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ABSTRACT

This paper describes preliminary results of a study designed to explore the issue of translating geometric ideas from a diagrammatic mode to a verbal statement, and vice versa. The research model calls for comparisons of responses under 16 different conditions; these conditions are defined by question type and the nature of the translation to be made. Testing materials were designed to be representative of the statements and diagrams contained in modern geometry textbooks. As a part of the test development, high school students rated items according to difficulty. Thirty-two subjects were randomly selected from a set of fifty volunteers. Preliminary analyses of data are reported. Among other findings, the author reports "that on two fairly straightforward tasks of recognition, students who have recently studied Euclidean geometry are averaging a 30 percent error rate," and suggests that these students are unsure about the underlying structure of geometry. (SD)

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Within- and Between-Mode Translation in Euclidean Geometry

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Paper presented at the American Educational Research Association Annual Meeting, held in Washington, D.C., March 30-April 3, 1975.

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Within- and Between-Mode Translation in Euclidean Geometry

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For the past few years, a number of us have been investigating the psychological basis for learning geometry. Decisions for curriculum construction, teaching and evaluation of learning in geometry, as in most other subject-matter areas, are arrived at by considering the logical structure of the content, general principles of good teaching practices, and the insights gained from experience by teachers of the subject. As a result of the efforts of a substantial number of mathematicians and mathematics teachers over the past fifteen years, geometry curricula have changed considerably. The new curricula, though varying in detail, agree on the following characterization of the subject matter:

1. A geometry system is a formal system for the representation of the properties of space.
2. A geometry system is defined by a set of primitive elements (e.g., point, line, plane), a collection of basic assumptions about these elements (postulates), and the operations, or "legal moves", within the system. A geometrically "legal move" is one which preserves certain specified properties of the elements and relationships between elements.
3. By changing assumptions and operations, a large number of different geometries can be developed, of which Euclidean geometry is one.
4. The geometric system is one of several alternative ways of representing relations in space.

This characterization leads by implication to a set of "shoulds" for teaching which tend to emphasize understanding the structure of geometric systems (geometries), being able to reason deductively, and being able to see relations between geometric, algebraic and analytic representations of the same ideas.

The current emphasis on structure in teaching geometry raises a number of interesting questions related to learning. First, does the teaching of structure necessarily imply that structure will be learned? Framed somewhat differently, the question becomes: how good is the fit between the structure of the subject-matter and the cognitive structure of the student? Two studies carried out recently (Elman 1973a, b) shed some light on this question. In one study, subjects were asked to sort 40 geometry diagrams into as many as 6 groups based on similarities between items. Descriptors which subjects gave for their groups were also recorded. The results indicate that, at least at a gross perceptual level, subjects tended to discriminate and to use four categories - open figures, triangles, circles and quadrilaterals - in sorting geometry diagrams. Sex of the subjects and experience with geometry did not appear to affect performance, although females, when given a free choice, were more likely than males to choose the above categorization plan. The finding of a preference for the above model has been replicated with another sample. In the second study, the subject was given 12 items, each consisting of a target along with a set of 7 alternatives representing four types of transformations - no change; rotations of 60 degrees, 180 degrees, and 270 degrees; a 30 degree projection into the plane; and embedding in a circle and in an irregular polygon. Subjects were asked to rank the alternatives for each item in order of their similarity to the target. The most striking result of the study was that any change in representation makes a difference. Results over all subject-item sequences indicated that the order: No Change, Rotation, Embedding, was used more frequently than any other. Experience with geometry affected the rank order assigned to the perspective alternative, with naive subjects judging it as most like the target, and experienced subjects as least like the target. The finding that any change makes a difference is hardly novel. However, it points out the need for taking the problem of representation in geometry more seriously from the point of view of instruction. The observation of differences in ranking attributable to experience deserves to be replicated, as it may provide evidence concerning the nature of changes in one's representation of spatial relations as a result of formal instruction in geometry.

Another interesting question concerns the issue of translation from a diagram to a verbal statement and vice versa. If we regard the learning of geometry as in part the learning of a new language, what are some of the critical variables contributing to "fluency" in geometry? The present study explores this question.

EXPERIMENTAL DESIGN

The basic design is a 2X2 within-subjects factorial design, with input and output mode of the stimulus as the two two-level factors:

INPUT MODE	OUTPUT MODE: Diagram	OUTPUT MODE: Statement
Diagram	D-D	D-S
Statement	S-D	S-S

This design defines four cells, designated in the above figure as D-D, D-S, S-D, and S-S. Cells D-S and S-D contain conditions which are generally viewed as requiring translation from one representational system to another. Under the D-S condition, a subject sees a geometry diagram and responds with, or to, a corresponding verbal statement; under S-D, the stimulus is a verbal statement of some geometric state of affairs, and the response required is in the form of a diagram. Cells D-D and S-S, within-mode translation, could be termed more appropriately paraphrase conditions. For example, given a statement, the subject is required to select from a set of statements the one which most closely approximates the statement he has just seen.

Subjects are asked to make two types of response: Recognition and Production. Under the Recognition condition, the subject is required to make a same/different judgment (Y/N) or to choose from a set of alternatives the one most like the original stimulus (Multiple Choice). The Production condition likewise has two levels - cued production and free productive recall. Under the cued production condition, the subject studies the stimulus item. He is then given four cues, in the same mode as that in which he will be asked to recall the stimulus item. He is asked to judge each cue as "Helpful" or "Not helpful" in prompting recall. He then turns the page and produces a statement or diagram containing "essentially the same information" as the item studied.

The four response types taken together with the four Input/Output combinations define 16 different combinations which may be thought of as representing a large part of what goes on in learning geometry. Since we are attempting to make inferences from this study to geometry learning, an elaboration of these 16 conditions is appropriate.

Response Type		I/O Mode: Same		I/O Mode: Different	
Recognition	Y/N	D-D	S-S	D-S	S-D
Recognition	M C	D-D	S-S	D-S	S-D
Recall	Free	D-D	S-S	D-S	S-D
Recall	Cued	D-D	S-S	D-S	S-D

Table 1: The 16 testing conditions defined by the basic design.

Table 1 represents one way of organizing the combinations of testing conditions to be investigated. One major distinction is drawn between cases where input and output are in the same mode versus cases in which input and output modes are different. In cases where input and output modes are the same (i.e., both are in the form of a verbal statement (S-S), or of a diagram (D-D)), the study examines as individual's facility within each mode. Facility is taken to mean performance under each response condition. Specifically, Yes/No Recognition requires the subject to recognize a statement or diagram as one which he has or has not just seen. This testing condition provides information about whether diagrams or statements about diagrams are remembered over a brief delay. The design enables one to attribute Y/N recognition performance to specified subject and materials variables, but does not identify the analytic processes which may result in confusions between "old" and "new" items. The purpose of the Multiple Choice recognition task is to test hypotheses about strategies for processing information in geometry statements and diagrams. The assumption is that subjects have criteria for assessing "sameness" which are somewhat flexible - that is, within certain limits, an individual will accept, say, statement B as an adequate paraphrase of statement A (or diagram B as an adequate approximation of diagram A). The multiple choice alternatives were constructed so as to shed some light on what these limits are, and how well they correspond to what are accepted as adequate limits by geometers. Phrased another way, we are interested in determining the class boundaries which subjects use to define geometry concepts, represented verbally and in diagram form, and to compare these with class boundaries accepted in geometry.

The productive recall conditions provide a somewhat more stringent means of investigating information storage and retrieval processes within a given mode. Problem solving in geometry demands that the student not only recognize what he has encountered before, but that he also recall critical information provided in previously encountered statements or diagrams in order to be able to apply that information to new problems. The free and cued production conditions are analogous to the Y/N and M C recognition conditions, in that the first of these examines what is retained as a function of design variables (materials and subject variables), while the second, cued production, enables the experimenter to test hypotheses about which cues, and how many, are useful in assisting the subject to reconstruct an appropriate representation.

In cases where input and output modes are different (i.e., input in the form of a statement and output in the form of a diagram, or vice versa), the study is concerned with the "what" and "how" of translating information from one mode to another. Y/N Recognition, the weakest test, asks if a subject can recognize whether a given statement, say, contains essentially the same information as a corresponding diagram. Multiple Choice recognition provides a setting for the manipulation of response alternatives. Appropriate design of alternatives should enable the experimenter to search for systematic additions, deletions and transformations which occur in the process of translating information from one mode to the other. In cued production, partial information about the required response is manipulated to assess the effects of different kinds and amounts of information on the subject's ability to translate a diagram into a verbal statement, and vice versa. Finally, the free production condition examines translation skills when no external support is provided.

MATERIALS

The selection of materials is critical to the study. Materials must be representative of the range of diagrams and statements which occur in geometry textbooks. Furthermore, adequate control must be maintained over the difficulty of the materials, as this source of variability may affect subjects' performance to the point of obscuring common processing patterns. Preliminary to the major study was the task of controlling variability due to the difficulty level of the materials.

Difficulty has been operationalized differently for diagrams and statements. In the case of diagrams, a miniature experiment was conducted. Eighty diagrams - twenty each of open figures, triangles, circles and quadrilaterals - were selected by stratified random sampling from current standard geometry textbooks. Two forms of the stimuli were constructed. The marked form contained markers of the kind usually used in geometry to identify equal angles, equal line segments, perpendiculars, parallel lines and the like. The unmarked form used the same drawings, but without markers. The 80 diagrams in each form were divided into 4 sets each containing 5 diagrams from each of the 4 categories mentioned above.

16 high school geometry students of above average ability rated different combinations of items as to comparative complexity, placing items along a meter stick provided as a scale. The distance from 0 cm. to the midpoint of the item was used as a measure of an item's relative simplicity or complexity. On the basis of these judgments, 4 items from each category (2 of the most simple and 2 of the most complex) were chosen as diagram stimuli for the major study.

For the 16 diagrams, statements were written containing the equivalent information. The task of controlling for difficulty level of the verbal materials was dealt with as follows. Two 5% samples of text were taken from each of the two source textbooks. A count was made of all geometry terms encountered in the samples. The stimulus statements were then screened for occurrences of high- and low-frequency terms, and replacements were made where appropriate to increase or reduce the difficulty level of the vocabulary.

Thus ease (or difficulty) for diagrams is defined operationally as phenomenal simplicity as judged by high ability high-school geometry students. For statements, ease is a function of frequency of vocabulary used in the description. (Incidentally, there is a high correlation between phenomenal complexity of a diagram and the number of words required to describe the diagram; thus difficulty of statements is confounded with number of words, or can be seen as a function of description length.)

Category. Previous studies (Elman 1973a, b) have indicated that geometry designs tend to be categorized as open figures, triangles, circles or quadrilaterals. This variable was included in the present study as two two-level variables to obtain another measure of control over variability due to materials.

Form. Two forms of each Ease x Category combination were included as a materials replicate.

Order. To avoid peculiarities due to the ordering of materials in the test booklet, four order variables were included in the design to balance the levels of the materials variables over test items. The experimental and order variables were crossed to create a 2^{11} factorial within-subjects design. This portion of the design was fractionalized, and the fraction broken into 8 blocks (or test booklets) of 16 items each.

Subjects were 32 high school juniors and seniors who had completed the high school course in Euclidean geometry. They were selected so as to have a sample which was balanced with respect to sex and geometry achievement. Subjects provided information about the number of other math or math-related courses they had studied, and were administered a pretest on Spatial Ability (The Paper Folding Test - Vz-2, from the French kit). Amount of other math, spatial ability and verbal ability (SCAT stanines) were collected for inclusion as covariates.

PROCEDURE

Some 50 youngsters who volunteered for the study were pretested on the Paper Folding Test - Vz-2 from the French kit. The test is in two parts, each part being timed at 3 minutes. From this group 32 students were selected as experimental subjects. 16 were A students in geometry, 8 boys and 8 girls; the other 16, 8 boys and 8 girls, had passed their geometry course with a B or C average. Subjects were paid $2 for participating in the experiment.

Subjects were tested in groups of four or less, in a large school room. There were never more than two subjects for each tester present.

Each subject was provided with a test booklet, pencil, ruler and circle template. The subject was asked to read the cover sheet which provided general information about the task, while the tester read the same information aloud. The general instructions informed him that he was participating in a study to find out how people who have recently studied Euclidean geometry perform certain tasks. He was advised that he would be encountering some items. Each item would consist of a diagram or statement. He was to study the item carefully; when he was ready, he was to turn the page, read the instructions and follow them to the best of his ability. He could spend as much time on an item as he wished, but he could not turn back once he had turned a page. The cover sheet also included a list of common geometry markers and their meaning. The tester then answered questions about the instructions, and the subject proceeded with the task. Subjects took about 30 minutes to complete the task; no subject took longer than 45 minutes.

PRELIMINARY RESULTS

Owing to the complexity of the design and the extensiveness of the data base, we are still in the process of analyzing data. However, some preliminary findings of a descriptive nature can be reported at the present time.

First, subjects in the study had taken 2 courses in mathematics in addition to geometry, on the average. The breakdown on the basis of achievement is a median of 1 course for low achievement subjects and 3 courses for high achievement subjects. Girls averaged about 1 or 2 additional courses, while boys averaged 2 or 3. The mean verbal ability stanine for the sample was 7 (6 for low achievement and 8 for high achievement subjects), with essentially no sex differences. On the pretest of spatial ability, the sample mean was 12.7 (out of a possible score of 20), with low achievement subjects averaging 11.3 and high achievement subjects averaging 14.1. Low achievement males achieved substantially higher than their female counterparts (13.1 vs. 9.5). There were no sex differences for spatial ability in the high achievement group.

In scoring and analyzing the recognition and production data, certain criteria had to be established for the acceptability of subjects' responses. In the case of recognition data, the eight response alternatives for each multiple choice question were designed according to the plans in Tables 2 and 3.

EUCLIDEAN TRANSFORMATION

	Standard Orientation	Rotation
NON-EUCLIDEAN TRANSFORMATION	1. No change	2. Change orientation
	3. Add information	4. Change orientation + Add
	5. Delete information	6. Change orientation + Delete
	7. Non-rigid transform	8. Change orientation + Transform

Table 2. Plan for generating multiple choice alternatives for diagrams.

	LEGAL CHANGE	
	No Change	Paraphrase
	1. No change	2. Paraphrase
ILLEGAL CHANGE	3. Add information	4. Paraphrase + Add
	5. Delete information	6. Paraphrase + Delete
	7. Replace concept	8. Paraphrase + Replace

Table 3. Plan for generating multiple choice alternatives for statements.

In both cases, alternatives 1 and 2 are acceptable representations of the information in the stimulus item. A subject's response was given a score of zero if he correctly identified an acceptable response as "the same" and an unacceptable response as "different"; otherwise, the response was scored as 1.

Each response for a Yes/No item was one of the four possibilities presented in Table 4. Scoring followed the same procedure as that for

YES/NO RECOGNITION - DIAGRAMS

1. No change	2. Change orientation
3. Non-rigid transform	4. Change orientation + Transform

YES/NO RECOGNITION - STATEMENTS

1. No change	2. Paraphrase
3. Replace concept	4. Paraphrase + Replace

Table 4. Plan for generating Yes/No recognition alternatives for diagrams and statements. Subjects are asked to judge whether the alternative given contains the same information as the standard. Alternatives 1 and 2, by standards of Euclidean geometry, contain the same information as the standard; alternatives 3 and 4 are different.

multiple choice responses. In both cases, the proportion of incorrect responses, averaged over subjects, was the dependent measure.

Preliminary data reduction was performed to obtain means for within-subject main effects and selected two-way interactions, using a FORTRAN program written especially for that purpose.[1] These means were used as dependent measures in several SPSS BREAKDOWN runs. Some of the more interesting findings follow.

Between-subject effects

For Yes/No Recognition items, the average poportion of unacceptable responses is 0.297. The error rate for Multiple Choice items is slightly lower - 0.282. There are no sex differences or differences due to geometry achievement for the multiple choice items. There are likewise no sex differences for the yes/no items. However, the error rate of low achievement subjects for the yes/no items is substantially higher than the error rate of their high achievement counterparts (0.360 vs. 0.235).

Within-subject and mixed effects

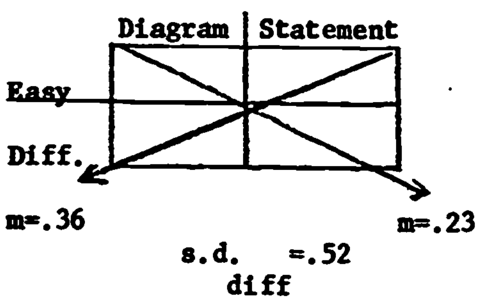
Of the main effects and two-way interactions, none seemed to amount to much. We looked at the effects of Ease, Form, Category, Input Mode and Output Mode for both yes/no and multiple choice items, and at all of the two-way interactions formed by pairwise combinations of these factors. We also examined the effects of all these sources in interaction with sex, achievement and sex by achievement; and at the effects for the eight multiple choice alternatives. We were particularly interested in the Input x Output interaction, as this effect would indicate whether

1 I am indebted to Dr. Len Fisk, Postdoctoral Fellow, for his assistance in writing the FORTRAN program used in preliminary screening of the data.

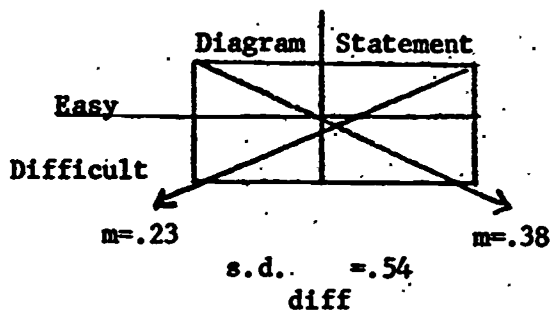
there were notable differences between same-mode and cross-mode processing for recognition.

None of the main effects for either type of recognition response accounted for any appreciable amount of the variability in scores. None of the two-way interactions, including Input x Output, made any difference in error rate for multiple choice recognition. For yes/no recognition, only two interactions seemed to make any difference - Ease by Input Mode and Ease by Output Mode:

Ease by Input Mode



Ease by Output Mode



Two-tailed t-tests performed on the differences of the means barely attained significance at the .05 level (n=32 in both cases); and the differences are in opposite directions for the two interactions. Hence we are reluctant to make much of these results.

Discussion

In brief, the results thus far indicate at least this much - that on two fairly straightforward tasks of recognition, students who have r recently studied Euclidean geometry are averaging a 30% error rate. This is a most striking figure. It suggests that the subjects, regardless of their achievement level, are somewhat unsure about what the underlying structure of geometry really is, and that their judgment about what constitutes "essentially the same information" in geometry differs from the formal interpretation of that phrase. More detailed discussion will be forthcoming when we have completed the analysis of the data.

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