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ABSTRACT

Nonadditive analogues of what is considered to be the essential mathematics of additive measurement are presented. The scope of this paper is defined when measurement theories are reformulated in such a way to assign transformations of numbers to transformations of natural objects. This paper is limited to theories of measurement in which empirical transformations are represented by affine measurement. (CK)

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NONADDITIVE ANALOGUES OF THE BASIC MATHEMATICS OF ADDITIVE MEASUREMENT

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NONADDITIVE ANALOGUES OF THE BASIC MATHEMATICS OF ADDITIVE MEASUREMENT[1]

I. INTRODUCTION

Recently there has been vigorous development of theories of additive measurement.[2] One reason for the success of these theories is the fact that the basic mathematics is very well understood. There is a kind of nonadditive measurement which is becoming important in psychology. This paper contains nonadditive analogues of what I consider to be the essential mathematics of additive measurement.

I.1 Measurement as the Study of Transformation and the Scope of the Paper

The scope of this paper is most easily specified when measurement is regarded in a special way. Measurement is ordinarily thought of as a matter of assigning numbers to natural objects or of embedding empirical relational systems in numerical relational systems (Suppes and Zinnes, 1963). However, many measurement theories are easily reformulated as studies of the ways to assign transformations of numbers to transformations of natural objects.

For example, we may either directly concern ourselves with the length of an object or we may think of an object as defining a transformation of the classes of equally long objects. The transformation defined by an object x is the mapping of classes induced by the mapping of objects to objects lengthened by concatenation with x.

In fact the usual theories of length are not formulated in terms of transformations. However, it would be routine to reformulate some of them (Suppes, 1951). Furthermore, there are theories such that, in their present form, the representation of transformations plays an important role. For two examples see Krantz's study of additive conjoint measurement (Krantz, 1964) and Levine's uniform systems (Levine, 1970).

In additive theories, the empirical transformations are characteristically represented by additive transformations of numbers $x \to x + \text{constant}$. In nonadditive analogous theories, the empirical transformations are represented by other transformations.

This paper is limited to theories of measurement in which empirical transformations are represented by the affine transformations[3] $x \to ax + b$. This sort of measurement will be called affine measurement. An example from signal detection theory is given in Appendix I. For other examples, see Levine, 1972.

I.2 Comments on the Use of Groups and Full Orderings Rather Than Semigroups and Partial Orderings

The additive transformations form a group with a full order; inverses and composites of additive transformations are additive transformations and the usual ordering on numbers induces an ordering on additive transformations in which any two transformations are comparable. However, the various measurement theories developed for understanding particular experimental situations rarely deal with all of the elements of the group. For example, in the measurement of length, the positivity of length leads one to consider fully ordered semigroups or groups in which only positive elements are compared. As an example from experimental psychology. in the use of disjunctive reaction time in sensory measurement (Falmagne, 1971) one compares moderately intense stimuli which are sufficiently different to be discriminated. In this setting it is natural to consider a partial order or to regard the transformations as elements of a local semigroup. There are a large number of complex and subtly different other partial systems.

The partial systems are absolutely essential for a complete understanding of any experimental problem. However, my study of the measurement literature convinces me that many of the most important ideas of additive measurement appear in their simplest form in fully ordered groups. Consequently in this paper the affine transformations will be regarded as elements of a fully ordered group.[4]

In both the additive and affine groups all of the inequalities in the group can be deduced from some of the inequalities. The affine group happens to contain a subgroup isomorphic to the fully ordered additive group with the following property: Every inequality in the affine group can be deduced from inequalities in the additive subgroup (Part V, Cor. 3). To expedite the translation of partial additive results into affine results we will study the problem of embedding a partially ordered group with a fully ordered subgroup in the fully ordered affine group. (See the first part of section V for details.)

I.3 Intended Role of the Main Result

The main result is a qualitative characterization of the subgroups of the group of affine transformations. This result is an analogue of a result attributed to Holder characterizing the additive transformations.

Holder's theorem, or more properly the ideas appearing in the various proofs of the theorem and its refinements, play an important role in additive measurement. They provide a core of familiar mathematics which enables one to quickly understand several very different measurement theories; they suggest experimental tests of measurement models and they suggest algorithms for computing numerical measurements.

For a nontechnical statement of the qualitative characterization see section II.1. For a more detailed statement see the final paragraph of Part V. For a discussion of how these results can be used to reduce affine measurement computations to additive measurement problems, see section II.2.

It is hoped that these results will play a role like Holder's theorem and accelerate the development of affine theories of measurement. To increase the accessibility of the results for nonspecialists, a discussion of ROC curves and the idea of using group representations as an alternative to curve fitting are offered in the appendix.

II. INFORMAL DISCUSSION OF RESULTS

In this section the mathematical results are informally presented and discussed. The mathematically sophisticated reader is advised to skip to section V.

An attempt has been made to avoid technical terms. Some of those which could not be avoided are defined in the next section.

Suppose $\underset{\sim}{G}$ is an ordered group defined either by a psychological theory or by experimental observations. Suppose further there is reason to believe that $\underset{\sim}{G}$ is structurally like[5] a subgroup of the affine transformations $x \to ax + b$. The results may be viewed as an attempt to solve the following three problems:

1. Experimental Verification: Find qualitative conditions subject to experimental test which are logically equivalent to the assertion that the given group is structurally like a subgroup of the affine group.
2. Measurement or Representation: Describe a procedure for measuring the slope and intercept parameters of the elements of $\underset{\sim}{G}$.
3. Uniqueness: Discover the extent to which the numbers assigned by the measurement procedure are unique.

II.1 Experimental Verification

By a direct calculation it is easy to show that if $f(x) = ax + b$ and $g(x) = cx + d$ then fg equals gf if and only if $b(1 - c)$ equals

$d(1 - a)$. From this formula it follows[6] that if f,g,h are any three affine transformations other than the identity transformation $f(x) = x$, then

$$fg = gf \text{ and } gh = hg \text{ imply } fh = hf .$$

In other words in the set of affine transformations other than the identity transformation the relation

$$f \text{ commutes with } g$$

is a transitive relation.

This gives a simple, testable necessary condition on the elements of $\underset{\sim}{G}$ other than the identity: commutativity is a transitive relation. This is the characteristic property of groups structurally like the affine group. When adjoined to conditions taken from the theory of additive measurement, this transitivity is both necessary and sufficient.

II.2 Measurement Results

To measure the elements of $\underset{\sim}{G}$ it is necessary to associate each element g of $\underset{\sim}{G}$ with a pair of numbers $\alpha(g)$ measuring slope and $\beta(g)$ measuring the intercept parameter. This task has been reduced to additive measurement. The key idea is to find an additive subgroup $\underset{\sim}{G}'$ of $\underset{\sim}{G}$ and study the way the elements of $\underset{\sim}{G}$ transform the elements of $\underset{\sim}{G}'$. The subgroup is called the derived group of $\underset{\sim}{G}$. It is defined in the next section IV.

Suppose that ψ is a monotonic, number valued isomorphism of $\underset{\sim}{G}'$ into the additive real numbers. The problem of computing some such ψ is an extensively studied additive measurement problem (Krantz, Suppes, Luce and Tversky, 1971, especially sections 2 and 9.4). Then the following steps define some functions α, β with all the desired properties. (The rationale for these steps is given in section V. In the sequel $[f,g]$ abbreviates $f^{-1}g^{-1}fg$.)

1. Choose an x and y in $\underset{\sim}{G}$ such that x and y do not commute.
2. Let g_1 be $[x,y]$ if $\psi[x,y]$ is positive and $[y,x]$ if it is not.
3. Let g_0 be x if y commutes with g_1 . Otherwise put g_0 equal to y .
4. Define α by $\alpha($ $\psi(gg_1g^{-1}) \div \psi(g_1)$.
5. Define β by $\beta(g) = \psi([g_0,g]) \cdot \alpha(g)$.

II.3 Uniqueness Results

In additive measurement, the measure of a transformation is essentially a number and this number is unique up to a certain kind of linear transformation. In affine measurement, each transformation is represented by a pair of numbers and these numbers are unique up to a certain kind of linear transformation. In particular, if α, β define a representation of $\underset{\sim}{G}$ then α', β' also do if and only if for some positive k and real t

$$\alpha' = \alpha$$

$$\beta' = k\beta + t(1 - \alpha) \quad .$$

To relate this to the usual classification in measurement, α is an absolute scale and the ratio $\beta/(1 - \alpha)$ is an interval scale.

III. NOTATION, DEFINITIONS, ABBREVIATIONS

Partial order: reflexive, antisymmetric, transitive relation

Full or linear order: a partial order which is also connected; i.e.,

$$a \geq b \quad \text{or} \quad b \geq a$$

Monotonic function: order preserving function; i.e.,

$$x \geq y \quad \text{implies} \quad f(x) \geq f(y)$$

Strictly monotonic function: monotonic plus

$$f(x) \geq f(y) \quad \text{implies} \quad x \geq y$$

Ordered group: a group with a partial order such that all the mappings defined by left multiplication and right multiplication are monotonic

Archimedean ordered group: an ordered group in which

$$a \leq aa = a^2 \leq b \quad \text{implies} \quad b \leq a^n \quad \text{for some iterate } a^n \text{ of } a .$$

Commutator: an element g in group which can be written in the form

$$g = x^{-1}y^{-1}xy \quad \text{for some group elements } x \text{ and } y$$

$[x,y]$: abbreviation for $x^{-1}y^{-1}xy$

Derived group: The smallest subgroup of a group containing all of the commutators. Equivalently, the set of products of commutators.

G' : abbreviation for the derived group of G .

Affine group: any one of the three ordered groups defined immediately below.

<u>Affine group of transformations</u>: The ordered group $\underset{\sim}{A}_T$ of monotonic increasing transformations of the reals with product defined by comp tion and order given by $f \geq g$ iff $f(x)$ is eventually as large as $g(x)$ (i.e., there is some x_0 such that $x \geq x_0$ implies $f(x) \geq$

<u>Affine group of matrices</u>: The ordered group $\underset{\sim}{A}_M$ of matrices $\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$: a is positive$\}$ with $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \succsim \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}$ iff

$$(a = c \text{ and } b \geq d) \text{ or } a > c$$

and product given by matrix multiplication. The mapping $[x \to (ax +$ $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ is easily shown to be a strictly monotonic isomorphism.

<u>Affine group of pairs</u>. The anti-isomorph $\underset{\sim}{B}$ of the affine group of matr

$$B = \{< a,b > \ : \ a \text{ is positive and } b \text{ is real}\}$$

with multiplication $< a,b > < c,d > = < ac, bc + d >$ and lexicograph ordering. It is the group one gets from the affine group $\underset{\sim}{A}_T$ if fun tions are written to the right of their argument; i.e., if the affine transformation f evaluated at x is denoted by $(x)f$ rather than $f(x)$. It is used only to simplify notation.

<u>Hölder's Theorem</u> (Fuchs, 1963, page 45): If $\underset{\sim}{G}$ is a linearly ordered arch medean group, then there is a strictly monotonic isomorphism into the group of real numbers with addition and the usual ordering.

<u>Hion's Theorem</u> (Fuchs, 1963, page 46): If u and v are strictly monotonic real valued isomorphisms mapping $\underset{\sim}{G}$ into the additive reals, then for some positive number a,

$$u(g) = av(g) \text{ for all } g \text{ in } \underset{\sim}{G} .$$

IV. FORMAL STATEMENT OF RESULTS AND PROOFS

The mathematical problem arising in psychology is to find all isomorphisms of an ordered group G into the affine group. Instead of directly attacking the problem of monotonically embedding a fully ordered group $\underset{\sim}{G}$ in the fully ordered affine group $\underset{\sim}{A}$ we ignore most of the inequalities on $\underset{\sim}{G}$ and study the isomorphisms $\psi: \underset{\sim}{G} \to \underset{\sim}{A}$ such that ψ restricted to the derived group is monotonic. The mathematical justification of this approach is corollary 3 which shows that there is a sense in which all the inequalities in $\underset{\sim}{G}$ can be deduced from inequalities in $\underset{\sim}{G}'$. The scientific justification is the fact that, in most applications, direct experimental verification of certain inequalities involving elements outside of $\underset{\sim}{G}'$ is practically difficult or scientifically questionable. A final consideration in defense of this indirect strategy is that it gives a simple theory with a sharp separation between additive and nonadditive ideas.

The concern for isomorphisms with monotonic restrictions rather than monotonic isomorphisms leads to the study of partially ordered groups with linearly ordered derived groups. Later the additional condition that $\underset{\sim}{G}'$ be archimedean will be imposed.

In the applications, the affine group enters as an ordered group of matrices $\underset{\sim}{A}_M$ or an ordered group of real functions $\underset{\sim}{A}_T$. (See preceding section for definitions.) However, notation and computations are simplest when the affine group is regarded as a set of ordered pairs $\underset{\sim}{B} = \{ \langle a,b \rangle : a$ is positive and b is real$\}$ with multiplication $\langle a,b \rangle \langle c,d \rangle = \langle ac, bc + d \rangle$ and order induced by the anti-isomorphism Θ,

$$\theta < a,b > = \begin{pmatrix} 1/a & -b/a \\ 0 & 1 \end{pmatrix} .$$

With θ all results for $\underset{\sim}{B}$ can be immediately translated into results for $\underset{\sim}{A}_M$ and $\underset{\sim}{A}_T$.

We begin with a uniqueness theorem. See corollary 2 for a less complicated version of it.

Uniqueness Theorem: *Let $\underset{\sim}{G}$ be a nonabelian partially ordered group with linearly ordered group $\underset{\sim}{G}'$. If ϕ and ϕ'*

$$g \to < \alpha(g), \beta(g) > = \phi(g)$$

$$g \to < \alpha'(g), \beta'(g) > = \phi'(g)$$

are group isomorphisms of $\underset{\sim}{G}$ such that for all $u, v \in \underset{\sim}{G}'$

$$u \geqq v \ \text{iff} \ \phi(u) \geqq \phi(v) \ \text{and} \ u \geqq v \ \text{iff} \ \phi'(u) \geqq \phi'(v)$$

then

$$\alpha' = \alpha$$

and for some positive k and real s

$$\beta'(g) = k\beta(g) + s[1 - \alpha(g)]$$

for all g in $\underset{\sim}{G}$.

We prove the equivalent assertion: If $\phi = \langle \alpha, \beta \rangle$ is a group isomorphism of one nonabelian subgroup $\underset{\sim}{G}$ of $\underset{\sim}{B}$ onto another such that the restriction of ϕ to $\underset{\sim}{G'}$ is monotonic then ϕ is of the form

$$\phi(\langle a,b \rangle) = \langle a, kb + s(1 - a) \rangle$$

for some positive k and real s.

<u>Proof</u>: Frequent use is made of the formula

$$(*) \qquad [\langle a,b \rangle, \langle c,d \rangle] = \langle 1, (1 - a)d - (1 - c)b \rangle$$

and of the fact implied by the monotonicity of ϕ restricted to G' that for some positive k and all $\langle 1,d \rangle$ in $\underset{\sim}{G'}$, $\phi(\langle 1,d \rangle)$ equals $\langle 1,kd \rangle$. The second assertion follow's from Hion's theorem.

From * every x in $\underset{\sim}{G'}$ and $\underset{\sim}{B'}$ is of the form $\langle 1,d \rangle$. Since $\underset{\sim}{G}$ is not abelian, for some $d \neq 0$, $\langle 1,d \rangle$ is in $\underset{\sim}{G'}$. To show for $x = \langle a,b \rangle$, $\alpha(x)$ is a we use the formula $[\langle a,b \rangle, \langle 1,d \rangle] = \langle 1, d(1 - a) \rangle$. Since $\phi([x,y])$ equals $[\phi(x), \phi(y)]$ and $\beta \langle 1,d \rangle$ equals kd for all $y = \langle 1,d \rangle$ in G' we have $\phi([x,y]) = \langle 1, kd(1 - a) \rangle = \langle 1, kd[1 - \alpha(x)] \rangle$ and $\alpha(x)$ is a.

Even for those $y = \langle 1,d \rangle$ not in $\underset{\sim}{G'}$, $\beta(y)$ equals kd. For choosing some $x = \langle a,b \rangle$ with $a \neq 1$ in nonabelian $\underset{\sim}{G}$ we have $[x,y] \in \underset{\sim}{G'}$ and $\beta([x,y])$ equals $kd(1 - a)$. But $\phi([x,y])$ also equals $[\phi(x), \phi(y)] = \langle 1, \beta(y)(1 - a) \rangle$.

If $x = \langle a,b \rangle$ and $y = \langle a,b' \rangle$ are both in $\underset{\sim}{G}$ then from

$$\phi(x^{-1}y) = \phi \langle 1,b' - b \rangle \text{ equals } [\phi(x)]^{-1}\phi(y) = \langle 1, \beta(y) - \beta(x) \rangle$$

follows $\beta\langle a,b\rangle = \beta\langle a,b'\rangle = k(b - b')$. Consequently $\phi(\langle a,b \rangle)$ equals $\langle a, kb + t(a) \rangle$ for some real function t. If $x = \langle a,b \rangle$ and $y = \langle c,d \rangle$ then from $\phi([x,y]) = [\phi(x), \phi(y)]$ follows $(1 - a)t(c) = (1 - c)t(a)$. Consequently $t(a)$ is proportional to $(1 - a)$ and

$$\beta \langle a,b \rangle = kb + s(1 - a)$$

for some constant of proportionality s. This completes the proof.

Since the group of matrices (and the corresponding group of linear functions $x \to ax+b$) is of considerably more interest than the group $\underset{\sim}{B}$ the uniqueness theorem is rephrased.

Corollary 1: *If ϕ, ϕ' given by*

$$g \to \begin{pmatrix} \alpha(g) & \beta(g) \\ 0 & 1 \end{pmatrix} = \phi(g)$$

$$g \to \begin{pmatrix} \alpha'(g) & \beta'(g) \\ 0 & 1 \end{pmatrix} = \phi'(g)$$

are isomorphisms of a nonabelian partially ordered group with linearly ordered derived group G' such that ϕ and ϕ' have monotonic restrictions to G' then

$$\alpha = \alpha'$$

and for some positive k and real s

$$\beta'(g) = k\beta(g) + s[1 - \alpha(g)]$$

for all g in $\underset{\sim}{G}$.

Proof: Computation with bijection θ .

The uniqueness theorem has a qualitative statement. It is given as a corollary.

Corollary 2: If an isomorphism of one nonabelian subgroup $\underset{\sim}{G}$ of the affine group onto another has a monotonic restriction to the derived group $\underset{\sim}{G}'$ then it has a unique extension to an inner automorphism of the affine group.

Proof: If $\phi < a,b > = < a, kb + (1 - a) s >$ then $\phi < a,b > = < k,s >^{-1} < a,b > < k,s >$. The uniqueness can be verified by a computation or deduced from the transitivity of commutativity described in the existence theorem below.

There is a sense in which all of the ordinal information about $\underset{\sim}{G}$ regarded as a subgroup of the affine group is carried by the derived group $\underset{\sim}{G}'$. It is made explicit in corollary 3.

Corollary 3: If $\underset{\sim}{G}$, $\leqq$ is a nonabelian partially ordered group with linearly ordered derived group then there is at most one extension of $\leqq$ to a full order on $\underset{\sim}{G}$ such that $\underset{\sim}{G}$ with the new order is isomorphic to a subgroup of the affine group.

Proof: If ϕ and ϕ' are isomorphisms with monotonic restrictions to G', then by the preceding corollary there is an x such that $\phi'(g)$ equals $x^{-1}\phi(g)x$ for all g in $\underset{\sim}{G}$. Since the affine group is an ordered group, $\phi(g) \leqq \phi(h)$ iff $\phi'(g) \leqq \phi'(h)$ and all the isomorphisms ϕ induce the same full order on $\underset{\sim}{G}$.

Since the derived group of the affine group is the group of translations, the linearly ordered derived groups of $\underset{\sim}{G}$ embeddable in the affine group will always be archimedean. For this reason (and to more completely isolate additive measurement concepts) attention is now restricted to groups with archimedean derived groups.

Existence Theorem: Let $\underset{\sim}{G}$ be a nonabelian ordered group with partial order $\leqq$ such that $\underset{\sim}{G}'$ ordered by the restriction of $\leqq$ is a linearly ordered, archimedean group. Then there is a group isomorphism ϕ onto a subgroup of the affine group if and only if commutativity is a transitive relation in the set of elements of $\underset{\sim}{G}$ other than the identity. Furthermore, when there is an isomorphism it may be chosen so that the restriction to $\underset{\sim}{G}'$ is strictly increasing.

Necessity: The necessity of the condition may be estabilished by a routine calculation. Alternatively one may reason geometrically by regarding the affine group as the group of transformations $x \to ax + b$. Two such transformations commute when they have the same fixed points.

Sufficiency: By Holder's theorem there is a strictly increasing ψ mapping G' onto a subgroup of the additive reals with the usual ordering. An isomorphism is constructed with ψ and two elements g_0 and g_1 of $\underset{\sim}{G}$ chosen so that g_0 doesn't commute with g_1, g_1 is larger than e the identity of $\underset{\sim}{G}$ and g_1 is in $\underset{\sim}{G}'$.

To obtain g_0 and g_1 let x and y be any two elements of non-abelian $\underset{\sim}{G}$ which don't commute. Put g_1 equal to the maximum of $[x,y]$ and $[y,x] = [x,y]^{-1}$ in linearly ordered $\underset{\sim}{G}'$. Since commutativity is transitive we may take g_0 to be either x or y and have $[g_1,g_0] \neq e$.

Since $\underset{\sim}{G}'$ is normal and $\underset{\sim}{G}$ is an ordered group $g^{-1}g_1g$ is in $\underset{\sim}{G}'$ and $g^{-1}g_1g$ is larger than e. Consequently the formula

$$\alpha(g) = \psi(g^{-1}g_1g) \div \psi(g_1)$$

defines a mapping of $\underset{\sim}{G}$ into the positive reals. Next it is shown that α is a homomorphism.

Since $\underset{\sim}{G}$ is an ordered group, the mapping $x \to g^{-1}xg$ is a strictly increasing homomorphism of $\underset{\sim}{G}'$. Consequently for each fixed g $x \to \psi(g^{-1}xg)$ is also a strictly increasing homomorphism into the additive reals and there is some positive k such that

$$\psi(g^{-1}xg) = k\psi(x) \quad \text{for all} \quad x \in \underset{\sim}{G}' \quad .$$

It follows that for all $g \in \underset{\sim}{G}$ and u, $v \in \underset{\sim}{G}'$

$$(*) \qquad \psi(g^{-1}ug)\psi(v) = \psi(u)\psi(g^{-1}vg) \quad .$$

In particular, for all f and g

$$\alpha(f)\alpha(g) = \psi(f^{-1}g_1f)\psi(g^{-1}g_1g) \div [\psi(g_1)]^2$$

$$= \psi(g^{-1}f^{-1}g_1fg) \div \psi(g_1)$$

$$= \alpha(fg)$$

and α is a homomorphism.

In order to obtain a mapping into the affine group of the form

$$g \to \phi(g) = < \alpha(g),\ \beta(g) >$$

it is necessary to choose a real valued mapping $x \to \beta(x)$. For ϕ to be a homomorphism it is clearly sufficient that $\beta(fg)$ equal $\beta(f)\alpha(g) + \beta(g)$.

Since $\underset{\sim}{G}'$ is abelian for any two conjugate elements f and $g = h^{-1}fh$ in $\underset{\sim}{G}$ and u in $\underset{\sim}{G}'$ we have $f^{-1}uf$ equal $g^{-1}ug$. In particular for all f,g in $\underset{\sim}{G}$

$$g^{-1}[g_0,f]g = g_0^{-1}g^{-1}g_0[g_0,f]g_0^{-1}gg_0$$

$$= g_0^{-1}(fg)^{-1}g_0fgg^{-1}g_0^{-1}gg_0$$

$$= [g_0,fg][g,g_0] \quad .$$

Thus, from * we have

$$\psi(g^{-1}g_1g)\psi([g_0,f]) = \psi(g_1)\{\psi[g_0,fg] + \psi[g,g_0]\} \quad .$$

Rearranging terms and using $\psi([x,y])$ equals $-\psi([y,x])$ gives

$$\psi[g_0,fg] = \alpha(g)\psi([g_0,f]) + \psi([g_0,g]) \quad .$$

Thus if β is defined by $\beta(g) = \psi([g_0,g])$, then

$$\beta(fg) = \beta(f)\alpha(g) + \beta(g)$$

and the mapping $g \to \langle \alpha(g), \beta(g) \rangle = \phi(g)$ is a homomorphism.

To show ϕ is one-to-one, we compute its kernel $\{g \in \underset{\sim}{G}: \psi(g) = \langle 1,0 \rangle \}$. If $\phi(g) = \langle 1,0 \rangle$ then $\psi([g_0,g])$ equals $\psi(e)$ and $\psi(g^{-1}g_1g)$ equals $\psi(g_1)$. Since ψ is one-to-one it follows that x commutes with both g_0 and g_1 . Since commutativity is transitive on the elements of $\underset{\sim}{G}$ other than e and g_0 doesn't commute with g_1 , g can only be e .

Since ϕ restricted to $\underset{\sim}{G}'$ is clearly strictly increasing, this completes the proof.

To obtain the equations offered in the recipe of the preceding section one simply computes Θ of the ϕ used in the proof. The fact that α is a homomorphism justifies the simplification $1/\alpha(g) = \alpha(g^{-1}) = \psi(gg_1g^{-1}) \div \psi(g_1)$.

In order to have a result valid for both abelian and nonabelian group let $\underset{\sim}{C}(\underset{\sim}{G}')$ denote the centralizer of $\underset{\sim}{G}'$,

$$\underset{\sim}{C}(\underset{\sim}{G}') = \{g \in \underset{\sim}{G}: gg' = g'g \text{ for all } g' \in \underset{\sim}{G}'\} \quad .$$

Since $\underset{\sim}{G}$ is abelian implies $\underset{\sim}{C}(\underset{\sim}{G}')$ equals $\underset{\sim}{G}$ one may include Holder's theorem in the following easily proven generalization of the existence theorem.

Let $\underset{\sim}{G}$ *be an ordered group with linearly ordered, archimedean* $\underset{\sim}{C}(\underset{\sim}{G}')$. *Then* $\underset{\sim}{G}$ *is isomorphic to a subgroup of the affine group iff commutativity is a transitive relation in the set of elements other than the identity of* $\underset{\sim}{G}$.

APPENDIX: REPRESENTING ORDERED GROUPS AS AN ALTERNATIVE TO CURVE FITTING AND MAXIMIZING GOODNESS OF FIT

In earlier papers (1970, 1972) it was shown that there are many psychological theories that assume that theoretical or empirical curves either have the same shape or can be transformed to the same shape. Frequently the psychological reasoning deals primarily with the relationship between curves and only incidentally with the shape of the curves or the form of the transformation. Unfortunately, the usual statistical procedures for quantifying the relation between curves presuppose that the shape of the curves or the form of the transformation are known. The results in this paper are part of an attempt to quantify the relation between curves without making assumptions about shape, form of transformation or other assumptions lacking psychological content. Signal detection theory will now be used as a vehicle for discussing this research and as an illustration of affine measurement.

Signal detection theory (SDT) is well known and widely applied by experimental psychologists. Conditions for experimental data to be compatible with the basic psychology of SDT are now known (Marley, 1971). If the normality assumptions which are generally tacked on (Green and Swets, 1966, especially chapter 3) to the basic psychology are true, then the results in this paper are applicable. The application of the results is especially straightforward for SDT data.

The experimental data for SDT is generally a set of points on an ROC curve; i.e., a set of pairs of numbers x,y with

y = the proportion of yes responses when a signal is present and

x = the proportion of yes responses when only noise is present.

It is generally assumed that all of the pairs $\langle x,y \rangle$ on an ROC curve lie on a straight line on probability paper. More formally, it is assumed that there exist numbers σ and d' such that for all $\langle x,y \rangle$ on a given ROC curve

$$* \qquad \sigma n(y) + d' = n(x)$$

where n is the inverse of the normal probability cumulative distribution. The most common way to estimate the parameters is to replot the data points on probability paper and compute the best fitting straight line.[7]

Notice that this procedure presupposes the validity of normality assumptions in SDT. There is no generally accepted a priori argument for normality assumptions. Our data are usually not sufficiently reliable to reject the assumptions. The assumptions appear to be tolerated because there is not much interest in the exact shape of the curves and because there does not seem to be a convenient way to estimate the parameters of psychological interest without making some equally specific assumptions.

It seems desirable to have a direct method for estimating the parameters. Group representations provide such a method. Some researches in collaboration with Mr. David Saxe (Levine and Saxe, 1973) contain evidence that direct methods can be made to be competitive with existing methods for obtaining precise parameter estimates from data.

To see the relevance of group representations note that * implies

$$y = n^{-1}[an(x) + b]$$

for $a = 1/\sigma$ and $b = -d'/\sigma$. Consequently each ROC curve is associated with a function $x \to n^{-1}[an(x) + b] = f(x)$. A set of such functions generate a group of transformations of the numbers between zero and one.

The fact that permits one to drop normality or similar parametric assumptions is this: The product $x \to f[g(x)] = fg(x)$ of two transformations f and g is defined geometrically in terms of the graphs of the transformations and without reference to the equations of the transformations.

For the purposes of this discussion a representation of such a group is a matrix valued mapping

$$f \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where the numbers in the matrix assigned to the product fg is equal to the product of the matrix assigned to f times the matrix assigned to g. For example there is the mapping

$$n^{-1}[an(\cdot) + b] \to \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} .$$

The general strategy we are using for direct estimation of parameters is this:

1. Use empirical data points to define transformations of the unit interval.
2. Represent the transformations of the unit interval by matrices.
3. Calculate the psychological parameters from the numbers in the matrices.

Step one will be dealt with later in separate papers. It is likely that slightly different interpolation procedures will be appropriate for each type of data. Results in this paper are being used to carry out steps two and three. In section II.2 a process is described for reducing the problem of calculating the representing matrices to a standard problem in additive measurement. Section II.3 contains the uniqueness theorems which make step three possible.[8]

The results in this paper show that there is a sense in which psychologically significant parameters can be calculated without making normal, logistic, Poisson or similar parametric assumptions.

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FOOTNOTES

[1]The results in section II.1 and II.2 were presented at the Mathematical Psychology Meetings, Stanford, 1968 in a paper entitled "An analogue of Holder's theorem for measuring brightness contrast and mental test scores."

[2]For a recent work with an extensive bibliography see Krantz et al., 1972. For another point of view see Pfanzagl, 1968.

[3]Only increasing transformations are studied. This means that the parameter a is positive. In the sequel this parameter will be called the slope; the other parameter b will be called the intercept.

[4]The ordering of affine transformations used in this paper is $f \leq g$ if and only if $f(x)$ is eventually larger than $g(x)$ in the sense that for some N, if $x > N$ then $f(x) \leq g(x)$. With this order, it is easy to show that the affine group is a fully ordered, nonarchimedean group. For further comments on this ordering see section V.

[5]In the following discussion "structurally like" or "isomorphic" means "can be placed in one to one correspondence in a way which preserves all qualitative properties." Since the only qualitative properties of $\underset{\sim}{G}$ are its multiplication and ordering this means there is a 1 - 1 function ϕ defined on $\underset{\sim}{G}$ and taking values in the group of affine transformations such that $\phi(fg) = \phi(f)\phi(g)$ and $f \leq g$ implies $\phi(f) \leq \phi(g)$.

[6]To prove this define a function $\Theta(f)$ by $\Theta(f)$ equals $b/(1 - a)$ if $a \neq 1$ and $\Theta(f)$ equal infinity otherwise. Then use $\Theta(f) = \Theta(g)$ implies $fg = gf$.

[7]Much more sophisticated methods are available. The basic logic all seems very much the same: assume the curves have normal, logistic or some other particular shape and select parameters to maximize a criterion of goodness of fit. For a recent reference see D. R. Grey and B. J. Morgan, 1972. For a practical and general procedure making very weak assumptions about shape but with no allowance for transforming to the same shape see W. H. Lawton, E. A. Sylvestre, and M. S. Maggio, 1972.

[8]There are many different representations of these groups as groups. However, when they are treated as <u>ordered</u> groups there is essentially one (see II.3) representation. The ordering is discussed in footnote 4 and in section V.