DOCUMENT RESUME

ED 092 384 SE 017 945

AUTHOR Branca, Nicholas A.
TITLE Learning Mathematical Structures.
PUB DATE Apr 74
NOTE 22p.; Paper presented at the annual meeting of the American Educational Research Association (Chicago, Illinois, April 1974)

EDRS PRICE MF-$0.75 HC-$1.50 PLUS POSTAGE
DESCRIPTORS Activity Learning; College Mathematics; *Concept Formation; Games; *Learning Theories; Literature Reviews; *Mathematics Education; *Research; Secondary School Mathematics; *Thought Processes
IDENTIFIERS Group Theory; *Mathematical Structure

ABSTRACT

This paper describes some game situations used to study how subjects learn mathematical structures, in particular the structures of the cyclic groups of orders 2 and 4 and the Klein-four group. A series of experiments are reviewed and the methods used to determine whether subjects did learn the structures are discussed. Differences in strategies, conceptualizations, and interpretations are presented. Possible effects of such factors as age, sex, ability, and method of presentation are also discussed. (Author/LS)

10.02

SE

Learning Mathematical Structures[1]

Nicholas A. Branca

The Pennsylvania State University

Mathematical structures have been a major component in recent reforms in the school mathematics curriculum. Projects such as the University of Illinois Committee on School Mathematics (UICSM) and the School Mathematics Study Group (SMSG) have created innovative programs that emphasize the structural aspects of mathematics. Much of this new emphasis stems from relatively recent developments in pure mathematics. Begle has traced the origin of recent changes in the school mathematics curriculum to the work of Abel and Galois.

> They demonstrated that an examination of the overall structure of a mathematical system, in contrast to computations with the individual elements of the system, was a very powerful mathematical tool and could lead to solutions of problems not otherwise solvable. . . .Consequently, when mathematicians joined with high school teachers in an attempt to improve the secondary school curriculum, their first inclination was to apply to the subject matter normally taught in these schools the same point of view towards mathematics, an emphasis on structure, which had proved successful not only in mathematical research but also in mathematical education at the university level. When it was discovered that this could be done successfully for the secondary school curriculum, the reform movement continued and revised the elementary school curriculum in the same spirit (Begle, 1968, p. 45).

More recently, recommendations have been made for the introduction of even more abstract mathematical structures into the curriculum. Radically revised curricula along the lines suggested by the Royaumont Seminar (Organization for Economic Co-Operation and Development, 1961); the Dubrovnick Report (Organization for European Economic Co-Operation, 1961); and the Cambridge Conference (Educational Services Incorporated, 1963); have been produced and implemented both in this country and abroad.

[1]Paper presented at the annual meeting of the American Educational Research Association, Chicago, April, 1974.

The Secondary School Mathematics Curriculum Improvement Study (SSMCIS), has reconstructed: "the entire curriculum from a global point of view eliminating the barriers separating the traditional branches of mathematics and unifying the subject through study of its fundamental concepts (sets, relations, operations, mappings) and structures (groups, rings, fields, and vector spaces)," (Fehr and Fey, 1969).

In Europe, materials have been produced by Papy (1965), and Kristensen and Rindung (1964), and unified, structure-oriented mathematics curricula have been adopted, notably in Belgium, Switzerland, and Denmark. On the elementary and junior high school level, Dienes has created games based on mathematical structures such as groups, fields, and vector spaces, and has employed the games in mathematics projects in England, Australia, and Canada.

In spite of these trends, however, remarkably little psychological research has been carried out on the processes of learning about complex mathematical structures and this has not gone unnoticed by mathematics educators. Within the last decade especially, many calls for research on the learning of structures and patterns have been made. At a conference held in Greystone, New York, in 1965, Stone spoke of the need for studies of pattern perception and assimilation in mathematics (Fehr, 1966). The National Conference on Needed Research in Mathematics Education held in Athens, Georgia, in 1967, also emphasized the need for research on mathematical thinking (Pingry, 1967). More recently, the SMSG Panel on Research emphasized the importance of the structure of mathematics in its Final Report outlining research programs on the teaching and learning of mathematics (SMSG, 1972, pp. 5-9).

Dienes and Jeeves recognizing this need for research in mathematical thinking, have initiated studies which begin to investigate the processes of learning complex structures. In a 1965 monograph entitled, Thinking in Structures, they reported empirical observations of subjects learning mathematical structures and found some evidence of regularities in the performance of those subjects.

The initiative for the methods used by Dienes and Jeeves came from the work of Piaget, from the writings of Bartlett on thinking, and more immediately from the work of Bruner.

The breakdown of the process of thinking into component parts and the study of strategies used in sorting stimuli into significant classes were undertaken by Bruner, Goodnow, and Austin in a series of experiments in which they sought: "to describe and in some small measure to explain what happens when an intelligent human being seeks to sort the environment into significant classes of events so that he may end by treating discriminably different things as equivalents," (Bruner, Goodnow and Austin, 1956, p. viii). Each experiment possessed six basic elements that Dienes and Jeeves subsequently used in their experiments:

1. There is an array of instances to be tested. . .
2. With each instance, or at least most of them once the task is underway, a person makes a tentative prediction or decision. . .
3. Any given decision will be found to be correct, incorrect, or varyingly indeterminant. . .
4. Each decision and test may be regarded as providing potential information. . .
5. The sequence of decisions made by the person may be regarded as a strategy embodying certain objectives [such as]
   (a) to maximize the information gained from each decision and test of an instance.
   (b) to keep the cognitive strain involved in the task within manageable or appropriate limits and certainly within the limits imposed by one's cognitive capacity; and

(c) to regulate the risk of failing to attain the concept within a specifiable time or energy limit and to regulate any other forms of risk consequent to making a decision and testing it. A sequence of decisions or a strategy may be evaluated in light of these objectives whether the subject "intends" these as his objectives consciously or not. Strategies are not here considered as conscious or deliberate behavior sequences. . .

6. Any decision about the nature of an instance may be regarded as having consequences for the decision maker (Bruner, et. al., 1956, pp. 233-234).

Bartlett, on the other hand, attempted to analyze the whole cognitive process rather than its parts. He defined thinking as "the extension of evidence in accord with that evidence so as to fill up gaps in the evidence; and this is done by moving through a succession of interconnected steps" (Bartlett, 1958, p. 74). Bartlett, summarizing his main points, suggested that: "intelligence may be related to the amount of information (items) required to achieve a gap-filling which is most uniform throughout the operator's cultural group. The most intelligent may be those who, with the smallest amount of information (items) produce that for which others need more information."

Dienes and Jeeves, rather than studying the whole process as Bartlett had done, or the parts as Bruner had done, considered the whole process "through the detailed ways in which its parts are put together" (Dienes and Jeeves, 1965, p. 12).

During the late fifties, Dienes conducted a study of the process of concept formation in children. Working with the Leicestershire Mathematics Project, Dienes attempted to determine the psychodynamics of the concept formation process. He described his methods as: "Akin to Piaget's own procedure, i.e. the individual observation of a large number of subjects performing some concept formation tasks, devised with a view to obtaining some information on certain particular modalities, and administered in such a way that the successive stages of the concept formation process can be systematically observed" (Dienes, 1959, p. 11). One of the tasks Dienes devised was based on the mathematical theory of groups. The children were presented with a diagram of a dance floor having three fixed circles of different colors painted on it. There were six dance steps possible in which dancers moved from one circle to another. Geometrically, the three circles on the dance floor corresponded to the three vertices of an equilateral triangle. The six dance steps represented the three rotations of 0, 120, or -120 degrees about the center and the three reflections of the triangle in its medians. Thus the structure of the game was isomorphic to the structure of the dihedral group of order six.

The research was described by Dienes as a preliminary work, and questions were raised that set the foundation for the later experiment with Jeeves:

> What patterns occur together i.e. what various forms of concept organization, as measured along dimensions yet to be experimentally determined, tend to occur in the same people?. . .The present experiment, with its tentative conclusions, is an attempt to begin this process of sorting out typical concept organizational patterns, together with their connections with other aspects of the personality. . . If a process were available for the objective determination of such groups of patterns, it would then be possible to state what these groups had in common and exactly how they differed from one another and corresponding work to what has been done for boys and girls in this study, could be undertaken for the groups corresponding to the different patterns. The original intention was to do just a comparative study. . .it was only when it was decided that the available methods were unsuitable. . .that it became inevitable to narrow down the scope of the present enquiry. It is hoped that by putting the results of this preliminary work with the problems, mathematical as well as psychological, that it poses, before the public, some research workers will be stimulated to undertake enquiries with a view to enlarging our present very scanty knowledge in this most important field (Dienes, 1959, pp. 64-65).

The extent to which the subjects succeeded on the tasks involving the group structure posed interesting questions about how children learn mathematics. Dienes subsequently attempted to coordinate learning with the psychology of thinking and with the structures pecilliar to mathematics. His purpose was to promote the early devulopment of a theory of mathematics learning. A crucial view Dienes expressed is that "mathematics is based on experience; it is the crystallization of relationships into a beautifully regular structure, distilled from our actual contacts with the real world (Dienes, 1960, p. 11). Dienes questioned the stimulus-response theory of learning. He claimed that a stimulus-response explanation of mathematics learning is inadequate since "the accent in mathematics is more on structure and less on content." To Dienes, the very essence of mathematical thinking is pattern and structure. Using evidence from the work of Piaget, Bartlett, the Cognition Project at Harvard, and the Leicestershire Mathematics Project, Dienes sketched a skeleton theory of mathematics learning that consisted of the: Dynamic Principle, the Constructivity Principle, the Mathematical Variability Principle, and the Perceptual Variability Principle.

Dienes called for an investigation of mathematical thinking which studies the constructive process while it is taking place. To him: "The problem of learning is essentially how to find a kind of 'best fit' between the structure of the task and the structure of the person's thinking. For the process to be explained by any kind of intelligible theory, both these structures must be taken into account and at least some attempt made at quantitative description" (Dienes, 1960, p. 39).

The question of how such learning took place and why it could not be explained by existing learning theories led Dienes to join Bruner at the Harvard University Center for Cognitive Studies. Together they established the Harvard Mathematics Learning Project during the academic year 1960-1961. Starting from Dienes' theories and his experimentation at Leicester, Bruner and Dienes set out to explore the cognitive processes involved in learning complex mathematical structures. The work was exploratory and after the termination of the project, Dienes together with Jeeves began to formulate some of the problems encountered "in ways in which they could be put into experimental paradigms (Dienes and Jeeves, 1965, p. 13). Through their collaboration, Dienes and Jeeves consolidated much of the previous research on strategies of thinking and isolated some fruitful problems for future research.

The first, and major, study they devised concerned the emergence of structures in terms of which we think. The main question was, "How do we sort the apparent chaos of our environment into anything like order?"

More specifically:

> What individual strategies are distinguishable, and do these naturally subdivide into types?
>
> Under what conditions does transfer occur between structures?. . .
>
> Under what circumstances are structures recognized as forming parts of other, more extensive structures?. . .
>
> Under what circumstances will structures be generalized into more extensive structures, comprising the one already known?. . .
>
> With what kinds of properties must we endow a structure A, and not a structure B, so that given the evidence for B, the structure A will be expected?. . .

Are the answers to any of the above questions different for children and adults, for males and females, or for children of different ages, or for adults of different ages? (Dienes and Jeeves, 1965, pp. 13-17).

To answer these questions, tasks based on mathematical group structures were devised. Briefly a group is defined as an operational system (a set of elements with a closed binary operation defined over it) possessing the property of associativity, an identity element, and an inverse for each element (also referred to as the roundness property). More specifically, a set [A, B, C. . .] exists such that the following hold:

(i) For any elements A and B, $A \cdot B$ (A operating on B) produces some unique element contained in the set. This defines an operational system.

(ii) For any elements A, B, and C, $(A \cdot B) \cdot C = A \cdot (B \cdot C)$, or the result of operating on C with the result of A operating on B is the same as the result of A operating on the result of B operating on C. This property is called associativity.

(iii) There exists one element of the set, called the identity element I, such that for any element A of the set

$$I \cdot A = A \cdot I = A$$

That is, the identity element operating on any element of the set results in that element, and any element operating on the identity element results in that element.

(iv) For each element A in the set, there exists an element A^{-1} such that the operation of either one on the other results in the identity element. That is, $A \cdot A^{-1} = A^{-1} \cdot A = I$. A^{-1} is called the inverse of A. This property is called roundness.

It can be shown that there is only one two-element group structure and only two four-element group structures. Dienes and Jeeves used tasks based on the two-element group and both four-element groups in their experiment. The structure of these groups can be illustrated by tables showing the result of any element operating on any element (See Tables 1, 2, and 3). For a detailed account of the structure and properties of groups and the nature of group theory, see Groups, by Georges Papy, (1964).

The tasks consisted of two identical sets of cards (the elements of the group), and a simple exposure apparatus. The experimenter and subject each had a set of cards, and the experimenter used the exposure apparatus to display a card to be operated on by the subject. Subjects were directed to play any card they chose and to predict what the result would be. The experimenter then displayed the card that was the result of the operation as defined by the group structure. This card would then be acted upon in the next instance. For example, if the Klein four-group structure were being used, the experimenter and subject would each have a set of four-colored cards, one with the properties of the identity element, the yellow card, and the others, the orange, blue and green cards, with the properties as indicated in Table 3. If the orange card were displayed, and the subject played the green card, then the experimenter would next display the blue card. The next instance would then begin with the blue card displayed. An intricate design for the study was devised, and Dienes and Jeeves explained it as follows:

> We thought that the order in which the two-game and the four-game were presented would make a difference to the performance and so the subjects were divided into 2 groups: the 2-4 subjects and the 4-2 subejcts. We also thought that the use of identical or different symbols was relevant and so when passing either from the two- to the four-game or from the four- to the two-game, half the subjects had their symbols changed and the other half did not. . .

> In order to answer developmental questions, children, whose average age was eleven and a half, were used as well as adults. . . We also decided that with the adults a distinction should be made between a free selection of strategies. Therefore, some adult subjects had their cards played for them by the experimenter, others being able freely to select them. These were the reception and selection subjects respectively. In order to answer questions about possible sex differences, some subjects were male and some were female, both in the case of adults and in the case of children. In order to test the difference in performance between two different structures, half the children were given the M4 group and the other half the Klein group (Dienes and Jeeves, 1965, pp. 34-35).

Each subject played two games, one with the two-group structure and the other with either the M4 or Klein group structure. The experimenter in each case explained the instructions for the game and played according to the method previously described. A detailed record of the cards displayed, played, and predicted was kept, and subjects' explanations of how the games worked were recorded. For the two-game, the explanations (called evaluations) were classified as either Operator, indicating that the subject considered the card played as operating on the card displayed, or Memory, indicating that the subject merely memorized the combinations and their results. When each of the four-games was played, a Pattern evaluation, indicating that the subject saw the game as composed of patterns, was also observed. In addition to pure Operator, Pattern, and Memory evaluations, combinations of these three were also observed.

From the records of the cards displayed, played, and predicted, the experimenters calculated various types of errors the subjects made and related these to hypotheses they could have held. To account for the evaluations, Dienes and Jeeves devised an operator strategy score and a pattern strategy score by measuring patterns in the choice of cards that corresponded to the respective evaluation types. Major conclusions of the study were that:

> there is considerable evidence of the existence of a positive relationship between measured strategies and subjects' evaluations. Those subjects who evaluated the tasks operationally tended to have higher operator scores than those who did not. Those subjects who evaluated the tasks in terms of patterns tended to have higher pattern scores than the remainder of the subjects.
>
> It is more advantageous to begin with the more complex task and follow this by the simpler task, from the point of view of explicit evaluation in both these tasks. The procedure of throwing subjects in at the deep end appears to pay off both at the deep end and at the shallow end (Dienes and Jeeves, 1965, pp. 75-76).

Dienes and Jeeves also presented evidence for a hierarchy of evaluations (Operator - Pattern - Memory, in descending order of efficiency) that they claim was validated by its relationship with the number of instances that the subjects who freely selected cards required to solve the task. They also found a relationship between the hierarchy and the total number of errors that the reception subjects made. Pattern evaluations occurred more frequently than Operator evaluations, and the latter were more in evidence in the M4 task than in the Klein task. Dienes and Jeeves speculated that the uniformity of the roles of the different operators in the Klein task makes the emerging structure appear more as a pattern than as a set of operational relationships. They called for more research on this point. Another important finding was no correlation between intelligence-test scores and measures of performance on the group tasks.

A number of additional research studies have been influenced by the work of Dienes and Jeeves. One study (Branca, 1971) investigated the consistency of subjects' evaluations and strategies across group structured tasks. One hundred subjects, selected from a private residential summer school for junior and senior high school girls, each performed three tasks; a Color Game, a Light Game, and a Map Game.

The Color Game duplicated, as closely as possible, the four-game with the Klein group structure used by Dienes and Jeeves (1965).

The Light Game embodied the Klein group structure in a switch-light apparatus. A light bulb and a four-pole double throw switch were located at each vertex of a square. The switch-light combinations were labeled Saturn, Mars, Venus, and Jupiter. One bulb was lit initially. By throwing a switch, the subject caused that bulb to go out and then one of the four bulbs to light. A play consisted of throwing a switch and predicting which bulb would light. The object was to learn the rules of the game. The Light Game was isomorphic to the Color Game, with the bulb that was lit playing the part of the card in the window and the switch that was thrown playing the part of the card the subject played. Saturn, Mars, Venus, and Jupiter corresponded to Yellow, Orange, Blue, and Green, respectively.

The Map Game consisted of a miniature car and a map of the United States on which were marked five cities (Birmingham, Chicago, Denver, San Francisco, and Washington) connected by eight fictitious highways. (A highway was defined as linking two and only two cities). The car was placed on the map at one of the cities. The subject chose a city as her first destination and then was told which city she would encounter first on her journey if she were to traverse the least number of passable highways enroute (the condition of the highways was not initially known by the subject--they could be closed, open in one direction only, or open in both directions). After the subject had moved to the first city on her route, she was then free to choose any destination again, predicting which city she would then visit first. The object was to learn the rules of the game, that is, the condition of each highway. The task has a network structure, not a group structure, as can be seen in Table 4 but like the other two tasks, 16 outcomes of a binary operation must be found. The operation was traveling from the departure city to the destination city, the outcome was the city first visited, and the 16 starred outcomes in Table 4 allowed the subject to infer the condition of each direction of the eight highways.

The tasks were presented in individual interviews, one task per interview, at intervals of approximately two weeks. The order of the tasks was the same for all subjects: Color Game, Map Game, Light Game. The interviewer kept track of the subject's moves and predictions in learning each game and the evaluation she gave at the end of the interview of how the game worked.

Each subject played each game until she thought she knew all of the rules. She was asked to give the rules and was encouraged to continue playing if she overlooked or forgot some of them.

The major results of the study indicated that subjects did describe the group-structured tasks according to the three fundamental evaluations found by Dienes and Jeeves and support was obtained for the relative frequency of evaluations and the hierarchy of evaluations identified by them. Unfortunately, however, the strategy scores did not show the expected relationships to the evaluations. Many differences between the two studies could account for this discrepancy and it was conjectured that the strategy scores, as Dienes and Jeeves defined them, are insensitive to the strategies subjects may be using since in Dienes and Jeeves' tasks the subject is constrained at each move by the outcome of the preceding move.

A second study (Branca, in press) compared the constraining game playing situation used by Dienes and Jeeves with a modified free choice situation. The modified free choice game playing situation consisted of the same rules and procedure as the original color game. However, rather than have the displayed card be determined by the previous play, the subject was given free choice of the displayed card as well as the card he played. Thus for each play a subject chose a card to be displayed, then a card to play, and finally made a prediction of the outcome.

A randomly selected half of a group of 36 experienced mathematics teachers performed the Color Game described above under the original restricted game playing situation conditions. The other half performed the same task under the modified free choice conditions.

The results of the study indicated the superiority of the free choice game playing situation over the original restricted game playing situation. Subjects who played the free choice game gave more successful evaluations than those who played the restricted game (17 vs. 12), took fewer trials to learn the rules of the game, and were more systematic in playing the game, reporting the outcomes, and recalling what strategies they had used.

This latter point is important since it was the conscious application of strategies that the study attempted to investigate. Allowing the subject to choose both elements of the binary operation creates a situation in which sixteen pairings are possible at each play compared to four which is the case in the original situation. Being able to determine both cards allowed the possibility of testing for commutativity, an option not open in the restricted game, and also to systematically investigate pairs of cards in any order desired. The remarks made by subjects at the conclusion of the free choice game indicated a conscious effort at these strategies. However, the small number of subjects in each instance precluded any generalizations.

The recent study by Kellogg (1973) also investigates the effect of the task format upon the degree of learning, again comparing performance when the learner's freedom to explore the structure of the tasks was restricted by previous trials with performance when the learner was allowed to examine any part of its structure at any time during the learning of the task. In addition, Kellogg examined the effect on the degree of learning of the order of presenting two structurally related learning tasks: one order (shallow-end) presented the simpler task first followed by the more complex one; the other order (deep-end) presented the more complex task first, followed by the simpler one.

The tasks used in the study were based on mathematical groups of three, four, and five elements, and were given to subjects in the form of games, presented by a computer, each subject playing at a cathode-ray tube console. Elements of the group were represented on the cathode-ray tube screen by geometric figures; on each play a subject predicted what figure would result form a pair of other figures; the computer then responded with the correct result, in accordance with therules of the group operation. Learning of the task consisted of learning the results of the possible cominations of figures; performance was measured by testing a subject after learning on his knowledge of the combinations. The four-group task was given first as a training task, and the order of the three- and five-group tasks was varied, with the three-five order forming a shallow-end task sequence and the five-three order a deep-end sequence. Two task formats were used: one, called the state-operator-state format, closely parallelled Dienes and Jeeves' task format; the other, called free-choice, allowed the subejct complete freedom in exploring the task structure.

Classifications of subjects' responses on a questionnaire, using a classification scheme devised by Dienes and Jeeves, supported Dienes and Jeeves' and Branca and Kilpatrick's findings that subjects tend to be consistent from one task to another in how they describe the tasks.

Subjects given the deep-end task sequence did not improve their rankings from the second to the third tasks in comparison with subjects given the shallow-end task sequence ($p=.05$), for either of the two task formats. Consequently, no evidence was obtained for a deep-end effect for adults. Performance scores were higher for those subjects given the free-choice format tasks than for those given the state-operator-state format tasks, for both shallow-end and deep-end subjects, on both the three-group ($p<.01$) and the five-group ($p<.05$) tasks. A possible explanation of the greater difficulty of the state-operator-state tasks is that the subjects found it difficult during learning to frame and test hypotheses about the structure; whereas on the free-choice format tasks, the subjects could apply their individual learning strategies without hindrance. This is consistent with the results and discussion reported by Branca.

Another study which supports the notion of consistency of strategies across embodiments (models) was conducted by Chilewski (1973). Using concrete materials, the Klein group and the modular (4) group were each represented by two types of embodiments - a Static model (a game using fixed objects) and a Permutation Model (a game using transformations). Fifty-two fourth grade children were randomly selected and assigned to one of the following treatments: (a) Those exposed to the Static Model first then the Permutation Model of the Modular (4) group; (b) those exposed to the Permutation Model first then the Static Model of the Modular (4) group; (c) those exposed to the Static Model first then the Permutation Model of the Klein group; and (d) those exposed to the Permutation Model first, then the Static Model of the Klein group. For independent sample of subjects, consistency of strategies across the Modular (4) group and the

Klein group and across model types was found. For individuals across tasks, individual consistency across model types was founded. The order of presentation of model types had no significant effect.

All of the above studies indicate that in learning mathematical structure, students can and do perceive them differently. These perceptions, given as evaluations, are consistent across different embodiments of the same structure. Although these evaluations indicate how students perceive the structures, they do not give the total picture. Missing is information on how students organize these structures in their memories while learning, and the relationship between that organization and measures of achievement. The remaining papers will look at that aspect of the question.

BIBLIOGRAPHY

Bartlett, Sir Frederic. Thinking. New York: Basic Books, 1958.

Begle, E. G. "Curriculum Research in Mathematics," Research and Development Toward the Improvement of Education, ed. Herbert J. Klausmeier and George T. O'Hearn (Madison, Wisc.: Dembar Educational Research Services, 1968), 232-244.

Branca, Nicholas A. Strategies in Learning Mathematical Structures. (Doctoral Dissertation, Teachers College, Columbia University, New York), Ann Arbor, MI: University Microfilms, 1971, No. 70-26765.

Branca, Nicholas A., and Kilpatrick, Jeremy. "The Consistency of Strategies in the Learning of Mathematical Structures," Journal for Research in Mathematics Education, Vol. 3, No. 3, May, 1972, pp. 132-140.

Branca, Nicholas A. "A Study of Experienced Mathematics Teachers' Learning a Mathematical Structure," Australian Mathematics Teacher, in press.

Bruner, Jerome; Goodnow, Jacqueline; and Austin, George. A Study of Thinking. New York: John Wiley and Sons, 1956.

Chilewski, Roger J. Consistency of Strategies Across Mathematical Structures and Embodiments. Unpublished Doctoral Dissertation, Northwestern University, 1973.

Dienes, Zoltan P. Building Up Mathematics. London: Hutchinson Educational, 1960.

Dienes, Zoltan P. Concept Formation and Personality, Leicester: Leicester University Press, 1959.

Dienes, Zoltan P. and Jeeves, Malcolm A. The Effects of Structural Relations on Transfer. London: Hutchinson Educational, 1970.

Dienes, Zoltan P. and Jeeves, Malcolm A. Thinking in Structures. London: Hutchinson Educational, 1965.

Educational Services Incorporated. Goals for School Mathematics, The Report of the Cambridge Conference on School Mathematics. Boston: Houghton Mifflin Company, 1963.

Fehr, Howard F. (ed.). Needed Research in Mathematical Education. New York: Teachers College Press, 1966.

Fehr, Howard F. and Fey, James. "The Secondary School Mathematics Curriculum Improvement Study," American Mathematical Monthly, LXXVI (December, 1969), 1132-1137.

Kellogg, Howard M. Task Format and the Deep-end Hypothesis. Unpublished Doctoral Dissertation, Teachers College, Columbia University, 1973.

Kristensen, Erik and Rindung, Ole. Matematik I, II, III. Copenhagen: G.E.C. Gads Forlag, 1964.

Organization for Economic Co-Operation and Development. New Thinking in School Mathematics. Paris: OEEC, 1961.

Organization for European Economic Co-Operation. Synopses for Modern Secondary School Mathematics. Paris: OEEC, 1961.

Papy, Georges. Groups, trans. Mary Warner. London: Macmillan and Company, 1964.

Papy, Georges. Mathematique Moderne. Paris: Didier, 1965.

Pingry, Robert. "Summary of Discussion," Proceedings of National Conference on Needed Research in Mathematics Education, Journal of Research and Development in Mathematics, ed. Joseph R. Hooten, Jr. (1967), Vol. 1, No. 1, 44-46.

School Mathematics Study Group, Newsletter No. 39, Final Report of the SMSG Panel on Research, Stanford: August, 1972.

TABLE 1

STRUCTURE OF THE
TWO-GROUP

Operator	Operand	
	I	A
I	I	A
A	A	I

TABLE 2

STRUCTURE OF THE CYCLIC
OR M4 GROUP

Operator	Operand			
	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	A	I
C	C	B	I	A

TABLE 3

STRUCTURE OF THE KLEIN FOUR-GROUP

Operator	Operand			
	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	I	A
C	C	B	A	I

ISOMORPHIC STRUCTURE FOR THE COLOR GAME

Card Played	Card Displayed			
	Yellow	Orange	Blue	Green
Yellow	Yellow	Orange	Blue	Green
Orange	Orange	Yellow	Green	Blue
Blue	Blue	Green	Yellow	Orange
Green	Green	Blue	Orange	Yellow

TABLE 4

Outcomes of Choosing a Destination City

from a Departure City

Departure City	Destination City				
	Birmingham	Chicago	Denver	San Francisco	Washington
Birmingham	Birmingham	Washington	San Francisco*	San Francisco*	Washington*
Chicago	Denver	Chicago	Denver*	Denver*	Denver*
Denver	Birmingham*	Chicago*	Denver	Birmingham*	Birmingham*
San Francisco	Denver	Denver*	Denver*	San Francisco	Denver
Washington	Birmingham*	Chicago*	Chicago*	Birmingham	Washington

*An outcome that can be used to infer the condition of the eight highways.