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ABSTRACT

This paper presents the results of three experiments studying routine problem-solving tasks in simple addition and subtraction. Indications are that children tend to solve such problems by internalized counting procedures which may be learned independently as a consequence of practice in problem solving. Brief descriptions of exploratory studies concerning word problems and sequential rules are also included. (LS)

Final Report

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BASIC PROCESSES IN SIMPLE PROBLEM SOLVING

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ABSTRACT

The research described in this report is concerned with the processes used in performing simple, routine problem-solving tasks that are either identical or very similar to tasks encountered in the school. The bulk of the report is devoted to describing three experiments on simple addition and subtraction. The basic approach in all three experiments involves the collection and analysis of reaction time data. The results support the notion that children tend to solve such problems by means of internalized counting procedures, which are usually the most efficient possible of a given class. There is also evidence that these procedures are not explicitly taught, but rather learned independently by the child as a consequence of practice in problem-solving. Also included in the report are briefer descriptions of an exploratory study of word problems and of a formal analysis, involving the application of automata theory, of the processes involved in the induction of sequential rules.

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Introduction

The research described in this report was concerned with simple, routine problem-solving tasks that are either identical or very similar to tasks encountered in the school. Especially in teaching basic mathematical skills, such problems form an important part of the normal school curriculum. They also occur frequently in intelligence tests. However, little is currently known about the processes used in solving them.

Broadly speaking, this research represented an attempt, in the context of certain highly specific tasks, to begin to answer the following questions:

1. To what extent is it possible to formulate a well-defined model which describes the precise process an individual is using in solving a given problem?
2. What features do models for different problem-solving tasks have in common? To what extent are the same processes used for different tasks?
3. How do individuals differ in the processes they use to solve a given problem?
4. What role does the training procedure play in determining the process used on a given task?

The bulk of this report will be devoted to describing three experiments on simple addition and subtraction. Also included will be briefer descriptions of an exploratory study of word problems and of a formal analysis of the processes involved in the induction of sequential rules.

Chronometric Analysis

A classical technique for inferring the nature of the processes involved in performing a simple task is the Donders method of chronometric analysis as reformulated by Sternberg (1969). This method essentially consists of a set of techniques for analyzing reaction time data. In the simplest case, which occurs when a process can be broken down into a number of identical steps, the reaction time will be a linear function of the number of steps required to perform the task.

This approach can be used to investigate the processes used by children in solving simple addition problems (Groen & Parkman, 1972). In this research, a variety of simple counting models were proposed, all of which assume a counter with two operations, setting to a value and incrementing by one. A simple type of experiment sufficed to test these models, in which a variety of problems was presented and the time required to solve each problem was measured. Each model could be evaluated by fitting the observed reaction times to a linear regression line of the form

$$RT = a + bz \qquad (1)$$

where z is a structural variable defined by the model under consideration, and a and b are constants which can be interpreted as the mean time required to set the counter and the mean time required to increment by one.

The major finding of this series of experiments (performed prior to the period of the grant), which primarily used children in the early grades of elementary school as subjects, was that reaction times were a linear function of the smallest of the two numbers being added. Ties (problems such as 1+1 and 2+2) were exceptions to this pattern. Reaction times showed no discernible linear trend and tended to be faster than for non-tie problems. These results were consistent with the notion that most children solved addition problems of the form m+n, with m and n single digits, by a counting procedure in which the counter was set to max(m,n) and then incremented min(m,n) times. Ties could be viewed as being stored in some fast access memory.

The three experiments to be described next were concerned with two issues arising from these results. The first was whether similar analyses could be applied to other aspects of arithmentic. The second was how procedures such as these were learned in the first place.

<u>Subtraction</u>

Intuitively, it might be expected that the process used to solve subtraction problems would be a simple converse to the process used for addition, with incrementing replaced by decrementing. However, in research performed prior to the period of this grant though written up in final form during the grant period, Groen and Poll (1973) have provided indirect evidence that a more complex hybrid counting process is required. These investigators were concerned with the processes used by children in solving open sentence problems, which are simple linear equations of the form m+u=n where m and n are given and u is an unknown (m, n and u all being positive single-digit numbers). They found that reaction times for these problems could best be accounted for by a process that assumed a child either decremented or incremented, depending on which was the quickest procedure. Since open sentence problems are frequently used in the elementary school as a means of introducing subtraction, this raises the possibility that subtraction problems are solved by the same procedure. The purpose of this experiment, which was conducted in collaboration with Shirley Woods and Lauren Resnick of the University of Pittsburgh, was to establish the extent to which this is actually the case. (Woods, 1972)

The counting device postulated by Groen and Poll posesses three operations: setting, incrementing by one and decrementing by one. It is also assumed that there exists some mechanism for keeping track of the number of times the counter has been incremented or decremented.

There are at least five distinguishable ways in which such a device might be used to solve subtraction problems of the form m-n where m and n are integers between 0 and 9 and m is greater than or equal to n.

1. The counter is set to 0, it is then incremented m times and is then decremented n times. The solution is the final value in the counter.
2. The counter is set to m and then decremented n times. The solution is the final value in the counter.
3. The counter is set to n and is then incremented until m is reached. The solution is the number of times the counter has been incremented.
4. The counter is set to 0, it is incremented n times and is then incremented until m is reached. The solution is the number of times the counter has been incremented in this last stage.
5. Either Process 2 or Process 3 is used, depending on which is faster. This is the process that was found by Groen and Poll (1973) to give the only adequate explanation of the pattern of reaction times to open sentence problems.

The mean reaction time to a given problem can be predicted by assuming that the time required to set the counter is independent of the value to which it is set and that the mean incrementing time is equal to the mean decrementing time. Equation 1 then holds, with the structural variable z determined as follows:

Model 1. $z = m+n$
Model 2. $z = n$
Model 3. $z = m-n$
Model 4. $z = m$
Model 5. $z = \min(n,m-n)$

The same general procedure as was used by Groen and Poll (1973) sufficed to test these models. A variety of subtraction problems is presented, the reaction time is measured and Equation 1 is fitted to the data for each model. The present experiment differed, however, in that while Groen and Poll analyzed data averaged over subjects, the present experiment was designed so that individual subjects could be analyzed separately.

The subjects consisted of 40 children in the second grade and 20 children in the fourth grade of a school system in suburban Pittsburgh. The apparatus consisted of a small wooden box connected to a timer and a response panel consisting of a horizontal row of 10 buttons numbered from 0 to 9. The side of the box facing the subject contained a ledge on which index cards could rest and behind which a photo-electric cell was housed. Slidin a card in place on the ledge activated the timer, which stopped as soon as the subject responded by pressing a button on the response panel. The experimenter could then record the response that had been made and the reaction time to the nearest hundreth of a second. Stimuli consisted of 54 subtraction problems, printed on index cards (a single problem to a card). The problems were displayed in the form m-n=_ where m and n were single digit numbers with m always greater than n. The subject's task was to press the button on the response panel corresponding to the correct answer. The entire set of 54 problems was presented on each of 5 consecutive days.

The data analysis was based on the last 4 days of the experiment, the first day being regarded as a practice session. It consisted of a set of regression analyses of the reaction times of individuals' correct responses, by means of which the goodness-of-fit of each model to the data could be evaluated. Two criteria were used to evaluate the models: (a) whether a model yielded a slope significantly different from zero (b) which model yielded the maximum value of R-squared (the proportion of variance accounted for by the regression line). The outcome of this analysis was that all the children in the fourth grade were best fitted by Model 5, as were 30 of the 40 children in the second grade. Of the remaining subjects in the second grade, six were best fitted by Model 2 and four were not fitted by any of the five models (ie. none of the models yielded slopes significantly different from zero at better than the .05 level of significance). Since Model 5 is the same as the model found by Groen and Poll (1973) to provide the best fit of reaction times to open sentence problems, it can be concluded that the results of this experiment provide considerable support for the notion that children do, indeed, solve subtraction problems and open sentence problems by the same process.

Teaching an Addition Algorithm

If, as claimed by Groen and Parkman (1972), children actually do solve problems by setting a mental counter to the largest of the two numbers being added and then incrementing in the smallest by "ones", then a major question is raised. Such an algorithm is not systematically taught in the school. Indeed, there is no evidence that any specific algorithm is taught. The purpose of this experiment (Groen, Resnick & Silverthorne, 1972), conducted in collaboration with Lauren Resnick of the University of Pittsburgh, was to examine the reaction times of young children when they were taught a specific algorithm.

The algorithm taught was not the algorithm that children appeared to be using in Groen and Parkman's study, which seemed to be impossible to teach explicitly, but a much simpler one. Children who used this algorithm would be expected to have reaction times proportional to the sum of the two numbers being added rather than the smallest of the two numbers. Part of the purpose of the study was to investigate whether, as a result of extensive practice in addition, reaction times proportional to the smallest of the two numbers would spontaneously appear.

The subjects were 7 children from a racially mixed nursery school in Pittsburgh, with an average age of 4.10 years at the beginning of the experiment. Pretests established that they knew how to count, and could recognize the numbers from 1 through 9, but did not know how to add. They participated in the experiment (conducted on the premises of the Learning Research and Development Center at the University of Pittsburgh) during two 15-minute sessions each week over a six-month period.

The initial training procedure for addition made use of wooden blocks. Each child was shown an addition problem, displayed in row form, and printed on a card. The problems used in the experiment were all single-digit problems, with each digit less than or equal to five. The child had in front of him a pool of blocks. He was told to count out m blocks from the pool (for a problem of the form m+n), then count out n blocks, put them together in a pile, and count out how many there were in the pile. Thus, for 3+2, they counted out 3 blocks, then 2 blocks, then the 5 blocks in the total pile. A deliberate attempt was made to make this procedure independent of lexicographic order. Sometimes, the experimenter began with the number on the left of the plus sign and sometimes with the number on the right.

The children were trained in this procedure until they were able to solve all the problems correctly (this usually took about 8 sessions). After this, the main phase (the practice phase) of the experiment began. Each subject was required to solve the addition problems without using blocks. The stimuli were identical to those used in the initial training phase, and the subject was required to respond by means of the apparatus described in the Subtraction Experiment. The subject was initially not instructed in how to perform this new version of the task. If he told the Experimenter that he did not know how to do it, he was shown a version of the algorithm taught in the initial training phase that involved counting on the fingers. This involved counting out one of the numbers to be added on the fingers of one hand, counting out the other number on the fingers of the other hand, and then counting how many fingers had been "used up".

These sessions without blocks continued until the end of the school year, about 25 problems being given in each session. In each of these sessions, the reaction time of the subject to each problem was recorded. Also, the subject's gross overt behavior was recorded on videotape.

Four subjects remained in the experiment until the end of the school year (of the others, 2 dropped out for personal reasons and one was unable to solve addition problems). Of these 4 subjects, three were black and one was white. Two of the black subjects were females.

Only the data of these 4 subjects will be discussed here.

The main analysis was concerned with the practice phase only. Mean reaction times for correct responses were computed for each problem over five-session blocks, and two regression analyses were performed on the data in each block with the exception of Block 1 (where subjects were becoming familiar with the apparatus). One of these analyses computed the regression line of the form

$$RT = a + b.\min(m,n) \qquad (2)$$

while the other computed the regression line of the form

$$RT = a + b.(m + n) \qquad (3)$$

where m and n denote the two numbers being added.

The results of these analyses are shown in Table 1. For each subject, this Table gives the values of R-squared (the proportion of the total variance accounted for by the regression line) obtained by fitting Equation 2 (the Min Model) and Equation 3 (the Sum Model), and whether each regression line is statistically significant (on the basis of the standard F test for the significance of the slope parameter b). Also indicated are the mean proportion of errors for each block and the mean proprtion of covert responses. This latter figure was obtained by examining the videotape records for evidence of counting behavior. It is the proportion of problems in each block which the subject did not solve by overtly counting on his fingers.

The results in this Table indicate that the data of Subject 2 and Subject 4 (who began the experiment 2 months later than the other subjects) is consistently best fitted by the Min Model. The data of Subject 1 is initially best fitted by the Sum Model, except for the last two blocks, where the Min Model is superior. Neither model gives a consistent fit to the data of Subject 3. In other words, two out of the four subjects consistently exhibit the same pattern of reaction times as was found by Groen and Parkman (1972) in data obtained from first graders. One of the reamining subjects (Subject 1) begins by exhibiting a pattern of reaction times more consistent with the algorithm taught in the initial training phase of the present experiment, but makes a transition to the other pattern in the last two blocks. It should be noted that this transition is accompanied by an increase in the proportion of covert responses.

Assuming that reaction times conform to the Min Model because subjects add by setting a mental counter to the largest number and increment the smallest, it must be concluded from the present experiment that children discover this procedure by themselves, even when initially taught a completely different procedure. Moreover, it's emergence tends to be associated with the

TABLE 1

COMPARISON OF SUM AND MIN MODELS

Subject 1

Block	2	3	4	5	6	7
% Errors	15	9	3	3	6	5
% Covert	2	0	4	8	33	34
R^2 Sum	.78*	.45*	.79*	.69*	.50*	.40*
R^2 Min	.65*	.16	.38*	.57*	.59*	.63*

Subject 2

Block	2	3	4	5	6	7
% Errors	18	14	11	6	6	11
% Covert	40	57	57	76	99	100
R^2 Sum	.44*	.51*	.51*	.22	.23	.17
R^2 Min	.65*	.88*	.69*	.38*	.54*	.43*

Subject 3

Block	2	3	4	5	6	7
% Errors	4	3	5	11	12	7
% Covert	0	9	5	30	92	83
R^2 Sum	.14	.71*	.50*	.06	.05	.03
R^2 Min	0	.57*	.27	.13	.30	.10

Subject 4

Block	2	3	4	5	6	7
% Errors	25	12	6			
% Covert	73	61	94			
R^2 Sum	.23	.38*	.32			
R^2 Min	.54*	.41*	.65*			

* denotes slope significantly different from zero at .01 level

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cessation of overt counting behavior. It may be speculated that, as a result of extensive practice in addition, the child begins to internalize his solution process. An algorithm such as that taught in the initial training phase of the present experiment enforces too high a load on the subject's short term memory processes. On the other hand, the algorithm corresponding to the Min Model, that subjects end up using, is highly efficient in this respect.

Teaching Addition Tables

It has sometimes been suggested that the pattern of reaction times proportional to the minimum addend is due to the fact that there are more problems (in the set of all single digit addition problems) with a minimum addend of one than with a minimum addend of two, and so on. By the principles of classical learning theory, so this argument goes, one should expect reaction times proportional to the minimum adden because problems with the smaller minimum addend occur with greater frequency, and the child is exposed to them more often. The purpose of this experiment was to test this hypothesis and, more generally, to examine what occurred when a procedure analogous to practicing the multiplication tables was adopted in the context of addition.

The subjects were 12 children, aged about 4 at the beginning of the experiment, from a nursery school in the Pittsburgh area. The initial training phase was identical to that in the preceding experiment, except that sessions occurred at the rate of one per day rather than two per week. Sessions also occured at this rate in the practice phase. Apart from this, the practice phase differed from the preceding experiment only in that problems were presented in a systematic fashion with respect to the variable of frequency of presentation. In order to do this, the problems were classified as follows:

Type 1 1+1, 2+1, 3+1, 4+1, 5+1
Type 2 1+2, 2+2, 3+2, 4+2, 5+2
Type 3 1+3, 2+3, 3+3, 4+3, 5+3
Type 4 1+4, 2+4, 3+4, 4+4, 5+4
Type 5 1+5, 2+5, 3+5, 4+5

The presentation order can best be described by viewing succesive sessions as being grouped into blocks of four sessions each. Subjects were divided into two groups (Group A and Group B). The presentation order for each group was as follows:

	Group A	Group B
Block 1	All problems	All problems
Block 2	Type 1	Type 5
Block 3	All problems	All problems
Block 4	Type 2	Type 4
Block 5	Types 1, 2	Types 4, 5
Block 6	Type 3	Type 3
Block 7	Types 1, 2, 3	Types 3, 4, 5
Block 8	Type 4	Type 2
Block 9	Types 1, 2, 3, 4	Types 2, 3, 4, 5
Block 10	Type 5	Type 1
Block 11	All problems	All problems

If the frequency hypothesis were correct, one would expect that Type 1 problems should be the fastest for subjects in Group A whereas Type 5 problems should be the fastest for subjects in Group B. Table 2 shows the results of two typical subjects (Subject A was in Group A while Subject B was in Group B). It is clear from this Table that the performance of these subjects is not consistent with the frequency hypothesis. In fact, Type 1 problems are the fastest in both groups. Moreover, for Group B, Type 5 problems are the slowest, even though they have been presented more often than any other problem.

A more detailed analysis indicates that results such as those in Table 2 are due to two major reasons. The first is that subjects tended to learn how to add one before they learned how to add any higher numbers, and hence became able to solve problems involving adding one more quickly. The second is that when subjects learned an addition fact, they also tended to learn the same fact with the addends in reverse order (for example, when 4+1 was learned, 1+4 was learned as well). All this is consistent with the notion that these children actually are learning a counting algorithm of the kind considered in the preceding experiment.

Word Problems

Rosenthal and Resnick (in press) have investigated the performance of children on four types of word problems. The construction of these types of problem is illustrated in the following examples:

Type 1. If Paul started out with 5 boats and he bought 3 boats, how many boats did he end up with?

Type 2. How many boats did John start out with if he bought 2 boats and he ended up with 6 boats?

Type 1R. How many boats did Paul end up with if he bought 3 boats and he started out with 5 boats?

Type 2R. If John ended up with 6 boats and he bought 2 boats, how many boats did he start out with?

TABLE 2

MEAN REACTION TIMES IN MILLISECONDS FOR
EACH TYPE OF PROBLEM

Subject A

		Block										
		1	2	3	4	5	6	7	8	9	10	11
Type of Problem	1	98	74	105	--	57	--	47	--	45	--	42
	2	108	--	104	95	85	--	76	--	93	--	65
	3	120	--	106	--	--	70	66	--	100	--	79
	4	142	--	126	--	--	--	--	46	88	--	80
	5	134	--	133	--	--	--	--	--	--	33	64

Subject B

		Block										
		1	2	3	4	5	6	7	8	9	10	11
Type of Problem	1	96	--	94	--	--	--	--	--	--	16	16
	2	108	--	94	--	--	--	--	16	39	--	44
	3	100	--	122	--	--	38	76	--	45	--	28
	4	121	--	131	101	75	--	52	--	54	--	42
	5	125	110	127	--	109	--	59	--	45	--	85

An exploratory study (Groen, 1971) of reaction times of adults to these problems was undertaken with the goal of discovering how the processes used by adults differed from those used by children. The results suggested the existence of a developmental shift. Whereas Rosenthal and Resnick's fourth graders solved Type 2 problems faster than Type 2R problems, it was found that adults solved Type 2R problems more quickly than Type 2 problems. For both groups, Type 1 problems were solved the most quickly.

There was too much variability in the reaction times of these adult subjects to make possible a precise analysis of the kind used with childrens' addition and subtraction. In an attempt to obtain further insight, verbal protocols were collected in which subjects attempted to reconstruct how they had just solved given problems. An analysis of these protocols indicated that the crucial step in the solution process was the decision of whether to add or subtract, and that subjects tended to use mental imagery extensively in arriving at this decision. However, the precise specification of these solution processes and the developmental changes that take place is a problem that needs considerably more research before a satisfactory solution is found.

The Complexity of Sequential Rules

This research was concerned with the complexity of two kinds of sequential rules. One kind generates sequences of the type frequently found in series completion tasks and commonly used in standard tests of intelligence, such as

ABMCDMEFM...

The other generates so-called sequential concepts or sets of sequences such as

11, 101, 1001, 10001, 100001, ...

The motivation of this research stemmed from two sources. The first was the fact that the research on word problems described in the preceding section indicated that the techniques used successfully with very elementary problems such as single digit addition and subtraction had definite limitations where more complex problems were concerned. Hence there appeared to be a need for the development, at a purely theoretical level, of new ways of defining the complexity of processes. The second source was an attempt to relate Piagetan theory to some notions of information processing. It seemed clear, as an outcome of this exploratory work, that Piagetan theory attempted to define the complexity of general classes of tasks, but was to imprecise to enable detailed predictions of performance to be made. Sequential rules were chosen as the problem area of interest because of their relation to certain Piagetan problems of classification and seriation.

Because of its highly technical nature, this research will not be described here in detail. The basic idea was to apply a novel kind of automata theory (McNaughton & Papert, 1971). This theory allowed one to define the sequential rules that would be recognized by a device, quite akin to that postulated by certain theories of short term memory, that matches short segments of the sequence to internally stored templates. It has been shown (Groen, 1973) that this theory can be applied to classify the complexity of both kinds of seuential rules, and can also represent certain aspects of the induction process. Some applications of these notions to certain problems of hypothesis testing that appear in a more general context have been developed by Simon and Groen (in press).

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