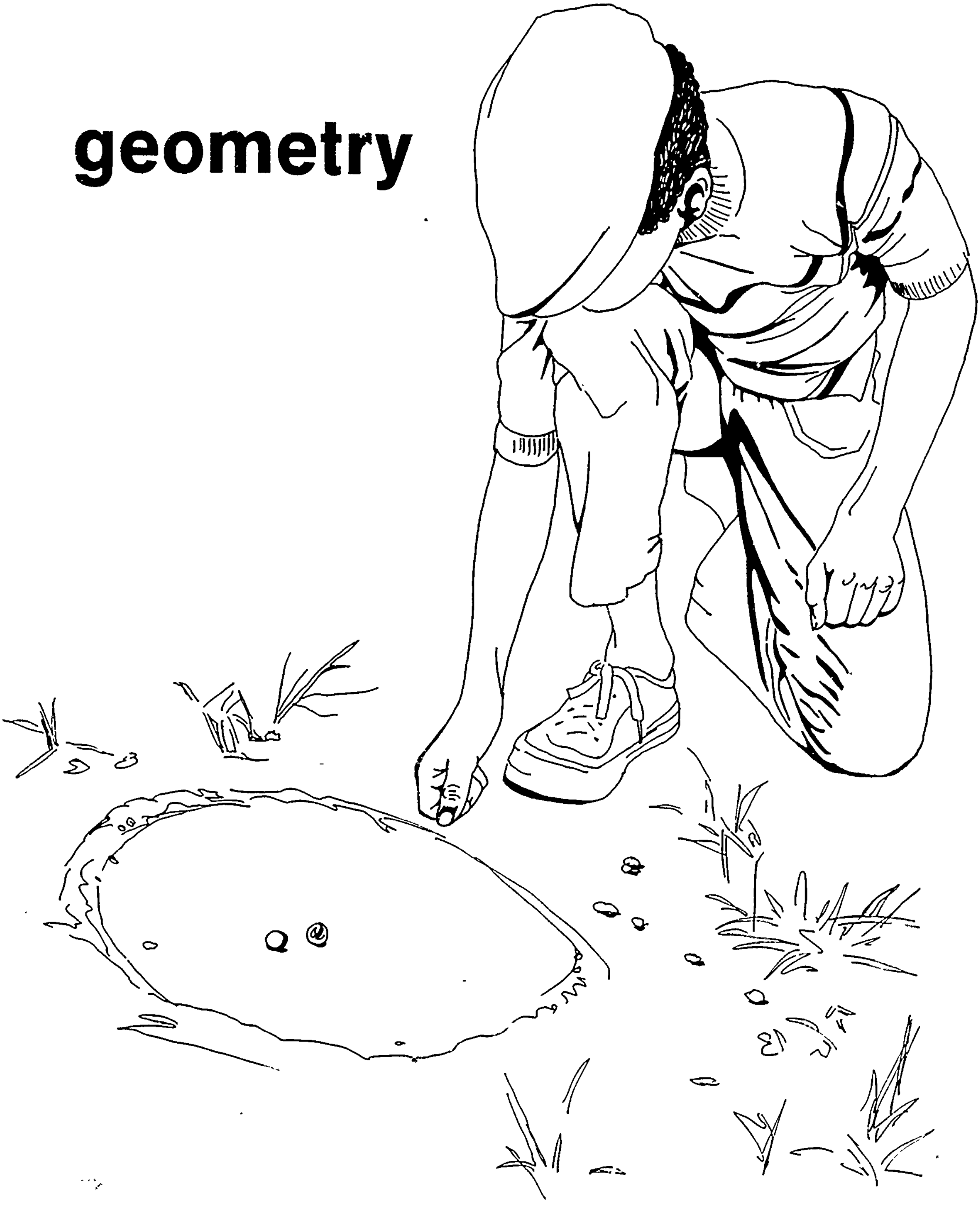
Transparencies and overlays may be used to help children check to see that the same amount can be named with two different fractions.

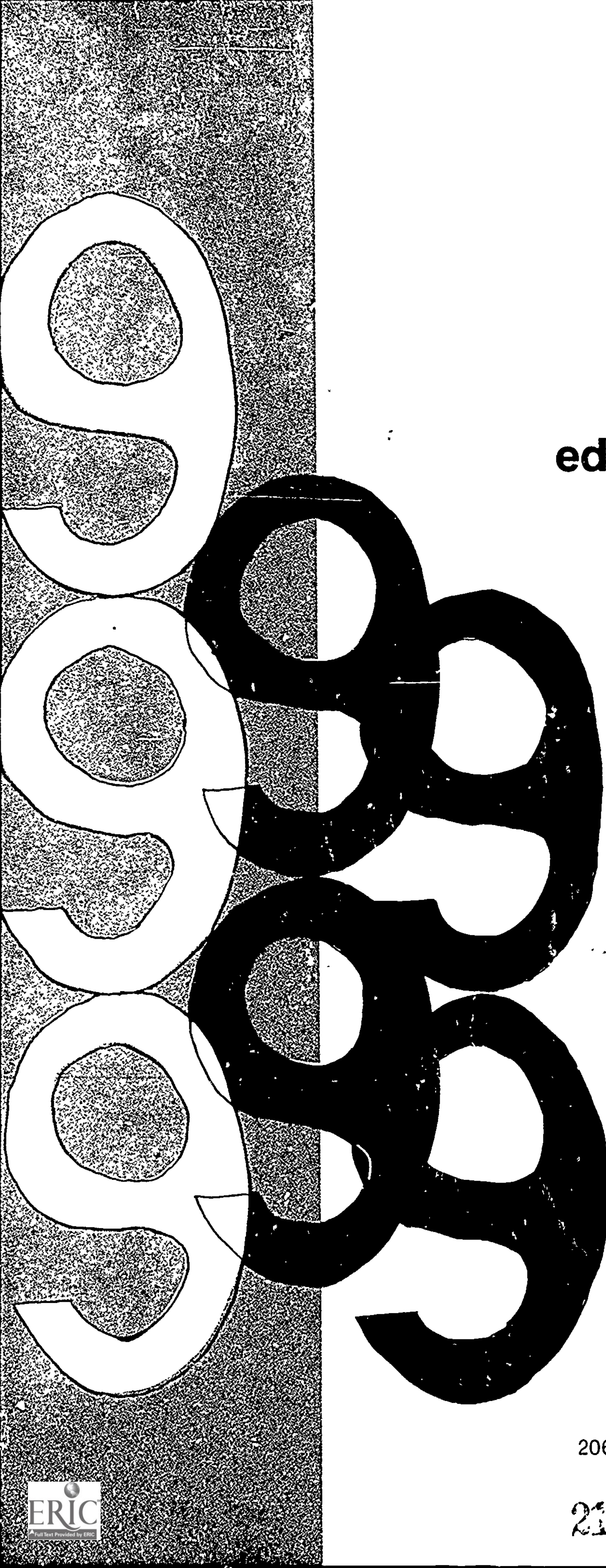
conclusion

There is much initial work on the concept of fractions that can be done with young children. A good understanding of fractions at this age helps in later work with computation. In this chapter a sequence was provided that emphasized the key notions underlying fractions. Concrete objects and their diagrams, oral language, and written fractions were connected to form the basis for understanding fractions.

Support for many of the ideas suggested in this chapter is found in ongoing research being carried out at The University of Michigan. The major investigators are Joseph Payne, James Greeno, Stuart Choate, Chatri Muangnapoe, Hazel Williams, Patricia Galloway, and Lawrence Ellerbruch.

geometry

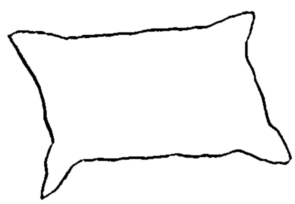




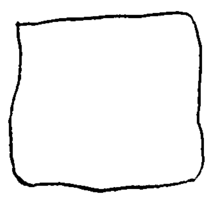
edith robinson

JUST as the child comes to school with some number notions, such as the ability to count by rote or tell how old he is. so does he come with some geometric, or spatial, notions. As Ilg and Ames point out in their book. *School Readiness*, a three-year-old can draw a circle stopping after a single revolution (p. 65), in contrast to younger children, who go round and round. Furthermore, a four-year-old can make a square with acceptable corners, and a five-year-old can copy an equilateral triangle (pp. 74, 80).

An examination of the pattern of development shows why these competencies can be considered to be evidence of geometric notions. To cite some examples. before age four, the child often reproduces a square by drawing it with "ears" at one or more corners:



Although we do not know for certain why the child draws the ears, we may conjecture that he notices the corners sticking out but is unaware of how they got there. His means of showing what he sees is to draw them in as separate entities. By about age four, he is able to draw the square somewhat like this one:

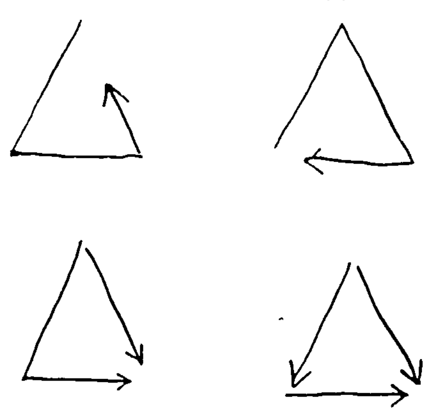


His drawing may be imprecise, but it does consist of four more or less straight strokes—two horizontal and two vertical—with no outward thrust to the corners. Moreover, a typical method of drawing the square develops, so that by age five, 60 percent of the girls and 52 percent of the boys begin at one corner (usually the upper left) and move the pencil in one continuous sweep, usually counterclockwise (■).

■ ●

There are variations—beginning at some other corner, going clockwise, or making two distinct marks as in ● above—but by age six, 80 percent of the girls and 72 percent of the boys use one continuous motion, with 58 percent of all using the method exactly as shown in ■ (Ilg and Ames, p. 76).

As the normative ages mentioned earlier suggest, the equilateral triangle is more difficult to execute than either the circle or the square; the problem appears to be that of reproducing the diagonal stroke. Although very young children (eighteen months or less) can imitate vertical and horizontal strokes with a crayon, a few children still have trouble with the oblique stroke at age six or seven (Ilg and Ames, p. 65). Thus, although only 15 percent of the five-year-olds *fail* at the task of copying the triangle, only 17 percent of them produce a well-proportioned figure (Ilg and Ames, p. 80). Unlike the square, no typical method of drawing the triangle is favored by as many as half the children; it may be made with one, two, or three separate strokes (Ilg and Ames, pp. 78-86).



Ilg and Ames give a more detailed discussion of children's performance on these tasks than is possible here. The few instances cited above were intended to illustrate the available evidence concerning what can be expected of preschool children with regard to geometry. In general, the Ilg and Ames studies support the following conclusions:

1. By age five, the majority of children can distinguish circles, squares, and triangles from one another without necessarily knowing the names; that is, there is visual form discrimination.
2. By age five, most children are aware, at least to the extent that they can "correctly" reproduce them, that the square and triangle have certain characteristics—namely straight sides, a fixed number of sides, and vertices (corners).
3. By age five, most children are aware that all these geometric figures are simple closed curves, for the drawings change with age. At first they show not-simple or not-closed curves and change to simple and closed as the child gets older. At two and one-half, a child's "circle" may look like one of these:

   At three and one-half, his circle is a simple closed curve, but his square may look like one of the following:

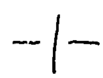
   

   At age five, the child draws both a circle and a square with one continuous motion, closing it up and not retracing; that is, he draws each as a simple closed curve.
4. By age five, most children appear to have some sense of the direction of a line, since the sides of the squares they draw are

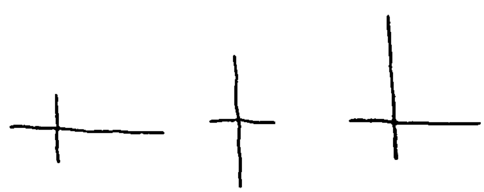
reasonably horizontal and vertical and the appropriate sides of their triangles reasonably oblique.

The preschool child also appears to be aware of intersecting lines, and again it is the work of Ilg and Ames that provides the evidence. When asked to copy a cross,

a typical effort by a 3-to-$3\frac{1}{2}$-year-old showed one of the lines being split (Ilg and Ames, pp. 70-73):



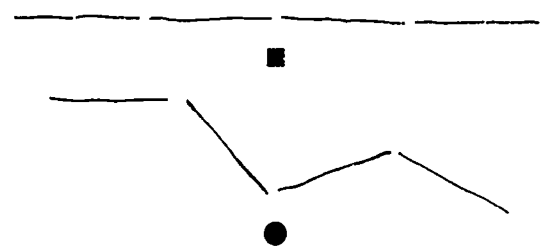
By age five, most children could execute the drawing with two intersecting strokes, although the resulting segments might not be bisected.



Not until age nine are the drawings made with both segments bisected by at least half the children, and both the change in the drawing from split (three segments) to intersecting (two segments) and the method of accomplishing this result argue that it is the *intersection* that the child is trying to reproduce. Apparently this is the feature of the model to which he is now attentive.

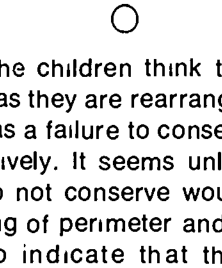
To say that a typical five-year-old has these geometric notions is not, of course, to claim that he can name them or give any sort of verbal description of what he sees. It is merely to say that he has some sort of perceptual awareness that he can reproduce in a drawing. Furthermore, by studying *changes* in the drawings, in which the type of error made by younger subjects disappears with age, one is led to the conclusion that the improvement is neither accidental nor solely attributable to better muscular coordination. As a matter of fact, drawing a square with ears is *not* easier than drawing a true square, and drawing the cross "split" takes *more* care than making two intersecting segments. It would seem, then, that the better drawings represent some new perception at work.

On the other hand, the studies of Jean Piaget and his followers provide strong indication of what *not* to expect of the preschool child. Most of these studies relate to the measurement aspects of geometry. For example, young children do not seem to recognize that the four sticks in ■ make a path of the same length when arranged as in ●.



Nor do they seem to recognize that the four pieces of paper in □ cover the same amount of space when arranged as in ○.

□



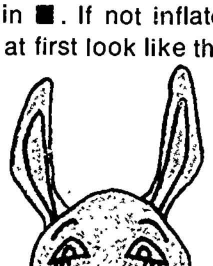
○

Apparently the children think that the pieces change size as they are rearranged. Piaget describes this as a failure to conserve length and area respectively. It seems unlikely that children who do not conserve would have much understanding of perimeter and area. Piaget's studies also indicate that the concept of "straightness" is slow to develop in children. For his subjects, ideas of straightness came at about age eight.

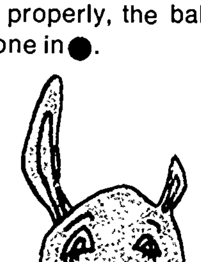
Usually the word *geometry* is used to catalog such topics as perimeter, area, congruence, perpendicularity, right angles, and the classification of polygons. That is, geometry has been thought of in terms of topics in high school geometry—so-called Euclidean geometry. Most of these ideas are based on measurement or straightness or both, and as indicated above, Piagetian research suggests that both of these concepts are slow to develop. Certainly they do not seem to be possessed by the child entering school Thus, one curricular issue raised by the research is the following: Should we begin early with a systematic introduction to Euclidean geometry, directing our efforts to speeding up the acquisition of the Euclidean concepts of straightness and measurement, or should we focus on geometric notions for which young children can be expected to have some readiness?

Experiments aimed at lowering the age at which children can conserve have so far had disappointing results. Several explanations for this are possible. Some of the results are probably due to a lack of knowledge about the exact mechanism by which conservation is attained. Knowledge is also lacking about the role that language plays in testing for conservation. Finally, the task itself may be impossible—that is, it may not be possible to accelerate the process. At this time, we simply do not know.

Fortunately these difficulties can be circumvented, for mathematics presents alternatives to Euclidean geometry. There are, in fact, many geometries, and in some of these length and straightness are not important. Many children will have had experiences with one such geometry; it is called topology. Consider a balloon with facial features printed on it, such as the one in ■. If not inflated properly, the balloon may at first look like the one in ●.



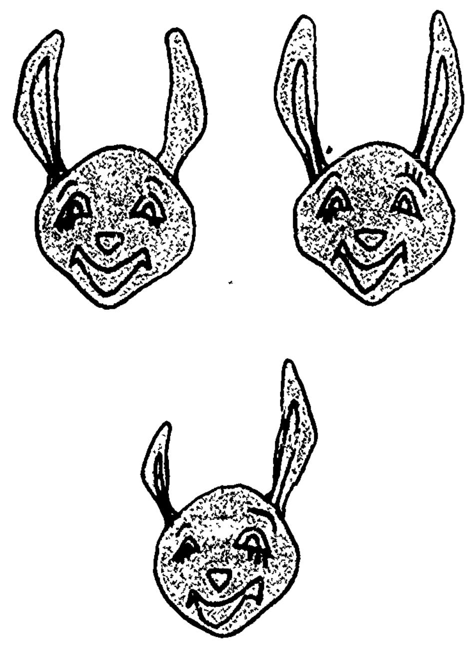
■



●

The child can notice that although one eye is bigger than the other, the lashes on both eyes are on the lower lid, the right eye is separate from the left, and so forth. In other words, certain aspects of the face cannot be changed by any amount of bad luck in blowing up the balloon short of bursting it.

Children can be led to discriminate between things that change and things that do not change without using words to describe what they see. For example, they could be asked which of these faces they might get by blowing up the balloon:



The teacher can tell from a child's choice which things he has noticed and which he has not noticed. An advantage in asking questions this way is that the child need not take anyone's word for anything; he can find the answers for himself, and differences of opinion can be settled by trying things out to see what happens.

Several important points about the study of geometry are implicit in the preceding illustration:

1. Certain *ideas* of geometry can be learned. For example, children can learn to anticipate that certain things will always happen

in certain ways and that some things will never happen in certain ways.

2. It is not necessary to use words to describe what always happens and what never happens. Pictures showing both possible and impossible events can be used, and the child can be asked to tell which is which. To sort such pictures successfully, the child must be attentive to certain visual cues. These cues will be right whether or not he knows the precise words for describing them. For example, if in the preceding display a child says that the first result cannot be obtained by blowing up the balloon because there are "no marks on one ear," the teacher knows he has noticed the right thing. Neither a more precise explanation nor a further generalization is necessary. Thus picture sorting can be used as an instrument for evaluating learning.
3. Learning by doing and settling differences of opinion by demonstration build the child's confidence and self-reliance. He can thus come to expect that geometry—and, it is to be hoped, all mathematics—describes reality; that he can master it successfully by himself; and that he does not always need someone else to show him what is correct or to legislate what is right and wrong.

The discussion that follows includes a description of a more structured learning situation than the balloon example. "Structured" here means that the ideas that can be learned in the various geometries have been identified and listed and that the representations suggested are those in which these particular ideas can be exhibited. Suggestions are also included for sequencing lessons and for evaluating the learning. The discussion is organized under three headings:

1. Geometry without straightness or length
2. Geometry with straightness but not length
3. Geometry with straightness and length

It is hoped that the suggestions given will serve as an impetus, prompting teachers to think of additional ideas for presenting these geometries to their own classes.

geometry without straightness or length

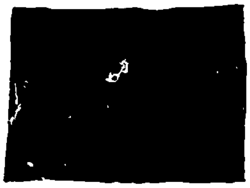
As indicated in the balloon example, preschool children probably have some experience with this kind of geometry, although customarily no attention is paid to that fact. Nevertheless, as we have seen, there *are* some geometric ideas to be learned: namely, that certain properties and relations will not change even though others, notably size and shape, will.

The basis for this kind of geometry is the notion of "elastic m[illegible]ons." If we momentarily restrict our attention to the plane, we can consider such things as stretching the plane, shrinking it, pushing humps in it, or making waves in it. In fact, we will think about doing just about anything to it except tearing it or folding it over on top of itself. Then we will study what happens to some subsets of the plane (a circle, for example) when any of these motions is going on. Clearly, if the plane is not torn, the circle will not be split, but it may end up being considerably larger or smaller and may not be a circle at all. If you think about the circle as being drawn on a broken piece of balloon, you can easily see that pulling, pushing, bending, and the like could produce some strange looking results. Yet with all this infinite variety, there are some predictable results. Before considering lessons for children, we will discuss each of these in detail.

some geometric properties that do not change

1. *Inside and outside.* If a simple closed curve is drawn on a piece of balloon, with an X inside (outside), there is no elastic motion—that is, no amount of pushing, pulling, and the like—that will result in the X being on the outside (inside).

2. *Closed and not closed.* If a closed curve is drawn on a piece of balloon,

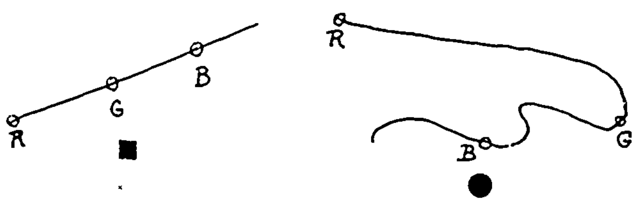


it cannot be made to be not closed except by tearing, and this is not allowed under the rules of the game.

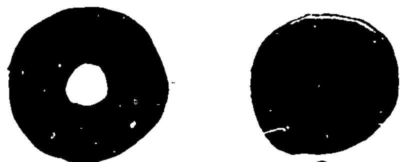
Likewise, a not-closed curve could not become closed without folding the balloon onto itself;

again, this is not allowed by the rules of the game.

3. *Order relations along a path.* If on a piece of elastic thread we paint three spots with different colors (R = red, G = green, B = blue), we might pull, push, stretch, and so forth, so that ■ looks like ● , yet the green patch will still be between the red one and the blue one in the following sense: in traversing the path, in order to get from the red spot to the blue one, we will always have to pass the green one.

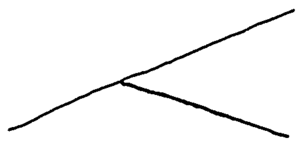


4. *Certain set relations, such as set membership, subset relations, and set intersections.* A piece of clay shaped like a doughnut (as in □) cannot be made into a piece of clay shaped like a disk (as in ○) without folding it onto itself, which is against the rules. The doughnut can, however, be pulled around and pushed in so as to take on the shape of a cup without tearing or folding onto itself.

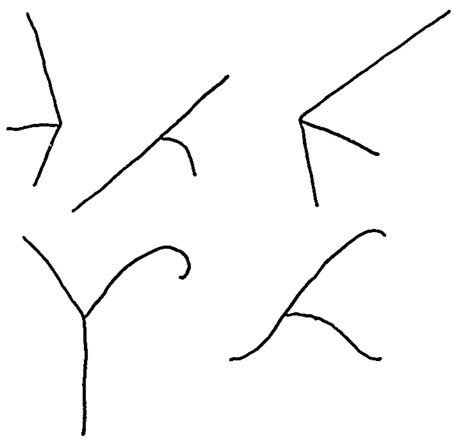


□ ○

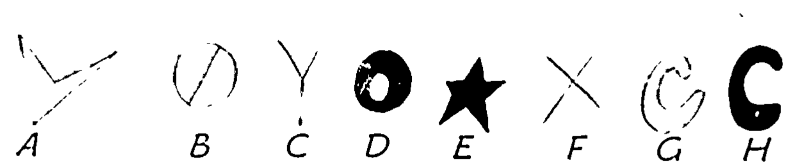
If the following figure is drawn on a piece of balloon, no matter how it is pushed or pulled, there will always be exactly one point of intersection.



Some possibilities for deforming the figure are shown here:

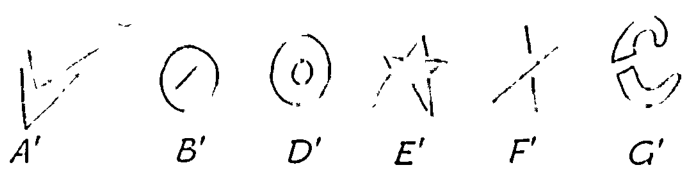


Regardless of the differences in their appearance, they all have three more or less distinguishable parts with exactly one point in common.



5. *Connectedness.* A geometric figure may be "all in one piece," in which case we say it is "connected." Some examples of connected sets (these are all plane figures) are shown in ■.

Some examples of figures that are not all in one piece, hence not connected, are these:



(Here *A'*, *B'*, *D'*, and G' are in two pieces, *F'* in three, *E'* in ten). Clearly, to change from connected to not connected requires tearing (against the rules); to change from not connected to connected requires folding onto itself (also against the rules).

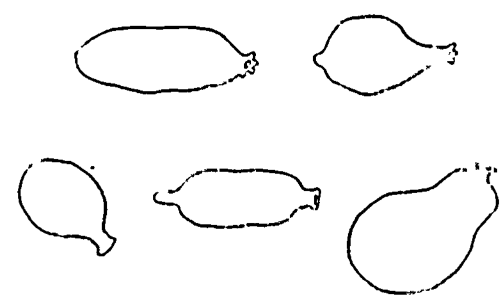
If the reader will refer back to the balloon face with which we began this discussion, he will notice that many of the properties just listed were exemplified in the face: there are connected and not-connected sets, intersections, and order in the eyelashes. Furthermore, it would be a simple matter to introduce a few examples of simple closed curves into the face; as a matter of fact, if we were to make the eyes look like this,

we would have two simple closed curves with something "inside" each. Although the five properties just listed may seem entirely too profound for small children when written in technical language, they are actually quite easy to exemplify, and studying them is quite within the capability of young learners. The technical vocabulary is for the teacher, not the pupil.

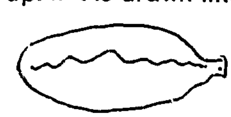
sequencing the lessons

Assemble the following teaching materials: Things that can be pushed and pulled without breaking are needed; these might be thin sheets of rubber—such as a piece of broken balloon, rubber bands, or elastic thread—and modeling clay.

► 1. The teacher might begin by blowing up a balloon and having the children watch for the different ways it can look. A long thin balloon works nicely for this because it can assume any or all of the following appearances, in addition to some others:



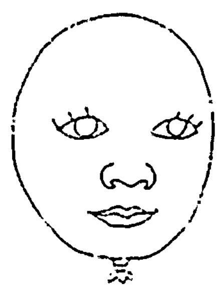
If a figure is drawn on the balloon, say a wiggly line (a felt pen can be used), the children can watch the change in the wiggly line as the balloon is blown up. If it is drawn like this,



blowing up the balloon could make the wiggly line look like any of these:

It might be instructive for the children to try to draw pictures on the chalkboard of some funny ways the balloon might look. The teacher might then ask, "Could it look like this balloon?" (No) "What's wrong with it?"

► 2. Draw a picture of a face on a balloon, perhaps similar to the one here:



Although the features need not exactly replicate the ones shown here, note that this example does incorporate the following from the list of properties that do not change:

(a) Something *inside* something; the eyes in the face illustrate this:

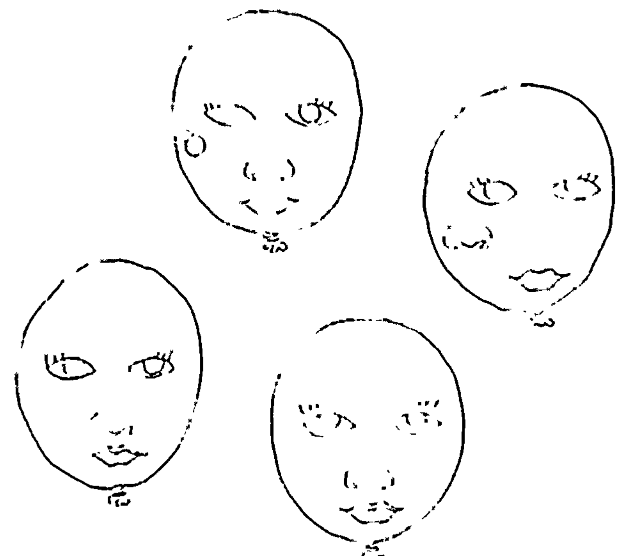
(b) Something *not closed*, such as the nose:

(c) Something involving *order*—the nose between the two eyes.

(d) Something *intersecting*, such as the eyelashes with the eyes, or the mouth:

(e) Something *connected* and something *not connected*, such as any of the features drawn on the face.

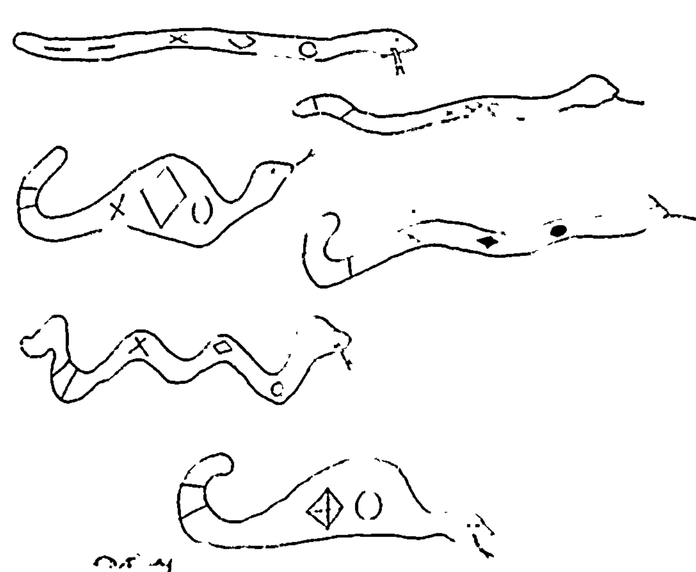
Then show the children some sketches of faces that are not possible, such as the following, and ask, "Could the balloon look like this? What's wrong?"



You will notice that a store-bought balloon with a face printed on it would do, provided that the features incorporate the five geometric properties that do not change.

► 3. Here is Sammy Snake:

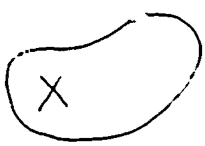
Imagine that Sammy is drawn on a balloon and that we blew up the balloon funny. "Who can draw a picture on the chalkboard to show how Sammy might look? Be sure to put in all his spots. Did George draw a good picture? Yes? Good. Who can draw a funnier picture? Could Sammy look like that? No? What's wrong? Who can make it right?" Continue with other volunteers. "Now I'll show you some I drew. Here are mine":

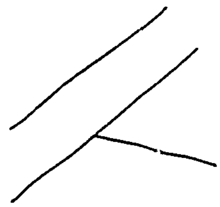


"Which of mine are right? What's wrong with those that aren't right?"

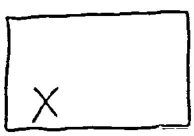
The preceding activities are not suitable for all children: they are rather juvenile for, say, third graders—but they illustrate that this kind of geometry is suitable even for kindergartners if the right sort of activity is chosen.

▸ 4. For older children draw figures that are more abstract on a piece of sheet rubber. For example:

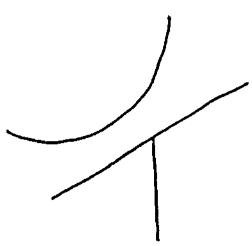




Have the children pull the rubber sheeting all sorts of ways to demonstrate how much they can change the figure. Let the children draw the various results on the chalkboard. Offer directions if necessary: for instance, ask if the first figure can be made to look like this:



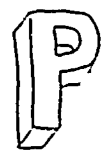
(With enough helping hands, it can!) Ask if the second figure can be made to look like this:



▸ 5. Here are some further suggestions for illustrating the properties of geometry without straightness or length:

Can (|) be made to look like (ı) ? (No)
Can △ be made to look like ○ ? (Yes)
Can △ be made to look like ▲ ? (No)
Can F be made to look like ⊢ ? (Yes)
Can F be made to look like Ƒ ? (Yes)
Can F be made to look like H ? (No)
Can ⫽ be made to look like ⟋ ? (Yes)
Can ⫽ be made to look like ⋋ ? (No)
Can [ ] be made to look like ⬡ ? (Yes)
Can [ ] be made to look like ☆ ? (Yes)

To illustrate the three-dimensional case, modeling clay is very useful; with it, for example, it can be shown that a ball, a sausage, a plate, a cup without handles, and a block (to list only a few) can be changed from one into the other in any order. To change a ball into a doughnut would not be possible under the rules; however, the doughnut can be changed into a cup with a handle, a basket with a handle, a sort of three-dimensional letter P, and so forth.



These suggestions should give the teacher an idea of the kind of focus that is required. If the teacher will experiment with balloons, rubber bands, elastic thread, and modeling clay, other possibilities will suggest themselves. The rules are simple—no tearing and no folding onto itself—and correct answers can easily be determined by direct experimentation.

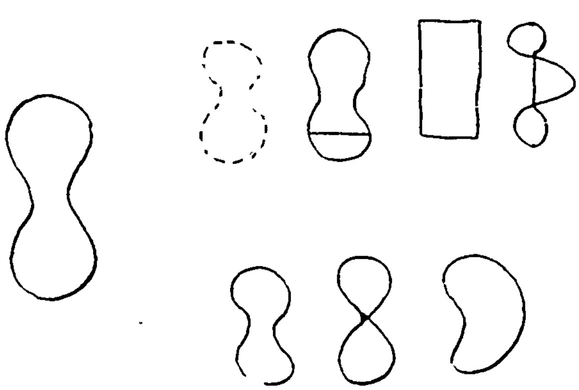
evaluating the learning

Evaluation should focus on the five properties listed at the beginning of this section, that is, inside and outside, closed and not closed, order relations along a path, set relations, and connectedness. However, the teacher should realize that children must first of all be able to recognize the pairs of figures that can be deformed into each other according to the rules. Such recognition can be helpful in other parts of mathematics. For example, the figures □ and △ can be transformed into each other under the rules of this geometry; when the children later study measurement, they will measure both in the same way—by perimeter. The figures □ and ■, however, cannot be changed into one another under the rules of this geometry, and they are not measured in

the same way. The will be measured by perimeter; the by area.

Sketching the results of experiments can be instructive for the children, and since straightness and length are not important properties in this geometry, the ability to replicate exactly is not required. Thus, a would be an acceptable form of deforming, whereas a would not. And for figures with identifying labels, would be an acceptable form of, whereas would not. The recognition that relative ordering always stays the same can be helpful when studying the number line; five, for instance, is always located between four and six regardless of whether a big number line or a small number line is used.

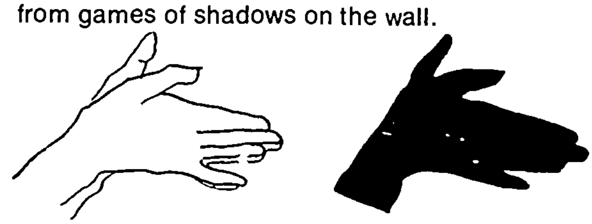
As suggested earlier, learning can be evaluated by means of pictures. The children can be asked, for example, to look at this collection of figures and pick out those figures that could be o tained from the first figure in the collection, keeping the rules for this geometry in mind.



The five that cannot be so obtained illustrate the properties of "not closed," "new intersections," and "not connected." Teachers will need to make collections of such pictures to accompany the figures they use in the activities. They will also need to collect some to go with figures *not* specifically used in classroom activities, since ability to predict in new situations is the ultimate test that learning has taken place.

geometry with straightness but not length

Shadow geometry provides a good representation of geometry in which straightness is important but length is not. At different times of the day a person's shadow may be shorter than, longer than, or the same length as, the person; so length is not preserved. Straightness is preserved, however, as can be seen from games of shadows on the wall.



some geometric properties that do not change

The children should learn through appropriate activities that this kind of geometry also has certain properties that do not change.

1. *Straightness.* A straight object will cast a straight shadow. A curved object, however, does not necessarily cast a curved shadow; a curved wire can be held so as to cast a straight shadow.
2. *Inside and outside.* A point inside a closed figure cannot be projected outside that figure, and vice versa. For example, if all points of the set lie in the same plane, it cannot have a shadow that looks like, although it can have a shadow that looks like or.
3. *Certain intersections.* If a *flat* wire loop like is used as a model, the shadow cannot look like, although it may look like, , or. Similarly, a flat figure like cannot have a shadow that looks like or. However, if the loop is *not flat* (that is, not all in the same plane), it could have a shadow that looks like.

4. *Openness*. In a restricted sense. A wire shaped like cannot have a shadow shaped like , although it can have a shadow shaped like or .
5. *Connectedness*. A shape that is in one piece cannot have a shadow that is in two or more pieces. For example, cannot have a shadow that looks like . However, an object in two pieces can have a shadow that is all in one piece. It is easy to see, for example, that two sticks can be held so as not to touch and yet cast a shadow like .

careful to trace only the shadow of the shape itself—for example, a could never be a shadow of any of these three shapes. To make sure that the children are learning to associate the shadow with the shape, bring out their tracings the next day and ask them to pick the shape from which each tracing was made.

►3. In subsequent lessons, use other specially prepared objects—for example, pieces of cardboard, some solid and some with holes. or or "Can you make the dent disappear? The hole? The tail?" Blocks also make good models. They can be in the shape of cubes, pyramids, cylinders, or cones.

sequencing the lessons

Assemble the following teaching materials: a source of incandescent (not fluorescent) light such as an old-fashioned goosenecked desk lamp, objects to cast shadows, such as wire figures, cardboard figures, and solids, white cardboard or paper on which the shadows can be seen clearly (newsprint works well), crayons for tracing shadows, and pictures for the evaluation of the learning.

It is important to note that the children need not know the names for any shape used as a model in order to learn from these activities. The ultimate test is whether they can successfully predict the shadows that could or could not be cast by a given object. The choices given them will, of course, have to be carefully selected because even adults might be hard pressed to say for sure which of the following could be the shadow of a wire like :

►1 Have the children hold objects between the light and the paper and try to make the shadows look different by turning the object. For this purpose, objects found in the classroom will do, such as large-sized paper clips, a pencil ("see how little you can make the shadow"), a piece of notebook paper ("see if you can make the holes disappear in the shadow"), and so forth.

►2. Next, have the children trace the various shadows. For this purpose, specifically prepared objects would be better, such as perfectly flat shapes fashioned from wire. Attach handles to these to prevent the children's hands from obscuring the shadow. Some good shapes to begin with are a , a , and a Let the children make different shadows with each object by turning the wire and have them trace each different shadow. The important thing here is that the children must be

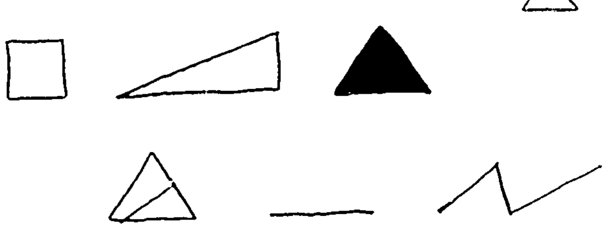
However, kindergarten children are able to recognize that they cannot get a shadow like from a flat wire like . A fair question for older children might be to decide whether can be the shadow of and whether can be the shadow of . Also, older children should be able to choose—from a cube, a pyramid, a cylinder, a cone, and a rectangular solid—all those shapes that could produce a shadow like .

Teachers will want to experiment a bit for themselves before beginning activities of the sort described above. However, it should be obvious that a teacher need not know all the possible shadows for a given object before embarking on such a sequence. Disputes can be settled easily by picking up a model and trying it out—the child who thinks can be a shadow of can determine for himself that this is not possible.

The suggestions above do not exhaust the possibilities for learning geometry from shadows. Teachers will think of other ideas for models as well as other ideas for presenting the geometry of shadows. For example, if the object is a tower of blocks, moving the light instead of the object will provide many of the same opportunities for observation.

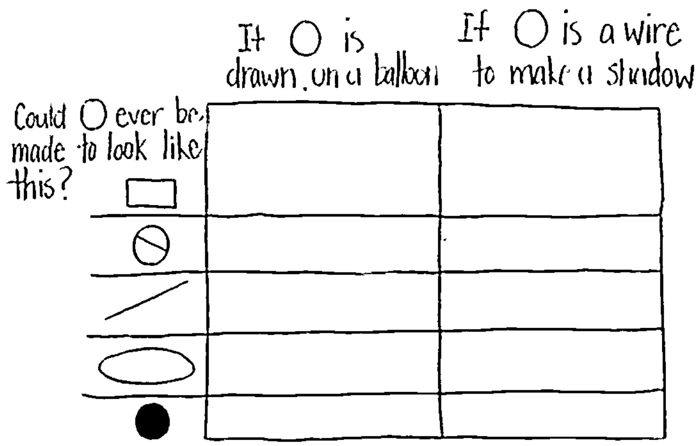
evaluating the learning

As with any learning activity, the teacher will want to make sure the children are learning and not just playing. The age and experience of the children, as well as their spontaneous comments, will dictate how often and at what points some sort of formal evaluation is desirable. As already indicated, this evaluation can take the form of presenting pictures and asking children to pick out from a given set of choices all those that could cast a shadow like the picture. The pictures presented should include some that can and some that cannot be shadows of the objects and should incorporate properties such as straightness, intersections, openness, inside and outside, and connectedness. These words, of course, need not be used, or even known, by the children. Successful learning should be measured in terms of the ability to predict, illustrated with a question such as this: Which of these could be the shadow of △?



Nor is it necessary to delay working with shadows until the children are mature enough to learn everything (whatever that may be!). Kindergarten children can work successfully with two-dimensional wire forms, the use of three-dimensional solids to cast shadows can be postponed until the second or third grade, when the "flattening out" that occurs in the shadow will make sense to them. Conceivably, experience with shadows of three-dimensional objects could help with perspective in drawings, thus geometry and art may be mutually reinforcing.

If the children have done both geometry on the balloon and shadow geometry, the following chart might be used to contrast the two. Think about a circular shape. This shape could be drawn on a balloon, or it could be a piece of wire made for casting shadows. Decide whether each shape shown down the left side of the chart could result, and write your answers in the chart.



(Answers: yes, no; no, no, no, yes; yes, yes; no, no.)

geometry with straightness and length

Geometry with straightness and length is the most familiar type of geometry to most adults. For example, one of the assumptions of high school geometry has typically been that "a geometric figure can be moved without changing its size or shape." In this section appear activities that illustrate the conditions under which this assumption is valid. This is the geometry of *nonelastic*, or *rigid*, motions.

some geometric properties that do not change

1. Length
2. Angle measure
3. Straightness
4. Properties that did not change in other geometries, that is, connectedness, set relations, and the like

sequencing the lessons

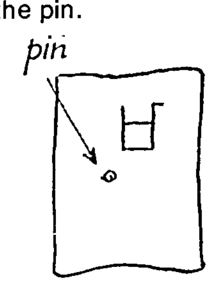
The first kind of geometry discussed in this chapter was based on elastic motions; the present geometry deals with three kinds of nonelastic motions. These are—

1. turns (sometimes called rotations);
2. slides (sometimes called translations),
3. flips (sometimes called reflections).

The children will need to learn about each of these motions.

Assemble the following teaching materials: pieces of cardboard on which to draw pictures; an overhead projector; thin paper or transparent overlay paper for the overhead projector; small mirrors; Instant Insanity blocks; and wooden models of pyramids, cylinders, and so on.

► Draw a picture on a piece of paper; place a pin in the paper as shown and *turn* (rotate) the paper about the pin.



The picture on the paper can appear in any of these ways:

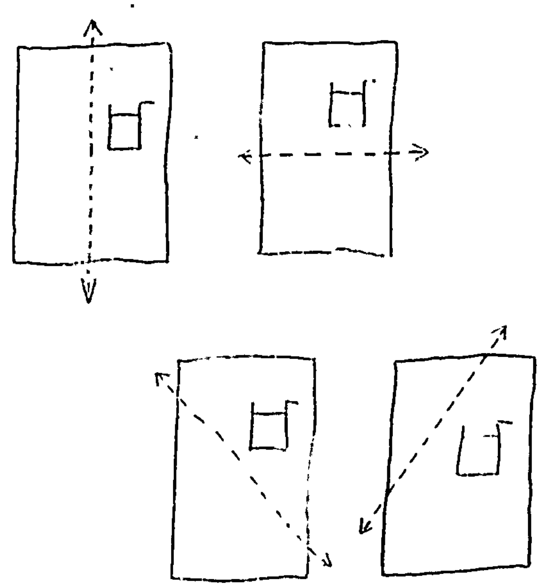
Experiment with the pin in different places to show that the paper can always be turned so that the picture looks like any of those above but that no amount of turning will make it look like the one in ■.

■

► Since the paper must be *flipped*, or turned over, for the picture to look like that in ■, draw the picture on transparent paper so the children can see that it will look like the one in ■ when flipped, or like any of the views in ●.

●

► To obtain the different appearances shown in ■ and ●, flip the picture along different lines, as shown by the dotted lines in these sketches. (Each of the figures can be made by flipping along some line other than the ones shown in this drawing.)



► Demonstrate to the children that to *slide* the picture does not change its appearance. A slide means that the paper is placed on some flat surface and pushed along that surface in

any direction, which will never give any of the results shown; the picture will always appear as it does in the first drawing, just in a different location in space.

The children should become acquainted with each of these three motions separately. It may be instructive for each child to have his own piece of paper with a picture on it so that he can turn it and thus be able to convince himself how that picture can appear.

► Have pictures to show on the overhead projector or sketch them on the chalkboard. Have the children decide in each case whether they can turn the paper so as to make theirs look like the one displayed. Include a few examples that cannot result from turns, so that the children begin to notice the cues that are associated with turns. For some children, this activity may need to be repeated, using other pictures. The choice of picture to use as the model will depend on the maturity of the children For a simpler one than the choice shown in earlier drawing, block letters of the alphabet can be used, such as P, Q, or J. Of course, circles, the letter T, and so forth, will not be good choices, since differences will not be obvious when the paper is turned.

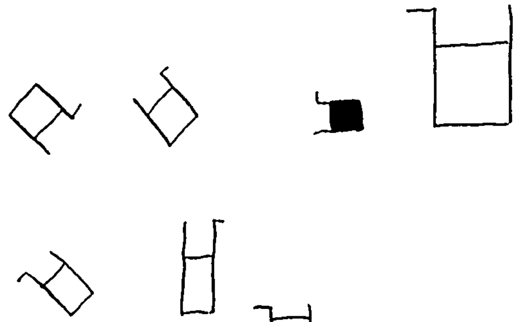
► Illustrate flips with pieces of paper, using the same type of sequence described for turns.

► Illustrate flips with mirrors. The mirror shows the image that would be obtained if the paper were flipped along the line on which the mirror rests.

There are two reasons for doing the slides last: (1) since the picture does not seem to change, the children might not understand what they are supposed to be noticing; (2) if slides are done last, many, if not all, of the children will realize that in contrast to the ones they have been studying, this motion produces no noticeable change except in location. Thus it may not be necessary to spend more than a few minutes on this motion.

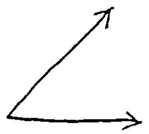
evaluating the learning

To determine whether the children can recognize the results of the three motions pictures can be used, as before. The children could be asked, for example, which of these pictures could be obtained from the original picture and which motion they would use to get it.

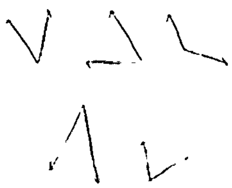


Asking the children why  could not be obtained will emphasize that no amount of sliding, turning, or flipping will make the picture bigger (or smaller) than it was before; likewise, none of these motions will add any features to the picture, and so  cannot be a result.

The real test of understanding these motions is whether the children have the ability to select correctly when the given picture is *not* one with which they have had practice. For older children, show a picture of an angle, such as this one:



Ask which of these angles could be obtained by turning, flipping, or sliding.

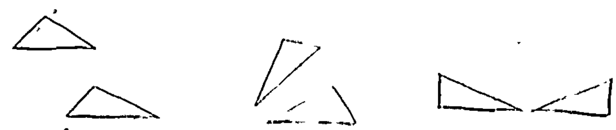


This test would not be appropriate for younger children, since the use of the arrows can be confusing. Experienced teachers know that the children's idea of congruence for angles is complicated by their attention to the lengths of

the sides If they have learned to recognize the results of turns, flips, and slides, they should be able to understand that the arrows mean only that the sides "go on and on." They should then be attentive to the cues that have served them in the previous exercises and be able to make the correct selections in this instance.

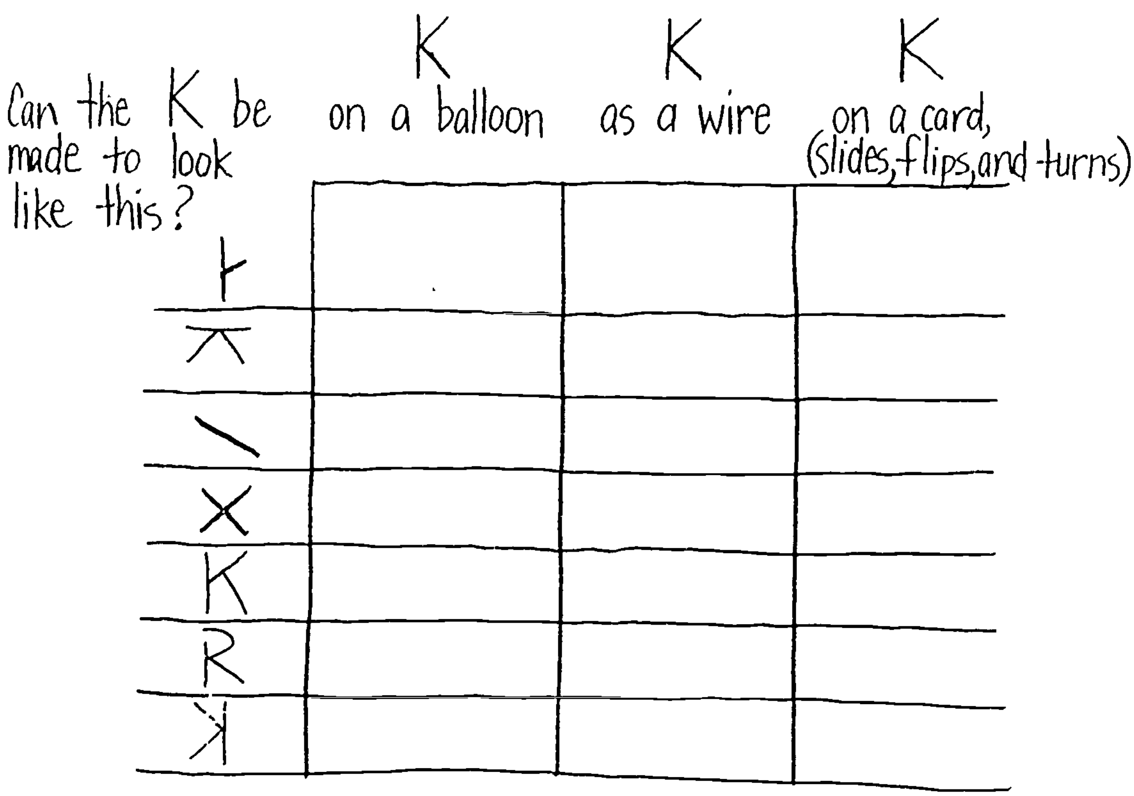
If the children have experimented with geometry on a balloon, shadow geometry, and slides, flips, and turns, you may want to contrast the results. For the chart below, the block letter K can be thought of as being drawn on a balloon, of being a shape for a wire; or being drawn on a card to be flipped, turned, or slid. Then decide whether each picture shown at the left of the chart could be the result, and fill in the chart (Note how carefully the picture must be drawn!) Answers: no, no, no; yes, yes, yes, no, yes, no; yes, no, no: no, no, no; yes, no, no; yes, yes, yes.

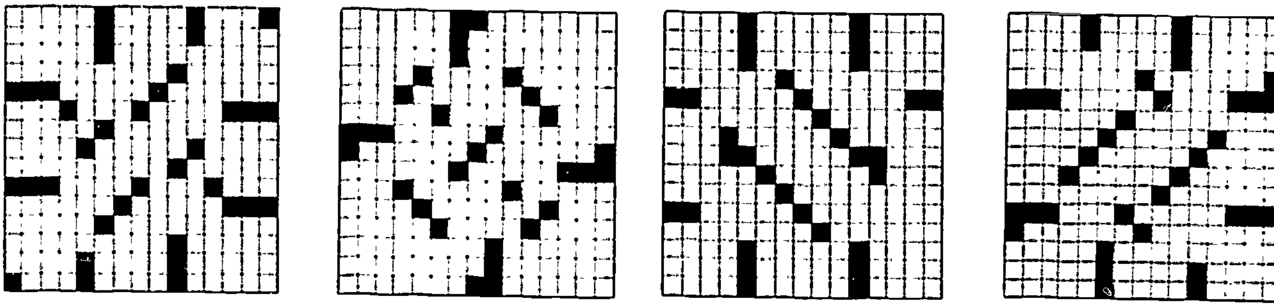
The exercise with angles illustrates the kind of long-range goal that can be achieved from a study of slides, flips, and turns. The idea of congruence for geometric figures can be described in words, but children adept at recognizing slides, flips, and turns will, in fact, have been selecting congruent figures. For these children, only the word *congruent* need be supplied, they probably already have the concept. Such children should very easily identify the pairs of figures shown here as congruent once they have been told the word.



A nonmathematical example that involves turns and flips is to be found in crossword puzzles. Examining the crossword patterns shown here can show how the pattern can be made to go onto itself by turning or flipping. If a single flip will do this, the figure is said to have a *line of symmetry*; if a half-turn (180°) will do it, the figure is said to have a *point of symmetry*.

Slides, flips, and turns are not motions restricted to figures in the plane. They are also





applied to three-dimensional figures, but are more difficult to visualize for most people. Intant Insanity blocks can help the children become acquainted with these motions because the faces of the blocks are painted different colors However, not all teachers will want to pursue the matter to this extent.

shapes and names for shapes

The emphasis in the three preceding sections has been on relationships among geometric figures· as a child experiments with different figures, he learns to examine them selectively for cues that will tell him whether they are alike in some sense. He sees, for example, that is just turned (rotated) and that is just stretched out. but that is not like under any of the rules. Furthermore, he learns to ignore or attend to some cue according to its relevance for a particular geometry—size is not a relevant cue in elastic motions, but it is relevant in flips and turns Moreover, he has derived his own cues through direct experience; no one had to tell him what to be attentive to and what to ignore. This practice in self-directed, selective scanning can stand him in good stead when certain geometric relationships are studied more methodically.

Nevertheless, it is true that in our environment certain configurations occur with sufficient frequency to require that children be able to recognize such similarity of shape. Some of these instances occur naturally—if children view grains of salt under a microscope, they can observe that they all have the same shape (cubical). Likewise, snowflakes are seen under the microscope to have hexagonal boundaries, although the interior patterns vary. If the children experiment with growing crystals (copper sulfate, magnesium sulfate, and citric acid are easy to grow), they can observe the regularity with which the crystalline structure of pure compounds is built up. Collectors of rocks and minerals can see identifiable crystalline structure in such minerals as galena, feldspar, and quartz—all cleave along well-defined planes when struck by a hammer so as to conform to that structure. Mica and shale exhibit strata, that is, parallel layers that are retained when a chunk of the mineral is broken. Plants also exhibit regularity of form. Trees, for instance, have shapes that fall into certain general categories—some are nearly spherical, others conical, cylindrical, or shaped like fat footballs. Their shadows are seen respectively as round, triangular, rectangular, or elliptical in outline.

Some degree of regularity of shape is also seen in that portion of the environment that is man-made. This is due to in part to certain practical considerations. A manufacturer of cereal, for example, must package his product, place the packages in cartons, and ship these cartons by truck or rail to various locations. The economics of the situation demand that he minimize the cost of packaging and also the waste space in transportation. Certain shapes meet the needs of his particular situation better than others. As a result, most of the cereal boxes that we see in the store are rectangular prisms, as well as the containers of many other

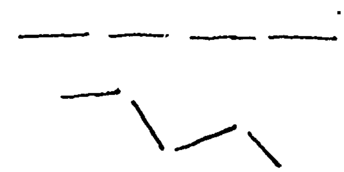
dry products. In architecture, aesthetic considerations play some role. yet here, too, practicality accounts for some degree of regularity. Local climatic conditions govern such matters as the pitch of a roof or the location of windows, for example. it has been found efficient in areas of heavy snowfall to have steeply pitched roofs. The mass production of millwork results in uniformity in the size and shape of window and door frames and the like. Bricks and building blocks in the shape of rectangular prisms fit together with no gaps. Floor tiles that are rectangular, hexagonal, or octagonal can also be fitted together with no gaps. Thus certain shapes recur over and over in both the natural and the man-made environment.

Children's general literacy should include the names for those objects that they encounter in their daily lives. so along with the names of animals, nearby towns, and colors, they should learn the names for the geometric shapes that they see. However, just as there is more to art than the ability to distinguish blue from green, so is there more to geometry than the ability to distinguish a square from a triangle. The practical problems that led man to choose regularity of form for the objects he builds or manufactures lie in the realms of science and measurement. Suggestions for introducing children to the measurement aspects of geometry are given in the following section and in the next chapter (see especially pp. 245–47).

premeasurement activities

At the beginning of this chapter it was pointed out that the concept of measurement, as applied to geometric figures, posed some difficulties for young children. Although measurement will be discussed in detail in the next chapter, this section will present some activities that can help children begin to understand some of the underlying ideas. These activities can be used independently of those suggested in the preceding sections. However, children who have studied the motions outlined earlier are likely to find these premeasurement activities much easier. For example, as mentioned earlier, one of the difficulties children have with measurement is in recognizing that the same collection of parts (four sticks, for example) is "the same size" when rearranged, provided the parts do not overlap. This state of affairs ensues, of course, because we rearrange by turning, flipping, or sliding, and not by any of the elastic motions. In the example of the four sticks shown at the beginning of the chapter, to get from

to

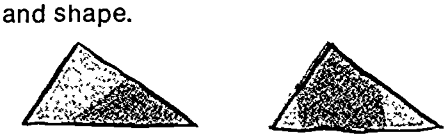


one stick is turned (rotated), and two others are rotated and slid.

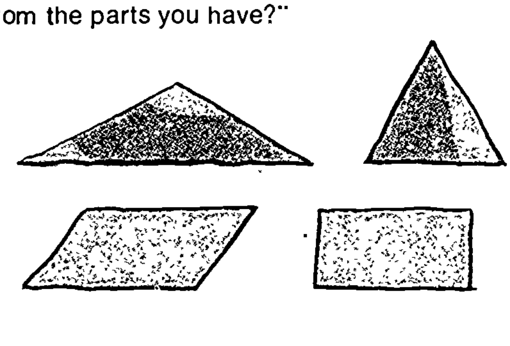
It will also be noted that in some of the activities already described, there has been opportunity to use such phrases as "larger than" or "smaller than" in an informal way. A given picture, for example, could not come from another by a rotation because it was "too big" or "bigger than" the original. Some of the activities below will provide opportunities to recognize two figures as being "the same size" without necessarily being the same shape.

assembling pictures from parts

▸ Begin with two-dimensional models. Provide two pieces of cardboard exactly the same size and shape.



Then ask, "Which of these shapes can be made from the parts you have?"

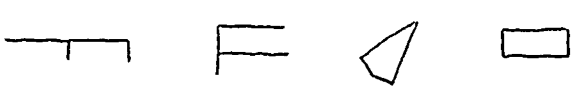


The correct answer, of course, is that all of them can be made. The follow-up question might be, "Which of the four shapes is the largest?" The recognition that all four must be the same size regardless of their appearance can go a long way toward helping the child accept the results of the measuring process.

The preceding activity could be a class discussion or an individual activity; the teacher must decide which way to use it on the basis of the age and experience of the pupils. It will be noted that all the figures are derived by sliding, flipping, or turning the parts. Children familiar with these motions may be able to recognize immediately that all four pictures can be made from the parts. Other children may need the experience of actually fitting the pieces together on a specially prepared pattern. This would need to be an individual activity. It is important that pieces and patterns be made with sufficient precision for them to fit together without overlap or gaps.

Other ideas for both pieces and designs will suggest themselves to the teacher. The pieces might be square-shaped, for example, and three or more used instead of two.

► Next, have the children work with one-dimensional models, with pipe cleaners as the parts. Two pipe cleaners shaped like ⌐ can be put together to form any of the following shapes:



Many other figures and arrangements are possible.

► For work with three-dimensional models, have the children use wooden blocks, the parts of these models, unlike the previous ones, cannot actually be superimposed on a pattern. Whether a picture of the pattern or a model made by the teacher is used, a bit of imagination is required on the part of the children. Here it might be a better idea to ask how many different patterns the children can make with, say, six blocks and then to discuss which arrangement is the "biggest." With different arrangements made by the children on display for all to see, the same idea—namely, that regardless of appearance, all must be the same size—can be demonstrated.

Activities of this type can focus attention on (1) what the figure is made of, which is important later in distinguishing perimeter, area, and volume; (2) size as an attribute distinct from shape; and (3) the process of "covering" patterns with no gaps and no overlap.

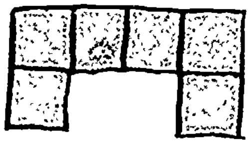
The process of covering would seem to be a very simple concept; yet in studying the measurement of perimeter and area, it is frequently imagined rather than actually carried out. Experience with the one- and two-dimensional examples in the preceding activities give the child practice without specifically calling his attention to it. If the patterns and pieces are sufficiently well made, the pieces have to be fitted together with no gaps and no overlap.

separating the figure to identify parts

Children are sometimes given patterns to cut out and paste together to make polyhedra. As an alternative, have them invent their own patterns.

► Permit the children to examine carefully a cube (which is hollow, not a solid block), and then ask them if they can make one of paper. "How many pieces will it take? What shape should they be?" Suggest that perhaps instead of six pieces, one piece might do if it were cut out the right way. Displaying a sheet of squared paper should make it obvious that the six pieces can be cut in one piece instead of separately. The children may need some assistance at the beginning, if so, display a piece like the one shown and ask if it can be folded to make a cube. Any child who thinks this will

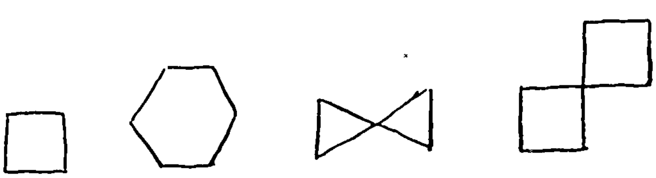
work should be encouraged to try it for himself to see why it won't work. By so doing, he can likely figure out how to make a pattern that will avert the difficulty.



▸ Give all the children a piece of squared paper and a pair of scissors and allow them to try to make a pattern that will work. Many ideas will have to be tested to determine which will and which will not work as a pattern. Since there is more than one pattern for the cube, some children will enjoy finding as many different ones as possible; the question of what shall constitute "different" patterns can challenge the best students. Refrain from telling a child whether his pattern will work, since trying a pattern that does not work can suggest to him what to try next in order to make one that will.

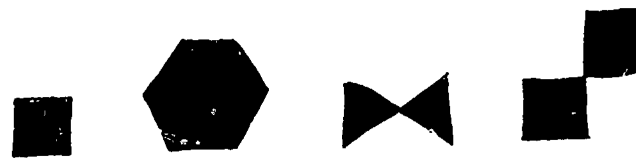
The cube, of course, is only one possible space figure for which children can devise patterns. Other possibilities are pyramids, rectangular solids, and even cones and cylinders.

▸ Make the following figures from pipe cleaners and ask the children which of the four shown can be made from just two pipe cleaners and what shapes the parts must have.



Since we are not requiring the two parts to be exactly alike, several answers are possible for each figure. Plenty of pipe cleaners should be available for such an activity so the children can test their ideas. Some children will select two pieces exactly alike, which gives the opportunity to view the first figure, for example, as the union of and , or the union of and . They could view it as two unlike pieces, such as and .

▸ Two-dimensional figures may also be used. For example, prepare pictures such as the following:



Have the children decide on two pieces of paper or cardboard to put together to make these figures and compare them with the pieces they put together with pipe cleaners to make the figures in the preceding activity. This activity can emphasize the differences in the figures—differences that are relevant to the study of perimeter and area.

Although space permits only a few examples of possible exercises, many variations of these activities can be designed to fit a particular class.

summary

The study of geometry has both mathematical and pedagogical goals. Pedagogically, it is an ideal subject for—

1. developing self-reliance, since the child can determine for himself whether his answers are right or wrong;
2. acquiring the ability to predict what will happen, since the visual cues that are present, and to which he learns to be attentive, allow the child eventually to perform "experiments" mentally and thus predict outcomes;

3. developing an expectation that mathematics makes sense, since by manipulating pictures and other materials, the child can see what always happens and what never happens. The follow-up discussion then agrees with his own observations—the discussion "makes sense" because it describes what actually happens.

Mathematically, the geometry presented in this chapter provides an opportunity for the child to become acquainted with certain geometric properties and relations without being concerned with vocabulary. The activities have provided many examples of properties such as connectedness, straightness, and closedness. They have also provided examples of two types of relations between figures: nonmeasurement relations, such as inside and outside, part-whole, set membership, subset relations, betweenness, parallelism, same shape as; and measurement relations, such as bigger than, smaller than, same size as, congruent to.

It cannot be overemphasized that these goals are long-range goals; the more immediate goals are specified for each activity suggested in this chapter. Since the goals are within the capabilities of young children, the activities are interesting as well as easy for them. When more formal instruction in geometry begins and a vocabulary needs to be developed, the children will already have a rich background of examples to associate with the words. Likewise, when formal measurement is begun, the children will have had first-hand experience with the attributes they are attempting to measure.

The following sources all contain activities that will supplement those given in this chapter.

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Phillips, Jo M., and Russell E. Zwoyer. *Slides, Flips, and Turns*. Motion Geometry, vol. 1. New York: Harper & Row, 1969.

Robinson, G. Edith. *Shadow Geometry*. Practical paper, no. 18. Athens, Ga.: Research and Development Center in Educational Stimulation, University of Georgia, 1969.

Shah, Sair Ali, *Bending and Stretching*. Athens, Ga.. Research and Development Center in Educational Stimulation, University of Georgia, 1969.

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references

Ilg, Frances L., and Louise Bates Ames. *School Readiness: Behavior Tests Used at the Gesell Institute*. New York: Harper & Row, 1965.

Piaget, Jean, and Barbel Inhelder. *The Child's Conception of Geometry*. New York: Harper Torchbooks, Harper & Row, 1964.

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measurement

18

edith robinson
michael mahaffey
doyal nelson

MEASUREMENT provides one of the most frequent needs for number in everyday life: we speak of the *number* of miles on a trip; the *number* of days of vacation; the *number* of cakes needed for some expected *number* of guests; and *numbers* inform us as to the temperature, air-pollution index, and wind velocity on any given day. Yet the teaching of measurement in the elementary school presents some of the most frustrating experiences that student and teacher encounter. There are difficulties with telling time, difficulties with converting from one unit to another, and difficulties with formulas.

In this chapter the nature of measurement is explored in some detail; described also are some studies concerning children's beliefs with regard to measurement. It is hoped that by so doing, readers are helped to diagnose difficulties and plan appropriate learning sequences for their own classes.

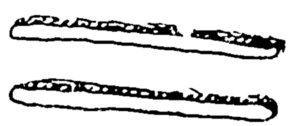
Children today enter school with vastly different backgrounds, not only among themselves, but different from the background of children a generation or so ago. Furthermore, the mobility of our population often enhances these differences. Thus one of the major tasks facing today's teacher may well be the establishment of a common pool of experience from which more formal instruction may sensibly proceed. It is with this in mind that we examine available evidence about what young children may or may not know about measurement and offer suggestions to help clarify possible misconceptions. Teachers will, of course, need to select and adapt these suggestions to their own particular classroom situation.

the nature of measurement

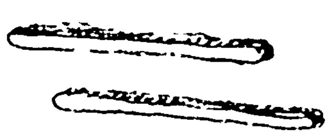
what are children's ideas of measure?

Many of the studies of Piaget relate to children's ideas of measurement, and some of these ideas may seem curious to an adult. For example, first-grade teachers may wish to try this simple experiment: Give every child a piece of modeling clay and have them roll it first into a ball and then into a worm. Then ask the children whether there is more clay in the ball or in the worm. It is surprising to some adults that the children often express a preference—some say there is more clay when it is in a ball, and some say that there is more when it is in a worm. If we stop and think about this for a bit, it seems clear that whatever the phrase "more clay" means to the children, it does not mean the same to them as it does to adults. Whether the children really believe that the amount of clay mysteriously changes with the change in form or whether they do not understand the question is not known. What is clear is that some misunderstanding exists, and that this misunderstanding could cause difficulty in the study of measurement. Piaget, incidentally, describes this misunderstanding by saying that the children do not "conserve" quantity.

There are other experiments with similar curious results. One requires two sticks of the same length. If the sticks are placed like this,

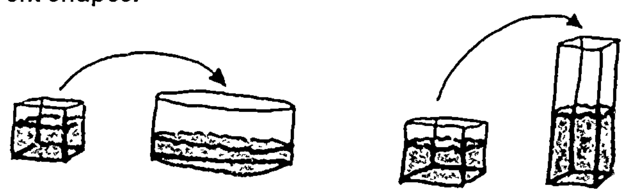


the children agree that they are "the same" (that is, neither is longer than the other). If, however, one of the sticks is moved slightly so as to look like this,



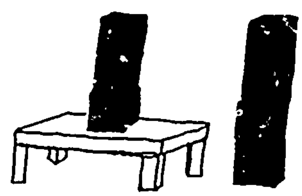
the children may now respond that the bottom one is longer (or shorter) than the top one. Here again, a moment's reflection leads us to question the child's understanding of what is being compared.

In another experiment, equal quantities of water are poured into two containers of different shapes:



When questioned, the children may say that one of the containers now has "more water" than the other.

For a fourth example, suppose two towers of blocks are built, one built on a bench:



These two towers are often reported to be "the same" (neither taller than the other).

In each of these instances, the difficulty seems clear: the children are not comparing the "right" things. What is not at all clear is how one gets around the difficulty. Certainly insisting that a child agree to the adults' judgment in these instances is no guarantee that his *belief* will change, and belief is the basis for understanding.

In the experiment with the blocks, it is tempting to try to convince the child by asking him to count the blocks in each tower. But this may not be as simple as would appear at first glance. For one thing, we would be asking him to accept *numbers* as *measures* of height in a way that is in direct opposition to his intuitive feelings in the matter. That is, we would be assuming that numbers are so well understood by the child that he would readily discard his perceptual judgment in the face of the overwhelmingly persuasive argument that "four is greater than three." Furthermore, this argument makes use of the fact that the blocks are

all the *same size*—a tower of four blocks could be shorter than a tower of three blocks! Clearly, some tacit assumptions are being made, perhaps at this point we should look more closely at the fundamentals of measurement.

what are the most basic ideas of measure?

Our discussion will focus on the following:

> In essence, measurement is a process by which *numbers* are assigned to objects in such a way that, for some given common attribute, an object having more of that attribute is assigned a larger number.

To illustrate, part of a school record might look like this:

George	*48*	*53*	*80*	*88*	*110*
Tom	*47*	*57*	*78*	*88*	*113*

An adult would readily accept the fact that five numbers are assigned to George (and five to Tom), along with the understanding that each of these numbers measures a different attribute But is it naturally clear to a child that a single object can have more than one attribute?

Furthermore, an adult would infer that Tom has less than George of whatever attribute is represented in the first column, but would *not* infer that George has less of whatever is represented in the first column than he has of whatever is measured in the second co'umn. The first inference is based on the observation that 47 is less than 48; however, the similar observation that 48 is less than 53 does not warrant any valid inference. That is, the appropriate inference is not derived solely from the numbers, but must be combined with an understanding of their role in the comparison process.

The process of measurement, then, gives quantitative information, and in our society we accept this use of number, but only in a very special way. For one thing, not all attributes of an object are quantified. Color, for instance, is not usually assigned a measure except in a highly technical sense. As a matter of fact, few of the attributes by which objects are perceptually identified, and hence distinguished from one another, are quantified. Yet these attributes are frequently the most familiar to young children. Moreover, many of the attributes that are quantified are difficult to describe in words, and to say that length is "how long something is" does not shed much light on the subject. Thus for the child, the first major task may be to learn to recognize the attributes that our society elects to measure. A partial list of these attributes, in alphabetical order, is given below; many, if not all, are studied in the elementary school.

area	speed
capacity	temperature
density	viscosity
distance	volume
length	weight
pressure	

If we contrast the items in this list with the world as the young child sees it, we see an even greater gap between the child and the adult. A typical preschool child, for example, identifies objects in terms of their use. "A hole is to dig" (from a title of a book by Art Linkletter), a ball is "to play with," an orange is "to eat." In the early stages, he may not recognize any way in which a ball and an orange are alike, although somewhat later he may say they are "both round." That is, young children tend to think of likenesses (common attributes) on the basis of—

1. use ("all toys," "you eat them," or "tools");
2. genera ("children," "grandmas," "pets," "flowers");
3. color ("all green");
4. form ("they're round," or "they're flat").

Now none of these is an attribute to which the average adult assigns a measure (except in the most technical sense, which does not concern us here), and it would certainly be a remarkable child who would claim that a ball and an orange were alike because both "had weight" or "had volume" or any other attribute on the earlier list. These attributes must then be learned and, compared to such characteristics as "use," "genera," "color," and "form," are not easy to observe.

In addition to the necessity for recognizing certain attributes, the assignment of numbers

as measures requires that objects having "more" of a given attribute be assigned larger numbers However, as indicated by the studies of Piaget cited earlier, children may not recognize "more" or "less" or even "the same amount" of some measurable attribute.

Surely good pedagogy requires that from the very beginning, all phases of measurement be accepted by the children as *reasonable* enterprises This would seem to imply that the most basic learnings would be—

1. the identification of those attributes that our society considers important enough to quantify;
2. some intuitive understanding of what "more" or "less" of those attributes should be.

In earlier years, children may have entered school with a greater understanding of both, because housewives baked from scratch, toys were improvised, and grocers measured out to individual order. Thus children had the opportunity to observe cups, scoops, scales, spoons, and the like used to measure out "just the right amount of" a wide variety of raw materials. Today's children may *hear* on television about "lower in phosphates" or "more concentrated," but there is no *visible* referent for the words. And the child who reports that his mother "lost ten pounds" may know less about measurement than the one who reports that he "outgrew his shoes" (the latter statement definitely has a referent).

In this first section we shall suggest some activities designed to supplement out-of-school experiences with measurement, and for this purpose the activities will focus primarily on the identification and comparison (in a relatively gross way) of some of the attributes in our first list.

Some readers may have questioned the separation of distance and length and the separation of capacity and volume in our list. Both distinctions were made for a reason. Although adults tend to equate distance and length, a distinction is nevertheless sometimes made, as, for example, in the phrase, "20 miles *as the crow flies*." In the case of children the situation is somewhat different, because it appears that distance is a harder notion for them to grasp than length. Teachers will need to take this into account when talking to pupils, as well as when planning learning experiences for them.

With respect to capacity and volume, the initial distinction will be as follows: Capacity will be considered to be an attribute of *containers*. The rationale for this is that it will be easier to compare two containers on the basis of "which holds more" than to compare two blocks of wood on the basis of the "amount of wood" they "contain."

Some readers may also have wondered why we omitted time from our list Time is not an attribute of objects and it will be discussed in a separate section.

As mentioned previously, the activities outlined in this section are to be thought of as suggestions for learning about measurable attributes and comparisons based on those attributes. In particular, exploratory activities of this sort should not be thought of as having precisely one objective per activity. On the contrary, rather than "closing in" on some specific item to be "mastered," their purpose is to *open up* new aspects of familiar situations for comparison and contrast. Nor will the teacher be able to assess the learning in the conventional manner, such assessment will have to come from listening carefully to children's conversations, observing any independent actions taken by them as they work, and noting the amount of transfer to new settings. We do not, then, offer a prescribed sequence that, when followed, will guarantee certain results. Instead, we describe a variety of contexts from which the teacher can select and adapt according to need. Some of these are extensions of regular classroom routine, and others are more contrived, but all offer children the opportunity to encounter measurable attributes and to make comparisons of the sort that motivates adults to engage in the enterprise known as measurement. Since these activities are foundational, the following guidelines seem appropriate:

1. Appeal to children's senses, specifically sight and touch, to learn to recognize measurable attributes.
2. Select settings with which the children can personally identify and to which they can bring knowledge of their own.

3. Use problem situations, that is, situations in which there is a need to acquire "enough" of something.
4. Provide experiences from which the children themselves can draw the conclusion that one thing has "more" (or "less") of some attribute than another; that is, provide an experiential basis for the use of appropriate measurement language.

Although these activities were written with the young child in mind, teachers of older children may find in them ideas that could profitably be taken up with their pupils. A final remark: Quotation marks have frequently been used to point up a naturally occurring word or phrase that is so familiar to an adult that he may not realize that it may be meaningless to a child. In some instances, the activity provides an opportunity for the child to learn what this word or phrase means to the adult.

activities to develop basic concepts of measurement

using commercially available materials

attribute blocks

These materials are used for a variety of learning experiences, one of which is recognition that a single object may have more than one attribute. Once the blocks have become familiar to the children and can be easily sorted according to any of their several attributes, their use can be extended to the introduction of comparison words.

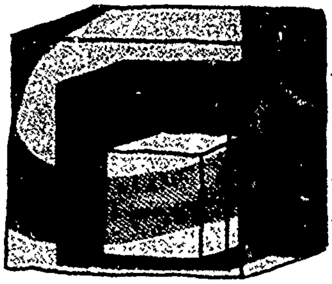
▸ For example, select a large, blue, thin, round block and a small, blue, thick, round block and ask the children how they are different. The children should be able to name two differences. If they do not know the correct words, provide them. The terms will have meaning because the children have a perceptual basis for reference. With children who already know the words, show them an empty spool or piece of dowel rod painted blue and ask them how it is different from the other two blocks. Noticing that it is the "thickest" of the three but the "smallest around" helps direct their attention to the different kinds of comparisons necessary for the understanding of measurement. The reader will note that in this example the question of which block is biggest is relative to the attribute in question. Since failure to realize that one object may be "bigger than" another in one respect but "smaller than" in another may be the basis for some children's difficulties with measurement, these blocks provide an opportunity for learning such discriminations. Scraps of wood or parts of discarded toys can be used along with the regular blocks for activities of this type. Those chosen should, of course, be the same shape and colored the same as in the commercial set so that neither color nor shape becomes the attribute being compared. If the children are old enough to be instructed to "pay no attention" to color, then of course many ordinary objects in the classroom (a piece of chalk, for example) may be used in comparison.

nests of boxes

As with attribute blocks, the children need to be completely familiar with the materials through free play. Once they have become adept at assembling the boxes correctly, their attention can be drawn to comparisons by size.

▸ For example, hold up two of the boxes (it might be wise to choose two "nonsequential" ones) and ask which is bigger. If the children do not know the word, ask them which would fit into which. Then show them a third box and ask the children which is the biggest of all and which is the smallest of all. Having the children order up various subcollections of the whole set may be instructive. Asking "Which box holds all the rest?" or "Which box will fit into all

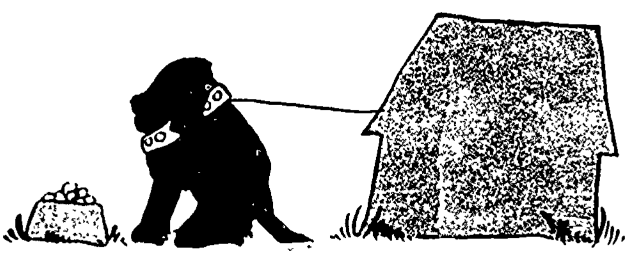
the rest?" may help. Care should be taken with this activity, however, lest the children conclude that the tallest container is *always* the biggest (it is here, of course). For this reason, the emphasis should be on comparing by "fitting in," and teachers may wish to follow this up immediately with some of the activities under "Capacity" on the following pages.



class projects

finding a rope for spot

For this project, a cutout or toy dog is needed, along with a doghouse, a dish of food, and a rope (piece of string) with which Spot is tied to the doghouse.



▸ Arrange the pieces so that Spot can't get his food. For one reason or another (the ingenious teacher can invent a reason), the problem is not to be solved by moving the dish of food but by finding the "right" rope for Spot. Some teachers might wish to provide several pieces of twine already cut and have the children choose a piece that would do, then try out their various choices to see which would work and why some will not. Other teachers might want to have the children cut off the "right" amount from a ball of twine. In any event, the parts of the display should be movable so that some experimentation can be done. Readers will note that there is plenty of opportunity here for comparisons and that the comparison words will arise naturally in the discussion: "What's wrong with the rope Spot already has?" and "What do we need then?" are examples of questions that will direct the discussion along measurement lines.

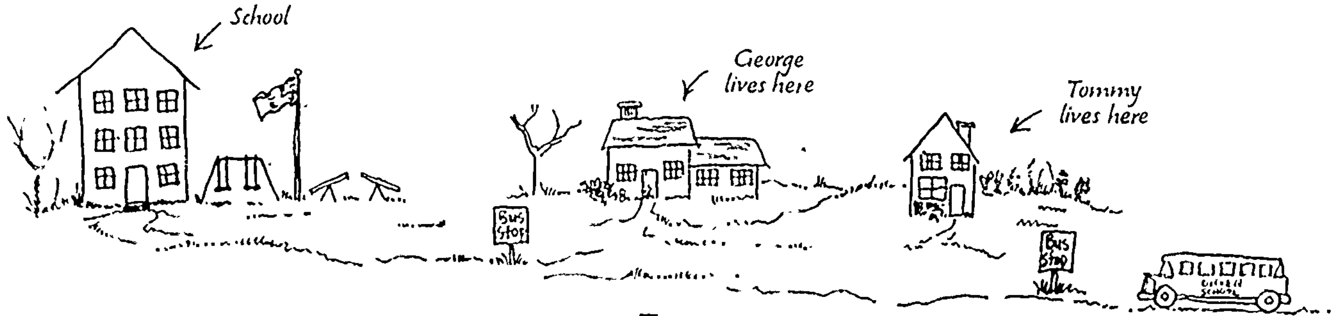
As an alternative, the display could involve, not a dish of food, but a flower bed and a rope that permits Spot to bury his bones in the flower bed (Mrs. Jones objects to this).

For older or otherwise more mature children, a combination of the two suggestions could be instructive. Once the "right" rope is found (long enough for him to reach his dinner but short enough to prevent him from burying leftovers in the flower bed), ask the children which is closer to Spot, his dog dish or the flower bed. If both of these objects are movable and the scenery can be rearranged, using the rope to compare distances becomes a reasonable solution. It should be noted that this solution is more reasonable as an outgrowth of a "simpler" problem or sequence of problems; that is, unless the teacher pursues the matter, some children might not realize that the rope does become a means of comparison. Naturally, teachers will not want to pursue the activity this far if the children have difficulty finding the right rope; for such children, selecting the "right" rope is sufficient when accompanied, of course, by a discussion of what is wrong with ropes that aren't "right."

making a model village

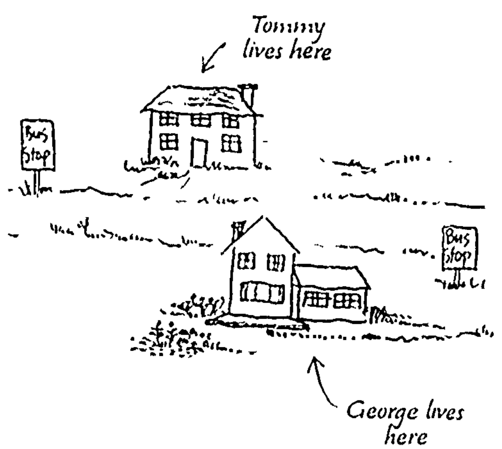
There are many possible variations to this project, but basically it requires a schoolhouse, some houses, and streets.

▸ Represent these on a large map (sheet of paper) or on a sandtable and use toy buildings or diagrams. The village could be a representation of the actual neighborhood or some imaginary one. Finally, the layout could be in city blocks or not, as the experience of the children dictates. The problem to be explored is "Who goes farther to school?" Thus several inhabitants of the village must be identified and their homes located on the map. The story for the village could involve riding the school bus,



■

in which case it would be wise to have all the characters ride the same bus. On the map, then, would be located the bus stop at which each of the characters gets on. In simplest form, the map might look like the one in ■. For younger children the story could be simple and the bus route shown as above, with no attention given to the fact that some children have farther to walk to the bus stop than others. Depending on the maturity of the children, more complicated arrangements are possible.



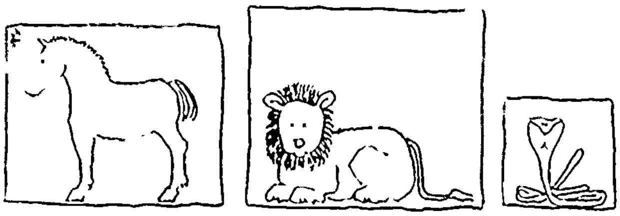
In these cases, the first question might be, "Who rides farther on the bus?" followed by "Tommy has to walk from his house to the bus stop, and so does George; if we figure all the way from home to school, does Tommy still have farther to go than George?" and finally, "How can we find out?" For still older children, two buses might be involved in the story—one coming to school from each direction, or one whose route is farther "south" on the map than the other. The emphasis should be on developing a concept of the length of a path and on the comparison of the lengths of two paths to give meaning to "longer" and "shorter." Thus the questions could be asked in terms of who has the longest way to go. The means of comparison here involves a part-whole relationship for the simple case. To ensure that the children can use this method of comparison, introduce a third character. "Pete rides the same bus, and he rides farther than either Tommy or George. Where do you think he gets on the bus?" Then select some possible locations, some right and some wrong, and ask the children if each of these could be the place where Pete gets on, and if not, why it is wrong. A fourth character might ride "farther than George but not as far as Pete." To make sure the children understand the attribute in question (*length* of a path), it could be imagined that George misses the bus one day and has to walk all the way to school; does Tommy still have farther to go than George? Or, if both boys miss the bus but Tommy runs while George walks, who goes farther to school (assuming, of course, that both follow the route of the bus)?

For the more complicated arrangement suggested above, the comparison of the lengths of the paths should include the walks to the bus stop. Tracing each path by "walking" one's fingers or a doll may help. The teacher may even have to say, "See, the walk is about the same for each boy." If this is necessary, the arrangement is too difficult for the children and should be abandoned.

cages for the animals

▸ Lay out a model zoo with several toy animals. Explain that each of the animals needs his own play yard fenced off, or each of the animals needs a cage for shelter. Big animals need bigger yards or cages. If a supply of boxes is made available, the chidren can select

what they think is the "right" box for a cage for each of the animals Cutting holes in the cages for doors can also be instructive, since the hole needs to be "high enough" and "wide enough" for the animal to walk through. If string or toy fencing is made available, the children can set this up to make the "right" sized pen. The latter problem may involve making the pen large enough for the cage selected with some spare room for the animal to exercise. Young children gain some good practical experience in "making things so they will fit" as well as in being introduced to some words such as "tall," "high," "wide," "wider," and the like, since these can come up naturally in conversation.



learning about specific attributes

capacity

▸ Since this is an attribute shared by containers and it appears easy for children to think of objects in terms of their use, the discussion might begin by asking the children to tell what a box is. If they say it's "to put things in," ask them to name some other objects used to put things in. Or begin the discussion by displaying several pieces of chalk, for example, and asking the children to name some things that could be used to put the chalk in. When it is clear that they are classifying objects as either being or not being containers, introduce the notion of relative size (capacity).

To do this, display several containers and ask the children which could hold some particular object (say, a baseball). The containers being displayed should have been chosen so that the object will not fit in all of them. Thus the children can be asked why the object will not fit in the containers not selected (too small). From this it is established that some containers are bigger than others. To reinforce the notion, ask the children to name things in the room "big enough to hold a football," or "too small to hold a football," and so forth, choosing objects of other sizes to ask about.

The activity above introduces the notion of relative capacity in rather a gross way, and the objective is to make children aware that containers do vary in size. The method used is not too reliable, since an orange, for example, will not "fit in" a soft-drink bottle, not because of its capacity, but because the neck of the bottle is too small.

▸ To advance the idea of relative capacity in a more productive way, select two bottles of different capacity, one short and fat, the other tall and thin (olive jars are a possibility). Fill one of the bottles with water (or, for less mess, use sand, salt, or cornmeal). Then ask the children to guess whether the other jar will hold all the water. Having them guess will give them a vested interest in the outcome, and a vote might be taken before proceeding. Then pour to see who is right. The bottles should be chosen so that the difference is "obvious" when using water for comparison, but not all that obvious just by sight. The outcome of the pouring will depend on which bottle has been filled first, but when the outcome has been observed, the children should be asked if they can now tell which bottle holds more, and why. If everyone does not agree and they cannot convince each other, repeat the experiment, being careful to begin with the same bottle. When everyone has been convinced, fill the other bottle and ask for a prediction on what will happen when you pour into the first bottle. The correctness of the prediction will be an indication of the children's understanding. Even if the prediction is correct, the water should be poured so that the children see that their conclusion is correct. To emphasize the method of comparison being used here, repeat the experiment with other pairs of bottles.

temperature

Differences in temperature can be detected by touch if the difference is large enough. On

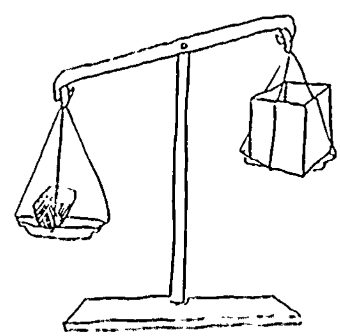
certain days, the difference in temperature between the classroom and the outdoors may be very noticeable.

► If the classroom is equipped with running water, differences in water temperature can be felt. Thus the environment provides opportunities for comparing the attribute called temperature. The way in which the behavior of the thermometer corresponds to the way the air feels can be very interesting to younger children. Since the column of liquid changes before their very eyes, they may want to try it in various parts of the room—over a radiator, near an outside door, and so forth—to watch it go up and down. Experience of this sort enables them later to accept instruments as extensions of the senses in comparison situations It should be noted that children do not need to be able to read the numbers on the instrument before using it; it is sufficient if their experience gives them the understanding that "the higher, the warmer" and "the lower, the colder." To check this understanding, ask them to predict what the thermometer would do if placed in some ice water or in some water heated on the stove. They should, of course, then be allowed to see for themselves that they were right, possibly with an assist from the school cafeteria. Predicting how high or how low the liquid will go is good preparation for later use of numbers, and the use of a china-marking pencil to compare day-to-day temperature changes (thus avoiding the reading of the scale) can anticipate the necessity for a reference point in other kinds of measurement.

weight

As with temperature, difference in weight can sometimes be detected by touch.

► Begin the discussion by asking the children to name some things in the room that they can lift, then some things that they cannot lift. Make a distinction, of course, between those things that can't be lifted because they are nailed down and those things that are "too big" to lift. Since small children use the term "big" in a nonspecific way (a big brother is probably older, a big house has more capacity, and so forth), this is the opportunity to teach the word "heavy" to describe bigness when the attribute of weight is intended. The experience of hefting enables children to become acquainted with the weight of objects; so a collection of objects should be provided for the children to heft and note the difference in the "feel." At first, these objects should be chosen so that they vary quite a bit, and teachers may need to practice a bit themselves in order to select a good representation. A stapler and an empty cardboard box, for example, might be a better choice than two books, because it is weight rather than volume that is under consideration. Having the children place several objects in order by weight can be helpful, especially if some of the lighter ones "look larger." Repeating the experiment with smaller and lighter objects (say, a bottle cap, half-dollar, etc.) refines the notion of weight. If a two-pan balance is available, these smaller objects can be placed on the balance and its behavior observed. Since the swing of the balance agrees with the "feel" of two objects being hefted, the use of the balance as an extension of the sense of touch becomes very reasonable. Thus paper clips, thumbtacks, and other objects whose difference in weight is too small to be detected by hefting can now be compared, using an instrument for that purpose.



height

The difference between height and length is chiefly one of orientation: If the alignment is horizontal, we frequently call the attribute "length"; if it is vertical, we call it "height." Since height is a very obvious way in which grownups are "bigger than" children, the introduction of

the attribute through the notion of height seems natural. The chief difficulty with the concept seems to be that children judge by looking at the tops of things without paying attention to the bottoms. They may think, for example, that a child becomes taller by standing on a chair.

► To isolate the attribute, begin by having two children stand back-to-back to see who is taller. Then have the shorter one stand on a chair and ask who is taller. If the children think the one on the chair is now taller, it *may* be sufficient to point out that we are not interested in the chair, just the children. If this is not convincing, it may be necessary to discuss the process of growth as the means by which children become taller. Comparing someone's present height with that when he was a baby or referring to outgrown clothes are ways of directing attention to the attribute called height. The problem may be linguistic, but the children need to know that their height does not change when they sit down, stand on tiptoe, or lie down. (We discount here, of course, infinitesimal differences due to gravity.) To check on understanding. ask the children which is taller, the wastebasket or some object on the teacher's desk—a chalk box, for example. Having several children line up by height or asking them to name some objects in the room that are taller than the teacher or shorter than the wastebasket are some other ways of checking on the children's understanding of height. (They may, for example, think of it only as an attribute of people.)

An out-of-school project might be to have each child find out who is tallest in his family, then next tallest, and so on. (Should we include the dog?)

Another project might be to make stick drawings of a mythical family: father, mother, school-aged child, and baby. Which should be tallest? Next? How high should we make the school-aged child? If the baby is drawn crawling, should the picture be as high as the other child? If we held the baby up on his feet, how high would he come in our picture?

If the children have already had some work with capacity, it might be instructive to display a collection of odd-shaped bottles and have the children first place them in order by height and then reorder them by capacity.

incidental problem solving

There are many opportunities for becoming familiar with the *need* to measure as well as with some of the techniques of measurement apart from activities designed especially for that purpose. The following list suggests some of these possibilities:

► *Cutting things to fit.* When decorating the room or making posters, let the children "measure off" the amount of crepe paper, twine, or newsprint needed for some purpose. (Some discreet guidance from the teacher can avoid excessive waste.) By not having everything precut or premeasured, children gain practical experience in determining how much is needed. For example, cutting a strip of paper to go around a fat Santa for his belt can make it reasonable that we sometimes need to know "how big around" something is; in addition, the children may be very surprised at how long a strip this takes! Problems of this sort also present natural opportunities for using phrases such as "too long" or "too short" in meaningful ways.

▸ *Putting things away.* Cleaning up the classroom provides opportunities to discover that some books are "too tall" to stand upright on the shelf, to choose from a collection of empty boxes one that is "big enough" to hold everyone's potato head, to pick out pieces of chalk or crayons that are "too short to save." Appropriate questions ("Why will this book stand up and this one won't?") and appropriate decision making ("Pick out to throw away all the pieces shorter than this") can help the children learn about comparing length, capacity, and the like.

▸ *Counting.* Rote counting need not always be practiced with taps of a pencil or special counters. Counting the cupfuls of sand needed to fill a pail or a dump truck enables the children to see one means for measuring capacity. The matter may simply be treated as an exercise in rote counting. However, for some children it might be instructive to fill two containers in this manner, count the cupfuls, and then ask the children which they think "holds more sand." The latter procedure, of course, anticipates the use of numbers for purposes of measurement and at the same time gives the teacher some evidence of the children's acceptance of this criterion.

some summary remarks

There are certain problems that arise in any activities approach to learning. For one thing, once children's curiosity has been aroused, they can pose unexpected (and sometimes embarrassing) questions. Careful preparation on the part of the teacher, including some role playing on his part, can often preclude this. Also, a bit more direction can sometimes be given to an activity if the teacher does part of it and has the children help. But as with any teaching strategy, one must weigh the disadvantages against the advantages. Activities of the sort described here do provide a common background from which the more precise aspects of measurement can be developed. The child, for example, who has tried to put two books on the shelf and found that one would fit and the other would not has an experience with "taller than" that makes the results of measurement with a ruler sensible—"Oh, I see, when you measure the taller one you get a bigger answer."

As mentioned earlier, it is not intended that children *master* any of the techniques involved in these activities. These techniques are introduced to make comparisons *meaningful*, not because they are always reliable. A Ping-Pong ball, for example, is smaller than a brick in volume, yet it could not literally be "fit into" the brick.

It was intended that the introductory activities with measurable attributes make use of the child's senses. Some attributes whose measure is studied much later—light-years, kilowatt hours, calories—require something of an act of faith. Perhaps these abstractions will be more acceptable to the learner if his first experiences with measure are seen as perfectly reasonable.

Almost no use is made of number in these activities. The assignment of numbers as measures derives from a felt need for more precision in comparison. This matter will be discussed in the next section.

The problem-solving approach requires that the child be permitted to make conjectures and then test them. Not only does this permit growth in self-reliance on the part of the child, it also lets him see that his mathematical experiences make sense when tested in the real world. For the teacher, children's comments and questions as they work will on the one hand provide valuable insight into sources of misconception and on the other give assurance that the children fully understand what they are doing.

the process of measurement

The process by which a number is selected to be called the *measure* of some attribute of an object is specific to the attribute in question. That is, we use one method for finding the number to be labeled "weight" and quite another to select the number to be labeled "height." Because of this specificity, it is easy in the class-

room to become so embroiled in the details of the process that some very basic notions are neglected. For this reason, it is important to explore some of those basic notions before turning to the processes themselves. This we did in the preceding section, and we shall now enlarge on those notions.

Although the process of measurement is specific to the attribute, there are some similarities (the use of a standard unit, for example) that enable us to make some sort of classification. The following outline may not be completely valid. Nevertheless, it will be useful for the purpose of instructing young children to consider that the determination of a number to be assigned as a measure may be carried out in one of the following ways:

1. Counting
2. Reading a scale on a calibrated instrument
3. Computation
4. A combination of the above

counting

To find the number this way, some object is taken as a unit and physically iterated against the object being measured. We use this method whenever we pace off a plot of ground. An individual's pace is taken as a unit, is applied successively to the plot, and the number of paces counted. The count is then taken as the measure of the length of the plot. Usually we think of this as giving a "rough estimate," possibly because we recognize that the pace may not be uniform, or possibly because we are not using a "standard" unit of measure. In some instances, therefore, we may elect to iterate a yardstick rather than the pace.

The simplicity of the method of counting makes it a natural one with which to acquaint children. Furthermore, the other methods listed are *refinements* of this in the sense that they were invented to introduce greater precision. As a consequence, we can often find ways of presenting the other processes first as counting. For example, determining the area of a rectangular region can be presented first as a process of iterating a unit. Now as adults, we tend to mistrust this method of approximating measure—we feel the pace may not be uniform or, if a yardstick is being used, that we lose a few inches as we pick it up and lay it down. That is, we sense a lack of "exactness" and hasten to show children how to read a fifty-foot tape so that they can measure more "exactly." However, the studies of Piaget cited earlier show that children may believe that *the length of the measuring stick changes* as it is shifted. If this is so, then we adults are worried about something that does not concern children in the least. Perhaps the pedagogical implication here is that we delay teaching more sophisticated methods until the children sense for themselves the "inexactness" that the method of counting entails.

reading a scale on a calibrated instrument

This is perhaps the most common method used for selecting numbers as measures. We "read off" the temperature on our thermometer, the air pressure on our barometer, our weight on a dial on our bathroom scales, and the length of some object on our ruler. Furthermore, we tend to place complete faith in the numbers we read off (except, of course, for those on the bathroom scales!). That is, we accept the instrument as accurate, we accept that it does measure what it is intended to measure, and we accept our ability to estimate subintervals on the dial. At first glance, it would appear that this process of estimation might present the greatest difficulty for children, since the user of the instrument must note the units of calibration—tenths, fifths, eighths, and so forth—and be able to interpret what "halfway between" two marks would mean. Subsidiary problems of establishing the correct line of sight and coping with the meniscus of liquids are also recognized. But for some children, the instrument itself may be baffling. He is handed a ruler, for example, and told that it is for measuring length. For some children this may be as informative as telling them a "plark is used to whiffle a thintch." Thus both the behavior of the instrument as a means to an end as well as the mysteries of its calibration may present difficulties for children.

computation

We compute to find a number to assign as a measure whenever we determine someone's age (subtract the year of birth from the present one), when we use any of the mensuration formulas (perimeter, area, volume, surface area), and when we calculate such measures as IQ. To many adults, this represents the most exact form of measurement. We do have confidence in our ability to compute, and since we can handle large numbers of decimal places, somehow we are convinced we are being more accurate. A moment's thought, however, should remind us that the measure of a side of a square must be determined before either area or perimeter can be computed and that there is *no formula for determining that number.* We must rely on our second method, the use of a calibrated instrument, with all its potential for human error to obtain the data to throw into our formula! Children's difficulties with mensuration formulas are well known. Perhaps these are derived from our zeal to eliminate human error in measurement, resulting in a rush to establish computational shortcuts. If we give the younger children time to learn what is meant by these things called perimeter and area, and give them time to appreciate the need for accuracy, many of these difficulties could be avoided.

combination of methods

The use of more than one method of computation was mentioned in the preceding paragraph. Another common example is determining one's weight on the scales in a doctor's office. Two separate readings must be made and the numbers summed. The computation here is mental, of course, but is computation nevertheless. We include this as a separate method merely for the sake of completeness.

What the studies of Piaget and his followers suggest is that the young child has almost no equipment for the appreciation of the degree of exactness in measurement that adults hold in such high esteem. Pedagogically, this implies that young children need to spend more time learning about those attributes that grown-ups wish to measure, more time comparing them in relatively gross ways, and more time iterating and counting units. When it begins to make sense, for example, that if one container holds three scoops of sand and another two, then the former is larger than the latter according to capacity (never mind the heights), then the children are distinguishing between two attributes and accepting number as a means of denoting relative size. But such understandings come about through experience, not through legislation.

The activities that follow are intended to create a smooth transition from the foundations of measurement to the more refined techniques of instrumentation and computation. Since the aim is to make those processes appeal to children as both reasonable and natural, the transitional activities will emphasize the process of counting and will incorporate a certain amount of variety.

Instructions such as "Measure the line segment below" assume that children accept numbers as measures, an assumption that may not be warranted in fact. In order to promote this acceptance, in some of the activities the children assemble configurations according to real-life data and then observe which is largest. The relative size is then associated with the number of pieces *used by the children* (not "contained in"—an important distinction).

In other activities, the usual instructions for measurement exercises are essentially reversed. Instead of being asked to find the measure of a given object, the measure will be given and the children asked to assemble as many configurations as possible having that measure.

The calibration of instruments is related to the way in which they operate. Larger numbers are to be associated with "more" of the attribute in question. Thus on a thermometer the larger numbers are higher up because the warmer the temperature, the higher the column of liquid. In the foundational activities some suggestions were given for observing the behavior of some measuring instruments, in this section we offer some suggestions for children placing numbers on instruments.

It is also part of this transitional phase that children learn that the number assigned as a measure is not only related to the amount of the

attribute but also to the size of the unit. Some activities are suggested for this purpose.

Finally, some activities are suggested for learning about the attributes called "perimeter," "area," "volume," and "surface area." These were placed last because they are more abstract than, say, length or weight. The aim is to distinguish between perimeter and area on the one hand and between volume and surface area on the other. Although it is not necessary, and may not be desirable, to use the words *perimeter, area*, and so forth, it *is* important for these activities that children believe numbers can convey information about relative amounts of some attribute.

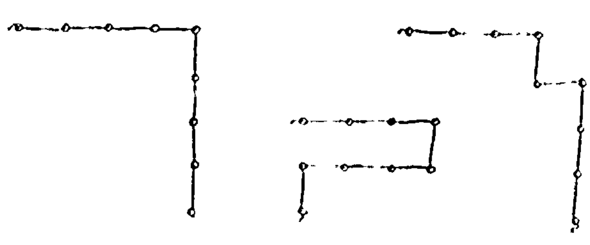
As before, the list of activities is provided as a source of ideas, and teachers are encouraged to devise variations to suit the needs of their own classes.

transitional activities

accepting counting as a measure

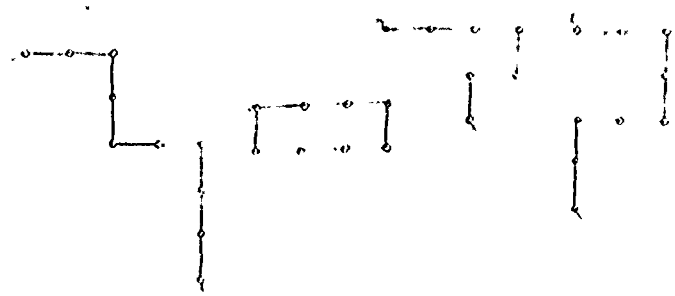
making designs on the geoboard

Since a rubber band stretches and can change length, a piece of string should be used for this activity. The string should be prepared so as to be some convenient number of spaces long (let us say 8), with loops on either end to be fastened to pegs. The string can then be used to experiment with making different "designs" or "paths" or whatever label the teacher wishes to use. Some possible arrangements of an eight-space string are shown here:



▸ Ask the children which path is longest and why they think so. Their answers will give valuable information about how well they "conserve." If some think one path is longer than another, it probably will not help to tell them that they are wrong. Instead, try the following.

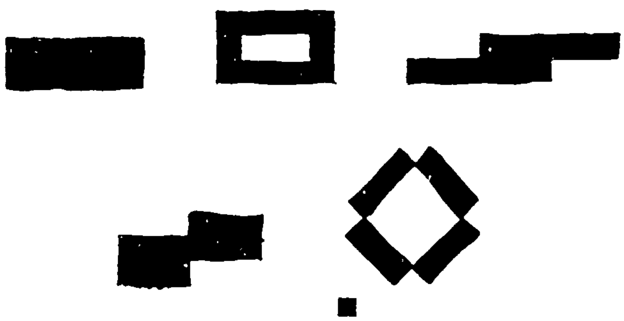
Draw each of these designs on the chalkboard or overhead projector and ask which the children could make using the same string:



Note that the first figure can't be made because the string isn't long enough, and the third figure cannot be made without doubling back, which should be ruled out by the teacher. The children should be encouraged to try out their conjectures with the string and geoboard so as to determine for themselves who is right. The trying out will entail some counting if these paths are to be reproduced exactly, and this may help some of the children who thought some of the earlier designs to be longer than others. In any event, the discussion accompanying this activity should be instructive.

making designs with pieces of paper

▸ Give each child four pieces of paper all the same size and shape; for example, rectangular pieces each 1 inch by 3 inches. Have the children arrange these into designs. The only rules are that each piece of paper must touch at least one other and there must be no overlapping. If such instructions are too difficult, printed patterns may be given out for the children to cover (graph paper helps in the preparation of such patterns). Some possibilities are shown in ■. Encourage the children to make as many designs as possible. Have each child choose his favorite and paste it on the paper; then display the designs. The question of interest here is, "Which design has more paper in it?" If the



children think this varies with the design, they do not conserve, and teachers will need to use their own judgment about how much urging to do in the matter. Some questions may help, as, for example, "Did you use all the paper for every design?" It may also help to take out another collection of four pieces of paper and fit these against each design in turn. The problem may or may not be linguistic. If it is not, carrying out some variation of this activity—different numbers of differently shaped pieces—at a later date may help to develop the notion that all designs made with the same collection of pieces of paper must have the same amount of paper in them. (It is not, of course, intended that the children verbalize this generalization.)

tangrams

The difference between tangrams and the materials of the preceding activity is that the pieces in a tangram are not all the same size and shape. They are commercially available, and of great interest to children. They also lend themselves to the same sorts of questions as those cited in the preceding activity.

This activity would be a three-dimensional analog of the second activity above.

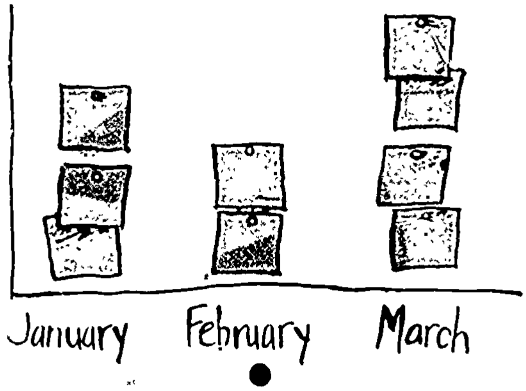
▸ Have the children make designs with blocks (let us say each child has six blocks) and encourage them to make as many different designs as possible. Then ask which design is bigger. If they think some design is bigger, ask where they got the extra blocks to put in it. Comparing designs made by individual children—"Could eveyone make a design just like Jane's with his own blocks? Let's all try."—may help the children see that the shape of a pile of blocks does not tell the whole story about its size. That is, piles of different shapes may have the same number of blocks in them. It is, of course, neither necessary nor even desirable to have the children verbalize the generalization. Asking appropriate questions can direct their attention to the phenomenon.

counting as a measure of height

▸ *Building towers.* If sets of counting blocks of various colors are available (all the same size, of course), place different numbers of three different colors in a box. Have the children count the different colors separately, and suppose there are six white ones, four green, and seven yellow. Ask the children which color they think would make the highest tower. When they have made their conjecture, stack the blocks up to check. Repeat with other collections. For one of these, it might be instructive to have the same number of two different colors. For reinforcement, prepare ahead of time three towers of three different heights, not ordered by height. (Placing a box over the dispaly will keep it from sight until you are ready for it.) Remove the box and ask the children which pile they think has the most blocks in it, then the next most, and the least number. Have them count to check.

▸ *Making a graph.* Each child should be asked if he knows when his birthday is. If all do, then a graph can be made as follows. Prepare a poster with the months arranged across the bottom. Give each child a square-shaped piece of paper and have each in turn go up and paste his paper in the appropriate place according to his birthday. Part of the resulting graph might look like the one in ●

(Lining things up neatly with no gaps or overlaps is nice, but a little sloppiness could be tolerated.) Then ask which month has the most



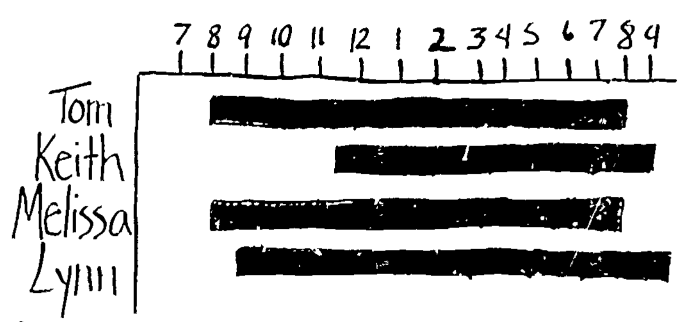
birthdays and how one can tell. The answers to the last question could be "taller" or "more squares." Sinc.. the differences in height may be quite obvious and are clearly related to the number of pieces of paper put up, this becomes not only an exercise in recording data on a graph but also a means of accepting counting as a measure.

counting as a measure of length

▸ *Pacing off.* There are many opportunities for pacing off. "Do you walk farther from the pencil sharpener to the door or from your desk to the door?" Step off both paths. Find out how many paces it is from the door of the classroom to the door of the library or the cafeteria or some other reasonable choice. For younger children this may simply be an exercise in learning a way of comparing two fairly long paths when one is not a subset of the other. For older children, the activity prepares the way for seeing the need for a standard unit of measure: George paces the room and gets one number, and Charles paces and gets another; or George paces and the children object that his steps weren't all "even." These situations can be the basis for a new problem. In the first case, the problem would be, "Why do the two boys get different numbers?" In the second, the problem is, "What could we do to get a more reliable number?" And, of course, pacing off the corridor or the playground could show the need for some larger unit.

▸ *Using a stick.* The janitor comes in to replace a light bulb and ducks his ladder to get it in the door. Why? Could we find out whether or not the ladder will go through upright without putting it against the door? We can't pace off the height of the door; so we need another way to compare the two. (The classroom flagpole, perhaps, or the window hook if there is one, might do.) Comparisons by use of a string might be a first suggestion, but what if we have no string handy? Problem situations of this sort can be proposed and the children somehow encouraged to use some object such as a stick to solve the problem. Exactness, of course, is not the goal here; rather, the goals are (1) accepting the method as a means of obtaining needed information and (2) learning the technique of successive application of the unit (iteration of the unit).

▸ *Making a graph.* Ask the children to report the time at which they went to bed Saturday night and the time they got up Sunday morning. Show these times on a large graph, and use this as an opportunity for children to cut strips of paper "the right length." (Freezer tape might be good for making strips.) This activity provides opportunity to explore several aspects of measurement, according to the age of the children and the kind of questions posed by the teacher. For younger children, it might be sufficient that they get the strips of paper the right length and can tell from the graph who got up earliest (latest) and who went to bed earliest (latest) as an adjunct to learning about telling time. For older children the question might be, "Can you find out from the picture who slept longest?" There are several ways in which this can be determined, one of which is to count the number of hours each child slept. The graph shown here is good for this purpose because of the irregularity. That is, if some try to tell by looking at the endpoints, they may have to be reminded that all children did not go to bed at the same time. (If a hint is needed, ask if they can tell from the picture how long Tom slept, how long Keith slept, and so forth.)



counting as a measure of weight

▸ Once the children understand how to use a two-pan balance to compare weight (see the activities in the first half of this chapter) and how to assess "the same weight as," a problem of the following type can be presented in a natural way: "The staple remover is heavier than a bottle cap; is it heavier than two bottle caps? Three bottle caps? How many bottle caps would it take to weigh the same as the staple remover?" Other combinations can be used depending on the availability of objects for comparison. This particular activity lends itself to extension to standard units if one-ounce or one-pound weights are available. It also lends itself to extension to the single-beam balance with a counterbalancing arm if that is available.

counting as a measure of capacity

▸ Explain that the water in the aquarium is getting low and some water must be added by the panful. Ask the children to guess how many panfuls it will take to "fill" the aquarium up to some designated height (to be marked with a china-marking pencil). This activity is particularly instructive because of the difference in the cross-section areas of the pan and the aquarium. If the children doubt that anything is being accomplished, the "beginning level" may also need to be marked. However the problem is handled, it will undoubtedly be surprising how many panfuls of water it takes to raise the height of the water level even a very small amount.

If small and large school buses are employed by the school system, the children may know how many seats are in their bus or how many children ride it. This information can be related to the "size" of the bus.

The relative sizes of two toy dump trucks may be compared by filling them with sand by the scoopful, by counting the blocks they will hold, or by counting the number of marbles that can be transported.

For an activity more directly related to standard units of measure, use individual half-pint milk cartons to fill quart containers, half-gallon containers, and the like. As each of these is filled, count the number of half-pint containers required.

relating the measure to the size of the unit

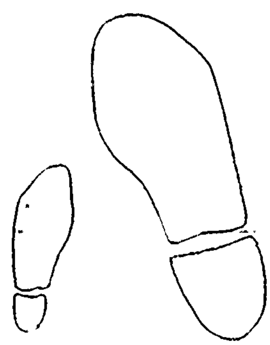
Many of the activities suggested above provide opportunities to observe that different numbers are also related to different units—the size of a "pace," for example. When the time arrives for this understanding to receive more emphasis, some of the activities can be redesigned for that purpose.

▸ To cite a simple example, fill a large bottle with water using first a juice glass and then a cup. Explain the difference in count. If the children have had experience with comparing capacity using a third container, the explanation should be easy.

▸ In Never Never Land there lives a giant who has a funny way of measuring his land. He doesn't pace as we do; he goes heel-and-toe along, like this:

In the same country there also lives an elf, who always does just the same thing as the giant When he tried to heel-and-toe the giant's garden, he got an answer different from the giant's. Why? Having some cutouts of the giant's shoe and the elf's shoe will help, for some actual numbers will be of use here. Let the children measure a "giant's garden" using both sets of shoes and record the numbers. "Suppose Charles's shoe (Charles is in the class) is used instead. Would we get either of the numbers we already have? What number do you think we might get?" See who can come closest

with his guess Other children's shoes (or feet, or cutouts of them) can be used.



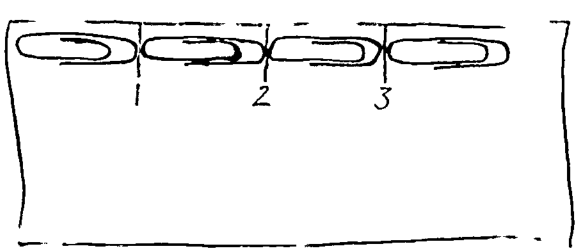
▸ Explain that in olden times the human body was used as a measuring instrument. Historians tell us that the rod as a unit of length was the length of the right feet, laid end to end, of sixteen men coming out of church on some predetermined Sunday. The children could be encouraged to reflect on the reliability of this as a "standard" measure, would this be the same regardless of the Sunday chosen? Experiments with various subcollections of the children could be informative here. Other "body measures" include the *span* (width of the expanded hand), the *cubit* (from the elbow to the tip of the middle finger, the *inch* (the measure of the top joint of the right thumb of the reigning monarch of England), and the *yard* (the length of the arm from fingertip to nose). Comparisons of the numbers obtained for different children's spans emphasize the relationship of the numbers to the size of the unit chosen. (Interested teachers are encouraged to consult books on the history of mathematics.)

calibrating instruments

▸ *Making a measuring jug.* Select a glass jar, such as an empty mayonnaise jar, for the measuring jug. Have the children pour in one juice glass of water. With a china-marking pencil, mark the level of the water. Then add another juice glass of water, mark the height, and so forth, making as many marks as you wish. The appropriate numbers could also be shown. If this measuring jug could then be used for some other experiments or classroom routine, so much the better.

▸ *Calibrating a two-pan balance.* If the two-pan balance has a dial, this can be covered up with masking tape. Have the children mark the empty position with an arrow or a "0." Then add one capful of sand and mark the spot where the pointer stops; add another capful and mark this, and continue as practical. A cap from a catsup bottle, or a cap from a fabric-softener bottle might be an appropriate size for a capful of sand. A cup might be too big, and, of course, a flat bottle cap could not as easily be filled "level" as a deeper one. It would probably be desirable to calibrate for both pans of the balance. The calibrated balance can then be used without piling on unit weights each time.

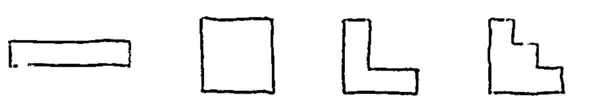
▸ *Making a ruler.* Give every child a strip of cardboard and have them mark off the length of one paper clip, two paper clips, and so forth. The use of paper clips is not mandatory, any unit being used by the class (elf shoes, for example) is appropriate. By adding the numbers, the children have their own rulers to measure objects with. The original cardboard need not be any integral number of units long; as a matter of fact, it might be instructive if the strips were not an integral number of units long.



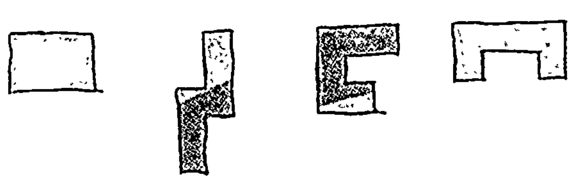
learning the attributes called perimeter, area, surface area, and volume

Some suggestions for discriminating among area and perimeter are to be found in the chapter on "Geometry" (chapter 9), and the following activities assume some rudimentary knowledge of geometry.

▸ Provide each child with a piece of paper ruled in one-inch squares and a collection of twelve or so 1-inch pieces of 1/8-inch dowel stick or pipe cleaner. Have the children use all their sticks to build a simple closed curve (or "fence" if they do not know the term) on the paper; the fence must be built along the lines on the paper. Have them trace the fence and then use their sticks to make another fence that looks different from the first, putting this in another place on the paper. Since there are many possibilities, asking "Who can make the most?" can encourage a bit of competition. The measurement questions are, "Which fence is the longest?" and "How many spaces are inside the fence?" Some of the children may jump to the conclusion that since the fences are all the same length, there must be the same number of "spaces" inside each fence. Having the fences outlined allows the children to check this idea by counting the spaces enclosed.



▸ On a sheet of paper ruled in one-inch squares, have each child shade in a "garden" that covers exactly six spaces. Again, there are many possibilities, and after several gardens have been laid out, some measurement questions should be posed. "Which garden covers the most space?" "If each were fenced using the sticks of the previous activity, which would require the most sticks?" As with the previous activity, the children may think that because all the gardens are the same size, they require the same amount of fence. Therefore, some sticks should be available for checking answers.

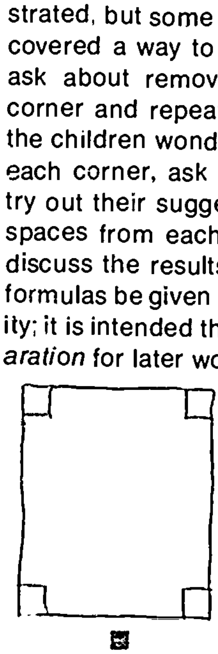


▸ *Making boxes.* Have a display of several different sorts of boxes. Some possibilities are an oatmeal or salt box, a rectangular cereal box, a shoe box or other box with a thin lid, a box with a thick lid such as pieces of hardware sometimes come in, a suit or dress box that opens up perfectly flat. "These boxes are all made of cardboard. We have some cardboard. Could we make some boxes like these? But we need a pattern." The children may be able to invent their own patterns, or it may be necessary to cut up the boxes to see how they are made. Picking out a big enough piece of tagboard, figuring out how to fasten pieces together, and learning the advantages of folding to avoid taping corners are all things that can be learned from this activity. Before tackling the cylindrical box, it might help to have a piece of mailing tube available. Make the first problem that of making ends for it in order to convert it into a box. With a little more experience the entire box can then be made. If the children seem interested, give them each a piece of construction paper (all the same size) and ask them to make the biggest box they can from the sheet. Then compare different children's products for size (have them guess first), using sand or salt to compare capacity. If they "weren't thinking about it that way," this is a natural opportunity to discuss such things as the amount of cardboard related to the cost of making boxes compared with the amount of "goods" various boxes would hold. Thus *both measures* (surface area and capacity) are needed in business; neither is "the right" one.

▸ *Making cubes.* Have the children make a pattern for a cube as described in chapter 9, using squared paper. (There is more than one possible pattern; so it is not necessary that all the children's be alike.) When they have found a pattern that works and become reasonably handy at assembling the cube, have them make a pattern *on squared paper* for a "bigger cube." This is definitely a nontrivial problem. It requires time on the part of the child—time to figure out how to make his pattern "bigger" and time to try out his patterns and revise them when they don't work. It also requires patience and help on the part of the teacher—patience so as not to give too much direction, and help in asking such questions as "What seems to be wrong?" and "What do you think you could do to make it right?" But the exercise is instructive because the children may be very surprised to discover how much bigger the pattern must be in order to make even the "next sized" cube. When all the children have finished, display a sheet of paper with outlines of the patterns of

the two different sizes on it and ask the children to count the number of spaces occupied by each pattern. This will reinforce numerically what they have discovered from inventing the patterns. The teacher might also wish to propose that the larger pattern be assembled into a box, with one side loose for a lid. Ask the children how many of the smaller cubes they think would fit into the box. The children might loan some of their little cubes to try it out and to see who is right.

▸ *More on making boxes.* One-inch squared paper and one-inch blocks, such as counting blocks, will be needed. The pieces of paper should be 7" by 8" each. Propose that an open box be made by first cutting one space from each corner of the paper and then folding, as shown in ■. Ask the children how many spaces are on their sheet (eight rows, each containing seven spaces), and then ask how many would be left for the box after removing one from each corner. The children should cut out the corners, tape up the sides, and then the teacher should ask how many blocks they think would go into their box. If they do not have any idea, borrow a box and have someone fill it with blocks while the other children watch. Next, propose that four spaces be cut from each corner (●). "How many would be left for the box?" With another sheet of paper, have the children make this box. "How many blocks will it hold? Should it hold more or fewer than the first box? Why?" Again, this may need to be demonstrated, but some of the children may have discovered a way to figure out the answer. Then ask about removing nine spaces from each corner and repeat the experiment. If some of the children wonder why you cut squares from each corner, ask what they think would work, try out their suggestions (why not remove two spaces from each corner, for instance?), and discuss the results. It is not intended that any formulas be given as the end result of this activity; it is intended that this activity be good *preparation* for later work with formulas.



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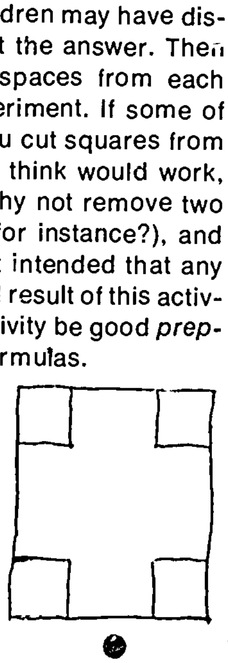


●

time duration

It would be artificial if we were to attempt to develop concepts of time in early childhood independent of the instruments used to measure time. The frequent use of time and clocks makes it reasonable to introduce clocks and calendars. At the same time we can help the child to develop a sense of time duration and to compare periods of time.

Children can engage in many activities that help them attain some idea of time duration. Some of these activities can be related to the clock and the calendar. Others are dependent on such devices as egg timers, simple water clocks, or candle timers. Still others involve simple comparisons independent of any kind of measuring instrument.

Following are a few examples of activities designed to help the child gain some idea of time duration. The teacher will want to develop others as well.

▸ *Races.* No special equipment is needed. This simple activity involves the running of races of various types. Make two lines about twenty yards apart on the playground or in the gymnasium. Teach the children to start "at the same time," using any method you wish. Also teach them to start "in the same place" by fixing their toes on the starting line. The child who gets to the finish line first is the winner. A discussion of what winning means in terms of time will establish that the first person to cross the line takes the least time.

Variations in the type of race should be made. Include changes in the start and finish line as well as in the kind of race. Insect races or turtle races are common variations.

The stopwatch can be introduced in racing activities fairly early. Children should see that there is a relationship between the distance the hand moves and the length of the race.

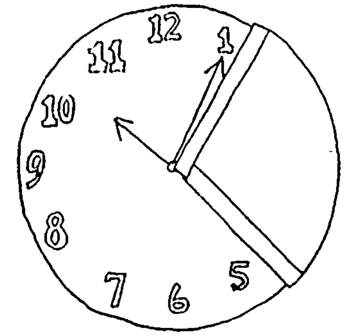
▸ *Beat the clock.* Provide an egg timer or a simple water clock. (A water clock can be made by punching a small hole in the bottom of a soup can. When it is filled with water, the water will take about the same time to run out each

time it is filled. If different periods of time are to be measured, the can can be graduated at the halfway or quarterway points. Shortening the length of time can be accomplished by enlarging the hole.) These devices can be used to answer such questions as "how many times can you walk from the front of the room to the back and return before all the sand runs out?" or "How many times can you bounce a ball before all the water is out of the can?" Have the child estimate when he thinks all the sand or all the water has run out. Turn the timer over and ask the members of the class facing away from the timer to put up their hands when they think there has been enough time for the sand or water to run out. They will have had to have had considerable experience with the time period measured by these devices before such estimating is attempted.

As a follow-up activity, punch holes of different size in three soup cans of the same size. Paint each can a different color. Call one can "short time," another "medium time," and the third "long time." This will help build the idea of a need for variation in lengths of time units, such as minutes, hours, days, weeks, and so on.

▸ *Time's up.* A large clock with a second hand and a flow pen with washable ink will be needed. The object of this activity is to have the child make estimates of short periods of time. For example, make a black mark on the clockface at twelve and another at six. Have the members of the class turn their backs to the clock. Give them a signal when the second hand is at twelve. When they think th[illegible] has reached six, ask them t[illegible] [illegible] who estimate too little will be able to wat[illegible] the rest of the time run out. Those who go over the time will be able to get a better notion of how quickly the hand moves.

▸ *How long?* A large clock, masking tape, and masks for the clockface with pielike open spaces are needed. To help the children get an idea of the relative motion of the minute and hour hands, affix the masks to the face of the clock. Set the children to some task and tell them to begin some other task when the minute hand disappears behind the mask. The hand and the edge of the mask should just coincide at the beginning of the activity.



Later on, use the hour hand instead of the minute hand. Mark with a strip of tape just where the hour hand is at the beginning of the day, for example. Then, just before the day ends, make another mark so that the child can see how far the hour hand has moved.

▸ Provide a calendar of the current month for each child and have it pasted in a book. At some time in a day, let the child make a mark in the spaces representing the day. This can be a basic activity for the development of a sense of the duration of a week, a month, or a school year.

Variations would involve the use of symbols to indicate the type of weather, the temperature, or some other aspect of the days of the month or the week.

Vocabulary such as "Wednesday, the sixth of November" should be used verbally whenever convenient. To help children get familiar with both the written and oral names, write the day of the week and the day of the month on the chalkboard each d[illegible] words orally

Stan[illegible] of measure—the metric system

Eventually, the children will need to learn those units of measure that are standard, or conventional, in contemporary society. As the United States moves toward the adoption of the metric system, it will become necessary for children to learn to use metric units in their measuring activities.

The metric system is a decimal system developed as an outgrowth of the French Revolution; the units are derived from nature rather than from the dimensions of some mortal monarch The following brief description is summarized under four headings.

length

The basic unit of length is the *meter*. Originally the meter was defined as one ten-millionth of the earth's quadrant (one-fourth of the length of a meridian) and was determined by survey. However, with advancing technology it has become possible to redefine the meter in terms of the wavelength of light. By so doing, it has become possible to reproduce the meter without the fuss of sending out an expedition. One meter is slightly longer than one yard. A meterstick is subdivided into 100 equal parts; each of these is called one *centimeter*. A centimeter is slightly less than half an inch. These two units are adequate for measuring lengths around the classroom or school building. For measuring such things as automobile trips, the *kilometer* is used. One *kilometer* is 1,000 meters, a length slightly more than half a mile.

capacity

The basic unit of capacity is the *liter*. It is the capacity of a cube measuring ten centimeters on an edge and is slightly more than the English quart. Liquids such as gasoline or milk are measured in liters. When a smaller unit is desirable (liquid medicines, for example), the *milliliter* is used. One milliliter is one one-thousandth of a liter, that is, the liter is subdivided into 1,000 parts, each of which is called a milliliter. Since a cube ten centimeters on an edge has a volume of 1,000 cubic centimeters, one milliliter is equivalent to one cubic centimeter.

weight

The basic unit of weight is the *gram*. It is defined to be the weight of one cubic centimeter of distilled water under certain specific conditions of temperature and atmospheric pressure (necessary because liquids expand and contract with changes in temperature). Since the gram is a very small unit (about the weight of a paper clip), the weight of a person would typically be given in *kilograms*. A kilogram is 1,000 grams, and hefting a two-pound bag of sugar gives a fair idea of the weight of one kilogram.

temperature

The basic unit of temperature is the *degree Celsius*. It is one one-hundredth of the difference between the freezing and boiling points of distilled water when the atmospheric pressure is that of sea level. Unlike units of length, capacity, and weight, no prototype of the unit can be housed in some bureau of standards. Instead, a thermometer would be calibrated by marking the level of mercury when the instrument is placed in freezing water and when it is placed in boiling water. The first mark is then labeled "0" and the second "100," and the interval between is subdivided into 100 equal parts Each of these parts represents "1 degree Celsius" on that particular thermometer; the scale may then be extended by laying off the desired number of units above and below the original marks and labeling them appropriately. It will be noted that if the thermometer were being calibrated as a Fahrenheit scale, the two original marks would be assigned the numbers "32" and "212" respectively, and the interval between would be subdivided into 180 parts.

The summary above is not intended to be a complete description of metric units, rather, it is intended to call attention to those units most useful to a beginner. The beginner needs only a few units, and those few should be ones that are appropriate for measuring things in his immediate environment. Relationships among the units in a single system or among different systems are not needed for the beginner: Thus, references to the customary (formerly English) system given in the summary above are intended to direct the reader's attention to the appropriateness of the unit for small, medium, or large quantities, they are in no way intended to suggest that young children be taught to convert from one system to another. Years of

experience give the typical adult the feeling that the customary units are somehow natural whereas the metric units are artificial and hard to learn without a prolonged introduction to the meanings of the prefixes "centi-," "milli-," "kilo-," and the like. What we must recognize is that for beginners, such as young children, *any* system of standard units of measure is artificial No system is more natural than any other, and the simultaneous introduction of the inch and the centimeter will cause no more confusion than the simultaneous introduction of the inch and the foot. For the beginner, having an appropriate unit and knowing its name will enable him to record the results of his measurement—after all, that's what the units are for.

The April 1973 and the May 1973 issues of the *Arithmetic Teacher* contain a number of articles on the metric system that many readers may find helpful.

summary

The overall aim of this chapter has been to develop in children an inner conviction that those activities to which we refer as measurement are reasonable means to practical ends. To accomplish this goal, we have described ways in which children can be made aware of certain attributes of objects that are useful to measure, and we have suggested ways in which children can be made aware of the usefulness of measuring these objects. We have suggested ways in which children can be made aware of what "more" or "less" of some attribute means, and we have shown how even young children can see that instruments can be extensions of their senses when making comparisons. In the matter of assigning numbers to quantify "more" and "less," we have emphasized the value of counting. We have stressed this because it seems futile to insist on a precision for which children have no appreciation. Furthermore, if—as the studies of Piaget indicate—children believe that a unit changes size as it is moved from place to place, neither complicated tools nor computational shortcuts will alter this belief. Finally, we have included activities in which two or more attributes of a set of objects were explored simultaneously. In so doing, preparation is made for recognizing the difference between, for example, the perimeter and the area of some region.

A problem-solving approach was used in all the activities because it provides the child with the opportunity to see the need for knowing "how much" of something and because it gives him the opportunity to experiment with various ideas that he may have. Presumably, he will come to accept as reasonable the conventional solutions to measurement problems. Although a problem-solving or activities approach may be considered time consuming, account must be taken of the time spent in review and reteaching under the usual plan of instruction. If several weeks are spent in each year for four years on the subject of, say, converting from one unit of measure to another, perhaps a few days spent pouring sand from one box to another or "walking" elf and giant shoes could reduce this time considerably. When viewed in this way, activities may not seem so time consuming after all. The goals of an activities approach are thus long-term goals, and "mastery" in the usual sense is not one of the goals. Consequently, the activities are often presented as a request for a prediction about what will happen or as encouragement to find more than one way to carry out a set of instructions. A child can hardly fail at either of these tasks, and so there is no stigma about not knowing the "right" answer. When the results are examined, there still need be no stigma. Initial "wrong" predictions were only guesses anyway, and no prize is offered for the greatest number of solutions. The examination of results is for the purpose of seeing what really happens, and this understanding can be assessed by asking for more predictions and checking experimentally. Thus understanding develops within the strictures of the real world, and it is in the real world that we have need of measurement.

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relations, number sentences, & other topics

henry van engen
douglas grouws

THIS chapter contains suggestions and activities from other mathematical areas that have relevance for mathematics programs at the early childhood level. Content areas included are mathematical relations; number sentences; and other topics, such as logic, probability, statistics, number theory, and the integers.

mathematical relations

There are many mathematical relations that are appropriately studied in the early elementary grades. In fact, many of the topics previously discussed in this book are in essence mathematical relations, although they may not have been so characterized. Topics such as *congruence* and *similarity* and ideas such as *more than, as many as, less than, one more than, is a subset of,* and *longer than* are all examples. This section describes the nature of mathematical relations and shows their fundamental role with topics in the elementary school.

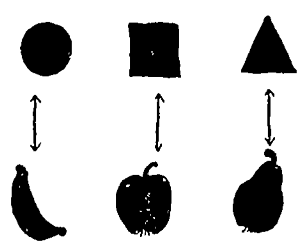
one-to-one correspondence

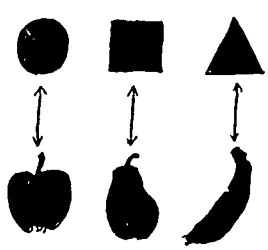
The notion that counting is the basic idea of arithmetic has been accepted and promoted for a long time by many people interested in elementary school mathematics. Counting is *not* the most basic idea of arithmetic! Ideas such as *one-to-one correspondence* and *more*

than are much more fundamental and are, in fact, prerequisite to a meaningful development of counting.

"One-to-one correspondence" and "more than" are basically mathematical relations and are not dependent on the concept of number but are the foundation for number. Determining that there are as many chairs in a classroom as there are children, for example, requires only that each child sit in a chair, that all the children have a chair to sit in, and that there are no unused chairs, that is, making a one-to-one correspondence between chairs and children. Similarly, if after this pairing there are some children who have no place to sit, it is known that there are more children than chairs. In neither case must the chairs or the children be counted.

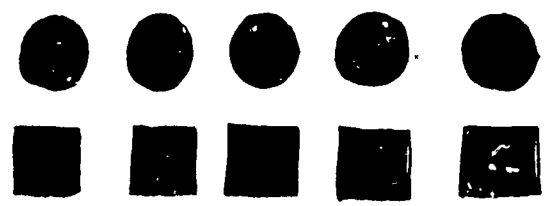
If the objects of two sets match one to one, there is more than one way to demonstrate it (unless the sets have only one object each). Two such pairings for sets of geometric figures and fruit are shown.

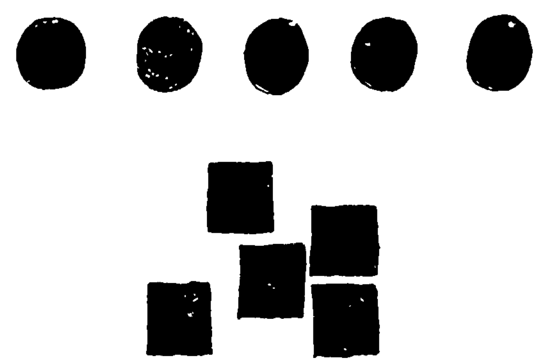




It is necessary that children come to realize that if two sets can be put in one-to-one correspondence, then the sets will always be in one-to-one correspondence regardless of the way the objects are matched.

The failure of children to recognize that two sets are in one-to-one correspondence regardless of how they are rearranged may result in inadequate number concepts and may also play a vital part in Piaget's conservation tasks. Consider the classic situation where squares and circles are arranged so that a one-to-one correspondence between them is visually obvious and then the squares are brought closer together. If the child says initially that there are as many squares as circles but changes his mind after the squares are brought together, then he cannot conserve one-to-one correspondence.



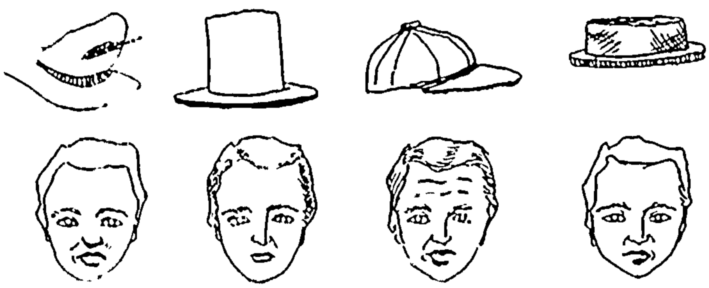


There are three important properties of the one-to-one correspondence relation, the *reflexive property*, the *symmetric property*, and the *transitive property*. Although the names of the properties are not needed by children in the elementary school, questions about relations that illustrate the properties are needed.

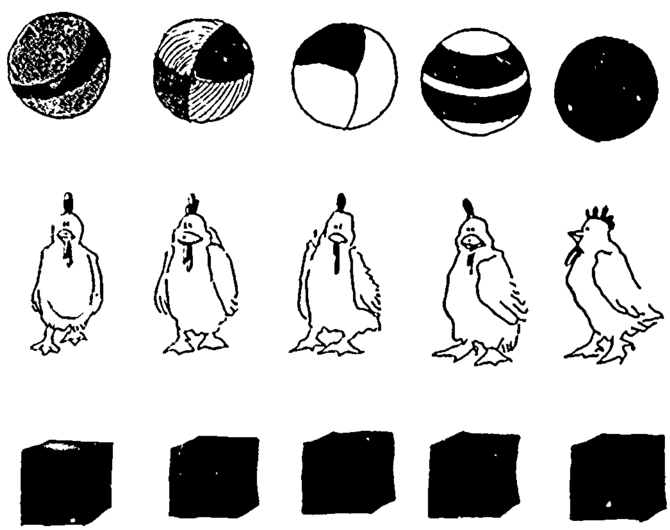
Reflexive property. It is quite obvious (at least to adults) that any set can be put into one-to-one correspondence with itself. The simplest pairing is to match each member of the set with itself. This property of one-to-one correspondence is referred to as the *reflexive property*. Its importance is best understood and appreciated when relations that are not reflexive are

considered. This property will be discussed further in such a context when the "more than" relation is examined.

Symmetric property. The symmetric property of one-to-one correspondence states that if set A can be put into one-to-one correspondence with set B, then set B can be put into one-to-one correspondence with set A. Put in simpler terms, if set A has as many members as set B, then set B has as many members as set A. For example, if there are as many hats as men, then there are as many men as hats.



Transitive property. The transitive property of the "as many as" relation states that if set A has as many members as set B and set B has as many members as set C, then set A has as many members as set C. For example, if there

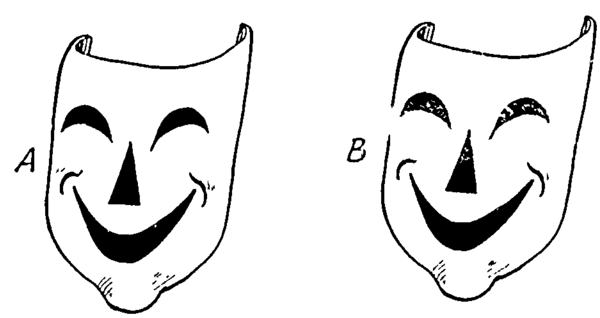


are as many balls as chickens and as many chickens as blocks, then there are as many balls as blocks. This result is immediate without resorting to physical matchings of any kind, once it is known that the transitive property holds. Of course, many physical matchings in an activity format are necessary before children grasp that one-to-one correspondence is transitive.

In summary, the "as many as" relation satisfies three important properties, namely, the reflexive property, the symmetric property, and the transitive property.

congruence relation

The idea of congruence is introduced in the schools after basic notions about size and shape have been established. This leads to the "same size and shape" relation, called the *congruence relation*. The congruence relation is used to compare and classify plane figures, such as triangles, squares, circles, rectangles, and pentagons. Through an examination of the properties of congruence, similarities between it and the one-to-one correspondence will become apparent.



Just as with one-to-one correspondence, there is a need to find an "action method" of determining if two figures are congruent. Congruence can be determined by attempting to superimpose one figure on the other. If figure A coincides with figure B, then figure A has the same size and shape as figure B; that is, A is congruent to B.

Notice that the congruence relation is reflexive, since any figure has the same size and

shape as itself. Further, note that if *A* has the same size and shape as *B*, then *B* has the same size and shape as *A*. Hence, the congruence relation is symmetric. The symmetric property can easily be developed in the classroom with activities similar to this:

► Use cutouts and make physical comparisons. Ask questions such as these: "Does mask *A* have the same size and shape as mask *B*? (Yes) "Jim, can you show that mask *A* has the same size and shape as mask *B*?" (Places *A* on *B*) "Do they fit exactly?" (Yes) "Does mask *B* have the same size and shape as mask *A*?" (Yes) "How do your know? Would someone like to put *B* on *A* and show that they fit exactly?"

So far it has been established that the reflexive and symmetric properties hold for both one-to-one correspondence and congruence. The "as many as" relation is also transitive. Does this property hold for congruence too? The answer, of course, is yes, and this can be demonstrated in the obvious way by using cutouts on a flannel or magnetic board.

equivalence relations

Many other relations, such as "has the same teacher as" and "has the same number of sides as," satisfy each of the three important properties: reflexive, symmetric, and transitive. Ask the children if they can think of other relations that satisfy all three properties. What about "is the same color as" or "lives on the same street as"?

Relations such as the ones just mentioned, which are reflexive, symmetric, and transitive, are given the special name *equivalence relations* Are all relations equivalence relations? Although many relations are equivalence relations, we shall later see that many important ones, such as "longer than" and "more than," are not equivalence relations.

One final characteristic of all equivalence relations cannot be overlooked. Any equivalence relation provides a means of forming distinct, nonoverlapping classes of objects. For example, if an envelope full of polygon-shaped cutouts is emptied onto the floor, we can form piles using the equivalence relation has the same number of sides as." All the cutouts in any pile will then be related (i.e., have the same number of sides). Further, any two cutouts from different piles will *not* have the same number of sides. This formation of distinct classes is an important characteristic of equivalence relations. Other examples that can be used to get across this compartmentalization idea intuitively are included in the illustrative developmental activities that follow.

► 1. Make two cutouts each of various geometric shapes. Attach one of each to a bulletin board within reach of the children. Fasten a piece of string or yarn to each of the remaining cutouts. Have the children attach the other end of the string to the cutout on the board that has the same size and shape.

Variation. Make a number of different cutouts of each shape. Fasten one of each to a bulletin board and place strings on the remaining cutouts. Proceed as before. When finished, ask questions about the cutouts attached to any given one on the bulletin board. "Do they all have the same size and shape?"

Activities of this type develop the congruence relation in a physical setting and also bring out intuitively the idea of equivalence classes.

► 2. Place on a flannel board in separate rows, or in scattered arrays, a set of flowers, vases, and watering cans. Ask, "Are there as many flowers as there are vases? As many vases as flowers? As many vases as watering cans?" After each question, have the children explain their answer and verify it by matching appropriate objects on the flannel board one for one by using yarn or string.

Variation. Place a number of flowers on a display board. Have the children place vases on the board such that there are as many vases as flowers. Repeat, but have them place more vases than flowers on the board.

These activities provide practice in actually constructing one-to-one correspondences and in looking at the properties of the relation.

▸ 3. Read stories to the children, such as "Goldilocks and the Three Bears" or "The Three Little Pigs." Use cutouts of the bears, pigs, houses, and so forth on a display board to illustrate parts of the story. Have the children name the sets of things that match the set of bears (pigs). Have them describe and verify their answers by matching cutouts one for one.

Variation. Place several different sets of objects on the display board. Have the children create stories about them. Follow up with appropriate questions like those suggested previously.

These activities develop the idea of one-to one correspondence and its properties. In order for the properties to be developed, however, questioning must be appropriately structured and at some time summarized.

▸ 4. Make about thirty cards with one, two, three, or four squares drawn on each card. Shuffle the cards and give one card to each child. Tell the children to find other children whose cards have as many squares as their card. Eventually (perhaps with some additional direction from the teacher) four distinct groups of children will be formed. Questions concerning cards from the same group and from different groups should be asked and then verified. "Will David's card have as many squares as Michael's card? Are you sure? How can we check?"

Variation. Place one geometric shape on each card and use the relation "same size and shape."

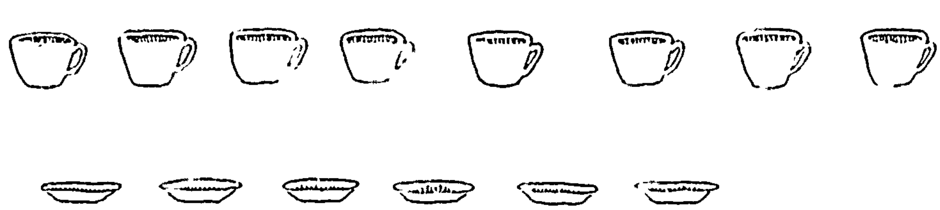
These activities develop the particular relation used and its properties. They also emphasize the partitioning into disjoint subsets induced by an equivalence relation. These activities will not work well for relations that are not equivalence relations.

order relations

In activities for the "as many as" relation, situations must be included sometime where the pairing of objects does not result in a one-to-one correspondence. In such situations, there are opportunities for the introduction of important order relations. Suggestions for teaching the "more than" order relation follow.

Consider a group of children investigating the "as many as" relation between sets, using the matching process previously mentioned. The teacher enters the scene and asks how things are going. The children quickly respond that "this is easy" or "this is cinchy." The teacher, rising to the occasion, asks them if set *A* (consisting of eight cups) has as many members as set *B* (consisting of six saucers). Almost immediately the air is full of responses: "yes," "as many as," "*A* has more than *B*," "*A* has two more than *B*," and perhaps others. The stage is now set for a discussion of the "more than" relation.

After some discussion, one of the children pairs the elements of set *A* with set *B*, indicates that two of the objects from set *A* do not have partners, and concludes, "It is not right to say *A* has as many as *B*." More discussion. Finally agreement is reached that in this situation it is appropriate to say that "set *A* has *more than* set *B*." Can we then also say that *B* has more than *A*, as we could with the "as many as" relation? The symmetric and reflexive properties are quickly ruled out. They are not satisfied by this new relation. But what about the transitive



property? Is it true that if A has more than B and B has more than C, then A has more than C? This calls for experiments!

David and Michael construct sets A and B such that A has more than B. Judy and Janet find a set C such that B has more than C. (Judy and Janet will have to make set C after David and Michael finish constructing set B.) Then everyone agrees that when A has more than B and B has more than C, it is certainly true that A has more than C, and so the transitive property works for the "more than" relation.

The "more than" relation is an order relation and not an equivalence relation. The transitive property is satisfied, but the reflexive and symmetric properties are not. Many seriation (ordering) types of activities can be used to develop the "more than" relation and its properties. These activities will also bring out the fact that the idea of the equivalence class is a unique feature of equivalence relations.

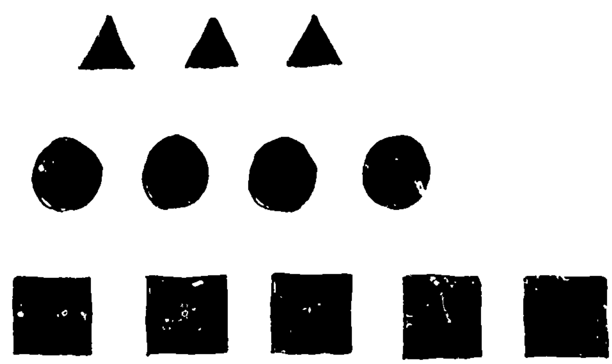
A special form of the "more than" relation is the cornerstone for the meaningful ordering of the cardinal numbers. This relation is the "one more than" relation. It can be developed in the classroom in much the same way that the "more than" relation is developed.

► Place cutouts of four children and three toys on a flannel board. "Are there as many children as toys?" (No) "Is there a child for each toy?" (Yes) "Can you show me?" (Yes, like this) "Is there a toy for each child?" (No) "Are there more children than toys?" (Yes) "How many more children than toys?" (One) "We say that the set of children has *one more than* the set of toys."

► Place cutouts of three men on a display board "Margaret, put umbrellas on the board so that there are *as many* umbrellas as men. Now who will place a cutout on the board so that there is one more man than umbrellas? What can you do so that there is one more umbrella than men?

Repeat with different materials and varied questions. For example, "How many men would get wet when there is one more man than umbrellas?"

The "one more than" relation is clearly neither reflexive nor symmetric, and this can be brought out in the same way as in the "more than" relation. But what about the transitive property? Is the "one more than" relation transitive? That is, if set A has one more member than set B and set B one more than set C, then will set A have one more member than set C? Showing this with cutouts allows the children to conclude quickly that the "one more than" relation is not transitive.



"longer than" and "as long as" relations

The relations "longer than" and "as long as" are important and have implicitly been a standard part of the elementary mathematics curriculum for years. The terms *as long as* and *longer than* can be thought of in terms of numbers, and this is true in most current programs. However, teaching these relations to young children in this way assumes that they have a stable concept of number and measure and certainly does little to connect the relations to the children's physical surroundings. Such an introduction, based on abstraction, is particularly disturbing, since these relations can so easily be defined operationally. For instance, rod A and rod B are laid side by side so that one end of A coincides with an end of B. If the second end of A extends beyond that of B, we say that rod A is "longer than" rod B. What are the properties of this relation? When rods of various lengths are used and appropriate questions are asked, children soon find

through actual experimentation that the "as long as" relation is reflexive, symmetric, and transitive. Further experimentation yields the conclusion that the relation "longer than" is neither reflexive nor symmetric but is transitive.

► 1. Cut strips of construction paper in different lengths so that there is one strip for each child and one strip to be used as a standard. Place the standard strip on a display board and have each child come forward and compare his strip to the strip on the board. The child should then describe what he finds, for example, "My strip is longer than this strip."

Variation. Use string or cord that will not stretch easily. Repeat the previous activity and follow by having children compare lengths with each other two at a time in front of the class. Have them describe what they find. This could be followed by ordering the children according to the length of their string.

These activities develop the "longer than" and "as long as" relation in a concrete setting. Depending on the follow-up questions used, they can also develop the symmetric and transitive properties of this relation.

► 2. Place on a flannel board a number of pieces of yarn varying considerably in length. Have the children order them by length.

Variation. Place several pieces of yarn on the display board in serial order, shortest to longest. Then conceal a piece of yarn behind your back and have the children ask questions ("Is it longer than the green yarn on the board?") to determine where the hidden piece of yarn should be placed to maintain the shortest-to-longest ordering.

If the children have had little experience with seriation activities, it would be best to use only three pieces of yarn initially. Further, in the very beginning it may be best to use strips of paper rather than yarn to avoid the necessity of holding the yarn taut.

general instructional strategy for relations

By now it should be clear that relations provide a means of examining objects with respect to a particular rule, attribute, or dimension. Real-world situations abound and should be used. A list of real-world situations that can properly be interpreted as a relation would include such relations as "is the brother of," "is taller than," "is the same color as," "belongs to," "was born in the month of," "is colder than," "is sitting beside," "has the same shape as," "lives on the same street as," "weighs more than," and so forth. Clearly, these are only a few of the everyday situations that can be conceptualized as a mathematical relation.

An overall instructional plan involves the following stages: introducing relations initially through familiar real-world situations, expanding the idea of a relation by studying important mathematical relations in situations where physical actions are employed by the children, and then refining the latter relations by examining them later in a more abstract framework. As the child's study of relations progresses through these stages, a meaningful symbol system for representing relations is carefully developed.

The opportunities for learning about relations should not cease at the end of the mathematics class period, never to be mentioned again; rather, as other topics (e.g., linear measurement) are studied, the idea of a relation should again be employed (e.g., "is longer than," "is shorter than," and so forth). The study of relations is also an ideal time for stressing the interrelatedness of various disciplines. For example, as part of a science-laboratory activity in which the relative hardness of various materials is tested by attempting to scratch one with the other, the idea of a relation entitled "is harder than" or "can be scratched by" could be brought out by asking appropriate questions of the students. Another possibility in science is the relation "will float in" when studying the idea of density. Still other possibilities would include the relations "has a higher melting point then," "has a lower freezing point than," "will dissolve in," "has a longer life span

than," "grows taller than," "migrate earlier than," and "is sweeter than." In social studies, relations such as "is the capital of," "receives more rainfall than," and "is closer to the equator than" could be considered. Interesting situations in other disciplines can be found if time and attention is given to the task. After many experiences in different areas of study, children will begin to recognize situations that can be conceptualized as a relation. As they mature intellectually, they will gradually capture the idea of a relation in a more general sense, that is, as a classifying and unifying concept.

number sentences

There is a strong need to represent and communicate ideas in mathematics, and because of the nature of mathematics these ideas can often be stated very precisely. Mathematical sentences provide a means of representing ideas precisely and compactly. This precision is achieved without lengthy verbal explanations and descriptions. This feature enhances the value of mathematical sentences as a means of representing mathematical conditions and stating properties and principles.

The ability to represent the mathematical conditions in a physical situation or verbal problem is well accepted as a desirable technique in any person's problem-solving repertoire. Symbolic representation should naturally follow the need to record physical phenomena or the results of experimentation and manipulation.

As the ability to represent physical situations with mathematical sentences develops, the role of a given sentence as a model for many situations should be emphasized. Following are examples of some fundamental open sentences involving the basic operations. These sentences are models for different physical situations.

1. Three children are at the front of the room. Two other children join them. How many are in the new set?
2. Three birds are on a fence. Two birds fly down and join them. How many birds are now on the fence?
3. Three apples are in a basket. Put 2 more apples in. How many apples are now in the basket?

$3 + 2 = \square$

$3 + 2 = \square$ is a model for all the situations in problems 1, 2, and 3.

Joining may also be thought of in the following situations, although the number of objects in one of the sets is not known.

4. David had 3 pencils, and then he got more pencils for his birthday. He has 5 pencils now. How many did he get for his birthday?

5. There are 3 marbles on the floor. John drops some more marbles on the floor. There are 5 marbles now. How many marbles did John drop?
6. Scott has 3 jacks. How many more does he need if he must have 5 jacks?

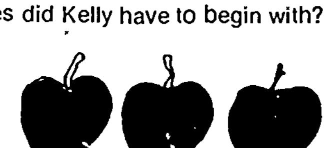
$3 + n = 5$

The sentence $3 + N = 5$ is a model for the situations in problems 4, 5, and 6. In fact, $3 + N = 5$ is a model for all situations where there is a set of 3 members and a set of unknown size is joined to it to form a set containing 5 members.

There is another situation involving the joining of disjoint sets where one set is of unknown size and a set of known size is joined to it.

7. There are some fish in the aquarium and 2 new fish are placed in the aquarium. There are now 5 fish in the aquarium. How many fish were in the aquarium at first?
8. Kelly has some apples. She needs 2 more apples so that she will have 5 apples for her party. How many apples did Kelly have to begin with?

$n + 2 = 5$

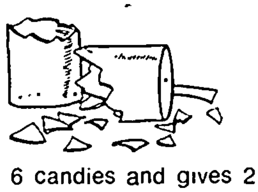


N + 2 = 5 is a model for any situation where we have a set of unknown size joined by a set containing 2 members, whereby the new set formed has 5 members.

There are many physical situations where a separating or removing action is involved.

$6-2=n$

9. If there are 6 glasses on the shelf and 2 fall off the shelf and break, then how many glasses remain on the shelf?



10. John has 6 candies and gives 2 of them away. How many candies does he have left?

The open sentence 6 − 2 = N is a model for both of the situations in problems 9 and 10 as well as for every other situation where there is a set of 6 objects and a subset of 2 objects is removed.

There are many physical situations that involve several sets all of the same size (i.e., equivalent sets) where the question of interest is how many in the new set when these sets are combined or considered in totality.

$3 \times 4 = n$

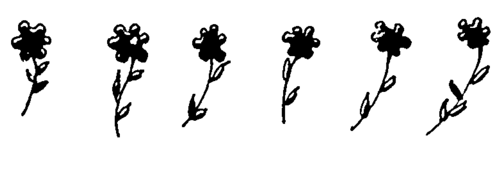
11. John has 3 boxes of toys. There are 4 toys in each box. How many toys in all the boxes?
12. An egg carton has 3 rows with 4 spaces in each row. How many eggs does the carton hold?

$3 \times n = 12$

13. Jim buys 3 packages of baseball cards. When he counts the total number of cards that he has purchased, he finds he has 12. How many cards were in each package?

$n \times 6 = 30$

14. Mary has a flower garden with 6 flowers in each row. She has 30 flowers in her garden. How many rows of flowers does Mary have?



The models for problems 13 and 14 show situations where the total number of objects is known but one of the other numbers is unknown.

The relationship just illustrated between mathematics and the physical world is an important one. As part of their study of mathematical sentences, children should learn to represent mathematical conditions in physical-world situations by mathematical sentences. As this skill is developed, an appreciation for the interrelationship of mathematics and the physical world is promoted. Making up instructional activities that focus on the use of a mathematical sentence as a means of recording activities with objects or diagrams is one way to develop this ability and to establish a meaningful relationship between actions on physical referents and number sentences. The development of this ability is part of a larger ability of representing the physical world by mathematical models.

After children become adept at writing an appropriate open sentence as a model of a physical situation or problem, they should have experience in "going the other way." For example, when given the open sentence 4 × 12 = N as a model, children should be able to translate the sentence into a real-life situation. This translation may take many forms. The child may be encouraged to express the idea with concrete objects by forming 4 sets of 12 objects each and combining them. Another possibility is to write a verbal problem to "go with" this open sentence:

John has 4 bags of marbles, and each bag contains 12 marbles; how many does he have in all? $\Big\}\ 4 \times 12 = n$

A creative teacher can devise other ways to translate from a number sentence to the physical situation.

Some further activities to help with these translations follow.

► 1. Put a number of verbal problems on index cards, one problem to a card. Have the children sort the cards into piles on the basis of sentence types. For instance, the following two problems would be placed in the same pile,

since $6 + \square = 10$ is a model for both of them.

John has 6 marbles. Bill gives him some marbles. John now has 10 marbles. How many marbles did Bill give to John?

Mary's bracelet has 6 charms on it. She receives some new charms for her birthday. She now has 10 charms. How many charms were given to her for her birthday?

The following problem would not be placed in the previous pile, since it represents a separating type of action, which is modeled by a subtraction sentence.

There are 10 boys playing football. Six boys leave and go home. How many boys are now playing football?

▸ 2. Put basic addition and subtraction facts on 3" × 5" index cards, one fact to a card. Place the cards in a box. Make available a quantity of manipulative materials. Have the children come forward, draw a card, read the number sentence on their card to the class, and then demonstrate a situation that represents the given sentence, using the available materials. For example, if the number sentence was $5 + 4 = 9$, the child could form a set of five blue disks and a set of four red disks and then combine the two sets into a single set. Perhaps at some point and for some children, a verbal explanation such as "I put a set of five with a set of four and there are nine in the new set" should be encouraged.

▸ 3. Have one child write a number sentence and another child translate the sentence into a physical situation, using the concrete materials present. For instance, if one child wrote the sentence $10 - 8 = 2$, then the other child could translate this into a physical-world situation by forming a set of ten and then removing a set of eight. Of course, forming a set of ten disks and a set of eight buttons and then matching each button with one disk, leaving two disks unmatched, would be a perfectly acceptable alternative to the former situation. The possibility of having more than one interpretation provides opportunities that should be capitalized on whenever possible. At the same time, it emphasizes the need for teacher supervision of activities in the classroom. Later, have the two children switch roles: the one who was writing begins to translate, and vice versa.

solving sentences

Number sentences without variables can be classified as true or false, as shown in these examples:

$2 \times 3 = 6$ (true)	$5 + 1 = 5$ (false)
$1 + 3 < 6$ (true)	$9 + 8 < 10$ (false)

Solving open sentences such as $3 \times N = 15$, $2 + \square < 6$, or $N \times 1 = N$ means finding the numbers that make true sentences.

As part of a child's introduction to mathematical sentences, practice should be provided in determining the truth or falsity of mathematical sentences. These experiences can and should be constructed in a way that maximizes the use of concrete materials for decision making and for verification. Further, this experience should provide a base for a sequence of carefully selected activities designed to introduce the notions of open sentence and solution set.

Some of the activities outlined here are built on the "longer than" relation. The means of introducing this relation and presenting its properties have been discussed previously. It is assumed that these activities will not be the children's first acquaintance with this relation or with the materials described.

▸ 1. Divide the children into small groups and provide each group with a set of rods of various sizes and colors. The lengths of the rods should differ only slightly, thus promoting physical comparisons; rods of the same length should be of the same color. Give each group a list of comparisons between rods, stated in the form of mathematical statements. The notation used in forming the sentences on the list should agree with the notation previously developed in the classroom. For example, sentences such as "Red $\xrightarrow{\text{longer}}$ Blue" could be used if such notation was familiar to the children. The task set for the children is to determine the truth or falsity of each of the sentences on the list by making comparisons between the two rods involved.

Variation: This activity can be expanded in many ways. For instance, ask the children to

expand the list of sentences by recording a specified number of true sentences and a similar number of false sentences about the rods and the "longer than" relation. Alternatively, one child could write additional sentences and the other children could determine their truth values Of course, the entire activity could be repeated with a different relation or different materials.

Possible objectives of the preceding lessons are manifold, and a long list of such objectives could be generated. One of the more important objectives, however, is the attainment of the ability to determine the truth or falsity of a given mathematical statement. This is an obvious prerequisite skill to the development of the concepts and skills associated with solution sets of open sentences.

► 2. The essence of this activity is to determine whether a given mathematical sentence is true, false, or open. It should again allow the children to use concrete materials in drawing their conclusions and should, perhaps, be restricted initially to the use of one relation. Let the relation be "as many as." Put materials like plastic counters into bags and identify each bag with a letter Print sentences such as "$A \xrightarrow{\text{as many as}} B$" on index cards. Have each child in the group working on this activity draw a card from the deck and try to determine if the sentence on his card is true, false, or open (i.e., cannot tell if it is true or false). His conclusion will be based initially on pairing the objects from the two sets identified in the given sentence or on realizing that only one of the two necessary sets has been specified. This type of activity can be extended to other relations and other materials. If the children have studied the basic addition facts, sentences such as $2 + 3 = 5$, $4 + 3 = 6$, and $4 + 1 = \square$ could be used. The concrete materials available for use might be a set of counters, Cuisenaire rods, or other similar materials.

Activities of this type provide a background that is helpful for students in the next activity and in their gradual development of the concept of a solution of a sentence.

► 3. Provide a set of index cards with an open sentence written on each. The open sentences should involve only one variable and should involve only relations and operations with which the children have had considerable experience. Give each child one card and ask each in turn to read the sentence on his card aloud. Assign each child an element (or have him choose one) from the domain of the variable in his sentence (e.g., a number). Have the children one at a time read their sentences, replacing the placeholder with the given number, and tell whether the resulting statement is true or false.

This activity, like the preceding ones, serves only as an illustration of an appropriate type of activity to fulfill the purpose stated. Other activities written in the same spirit are certainly needed to help children grasp the idea of a given element satisfying an open sentence.

► 4. This card game is to be played by a group of three to five children. On index cards, place open sentences with which the children have had experience. Use sentences that have only one number in their solution set—for example, $7 + \square = 15$. Distribute the cards to the children so that each child receives the same number of cards. To begin, instruct each child to turn up a card. The child whose card has the largest number for a solution wins that trick and places these cards under his stack. Play continues until one child has acquired all the cards. Rules will have to be developed as needed; for example, a rule concerning ties will have to be made.

Children's first experience with specifying solution sets should involve open sentences containing only one variable. Further, it is desirable that the domain of the variable contain only a small number of elements and that these elements be explicitly stated. The task can then be stated as, "Which of these elements make the sentence a true statement? Write them down." Most children will complete this activity by checking each of the specified replacements. Children need lots of experience in this type of activity, and the teacher should be sure

that the children become acquainted with the many different situations that exist.

The solution of sentences such as 7 + 5 □, 15 - 6 □, 7 × 9 □, and 60 ÷ 10 □ can be found directly by application of the operation shown in the number sentence. Sentences such as 8 - N 3, 4 + N 11, and 13 6 + N, however, are not solved directly and are more difficult for children. A perusal of contemporary mathematics textbooks at the primary level clearly indicates that much attention is given to developing the ability to solve the direct type of open sentence. This is in sharp contrast to the haphazard approach often taken to developing children's skill in solving open sentences that are not in direct solution form. Developing the ability to solve open sentences of the latter type may be the more difficult task. Some explicit attention can and should be given to solving such open sentences in the primary grades.

Children can often solve open sentences such as 8 - N 3 and 4 + N 11 without formal instruction on methods of solving. A great variety of methods is used. Most of these methods are informal and intuitive. For instance, a child might solve a sentence such as 8 N 3 by using a tallying procedure or by counting backwards; he might solve 4 + N 11 by "counting on." The use of such methods should not be discouraged In fact, a child using these methods is often displaying ingenuity and initiative. At some point, however, when much informal experimenting has been allowed, it may be desirable to suggest a more systematic way of solving these open sentences. The procedures suggested below are more systematic but are certainly not an abstract, formal approach to solving equations.

When addition and subtraction are meaningfully taught and are interrelated, children become very adept at identifying the known addend, sum, missing addend, missing sum, and so forth, in various open sentences They readily recognize that sentences such as N + 4 9, 5 + N 9, 9 N 5, and 9 - 4 N involve a known addend, a known sum, and a missing addend. Similarly, they become aware that open sentences such as 5 + 4 N and N 4 5 involve two known addends and that the sum is to be found. Carefully and regularly relating these sentences to appropriate physical representations can usually lead to the meaningful development of the two generalizations needed to solve *all* open sentences of these types. *In the missing addend situation, it is necessary to subtract the known addend from the sum to find the missing addend, to find the sum when both addends are given, the given addends are added.* Obviously, these generalizations could be taught by rote, but this is not the authors' intent. Rather, these principles should be meaningfully taught and developed.

A substantial number of multiplication and division open sentences not in direct form can also be systematically solved if several principles are known. The basis for these two principles is again the relationship between the two operations. If these operations are meaningfully related as suggested and detailed in a previous chapter, then children can grasp the idea that in open sentences such as 18 ÷ N 3, 18 ÷ 6 - N, 3 × N = 18, and N × 6 18, a product and a factor are known and the remaining factor is to be found. The principle needed to solve any sentence of this type, regardless of the size or type of numbers involved, is *Given the product and one factor, to find the missing factor, divide the given product by the known factor.* Sentences such as N ÷ 8 24 and 3 × 8 N are recognized by children as situations where the product is to be found and two factors are known. The principle appropriate for these conditions is, *If given two factors and the product is to be found, then simply multiply the one known factor by the other.*

In summary, children's ability to solve open sentences should be gradually developed. It should proceed from the informal to the more systematic, but not necessarily to the abstract in the primary grades. Many important open sentences involving the four basic operations can be systematically solved if several generalizations are meaningfully developed. The solution to other varieties of open sentences should be handled on an intuitive basis with plenty of opportunity allowed for solving them in a variety of informal ways. The systematic algebraic method of solving most open sentences should be deferred until the children can meaningfully operate on a more formal level.

other mathematical topics

Other mathematical topics belong in a comprehensive program. There are productive activities relating to logic, probability, statistics, and number theory that are quite appropriate for the primary grades. Many of the activities and problem situations in chapters 4 and 5 relate to some of these mathematical topics. As was shown in those chapters, approaches to these topics must be informal for young children.

logic

The most basic part of logic is recognizing whether a sentence that is a statement is true or false. (Sentences can be open and neither true nor false, e.g., $n + 3 = 7$. Sentences that can be classified as true or false are called *statements*.) Such statements often arise in classifying objects, in studying relations, and in working with number sentences.

The following activities focus on true and false statements.

► 1. Select a block from a set of attribute blocks, keeping it hidden from the children. Ask some child to make a statement about the hidden block. Prior to showing the block, record each child's attempt on the chalkboard and analyze it to see if it can be determined to be either true or false when the block is shown. That is, was a *statement* made or not? The sentences that are not statements should be erased or marked out. After this is completed, show the block to the children and help them judge each of the statements true or false.

► 2. Print simple statements on strips of tagboard, one statement on a strip. For example: Place two or three strips at a time on a display board. Have children say "true" or "false" as someone points to each strip.

John is taller than his father.

Sue has black hair.

2 + 3 = 5.

David is six years old.

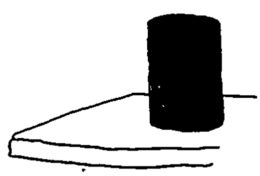
Jack has a sister.

12 - 3 = 10.

A blue block is on the table.

A square block is on the table.

Variations. Point to a false statement. Ask, "How can you change this to make it true?" Encourage replies such as "The block is *not* blue," "The block is red," "The block is *not* square," and "The block is round."



Use simple statements where some of the statements involve the word *not*. Occasionally, to emphasize the function of this word, statements such as "Mike is absent today" and "Mike is not absent today" should both be put on the display board at the same time.

Make simple statements using the words *all*, *there is*, and *none*. (*All* means every single one; *there is* means at least one; *none* means not a single one.)

► 3. Join two statements from the preceding activity with *and*. See if the resulting statement is true. (A statement with *and* is true only if both parts are true.)

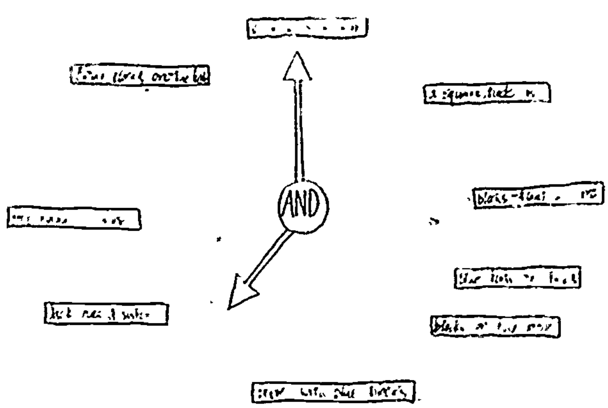
A red block is on the table. — *True*

A square block is on the table. — *False*

A red block is on the table. *and* A square block is on the table

False

Variation. Make a large clockface on the floor and place one statement in each hour's position. (The statements from the preceding activity could be used for this purpose.) Have the class decide orally the truth or falsity of each statement. On completion of this task, place a tagboard strip labeled "and" at the center of the clock. Move the hands of the clock so that each hand points to a statement. Have someone read the compound statement involving the connective word *and*. Then ask one child if the statement is true or false. If he answers correctly, he gets to reset the hands and choose from the volunteers another child, who must then judge the truth value of the new compound statement. Continue the game in this manner.



A variation with the clock is to label the center with the word ***not*** and use only one hand.

► 4. Begin with a false compound sentence containing *and*. Make one part of the compound sentence true. Ask, "Is the other part true or false?" (False)

The block is blue. *and* The block is round.

False

The block is blue. — *?*

The block is round. — *True*

Variations could include making the original compound sentence true and making one part true. Another variation would begin with a false compound statement with one part false, in this case you cannot tell for sure if the other part is true or false.

Other activities with logic involve the use of *or* and *if-then.* Such ideas are more difficult, and although they may be begun informally in the early grades, their full understanding is developed later.

At the primary level, the main objective with respect to the logical ideas must be intuitive experience. This experience should be based as far as possible on real-world situations with which the children can identify.

probability

Although there is substantial agreement that probability concepts should be included in the school curriculum, there is also some disagreement on when these ideas should be taught. These differences concern, in part, the age at which a child is sufficiently developed intellectually to comprehend concepts associated with chance and randomness. Concomitant with the consideration of this issue, attention must be given to the fact that there are large differences between the children in a given class as well as between classes. Teacher judgment and flexibility, therefore, are required if any portion of the sequence of probability activities suggested here is to be incorporated into classroom instruction.

An objective of an early lesson on probability should concern the ideas of certainty and uncertainty. Many everyday events are predictable, whereas others are uncertain. That the sun will rise tomorrow, that there will be no

school on Saturday, or that there will be cars on the streets tomorrow are events about which we can be quite certain. That it will be cloudy tomorrow or that there will be no absences from class tomorrow, however, are not as predictable; they are more uncertain. Children can be helped to gain an intuitive feel for these ideas by discussing such statements and questions.

► 1. On a display board, label one column "Certain" and another column 'Uncertain." Present cards one at a time to the class with such statements on them as these: "Tomorrow we will have two recesses." "It will be a clear day tomorrow." "Fifty airplanes will fly over the school today." "If a carton of eggs is dropped, all the eggs will break." "Two children will be absent tomorrow." As each card is presented, it should be discussed and placed in the appropriate column on the display board. Use of words such as *for sure, always, probably not, not very likely, impossible,* and *never,* to name a few, will certainly be useful in discussing the certainty or uncertainty of the event listed on each card. Note that under the "Certain" column, events that are both *very likely to happen* and *very likely not to happen* will be included.

Variation. Divide the display board into three columns representing "certain to happen," "uncertain," and "certain not to happen" categories. Repeat the previous activity, placing each card in the appropriate column after class discussion.

These activities develop the ideas of certainty and uncertainty and a working vocabulary for discussing situations that involve uncertainty in various degrees.

► 2. Use guessing games involving situations like the following:

> I am thinking of a whole number between 4 and 8. What number am I thinking of?
>
> I have a square block, a round block, and a triangular block in this bag. I am going to take one block out of the bag. What shape will it be?

In each of these situations possible outcomes may be listed. Questions concerning the certainty of a guess may be discussed. After one incorrect guess has been made, the previous questions can fruitfully be discussed again.

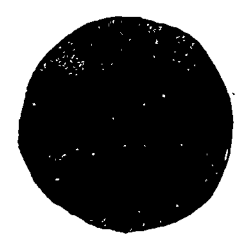
Variation. Use three blocks of the same shape, but vary the color of the blocks. To make a much more complex activity, put two square blocks of different colors and one block of each of the other two shapes into a sack and repeat the activity.

These activities introduce the idea of outcomes and provide additional practice with the ideas of certainty and uncertainty.

► 3. Use something such as a coin toss where there are just two outcomes. Ask questions such as these:

> When the coin is tossed, will the side that lands up always be either a head or a tail? (Yes)
>
> Will a head come up on the next toss? (Can't tell for sure)
>
> Can you be certain which side will come up the next time the coin is tossed? (No)

► 4. Make a spinner having more than two outcomes. "Suppose I spin this spinner; to what number could the spinner point? Is there another number to which the spinner could point? Any other? Could someone write down each of the numbers to which the spinner could point? How many different numbers would be on the list? Will the spinner land on a 3 next time? . . ."



Variations. To introduce the likelihood of various events, which is a cornerstone of all probability theory, make two spinners, one of them half red and half blue and the other three-fourths red and one-fourth blue. Explain that in his turn at some game a child moves forward one space if his spinner stops on the blue and moves zero spaces forward if his spinner stops on the red. "Which spinner would you choose?" Children very quickly realize that if both spinners are used, the game is not fair to whoever has the second spinner and that certain outcomes are more likely to occur on certain spinners.

Have two children use a spinner divided into five congruent sections, numbered one through five. Let one child receive a counter when he spins and lands on an odd number and the other child receive a counter when he spins and lands on an even number. Discuss possible outcomes for each child and favorable outcomes for each child. Discuss the fairness of the game in terms of who is most likely to win and why. Vary the activity with spinners that are not divided into congruent sections, such as those illustrated.

These kinds of games foster the development of most of the probabilistic notions previously mentioned if they are accompanied by probing questions and meaningful discussions.

► 4 Put one red block and one blue block into a cloth bag. Have teams take turns drawing a block from the bag. A point is scored each time a red block is drawn. Allow twenty draws by each team for a game. Follow up this game with a related, but more complex, game. This time use two bags, each containing one red and one blue block. On a given turn a block must be drawn from each bag. A team scores a point on that turn only if *both* blocks are red. Comparing the size of the scores between the two games leads to interesting discussions about the likelihood of two outcomes *both* happening.

Variation. Play the two-bag game, but let a point be scored whenever a red block is drawn from either bag. (A point is scored when a red block is drawn from bag 1 or from bag 2 or from both bags in a given turn.)

These activities bring out the fact that it is less likely that two events will both happen than that one of the two will happen.

The extent to which very many young children can go beyond the activities suggested here is questionable. Therefore, no further concepts or activities are discussed. For additional suggestions or ways to extend the ideas listed here, see the References at the end of the chapter.

statistics

Contrary to the reservations expressed concerning the teaching of probability, there is a great deal of evidence and support for the feasibility of introducing basic notions of statistics early in a child's school experience. These notions center on descriptive statistics and involve the processes of counting, collecting, measuring, classifying, comparing, recording, displaying, ordering, interpreting, and representing. Each of these processes is important, and they are all highly interrelated. These facts should be reflected in instruction—that is, each of these processes needs to be given explicit attention, and activities should be formulated in such a way that many of the processes are used in the activity. Clearly there are many appropriate projects and activities in which children can take an active part and in so doing develop the skills and concepts associated with the preceding list of processes. Rather than list suggestions concerning each process, some aspects of pictorial and graphical representation that incorporate many of these processes will be explored.

Graphical and pictorial representation are important means of communication and provide interesting ways of developing an abundance of mathematical concepts and skills. In order for a development of this topic to be most productive (in the sense of maximizing learning and interest), care should be taken to insure that the lessons proceed from the simple to the more complex and from the concrete to the less concrete. Giving children the opportunity to make choices wherever possible serves as a good motivational influence. Some *guidelines* for producing activities that develop representational skills follow. These guidelines are not to be interpreted as rigid rules.

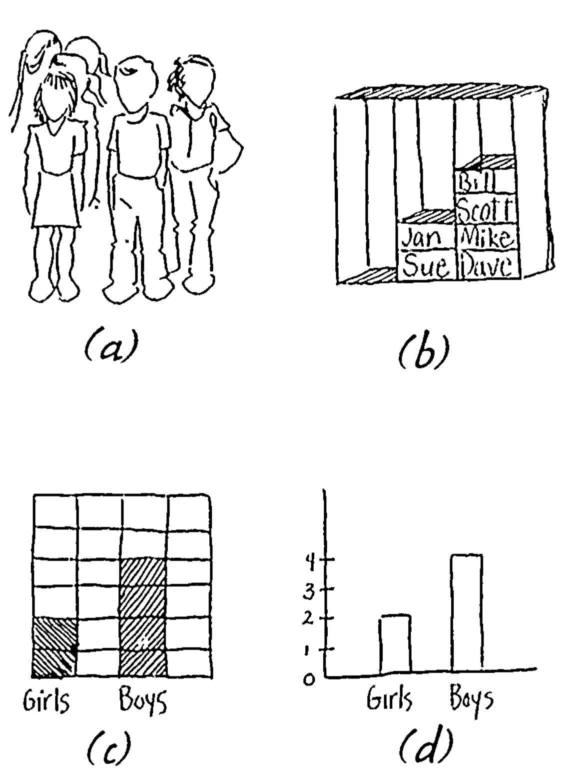
1. Focus initial work with pictorial representation on discrete variables (variables that involve only whole numbers). For example, have the children work with such interesting examples as the number of siblings, the number of children having lunch at school each day, the number of cars passing by the school during recess, the number of children absent each day, the number of birds seen on the way to school, the number of days in each month, and the number of pets each child owns. Later, continuous data should be examined. This could include such things as measuring (1) the height of a bean plant daily, (2) each child's height and weight, (3) inches of rainfall a week, (4) the perimeter of leaves, or (5) the area of classrooms.

2. Keep the number of categories small at first; in fact, the first representation of data should involve only two categories: present or absent, boy or girl, and similar dichotomies. Later, situations where a larger number of categories is required would be examined. Examples could include each child's favorite day of the week or the month of each child's birthday.

3. Avoid assigning more than one value to a category in early work with discrete variables. For example, recording the results of activities like rolling a die ten times should be represented initially by six categories, one for each possible outcome. Later, when it is clear that children have a good grasp of this one-value-to-a-category situation, record the results of throwing a die again, but this time record odd outcomes (1, 3, and 5) in one category (or column) and even outcomes (2, 4, and 6) in another category. Here a three-values-to-a-category situation is represented. Notice that assigning values to a category is generally required when representing data involving continuous variables.

4. Focus introductory work with representation on the one-to-one correspondence between the things being represented and the means of representation in the pictorial diagram. For instance, if the composition of a classroom is being represented according to sex, then each child must be represented in the diagram. Give each child a concrete object (e.g., block, matchbox, etc.) and use these objects in making the representation. This emphasizes the one-to-one correspondence between the objects in the representation and the children in the classroom. The specific representation might involve two rows of blocks or two sets of washers placed on two pegs. Label one row (or peg) "boys" and the other "girls."

5. Proceed in the representation from the concrete to the more abstract. Some definite stages in this progression are shown.



In (*a*) the composition of the classroom is communicated by a systematic lining up of the children in the class with boys in one row and girls in the other. In (*b*) each child places a matchbox with his name on it in the appropriate category. In (*c*) each child shades in a square in the appropriate column to represent himself on the grid type of diagram. In (*d*) the number of boys and the number of girls are first counted, and then a bar is used to represent the number in each category. Note that in this type of representation the scaling on the vertical axis plays a very important role.

6 Keep in mind the following important ideas associated with making graphs (pictorial representations):

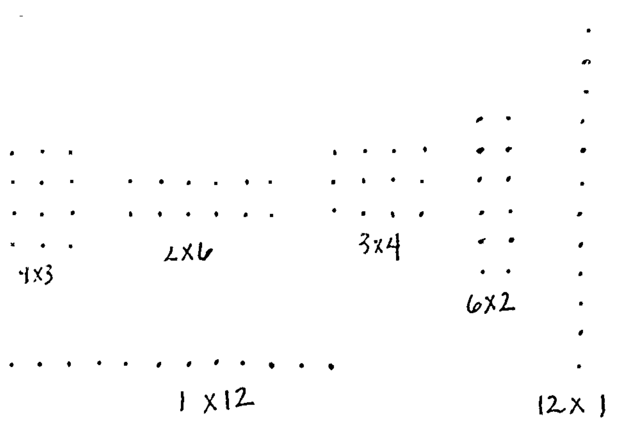
a) Labeling and interpreting graphs is important.
b) Using a common baseline and equal units is necessary if graphs are to communicate information clearly.
c) The order in which the categories appear on the baseline of the graph is sometimes important (i.e., when the baseline is scaled).

The number of opportunities for doing exciting, interesting and meaningful things with pictorial representation is practically unlimited. It is hoped that this topic will become a very popular one in the elementary school mathematics program.

number theory

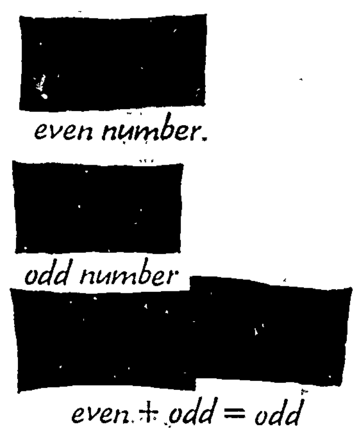
Few areas of mathematics offer more opportunities for children to make interesting discoveries than number theory. Fortunately, the basic tools needed to make these discoveries are neither elaborate nor highly abstract, and interesting activities involving concrete materials can easily be generated for teaching the basic concepts.

► 1. Have the children construct as many arrays as they can for a given number of objects. For instance, the following arrays could be formed for a set containing twelve objects:

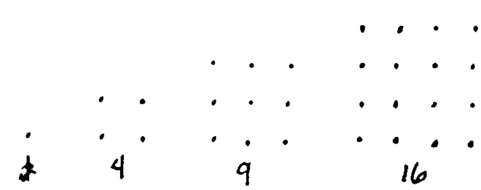


Variations. Ask questions like these: "Find a number for which you can build exactly four arrays." (6 is one; 8 is another) "For what number(s) can you build only two arrays?" (Any prime number: 2, 3, 5, 7, 11, . . .) "Are there certain kinds of arrays that can be constructed for every number?" (One-row or one-column array) "Which numbers have more than two ways to make an array?" (Composite numbers: 4, 6, 8, 9, 10, . . .) "Can an array having six rows be formed from a set of twenty-four objects?" (Yes) "Which numbers can be arranged to form a two-row array?" (Even numbers: 2, 4, 6, 8, . . .) "Which can't be put in a two-row array?" (Odd numbers: 1, 3, 5, 7, . . .)

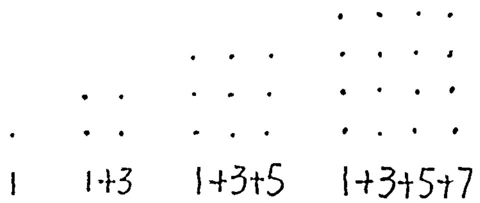
List the odd and even numbers. Choose two even numbers and ask, "Is the sum even or odd?" (Even) Repeat for "odd plus odd" (even) and "odd plus even" (odd). Verify by using arrays:



► 2. Use arrays to show square numbers. "For what numbers can a square array be formed, that is, an array having the same number of rows and columns?" (1, 4, 9, 16, 25, 36, 49, . . .)

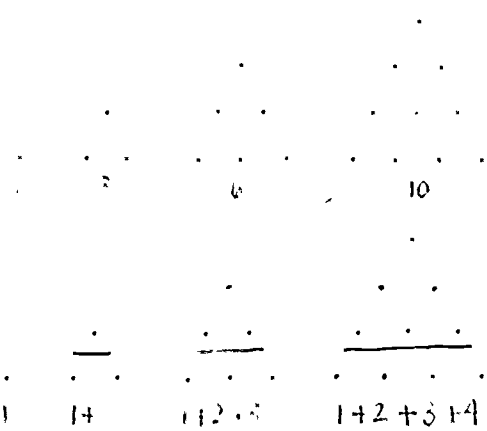


Variations. Ask, "How many do you add to each square to get the next one?" This shows the relation between square numbers and the sum of odd numbers.



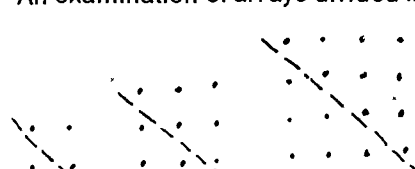
▸ 3. Make dots to show triangular numbers.

Variations. Relate triangular numbers to sums of whole numbers. Illustrate some of these numbers on the chalkboard and ask, "What will the next one be? Can you relate triangular numbers to sums of consecutive whole numbers?



What must be added to 1 to get the next triangular number? To 3 to get the next triangular number? To 6 to get the next triangular number?" Thinking of the first four triangular numbers in the following way makes seeing the relationship easy: 1 = 1, 3 = 1 + 2, 6 = 1 + 2 + 3, 10 = 1 + 2 + 3 + 4. The *n*th triangular number is clearly the sum of the whole numbers 1, 2, 3, . . ., *n*.

Another variation might be to relate triangular numbers to square numbers. "How are triangular numbers related to square numbers? Are there some triangular numbers that can be added together to form a square number?" An examination of arrays divided like this



should verify their guess that any square number is the sum of two consecutive triangular numbers. Children can often be guided to make this discovery and then intuitively verify it by demonstrations with counters similar to those just illustrated.

integers

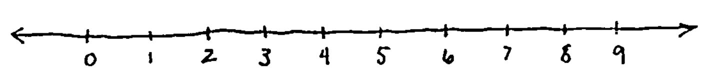
The idea of an integer is considerably more complex than that of a cardinal number. The set of integers consists of all the whole numbers and the negatives of the whole numbers. The integers can be denoted using set notation in this way: {. . ., ⁻3, ⁻2, ⁻1, 0, 1, 2, 3, . . .}. Long before a child formally studies this set of numbers, he has been exposed to many situations that can be considered representations or uses of integers. Focusing attention on such situations in the primary grades builds a foundation on which a more formal and detailed study can later be based.

Many physical situations familiar to children can be described using the integers:

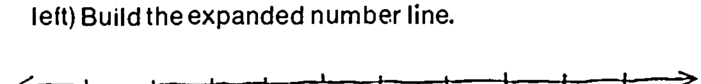
5 in the hole $^-5$
24 degrees below zero $^-24$
gain of 6 yards $^+6$
a hundred dollars in the red $^-100$

Each of the following activities, designed for the primary grades, is structured in a way that allows for "hands on" experience by the children and should be used in the classroom in this way.

▸ 1. When the child understands and can use the number line as shown, ask, "What kind



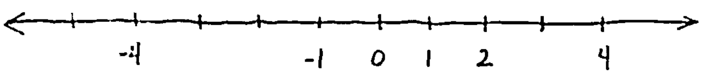
of numbers go to the left of zero?" (Negative, or "in the hole" numbers) "How could you show a score of five in the hole?" (5 units to the left) "A temperature of three below zero?" (3 to the left) Build the expanded number line.



Emphasize the unit distance between points.

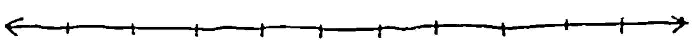
▸ 2. When playing games where both negative and positive scores are possible, such as the beanbag game, use the expanded number line in keeping score. If the number line is on the chalkboard, represent each child's score by a different mark.

▸ 3. Given a partially labeled number line such as the one shown, mark a specific point



on it, and ask for volunteers to write the correct "number" by the point.

▸ 4. Begin with an unlabeled number line like this:



Have a student come forward and label one of the marked points zero. Then have him call out an integer and select a volunteer to label the point associated with this integer. Continue the labeling of the number line in the same manner

▸ 5. Stage a weather show where a list of cities and their low temperatures are reported. Have some child display each city and its temperature as they are read. Ask questions about which city had the coldest temperature. Then ask comparison questions. Was Madison colder than Minneapolis? How much colder?

▸ 6. Ask questions about a rabbit who hops forward and backward on a number line and always starts hopping from zero. Emphasize that on the one hand if the rabbit jumps three hops forward, he will land on 3, and that on the other hand if he jumps two hops backward, he will land on 2. Then begin asking questions about where the rabbit will land if he jumps so many hops forward and then so many backward. Have the children move a rabbit on a number line to help them.

▸ 7. Use games where a postman delivers bills and checks. Represent, for example, the delivery of three bills by the integer -3. A delivery of four checks would then be associated with · 4. Go a step further to represent the delivery of two bills and one check by 1.

Teachers should feel free to experiment with any of these activities for the integers as long as intuitive experience, not mastery or abstraction, is the goal. The emphasis on integers in the grades should be kept under control, however, since the operations with these numbers can cause real difficulty, as any teacher of first-year algebra can testify.

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directions of curricular change

12

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THE glimpse of a different concept of preschool and kindergarten education for all has given impetus to change in the mathematics education of young children. During the last ten years, the sense of urgency that accompanied the belief in, and desire for, compensatory education as a treatment for societal ills has led to the rise of unprecedented resources for the development and implementation of curricular materials for the preschool and the early elementary school child.

The features of the mathematical materials developed within this context will be examined in this chapter. The quantity of materials alone argues against exhaustive cataloging of all developmental efforts or giving a complete and thorough description of more than a few programs. Consequently, the discussion of materials is focused on those program features that provide examples of some of the distinctive directions that have emerged in the development process.

This development has been multidirectional. At the preschool level the new emphasis on cognitive objectives has encompassed mathematical concepts and skills (Evans 1971; Kamii 1971). But few traditions existed for the design of materials for, or the teaching of mathematics to, the young child. Demand for programs and materials rapidly exceeded the capabilities of the available specialists in mathematics education for the very young. This has resulted in materials and methods being created by a large number of people with widely varying backgrounds. The programs developed represent

the differing philosophical, psychological, and pedagogical briefs held by this divergent group.

The range of differences in the *materials* developed for grades 1 through 3 is almost as broad as that of the programs. Stemming from a variety of causes, the differences represent reponses to dissatisfaction with materials created in the post-Sputnik reform period and reflect the influence of the stress on compensatory education. Clearly, the potential of the precursive experience in preschool has caused some authorities to reconsider tne traditional content dimensions of the curriculum.

Four categories of curricular development provide a classification scheme for the discussion that follows: activity learning, language development. mathematics, and compensatory education. In addition to these major directions of curricular development, a fifth section describes directions not readily subject to classification but of potential influence.

Programs are not necessarily classified according to their primary characteristics. Rather, the classification is made in the programs' exemplifying particular directions of curricular change. It should also be noted that many programs share characteristics with programs that exemplify other directions of change.

activity learning

Activity learning has a long, rich tradition in the teaching of arithmetic to young children. The trend toward activity-based instruction represents a return to a familiar tradition of using motor activities as a base for developing cognitive skills and abilities.

Three examples of activity-based instructional programs are provided in this section. The first, that of Montessori, is directed at children of ages three through eight and is the only example of a preschool mathematics program with a tradition dating from the turn of this century The second, the Developing Mathematical Processes program, attempts to incorporate recent psychological findings into the design of an activity-based program. The third, the British Infant School, is described more accurately as a movement than as a program. It illustrates how activity-based instruction can provide a different emphasis on the content of mathematics.

montessori

No effort to describe preschool practices in mathematics education would be complete without paying tribute to the influences of Maria Montessori (1870-1952). Not only do Montessori schools provide a backlog of experience, but a number of increasingly popular sets of manipulative materials (including Cuisenaire rods and Dienes blocks) have equivalent or nearly equivalent forerunners in Montessori materials. With the recent attention given to the work of Piaget, it is also in order to point out that a number of Maria Montessori's assertions about the nature of the child parallel those of Piaget (Sheenan 1969). For example. both Montessori and Piaget "emphasize the normative aspects of child behavior and development as opposed to the aspects of individual difference" (Elkind 1967, p. 536).

Several difficulties are involved when speaking in generalities about the Montessori approach to mathematics. In fact, some may consider it unfair, since the Montessori approach intends to deal with the whole child, not with a compartmentalized curriculum. Further difficulties in defining the mathematics program result from—

1. variance from school to school in the amount of play allowed;
2. variance from school to school in the amount of usage of classroom materials that would not be classified as "Montessori materials";
3. variance from school to school resulting from differences in Montessori teacher-training institutions.

However, the Montessori contribution to early childhood education is worthy enough of consideration to run the risk of some misrepresentation relative to any given Montessori school.

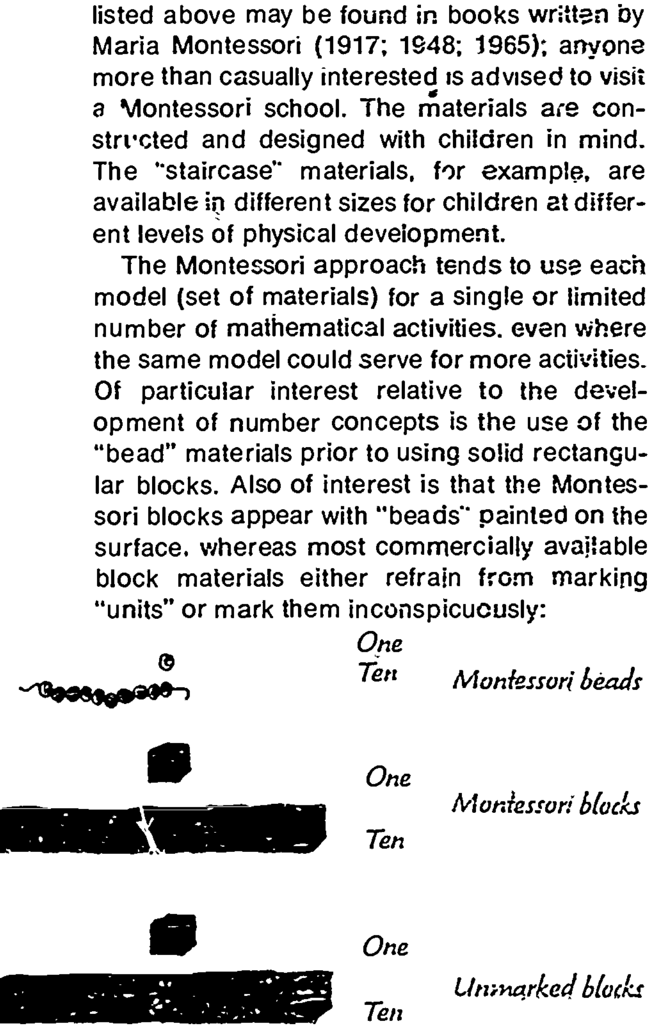
Materials common to most Montessori programs, listed in sequence according to their usage by pupils. include the following:

stairs (or number rods)
sandpaper numerals
spindle box
counters
colored bead material
bead frames
number frames
addition board for rods*
golden bead materials (base ten)
number (place value) cards
block material (base ten)
multibase bead materials*
multiplication board
abacus equivalent*
division board

*Place in sequence varies from school to school

Pictures or diagrams of most of the materials listed above may be found in books written by Maria Montessori (1917; 1948; 1965); anyone more than casually interested is advised to visit a Montessori school. The materials are constructed and designed with children in mind. The "staircase" materials, for example, are available in different sizes for children at different levels of physical development.

The Montessori approach tends to use each model (set of materials) for a single or limited number of mathematical activities. even where the same model could serve for more activities. Of particular interest relative to the development of number concepts is the use of the "bead" materials prior to using solid rectangular blocks. Also of interest is that the Montessori blocks appear with "beads" painted on the surface. whereas most commercially available block materials either refrain from marking "units" or mark them inconspicuously:



The bead materials afford the child the option of managing the learning tasks successfully without an appreciation for representing numbers by lengths; at the same time, the bead materials clearly allow for representations by lengths. The colored bead materials, as well as many commercially available block materials, allow for whatever payoff may come of associating numbers with colors.

The ease of transition to rational- and real-number concepts may be lost by the more explicitly discrete representations of number by beads. It is, perhaps, easier to perceive and deal with a 2 1/3-unit stick than with 2 1/3 beads. An attempt to determine an advantage or a handicap from using beads would suggest an investigation of questions like the following:

1. Do children perceive unmarked block materials as continuous, as discrete, or as either, depending on the task at hand?
2. If there are initial differences in perceptions, do the differences persist through the use of Montessori block materials?
3. If differences persist, do they appear to be a factor in success with subsequent learning tasks?

Perhaps more important to the program than the materials is the teaching methodology. Each set of materials is intended to be used in a tightly prescribed way. A child who uses materials according to the prescription is doing his "Montessori work"; a child who chooses to use the materials for purposes other than those intended is at "play." In some schools "play" is not permitted, and failure to use materials as intended results in their removal. In other schools a child at "play" may be asked to take the materials to a designated area in the room reserved for such activity. "Play" may or may not be overtly scorned, but in either event it is clearly second place to work.

The premium placed on "work" so defined may be, on the surface, puzzling in view of the claim by Montessori teachers that an important characteristic of the approach is the guarding of the child's "freedom." If the child is free, then clearly the meaning of freedom is something other than an absence of externally imposed values or a license to follow whims. Either there

is little freedom or, as is the case, the definition insists that feedom is gained by learning self-discipline relative to fixed values and tightly escribed activities. This latter definition is certainly not new. Plato would insist that a failure to control lower instincts is a sure path to slavery, Christianity insists that freedom is gained by submission.

A second possible source of confusion is the Montessori use of the word *discover*. Although Montessori teachers generally agree that the child "discovers" mathematics, the means of getting him from an initial demonstration of interest in a set of materials (by approach behavior) to the intended usage of the materials usually goes something like this:

1. The teacher asks the child if he would like to be able to work with the materials.
2. Given a yes, the teacher "names" the components or demonstrates the intended manipulation of the materials. Verbalization is kept to a minimum.
3. The child is asked to name indicated components or is asked to perform indicated subtasks to the total task.
4. The child is asked to identify components when given the name or is asked to perform the task.

There is a heavy dependence on the child's willingness and ability to imitate. Means like the above do not, however, evidence a strong commitment to the gestalt "aha."

"Individualization" in the mathematics program may be seen relative to pacing, point of entry with the materials, and one-to-one teacher/pupil interaction. Individualization as claimed by Montessori teachers does not imply the existence of a variety of acceptable ends or a variety of means to ends.

A key notion both to establishing the appropriate point of entry for a new pupil and to pacing is that of "readiness." In practice, readiness seems to mean simply approach behavior followed by an acceptable rate of advancement toward accomplishing the task. Beyond the intuitively easy decisions, there is no machinery for concluding that the child is ready for a given learning task other than observing his encounter with the task. Neither does there appear to be a formula for distinguishing between a child's being unwilling to follow the prescription and his being unable to follow it.

The importance of the notion of readiness to the Montessori approach derives from one of the Montessori claims. the best time to teach a pupil any given content is at the point of his being ready. Without some means of determining readiness external to encountering the learning task itself, this axiom reduces to saying that the best time to teach a pupil any given content is when the child is first able and willing to learn it. Beyond lacking research evidence that shows it to be valid, the statement is somewhat weak toward making the decision of "when" if any premium is to be placed on first experiences being successful experiences.

Among questions suggested by the above paragraphs are the following:

1. Is a heavy dependence on the child's ability and willingness to imitate appropriate to the teaching of mathematics?
2. Is the application of the Montessori definition of freedom a souce of security or a threat to the child? Is it an aid or a hindrance to sound and/or creative mathematical thinking?

developing mathematical processes

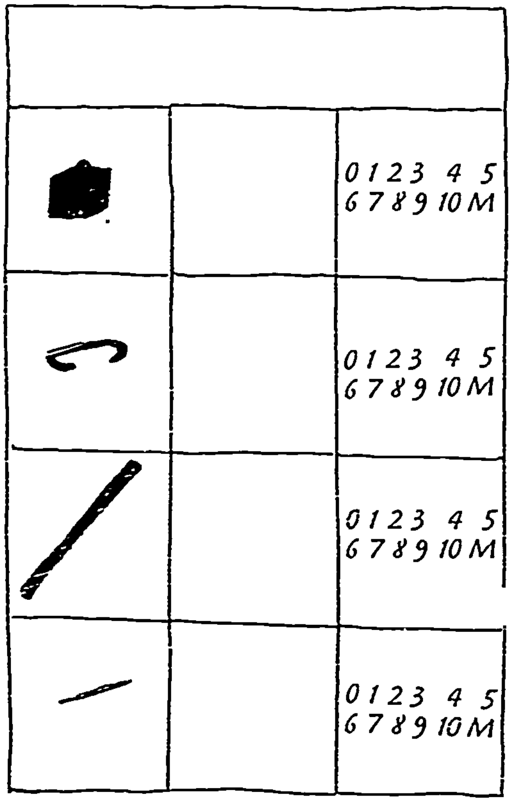
The Developing Mathematical Processes (DMP) program for grades 1 through 3, currently being developed, field tested, and evaluated by the Wisconsin Research and Development Center, offers an example of a tightly defined, sequential approach to activity-based learning (Center for Cognitive Learning 1971).

Each set of activities in the DMP materials includes a listing of "regular objectives" and "preparatory objectives." The regular objectives are those for which mastery is to be checked. These objectives are stated in behavioral terms, but they are not so given to precision in statement that they deny admitting certain goals as objectives. For example, one such objective reads, "Given a geometric figure constructed on a geoboard or from plastic or wooden objects, the child pictorially represents

that figure on specially prepared paper." Obviously a degree of subjectivity will be required in deciding if the child's picture is adequate: the child's familiarity with the geometric figure may be an important consideration, the degree of neatness the child usually brings to his work may be a factor in judging it, and some figures could likely be concocted to bring out errors. The preparatory objectives are those that can be pursued naturally in a given activity, that will be treated later, and that need not be checked for mastery until the later treatment.

The DMP materials may place an additional load on the teacher, since the required materials may not be readily available. In attempts to implement the program on a broader scale, this additional demand on the teacher may be a key factor to its acceptability. The introduction to the DMP materials encourages the teacher to develop a "pack rat" habit, cautions that for any given teacher some activities may "fail miserably," acknowledges that implementing the program is "a difficult and time-consuming task," and warns that "it may take several months with this program before (the teacher) and students are comfortable with it." A few sentences (admittedly selected for dramatic effect) from the directions given to teachers for prelesson preparations may serve to make the point: "For every cup of flour add one cup of salt and one tablespoon of cornstarch. The water should be added slowly as you stir. . . . Prepare cans by making holes of different cubes, frizzies, etc. . . . Using masking tape, make the letters A, M, E, F, H, K, N, W, and Z on the floor. . . . Borrow five balance beams from another teacher."

The flavor of the materials may be captured by an outline of the early treatment of linear measure. At the kindergarten level, the experiences (listed in order) include comparing lengths in pairs, "equalizing" lengths, ordering more than two objects by length, representing lengths by physical means, and representing lengths pictorially. At the primary level (K-2), children are given experiences in measuring lengths with arbitrary units. Suggestions to teachers include using toothpicks, drinking straws, Lots-a-Links, and Unifix Cubes. The worksheet accompanying this activity is the following:



In the space at the top of the worksheet, the child draws a picture of the object being measured. The object pictured in the column on the left is used to measure the object drawn in the space at the top; the "M" is to be circled for "more than ten." The middle column is for tally marks.

The instruction sheet for the teacher suggests that the children be encouraged to guess the number of units to be needed before they measure. This may serve two functions: First, it helps focus the children's attention on the goal at hand, second, over the long haul, it may develop their ability to detect unreasonable "answers" to mathematical problems. Standard units of linear measure ("inch, centimeter, etc.") are introduced after experiences with aribtrary units.

A three-step sequence for concept development is employed by the DMP materials. This sequence begins with physical experience, moves on to pictorial experience, and

culminates with a use of more abstract symbolism Insistence on precise vocabulary is delayed until the probability of the child's using the technical term before he has learned the concept has been minimized.

Provisions for individualization allow for alternate means to mathematical ends but not for alternate ends. Refraining from a wider allowance for individualization is justified on the following grounds: First, the program developers take seriously those implications of Piagetian tenets and research on the nature of the child that emphasize the normative aspects of child behavior and development. Second, present knowledge about individual differences is judged inadequate to make provisions for a variety of cognitive styles.

Viewing the wider range of activities carried on by the Wisconsin Research and Development Center, one could argue that the DMP materials are largely determined by prior research. Although it is too early to determine if such care pays off, it is comforting to find examples of curricular development that go beyond the practice of merely appealing to supportive tidbits from learning theory to justify a product after the fact. Findings on conservation and numerousness, for example, have been heeded in making curricular decisions.

One question suggested by the DMP materials is whether they embody an attempt to accelerate cognitive stage development. The hint that stage acceleration may be intended by the materials derives from the inclusion of the "preparatory objectives." If the function of such preparatory objectives is to provide what Ausubel would call "advance organizers," then it would seem reasonable to require confirmation of their being established. Whether such acceleration is possible has been an object for controversy and will remain an undecided issue unless someone proves convincingly that it is possible. Another question, perhaps more important, is whether or not such acceleration is desirable. For example, six-month-old kittens are winners over six-month-old children in a number of stage-related Piagetian tasks. Such observations at least suggest that the capabilities enjoyed by a later developmental state may be determined to some degree by the number of experiences encountered at the earlier stage. If so, stage acceleration certainly holds potential perils.

The several references to Piaget in describing the DMP materials were not meant to assert that the program is an example of one totally prescribed by Piaget's developmental psychology. Several of the programs that use Cuisenaire rods as the exclusive model for early training in mathematics also claim Piaget. In contrast to such, the DMP materials use a variety of models. A case can also be made for parallels between Montessori and Piaget on the appropriate treatment of the child. Again, the DMP materials and the Montessori approach differ regarding the provision for alternate means to a given end. In short, if learning theory as known today is prescriptive, many of its prescriptions appear to be somewhat vague. This is not to say that they are unworthy of being heeded; rather, it is to question the nature of such prescriptions.

A last, somewhat obvious, question raised by the materials is whether or not the elementary school teacher can be expected to accept the demands of using the materials.

british infant school movement

Informal classroom procedures are used in approximately one-third of the infant schools (ages 5 to 8) in Britain (Featherstone 1967). No single group or agency is responsible for generating and implementing the approach; rather, a variety of people in many settings have developed many techniques over a period of years. The Plowden report (Central Advisory Council for Education 1967), which described the state of British education in 1967 and proposed a guideline for reform, captures the essence of the attempts at reform. Because of the wide variation from school to school, the phrase *British Infant School* (BIS) is used as a generic label for the movement.

The BIS movement is more of a philosophy than a specific curricular development with scope and sequence. The BIS movement has been perceived, perhaps inappropriately, as a means to deal with "dehumanization" and

"relevance." As such, it has possessed a fascination for both the lay and the professional press on this side of the Atlantic and encouraged emulation.

The BIS program has three curricular thrusts that contribute to the strength and uniqueness of the approach. First, the activity orientation embeds mathematics in other subject matter contexts and provides an opportunity to use the skills and concepts of other subjects. Science and mathematics are unified as the child studies his environment. The conditions imposed by the individual and small-group instructional format require children to be somewhat self-directive in following written directions and require the use of reading skills. Children typically keep a diary of their activities. Representative activities can be found in Edith Biggs's *Mathematics for Younger Children* (1971), the Nuffield Mathematics Project materials (1967, 1969 [a] and [b]), and Harold Fletcher's teacher resource books (1970, 1971 [a] and [b]). The impact of these activities on children's reading and writing skills and on the learning of science may be of greater importance than the nature of the mathematics learned. Clearly longitudinal research can provide only information relative to this question of values, the answers are to be determined on other grounds.

The second curricular thrust, the treatment of geometric concepts, transcends the naming activities common to many American programs. Attention is given to developing visual-perception skills. The Nuffield *Environmental Geometry* (1969 [a]) uses the child's environment to help the child look for geometric regularity and constancy. Children experiment with the strength of various shapes not only in the plane by comparing the triangle to other polygons but also in three-space by comparing the support provided by cylinders, prisms, tetrahedra, and so forth. This geometrical experimentation is supplemented by measurement activities of a more scientific than mathematical character. These activities provide a setting for using computational skills.

A third feature of the BIS materials is the stress on collecting and analyzing data. The BIS materials are reminiscent of the recommendations of the Progressive Education Association's chapters in *Mathematics in General Education* (1940) concerning data and the function concept. BIS materials involve children with coordinate systems that are used to graph "nice" linear relationships stemming from problems concerning milk money, measurement, and a variety of other down-to-earth contexts (Nuffield Mathematics Project 1969 [b]). The materials do focus on the eventual experience of children with algebra, but the immediate payoff of the stress on the function concept is the graphical interpretation of data.

The preceding description of BIS material is content oriented. It is based on materials that do not describe a complete mathematics program but rather provide examples of the type and spirit of activities. The nature of the approach entails more freedom for both teacher and child. It seldom features activities by a whole, intact class. Children work in small groups and by themselves at different learning stations for several content areas. The teacher must keep careful records of an individual's progress to assure that mathematics is not favored to the exclusion of reading or geometric concepts to the exclusion of arithmetic. The application of the Montessorian idea of having "family" groupings of children of at least three age levels in one class and the fact that school entry is not just in the fall further confound the record keeping. Techniques for coping with these problems are discussed in Palmer's *Space, Time and Grouping* (1971).

Corresponding to the complexity of administration is the payoff of not needing a set of materials for each laboratory activity for each child. Only a limited number of children use a set of materials at a given time. Consequently, the BIS movement uses activity learning to a much greater extent than that allowed by typical American classroom practices.

A current, common assumption in the United States is that features of the BIS program can be lifted intact from one cultural setting and one school context and applied in another. Through premature and thoughtless misapplication, however, several characteristics of the BIS mathematics program may be sacrificed and lost. British schools, unlike those of the United States, have a tradition of early childhood education. Government inspector-advi-

sors and headmasters have a different style of operation. British school personnel use psychology—particularly developmental psychology—differently from their counterparts in the United States (Fogelman 1970). British children traditionally begin cognitive learning tasks earlier than American children. These facts provide a different context for implementing the BIS program (Elking 1967). More research in comparative education is needed before wholesale application of the BIS program is begun in another setting.

The preceding discussion portrays the spirit of the BIS approach. The philosophic base of the program is radically different from some of the rigidly structured materials found in the United States. Do children participating in such programs differ significantly in problem-solving skills and approaches? Do they differ in their application of mathematics?

Clearly the BIS approach places distinctively different responsibilities on the shoulders of teachers and demands considerable skill from them in designing a comprehensive, thorough mathematical experience for children. If the teacher is skilled and knowledgeable, the child could have a rich and rewarding experience, if not, not.

language development

Talk is a critical variable in teaching mathematics. Limitations on communication are one sort of pedagogical problem. An example of these limitations is found in the second-language problems of Mexican-American children. Their problems led the Southwest Educational Development Laboratory (n. d.) to develop bilingual tapes and materials for the implementation in grade 1 of the Individually Prescribed Instruction mathematics program, whose content and mode of instruction are independent of the language variable. Within this section are described two programs that were developed around different assumptions concerning the role of language. Rather than treating the limitations imposed by language or the lack of it, these programs were designed to use the characteristics of verbal interaction as a supporting, integral component of teaching methodology.

The DISTAR program uses language as a vehicle to help students habituate to mathematical responses. The Interdependent Learning Model program uses games extensively as a vehicle of instruction. The games are designed to take pedagogical advantage of the verbal interaction implicit in the gaming.

distar

The DISTAR arithmetic program, created by Siegfried Engelmann and Doug Carnine, is one of the more unique approaches to mathematics for the young child (Engelmann and Carnine 1970). The uniqueness stems partly from the mathematical content but most particularly from the instructional methodology. The instructional procedures detailed in the *Teacher's Guide* are built on philosophical and psychological foundations markedly different from those of the currently popular laboratory and British Infant School approaches. However, several learning difficulties of young children in arithmetic are identified, and strategies for coping with these difficulties are described. The teacher's precise control of her language is an important component of the instructional strategy.

The DISTAR arithmetic program evolved from the compensatory preschool education program developed by Carl Bereiter and Siegfried Engelmann at the University of Illinois in the mid-1960s. Features of this compensatory education program establish a basis for understanding the DISTAR Arithmetic Instructional System. The first task in planning was the identification of a limited set of objectives that the child should attain if success were to be expected in grade 1. Most of these objectives were oriented toward the acquisition of verbal skills. The second part of instructional planning was a search for "a practical way of getting all children to learning the specified content" (Bereiter 1967). The methods that evolved for the

single-minded pursuit of the limited set of objectives have created controversy in the community of preschool educators, particularly those subscribing to the whole child and the child development points of view. Beyond the scope of this book, the reader may turn to Bereiter and Engelmann (1967), Hodges (1966), and Hymes (1966) to examine the issues involved.

The instructional approach of the DISTAR arithmetic exhibits this same single-mindedness. Based on an extensive task analysis, such as that described in Engelmann's *Conceptual Learning* (1969), an efficient chain of skills and concepts leading to the content of the typical grade 2 arithmetic program was constructed. It consists of approximately four hundred lesson formats progressing slowly from matching tasks through addition and subtraction of numbers named by two digits and multiplication. Negative numbers are considered, but few geometrical concepts are developed. The rate of the presentation of the symbols of arithmetic indicates the pace of DISTAR. 150 formats must be accomplished before the symbols for the numerals through ten, addition, equality, and subtraction have been introduced.

DISTAR shares many of the features used in current approaches to the teaching of language. Pattern drill and practice procedures typical of the teaching of foreign language (based on behavior modification) are among those methods used in teaching mathematics. The hierarchy of the skills and concepts of arithmetic provides an efficient structure for the language-oriented teaching methodology. Arithmetic is a special form of language. The child develops this language by small, incremental steps.

Counting provides the basis for developing most concepts. Developed from matching experiences, the counting activities progress through two types of activities, labeled with the words *rote* and *rational*. The rote activities do not relate number to objects or events but are language drills. Two types of rote counting are critical to the later developments of addition and subtraction. The first type emphasizes counting *to* a number. The format of the introductory lesson of the rote counting sequence taken from the *Teacher's Guide* both exhibits the emphasis of the counting activity and characterizes the pattern drill and practice approach of the entire program. The teacher is provided a script (in regular type) and directions (in italics). For example (p. 29, fig. 14):

Task Counting to a Number

Listen to me count. Tell me what I count to.

a. 1,2,3,4,5,6-6. I counted to six.
b. 1,2,3,4,5,6,7,8-8. I counted to . . . *Wait.* Eight.
c. 1,2,3,4,5,6,7-7. I counted to . . . *Wait.* Seven.
d. 1,2,3,4,5-5. I counted to . . . *Wait.* Five.
e. 1,2,3,4,5,6. I counted to . . . *Wait.* Yes, what did I do? *Wait.*
f. 1,2,3,4,5,6,7,8. I counted to . . . *Wait.* Yes, what did I do? *Wait.*
g. 1,2,3,4,5,6,7. I counted to . . . *Wait.* Yes, what did I do? *Wait.*
h. 1,2,3,4,5. I counted to . . . *Wait.* Yes, what did I do? *Wait.*

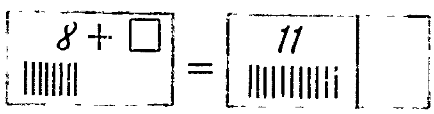
**To Correct* Repeat the counting sequence, saying the last number with emphasis. For example: 1,2,3,4,5,6-6. I counted to . . . Wait.*

The second type of rote counting emphasizes counting *from* a number *to* a number. Each of the remaining lesson formats on counting offers a variation of the stimuli associated with the counting response. The variations correspond to small increments in the skills of counting. Suggestions for eliciting group and individual responses are provided. These range from the control of pacing and cadence in oral counting to the consideration of appropriate reinforcement techniques. The suggestions are precise and often prescribe treatment for difficulties exhibited by learners.

Some teachers might find the tightly structured nature of the lesson formats uncomfortable and unrealistic. However, the careful task analysis of the increments in the hierarchy provide useful ideas and insights for those who do not accept the highly structured classroom approach. For example, many children do find difficulty in addition. Diagnosis may indicate an inability to conserve numerousness or a lack of understanding of the part-part-whole relationship. Or it may show difficulty with the skills of counting implicit in addition. For example, in solving the problem $9 + 4 \quad \square$, the child may first count the nine objects in the first set and then exhibit an almost allegiancelike propensity to one—he starts the counting process over

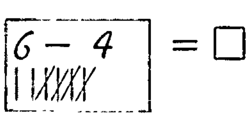
on moving to the set of four objects. The habituated process of counting *from* a number can provide linguistic support for the child having trouble associating the addition operation with the union of sets. The linguistic pattern provides but one entry into the child's establishing the desired conceptual connection.

Counting skills are the basic components in the chain of steps leading to the concept of equality and to the operations of arithmetic. Equality is identified as the single most important concept of the program. The counting here is rational instead of rote because objects are counted in the paradigms used in teaching the operations. The sets of objects to be counted are limited to "lines." In the paradigms used to teach the operations, the lines are in close proximity to the mathematical sentences used for the operations. Equality is considered as a charge to the child to produce the same number of lines on one side of the equal symbol as the other. For the sentence



the child counts and draws the eight lines associated with the 8 and the eleven lines associated with the 11. Using the skills previously established, he should (*a*) draw below the frame the lines that (*b*) correspond to the number necessary for counting from 8 to 11 and (*c*) place the correct numeral in the box. Early in the track of lessons for addition, the word *plus* is used both as a name for the operation and as a language cue for working the problem. In the problem above the child would "plus three lines." This distinctive verbal usage for operations pervades the program.

The paradigm for teaching subtraction is similar to that used for addition. Counting backward and crossing out vertical lines are associated with "minusing" as shown by the following:



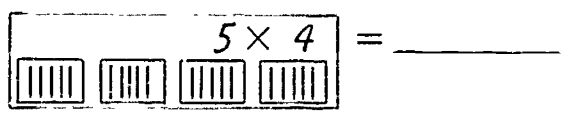
Steps in teaching are broken down into the following subactivities (Engelmann and Carnine 1970, p. 99):

1. Find the equal.
2. State the rule for equality.
3. Find the empty box.
4. Point to the side of the equal on which they start counting (the side without the frame).
5. Make the lines for the first group.
6. Minus the lines (crossout for the second group).
7. Indicate how many remain.
8. Indicate what numeral goes in the empty box.

Each of these steps corresponds to a previously acquired skill. The conjoining of the operation of subtraction and the act of crossing out provides the rationale on which the concept of negative number is developed.

Multiplication is also presented as a counting operation. The skills of group counting are carefully established before multiplication is encountered. The problem "5 × 4 = __" is decoded according to the following steps (p. 144):

1. The numeral 4 tells you to count four times. That means you must make four boxes.



2. The 5 × tells you that you are to count by groups of five. That means you are to put five lines in each box. . . .
3. You count the lines by fives.
4. Then you apply the equality rule. . . .
5. So the completed statement is

$$5 \times 4 = 20.$$

Children are to use a multiplication chart as a guide to counting groups. They are to touch the numerals above the appropriate groups as they count. If they are to count by groups of seven, then they are not to recite "seven, fourteen, twenty-one, . . ." but rather to touch those numerals as they count "one time, two times, three times, . . ." The *Teacher's Guide* states

that confusion arises for many children if they recite the numbers rather than the number of times they are supposed to count a group.

One of the distinctive features of the DISTAR arithmetic is the delay of emphasis on the concept of place value. A special symbol is used for the decades. Children do not encounter higher-decade addition and subtraction until late in the program. When children first encounter ten, the symbol 1^0 is used. Rather than fuss with the different role of the symbol 2 in the numerals 32 and 27, the children are to write 3^02 and 2^07.

In summary, Engelmann and Carnine have developed a rigid approach for teaching arithmetic in the early elementary grades. The lessons are based on a thorough analysis of the concepts and skills necessary for the next step in learning It would be easy to create a flow chart of ideas from the lessons. At many points in the program, shrewd insight into a difficulty of learning is exhibited by the nature of the small incremental step designed as treatment for the difficulty Linguistic and perceptual cues are carefully programmed into the instructional activities Some of these are extinguished or withdrawn later The *Teacher's Guide* provides several suggestions that teachers might find helpful for the handling of group activities. A testing program subsuming both diagnostic and achievement functions is a part of the DISTAR arithmetic. Used and evaluated in Head Start and Follow Through programs extensively. the program has been found successful in helping children acquire skills in arithmetic (Steely 1971).

The DISTAR program may have some severe limitations Several strong assumptions implicit to the DISTAR approach need to be tested. For example, DISTAR may be too efficient. Children have little opportunity to "monkey around" in a mathematical context. The role of mathematical play in developing intuition is not known precisely. Second, the perceptual models used for developing concepts are limited in type and in use. Sets of lines in close physical proximity to equations are used almost exclusively to establish operations. Do children need several models to increase the probability of transfer and application? Is proximity of the model and the symbols a critical variable in learning? Is the counting model used too exclusively? Finally, mathematics appears to be a language. The doing of mathematics becomes a problem of translation, of decoding and encoding. This is but one view of the nature of mathematics. Is this too limited a perspective to establish early in a child's experience?

interdependent learning model

The Interdependent Learning Model (ILM) uses a games approach to mathematical instruction. ILM is a Follow Through program designed by the Institute for Developmental Studies, directed by Martin Deutsch of New York University. Socialization is considered a primary variable in the learning of young children. Language and games are identified as two important components of the socialization of the young in any culture. The mathematical curriculum and methodology is designed to take advantage of the interactive nature of language in gaming. The word *transactional* is used to describe the value-laden, verbal negotiation between players that provides motivation, immediate feedback of the consequences of one's learning, and the socialization necessary to learning. The mathematical games are described as transactional (Gotkin 1971).

The purpose of the ILM program is to prepare the inner-city, disadvantaged child to cope with the demands of schooling in the later elementary and junior high school years. Learning and schooling are recognized as verbal and social activities. Thus, the instructional games are language games that provide children with the opportunity for social, verbal transactions as they apply the rules of the games. The game is a socializing activity that enjoys the advantage of not separating social and emotional development from cognitive learning and development.

The stress on language permeates the direct instruction portions of the program as well as the games. Accounting for less than 40 percent of the total program, the direct instruction materials are adapted from the work of Engelmann and Carnine. The scope and sequence of the curriculum is similar to the DISTAR pro-

gram but encompasses more traditional content areas of the early elementary school curriculum, such as geometry, telling time, Roman numerals, estimation, and measurement. Multiplication receives greater emphasis than in most typical pre-third-grade mathematics programs Negative integers are introduced. Although place value is introduced, no attempt to teach to completion is made. Counting provides entry into most subsequent arithmetic. Appropriate game formats were sought only after objectives were identified. Games of the lotto style are found throughout the program but receive particularly heavy use as a strategy for teaching correspondence. The rules and score-keeping procedures provide further opportunity for children to talk about the matching process and extend the use of the game to teaching addition and subtraction concepts. The use of the rules, and indeed the introduction of children to the rules, establishes the transactional mode of verbal interaction identified as the critical feature of the gaming approach (Winters 1971).

It is beyond the scope of this chapter to discuss each ILM game There are several useful games in the ILM materials that teachers can construct cheaply. A careful, summative evaluation of the transactional variable in the gaming approach has not been conducted to date. The expected outcomes relative to the verbalization and socialization objectives are important and extend beyond the achievement of mathematical concepts and skills. Using the transactional characteristics of games for these objectives and for those more specific to mathematics suggests many researchable hypotheses.

mathematics

comprehensive school mathematics program

The nature and uses of mathematics in our culture have determined the design of the Comprehensive School Mathematics Program (CSMP) for early elementary school children. The CSMP project collected groups of mathematicians and mathematics educators for the purpose of reflecting on the ideal content of school mathematics and attempted to translate these ideas into realistic materials for young children. Few development projects have stressed the study of the question of what mathematics to teach to the extent that CSMP has.

The CSMP effort at the elementary school level is relatively recent; in the mid-1960s CSMP concentrated its resources on the implementation of the Cambridge conference curriculum at the secondary school level (CSMP 1971). The first year of CSMP pilot materials in the elementary school was 1968. Since that moderate beginning, a complete K-12 curriculum has been outlined, and materials have been developed and field tested for grades K-3. The CSMP project was supported by the Central Midwestern Regional Educational Laboratory.

The selection of content was, and is, critical to the disciplinary orientation of CSMP. In order to free individuals for dreaming, the initial selection of content was to be unfettered by traditional notions of "what children can do" or "what teachers can teach" (CSMP 1971). Given a determination of what is important mathematically, the curriculum designers of CSMP then assumed the responsibility of creating or finding appropriate, effective materials and modes of instruction.

Three criteria were provided for judging what is important mathematically. These criteria—adequate coverage, the viewing of mathematics as both art and science, and utility—provided direction for content selection and were carefully detailed to stress the unity of mathematics in terms of both concepts (such as relations, sets, and mappings) and mathematical methods and tools (such as algorithms, mathematical logic, and heuristics).

These criteria led to content that is atypical of the majority of elementary school programs. The CSMP advisors and staff attempted to create materials that help children begin to develop concepts of probability and combinatorics. The CSMP program stresses the development of preliminary concepts of logic but does not reserve them to the linguistic- or

communication-skills component of the curriculum, which is typical of many Head Start and Follow Through programs. The instructional materials for logic at the first-grade level are independent of the instructional sequence in mathematics. The concept of function serves to unify the entire curriculum. Many of the devices and materials for teaching arithmetic concepts and skills help children develop an intuitive basis for the concept of mapping in geometry, another conceptual thread spanning the CSMP curriculum. Mathematical structures and the relationships between them and their real-world models were identified as concepts fitting the criteria.

Possessing content ideals is a long step from having a curriculum ready to implement. The CSMP staff, under the guidance of Burt Kaufman, elected to build instruction around the concept of individualization by means of activity packages, although the experience and influence of Mme Frédérique Papy tempered a total commitment to individualization (CSMP 1971). The activity packages use ideas from many different sources, such as Cuisenaire, Minnemast, Dienes, Nuffield, and Papy.

The CSMP program begins in the kindergarten and progresses through third grade. The packages are designed in sequence, with a spiraling of content. Special packages for enrichment are part of the program. Many activity packages have a corresponding remedial package, although the teacher is warned that spiraling provides a means of treating children who are unsuccessful in acquiring concepts on the first introduction to the concept. Many packages are cheap newsprint materials of a semiprogrammed nature. Some are used to diagnose whether a child should progress to the next activity or be channeled to a remedial package. Many of the packages are based on manipulative, laboratory materials. Color cues are used extensively. A gaming format is used with many of the packages.

One device, the Papy minicomputer, is used extensively. Recently introduced from Belgium, this device, through the CSMP use, is receiving one of the first extensive tests of its effectiveness in the United States. The device and its appropriate use are complex and exceed the scope of this chapter, the article "A Two-dimensional Abacus—the Papy Minicomputer" in the October 1972 issue of the *Arithmetic Teacher* (Van Arsdel and Lasky 1972) provides a description of the device.

Activities on the minicomputer involve children in placing counters on boards corresponding to place values in a base-ten numeration system. Embedded within each board is a limited binary system determining the value corresponding to counters placed on the board. The use of the minicomputer is constantly oriented toward minimizing the number of counters on the boards. The use of a particular problem orientation, such as this minimization task, as an integral part of the instructional strategies used to establish a number of different concepts and skills has not been researched thoroughly. Does this use of an overriding minimization problem establish a constructive orientation similar to that espoused by Zoltan Dienes (Dienes 1967)? Is this a critical factor contributing to the effectiveness of laboratory learning aids?

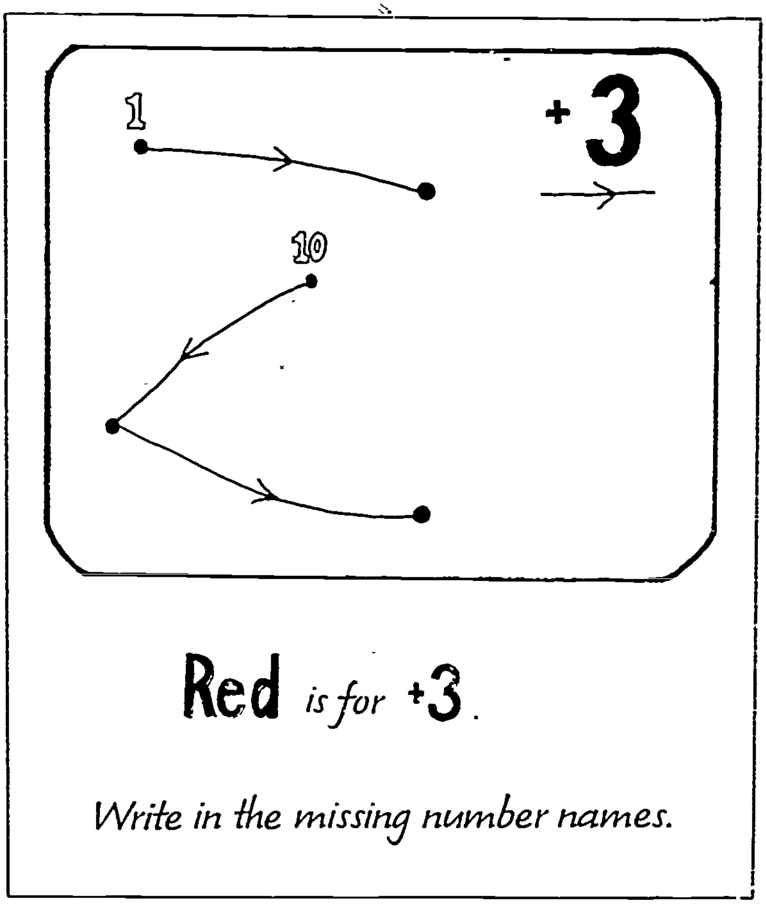
The CSMP approach to developing computational skill provides a primitive introduction to functions as mappings. Addition and subtraction are associated with arrows or mappings as shown by the addition exercise in ■ (an understanding of the concepts of multiplication and division is developed around similar mappings).

The "arrow" diagrams are used to develop a dynamic "feel" for operations as functions or mappings. The concept of the composition of functions develops nicely in this context, as well as the concept of the inverse function. This provides a treatment of fractions not so thoroughly grounded on the physical representation of fractional parts. Perhaps the greatest potential of the extensive use of mapping in establishing arithmetic concepts will be found in geometry rather than arithmetic. The behaviors children develop in the context of arithmetic are used by CSMP as an intuitive entry to performing slide and reflection transformation activities.

The CSMP materials are being field tested and evaluated. To date, the evaluative effort has been formative in nature, leading to the revision of materials and administrative procedures. Comparative, hard data of the pilot 1970/71 school year have not been published.

The year 1970/71 was the third year of implementation of the program, which was used in the classrooms of four schools for evaluative purposes. The variation of facilities and personnel make interpretation of data specious at best. but the need for careful evaluation is recognized by the CSMP project. Informal evaluation based on data collected up to midyear 1971 reports that children are enthusiastic about CSMP materials and that students who were able to work independently tended to work completely through great amounts of materials Pilot Study teachers. after training, were able to cope with the new tasks of the program, including a complex system of individualization (CSMP 1971). The cost of CSMP is. however, quite high.

The CSMP materials suggest a variety of research and evaluation problems. If the prognostic judgment of eminent mathematicians is accepted as justification for the introduction of nontypical content into the early elementary curriculum, the question of determining suitable means of instruction still remains. In teaching probability without a background of rational-number concepts, one is forced to concentrate on building a primitive intuitional base for future learning, an area of research not well explored. The Papy minicomputer shares many of the characteristics of the Cuisenaire rods and suggests comparable research and pedagogical problems. The task of seeking minimal displays of counters does impose a problem-solving orientation that pervades several activities. Few laboratory teaching devices establish a task-determinate mind set extending over a protracted time period. The impact of an extended problem task on learning needs



study The possibilities of exploiting the mapping experiences in the early grades warrant examination.

CSMP is the only development project that has given first priority to the question of the most appropriate content for the future (Lennart 1970) Answers to this presumptive question are of the utmost importance to society. CSMP has concentrated on determining whether it is possible to teach the identified concepts, but the project has not yet disseminated widely the results of its deliberations. Other mathematicians and educators involved in the task of curriculum design would benefit from the collective wisdom focused by CSMP on this question.

compensatory education

The DISTAR* and ILM programs had their roots in the federally sponsored efforts to educate the disadvantaged. The characteristics of these materials are testimony to the potential impact of the Head Start and Follow Through programs on the education of the young. In terms of the number of students alone, these programs would be judged significant to the field; in 1967, only three years after inception, there were 12,927 Head Start centers (Cicirelli et al 1969) But more significant for the future of early childhood education is the testing of distinctive approaches in the teaching of the young Many of the ideas being found productive in the education of the disadvantaged will be used in the classrooms of the more fortunate.

The implications of the Head Start and Follow Through programs are not clear for mathematics The growth of the number of programs for the preschool child from 1964 to the present has exceeded the capabilities of the mathematics education community to participate in the design of the programs. The evolution of programs has taken place with no genetic blueprint comparable to the 1959 *Report* of the College Entrance Examination Board (CEEB), which was so fundamental in shaping the secondary school curriculum (CEEB 1959). The impact of no curricular guidelines was compounded by the noncontent mind set of many specialists in the field of early childhood education. Ellis Evans, an authority in the field of early childhood education, states, "Childhood educators who are content-oriented seem to have been viewed with suspicion by those who feel such an emphasis implies a priority higher than children themselves" (Evans 1971). The important objectives of school readiness and socialization have appeared exclusive of content-oriented objectives.

The planning guidelines for Head Start programs reflect this bias. They do not emphasize conceptual objectives nor refer specifically to mathematics (Evans 1971). Indeed, six of the seven aims stated in the federal guidelines were noncognitive. These guidelines set the pattern for the majority of funded curriculum projects outside as well as inside government sponsorship. This is not to say that the aims of most Head Start programs are not important. Building responsible family attitudes, improving physical health, increasing self-confidence, and the like *are* important.

Head Start programs typically provide preschool experience for at least a summer but possibly up to a year's duration prior to school entrance. The broad, general goal is to provide learning experiences to supplement and compensate for the disadvantaged background attributable to the social context of the first years of life. Follow Through programs differ in that a concerted effort is made to modify the total environment of the child in addition to providing the preschool experience. It is assumed that further environmental planning will increase the probability that the gains made in preschool through the first years of elementary school will be sustained. Follow Through was authorized in 1968, and in 1969 nineteen Follow Through projects enrolled over 16,000 children (Evans 1971).

The implications of compensatory education for mathematics may well be realized in terms of the applications of more general characteristics of the programs. For example, parental involvement in the preschool program is an essential characteristic of many Head Start and Follow Through projects. The Florida Parent Education Program at the University of Florida

(Gordon 1971), the DARCEE Training Program for Mothers at George Peabody College (Barbrack, Gilmer, and Goodroe 1970; Cupp 1970; Gray 1971; Stevens 1967), and the Home-School Partnership Program at Southern University in Baton Rouge (Johnson and Price 1972) have each used and studied carefully the intervention of parents as a mechanism for teaching the disadvantaged. Typically the parent is incorporated into the instructional process and, for these three programs, may be involved in mathematical activities.

Behavior analysis is another means of instruction that pervades all aspects of many compensatory education programs. Skills and concepts are carefully analyzed to determine the order of teaching. Careful scheduling of reinforcement and rewards as well as the order of presentation of the concepts typifies most such programs. The DISTAR and ILM programs exemplify the extension of this instructional approach from behavior and language to mathematics Other projects making extensive use of this technique are the Follow Through project at the University of Kansas (Bushell 1970, Bushell 1971) and the Tucson Early Education Model program at the University of Arizona (1968).

Concern for helping children develop a more adequate self-concept is the primary aim of many programs. Behaviors labeled with words such as *initiative*, *self-direction*, *self-respect*, *curiosity*, and *commitment* are important determinants of instruction. Specific cognitive objectives are not as important as these behaviors, and mathematics may serve simply as the vehicle for acquiring such behaviors. Children typically participate in determining their own educational goals The Bank Street College of Education Follow Through program (n.d.) is typical of such programs. The attention to mathematics in such settings is incidental and often accidental. The present knowledge of readiness, efficiency, and the critical period for learning mathematics may be inadequate to require creating and implementing mathematics programs at the preschool level at the expense of realizing other goals. The absence of a definable program in mathematics need not imply that mathematics is either poorly treated or ignored. Mathematical instruction in such preschool settings may remain incidental, but it may also be high-quality instruction, even carefully planned instruction.

Two types of evaluation have been applied to Follow Through and Head Start programs—summative and comparative. Summative evaluation of Head Start indicates that Head Start generally has not been successful in generating growth in cognitive performance by children. The joint Westinghouse-Ohio University evaluation team for Head Start lists more than thirty evaluative studies agreeing with their finding that Head Start has had "only limited impact on cognitive performance of children" (Cicirelli, Evans, and Schiller 1970). This lack of payoff for the compensatory education dollar was perhaps predictable, the guidelines for Head Start proposals emphasized other than cognitive growth. The programs stressing cognitive objectives have tended to be more successful in promoting growth that was maintained two and three years later. Weber's *Early Childhood Education. Perspectives on Change* (1970) and Evans's *Contemporary Influences in Early Childhood Education* (1971) describe in detail some of these more successful Head Start programs.

Comparative evaluation efforts have resulted from the variation from program to program in instructional design and priorities of educational objectives. The integrity of the approach of various programs has been tested by comparative analysis. Soar and Soar (1972) in the NSSE yearbook devoted to early childhood education report systematic observations of teacher and pupil behaviors that indicate the purity and consistency of approach for five Follow Through projects. Kamii's analysis of the objectives of programs offers another example of the comparative evaluation efforts (1971). A basis for studying the long-term impact of distinctive preschool programs on subsequent schooling is being built by such studies.

Compensatory education has had a stimulating, salutary effect on preschool education. Based on a social pathology model of preschool education (Baratz and Baratz 1970) it has raised a variety of issues and problems (Rosenthal et al. 1968). The store of techniques and materials applicable to mathematics has been made richer by compensatory education.

other directions of potential influence

Two "programs" are discussed in this section, the television program "Sesame Street" and the phenomenon of local curriculum projects. Judgments of the effect of each are difficult. "Sesame Street" is not school based. The ineffable character of the viewing habits of children means that the relation of "Sesame Street" to mathematics teaching in the schools is difficult to ascertain. Curriculum projects that are locally sponsored are quite common. Most are bootstrap operations depending on teacher commitment. They are often directed toward laboratory learning or behavioral objectives, and the range of the quality of the materials is great.

"sesame street"

Preschool and kindergarten teachers should be acquainted with TV offerings like "Sesame Street." Although subtraction may not be adequately or purely modeled by a monster eating cookies, the monster's habits do provide a referent for the process of establishing an appreciation for subtraction. The reason for discussing "Sesame Street" in this context is not to pass judgment on the quality of instruction but to suggest its potential for providing points of entry and familiar examples. We refrain from making judgmental statements because we do not at present know what impact such TV programs may have on education in mathematics. The assumptions about learning on which the design of "Sesame Street" is based are discussed in the May 1972 issue of the *Harvard Educational Review* (Lesser 1972). Several questions, however, of special concern to preschool and early elementary education are raised:

1. Is a high degree of stimulation necessary or appropriate to attempt to educate smaller children? Do children develop a dependence on such stimulation that cannot be honored by the public schools?
2. Do the stimuli that are intended to hold interest but that are superficial to the intended learning interfere with the learning or change the nature of the learning? If so, do they leave a detrimental effect? (For a widely known example to help make the point, some kindergarten children who initially learned the alphabet by song receive a mild jolt on finding out that "L-M-N-O" is not a single letter.)
3. Is a child's attention span as fragile a thing as the "Sesame Street" programming assumes it to be? If not, does refraining to push for longer attention to a single stimuli set have an effect on the potential for a longer attention span at a later time?
4. Although both culturally deprived and advantaged children appear to profit from viewing "Sesame Street," do indications that the program does nothing toward closing the achievement gap between them suggest that its potential is restricted to complementing or clarifying experiences that involve the child more directly?
5. More specifically to the concern of this chapter, is the fact that "Sesame Street' viewers often learn to "say" things mathematical long before developing corresponding concepts an advantage or a disadvantage in learning mathematics? (For example, children learn to "say" the number names and repeat basic facts of addition before being able to "count" a small set of objects.)

The questions raised above are not solely the concern of "Sesame Street." They are among those questions that should be considered in any attempt to educate young children.

locally developed curriculum

A phenomenon spawned in part by the availability of government funds at the local level during the 1960s is that of the "locally developed curriculum." Other factors contributing to the development of this type of curriculum are a weariness over the number of new topics included in textbooks and the corresponding ne-

cessity to be selective; commitments to behavioral objectives that do not fit standardized achievement tests or the unit examinations provided by textbook publishers, and commitments to the proposition that relevance is gained by fitting the curriculum to the community setting.

By "locally developed" is meant "developed by a school system's employees under contract obligation." Patterns range from forming teams of curriculum developers with federal funds administered by the school system to appointing committees of teachers who assume responsibilities without additional reimbursement.

The search for "model" locally developed materials that did not enjoy federal funds was thwarted by the fact that such materials are often being considered for publication by commercial firms The better of these materials are likely to gain visibility without the aid of this chapter and are likely to undergo enough revision to advise against evaluating what is currently available.

A perusal of locally developed materials suggests that few can be named that will have a wide impact on early elementary school practices. The primary impact may remain at the local level, since development efforts characteristically demonstrate teacher commitment and enthusiasm on the part of those directly involved but not by others. In fact, the commitment and enthusiasm exhibited in creating even poor and mediocre materials suggests that the degree of the teacher's involvement may be the key to his success with them. Among the questions raised by this observation is the following: Given a choice between inferior materials with strong teacher commitment and superior materials with weak teacher commitment, which set most facilitates pupil achievement and the development of positive attitudes?

A number of local attempts to revise the mathematics curriculum have begun with listings of behavioral objectives as the first step. The most common scheme observed in arriving at the listing of objectives is that of outlining the content of several commercially available textbooks, then screening and ranking objectives according to the degree of their importance, and finally deciding on the flow diagrams for treating objectives. Many such attempts do go through the second step of creating a reservoir of test items for the objectives. However, rarely does the development of these curricular materials go beyond identifying specific page numbers in textbooks for the objectives and perhaps supplying short, teacher-made worksheets.

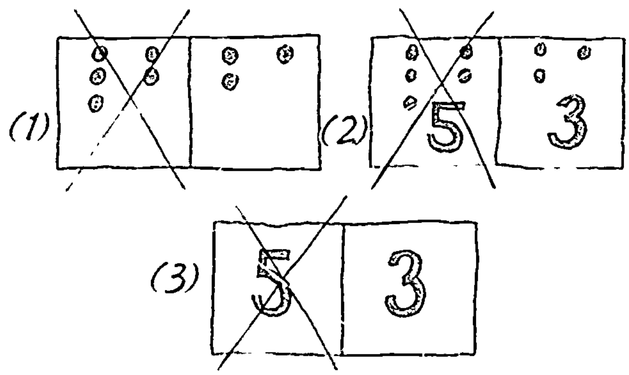
It seems safe to suggest that the primary advantages, if any, derived by the local school system from such attempts are a higher degree of teacher commitment to the mathematics program and the degree of teacher education resulting from the effort. The listings and rankings of objectives are usually no better than those suggested by the table of contents of any single commercial series; often they are inferior. For example, one such set of objectives fails to include establishing the area of a rectangle as length times width, thereby denying the option of treating the multiplication of rational numbers in terms of area.

One effort to develop a program locally which has taken care in creating instructional materials is that by the Winnetka, Illinois, schools (n.d.). The packaging format is that of short, consumable student workbooks with corresponding teacher manuals and examination items. Verbalization in the student materials is minimal. Many of the booklets treat a single concept or topic, with a number of key concepts (e.g., place value) covered by several booklets.

These elementary materials may be viewed as a compromise between extreme positions on teaching for understanding and teaching by repetitive drill, between practices that depend heavily on manipulative materials and practices that deal exclusively in symbolism, and between insisting that each child must be a discoverer and believing that each child is a machine to be programmed. The sequence and pacing of content does not require an early usage of precise mathematical symbolism or terminology in dealing with partially developed concepts. Nor, in attempts to foster concept formation, do the materials dart from one stimuli set to the next at a rate that might frustrate all but the brightest pupils.

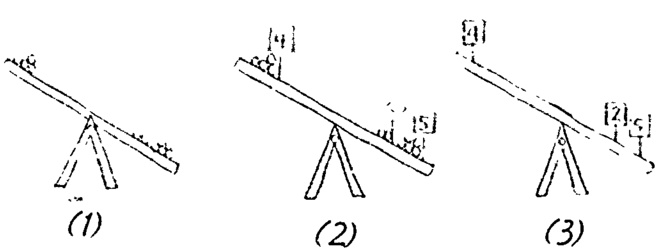
The early treatment of "less than" and "greater than" may serve to demonstrate the

style of the Winnetka materials. The first sets of exercises ask children to cross out the "greater" set:



Items in set 2 provide a transition from set 1 to set 3. Although the word *greater* is used with instruction for the first set. examples given with each worksheet may allow the assignment to be completed without recognizing the word.

Later in the unit the sequence from pictorial to numeral is repeated with the seesaw:



The unit on "greater than" and "less than" also devotes some exercises directly to the ordinality of counting numbers.

In summary, most attempts at curriculum development at the local school level tend to be oriented toward behavioral objectives or laboratory materials The strength of such attempts appears to derive from teacher involvement. The main concerns are the amount of money and time spent and the quality of the resulting instructional materials.

retrospect

The curricular development of the past decade has been extensive. Burgeoning enrollments, the change of perspective on objectives, new knowledge of children's learning, and plentiful resources have all contributed to encouraging creativity in designing mathematical experiences for young children.

Most program creators at least honor psychology by stating that their instruction is psychologically based; sometimes this is true. The influence of Gagnéan organization of objectives is evident in many programs. The increasing usage of active. manipulative instructional materials has contributed to the diversity of materials and to the significantly broadened range of quality of instructional approaches. The commitment to "psychologizing" the instruction of young children has led to the implementation of several instructional techniques that have not withstood the test of time. In some programs, such as with the transactional variable in the ILM games approach, the precise nature of the variable providing uniqueness has not been clearly identified. In brief, an evaluation of programs is often not enough; research to isolate and refine the operant factors of the approach is needed.

The preschool and kindergarten mathematics program deserves careful attention by the mathematics education community. Preschool programs have been established in many communities in the last five years. Most programs have no carefully designed mathematical component, since no set of traditions existed in this area and the quantity of appropriate materials is small. The mathematics education community can ill afford to ignore the stake it has in the new emphasis on preschool education. It is not known whether an informal, precursive exploration of mathematical topics or a formalized, hierarchical treatment directed toward specific goals has the greater potential for subsequent schooling. Clearly preschool and kindergarten education is more cognitively oriented than in the past. The directions for the future are being determined now. The compelling need for careful, comparative, longitudinal evaluation is evident.

The historic issues identified in chapter 2 are evident in the recent curricular thrusts. The premium to be placed on skill learning, the use of activity-based teaching strategies, and the

relation of mathematics to the rest of the curriculum remain significant problems.

The final generalization concerning developments in early childhood education is that federal funds have had a strong influence. The seminal ideas leading to unique program designs were developed, refined, and implemented only because resources were available.

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