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ABSTRACT

The relative power of the Mann-Whitney statistic, the t-statistic, the median test, a test based on exceedances (A,B), and two special cases of (A,B) the Tukey quick test and the revised Tukey quick test, was investigated via a Monte Carlo experiment. These procedures were compared across four population probability models: uniform, beta, normal, and double exponential. Sample sizes of (5.5), (10,10), (20,20), (5,10), and (5,20) were among those used. Results indicate the median test should be considered for distributions which certain outliers. The exceedances tests can be powerful alternatives to more standard procedures if the underlying distributions are platykurtic. (Author)

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# AN EMPIRICAL COMPARISON OF SELECTED TWO-SAMPLE HYPOTHESIS TESTING PROCEDURES WHICH ARE LOCALLY MOST POWERFUL UNDER CERTAIN CONDITIONS

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## INTRODUCTION

In the last few years, a great deal of information has been published regarding the robustness of the t-statistic and other normal distribution theory hypothesis testing procedures. In general, these procedures are remarkedly robust when the underlying assumptions are violated, especially with respect to control over errors of the first type. Exceptions occur when both the variances and sample sizes are unequal and under some conditions of rather extreme non-normality, primarily skewness. A very comprehensive review of the research on the robustness of the Student-procedure is reported by Hatch and Posten (1966).

While a great deal of research has been conducted on the robustness of the t-statistic, and a few of its distribution free competitors, this research has tended to focus on a rather narrow definition of robustness; i.e., the control over Type I errors. Violation of the assumptions necessary for the exactness of any hypothesis testing procedure also affects its control over Type II errors. Conditions of non-normality and variance heterogeneity, while not always detrimental to the performance of the t-statistics control over the nominal significance level, sometimes have a very noticeable effect on the t-tests power, especially relative to other hypothesis testing procedures, Pratoomraj (1970).

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Purpose of the Study

As is well known, many of the distributions which exist in educational and psychological research are non-normal in nature. Many carefully constructed standardized tests yield raw score distributions which are by necessity bounded and relatively flat or platykurtic in nature. Some of these distributions are, in fact, nearly rectangular or uniform, Brandenburg (1972). Another, contrasting, situation is one in which an occasional large measurement error will produce a highly disparate observation. This tends to create underlying distributions which are leptokurtic (peaked).

The major objective of this study was to investigate the relative power of four two-sample hypotheses testing procedures across four different underlying distributions for five variations of sample size. The four statistical procedures investigated were (1) the t-test (t), (2) the Mann-Whitney U test (U), (3) the median test (r), and (4) two variations of a test based on exceedances: a procedure described by Hajek (1969) which will be designated by (A,B) and a procedure recommended by Tukey (1959) referred to as (A+B).

Description of Statistics Investigated and Probability Models Sampled

In order to empirically determine the relative Type I error control and power of the various hypotheses testing procedures, four probability distributions were used as sampling models. Each of these distributions was continuous and symmetric, but each differed primarily in tail weight or degree of kurtosis = $K = E[(X - \mu)^4]/\sigma^4$. These distributions were: 1) the double exponential, 2) the normal, 3) the uniform, and 4) a lambda distribution (Ramberg and Schmeiser, 1972) with tail weight ($K = 2.3$) between the normal and uniform distributions.

Among the rank tests the two-sample median test is locally most powerful when the underlying distributions are double exponential. The double exponential distribution is characterized by its long (heavy) tails ($K = 6.0$). The Mann-Whitney U test is the uniformly most powerful rank test when the underlying distributions are logistic. The logistic distribution is somewhat lighter (or shorter) tailed ($K = 4.2$) than the double exponential, but still heavier tailed than the normal probability model ($K = 3.0$). It is well known that the t-statistic is uniformly most powerful if the underlying distributions are normal. The two tests based on exceedances (A,B) and (A+B) are each locally most powerful, for different alternatives, when the sample distributions are uniform ($K = 1.8$).

The test statistic for the (A,B) test used in testing $H_0$: $F(x) = F(y)$, against $H_1$: $F(x) > F(y)$, is the ordered pair (a,b) where a is the number of y's greater than the largest x, and b is the number of x's less than the smallest y. The (A,B) test assumes that the pairs are ordered by the following rule:

$$(A,B) > (A',B') \text{ if } \begin{cases} \text{either } \min(A,B) > \min(A',B') \\ \text{or } \min(A,B) = \min(A',B') \text{ and } \\ \qquad (A+B) > (A'+B') \end{cases}$$

$$(A,B) = (A',B') \text{ if } \begin{cases} \text{either } A' = A,\ B' = B \\ \text{or } \quad A' = B,\ B' = A \end{cases}$$

Then the pair (A,B) whose values (a,b) are ordered as above provides a one ended test of $H_0$: $F(x) = F(y)$. The (A,B) test is locally most powerful for uniform distributions with small mean differences.

The test statistic for the (A+B) test is $a + b$ where a and b are the same as for the (A,B) test. This procedure is locally most powerful for uniform distributions with "large" mean differences.

Graphs and density functions of the probability distributions sampled are given in Figure 1. Since these graphs do not clearly show the distinction in tail weights of the distributions, each distribution was rescaled to have the same median and .95 quantile as the standardized normal. This comparison of the tails of the four distributions is shown in Figure 2.

Procedures

The procedure used to generate the empirical sampling distributions of the hypothesis testing procedures investigated is described in the following steps:

1. Vectors of m + n elements, randomly drawn from each of the four population distributions, were obtained. The first m elements from an X-universe having mean $\mu_x$ and variance $\sigma_x^2$ and the remaining n from a Y-universe with mean $\mu_y$ and variance $\sigma_y^2$.

   Each of the m + n elements in the vector was obtained by generating a uniform random number between zero and one, which was regarded as a relative cumulative frequency of the uniform distribution. The random variable for each of the other distributions investigated (lambda, normal, double exponential) was then obtained through what amounted to an area transformation.

2. Five combinations of sample size (m,n)[(5,5); (10,10); (20,20); (5,10); (5,20)] and five values of $\Delta$ (0(1)4) were selected for investigation for each of the four probability models.
   $\Delta = (\mu_x - \mu_y)(\sigma_x^2/m + \sigma_y^2/n)^{-1/2}$

3. For each vector of m + n observations, the statistics t, U, r, (a,b), and (a+b) were computed. This procedure was repeated 1000 times for each combination of (m,n), population distribution, and $\Delta$-value.

4. For each of the above replications the test statistics were referred to their respective .05 two-ended critical values. Since critical values corresponding to a significance level of exactly .05 do not ordinarily exist for the rank type procedures, a randomization process was used which insured that each would have a nominal level of .05.

Results

The empirical Type I error and power values (times 1000) obtained for this investigation are presented in Table 1. In general, these results are consistent with predictions obtained from asymptotic theory. In the discussion which follows the various hypotheses testing procedures will be compared for the different population models sampled. Of the two tests based on exceedances all references will be to (A+B). Little practical difference existed between (A+B) and (A,B) and because of the simpler decision rule associated with A + B it seems to be the preferable procedure.

The results may be summarized by sampled distribution as follows:

1. Double exponential. Across the various sample sizes studied both t and U exhibit excellent power. There appears to be little reason to prefer either of these procedures although U was slightly more powerful for the larger equal sized samples. The most surprising result for this population model was the very poor performance of the median test (r). While this procedure is the locally most powerful ($\Delta$ small) of the rank tests for double exponential distributions, the only case in which it was in any way comparable to t and U was when m = n = 20. Considering the manner in which the (A+B) procedure is defined it performed surprisingly well except for m = n = 20.

2. Normal. As was expected, t was the superior procedure for this case. However, as is well known, the Mann-Whitney statistic performs very well when the underlying distributions are normal. Once again (r) was inferior to (A+B) except when m = n = 20.

3. Lambda. For this relatively flat population model (K = 2.3), t was superior to all statistics investigated. There appears to be little reason to prefer either U or (A+B) and both would seem to be reasonable alternatives to the t-statistic. The median test is noticeably less powerful than any other across all sample size combinations.

4. Uniform. (A+B) and t are the preferable procedures for distributions of this type. The t-statistic is slightly more powerful than (A+B) in the $m = n = 5$ case and there appears to be little difference between the two methods for $m \neq n$. For the larger equal sample sizes (A+B) is the superior method, markedly so in $m = n = 20$ case. Although less powerful than t and (A+B), the U statistic performs reasonably well for rectangular distribution types. This is especially true relative to r which is markedly inferior to all procedures.

Selected results from Table 1 discussed above are illustrated in Figures 3 through 7.

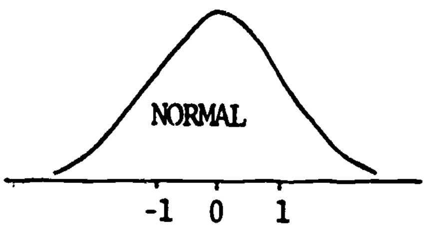
In summary, it appears that t is probably overall the superior statistic although for "heavy" tailed distributions U is a very competitive alternative and for "lighter" tailed underlying densities the tests based on exceedances are attractive alternatives, especially (A+B) because of its simplicity. With the exception of large samples from leptokurtic population models the median test has little to offer relative to the other procedures investigated.

TABLE 1

Empirical Power Functions for Each of Five Test Statistics (T') When Sampling from Selected Probability Models

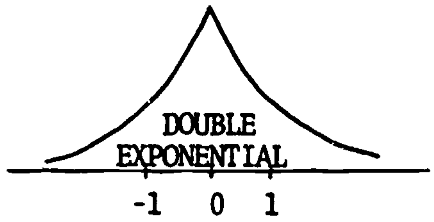
$(\Delta = (\mu_y - \mu_x)(\sigma_x^2/m + \sigma_y^2/n)^{-1/2}, \alpha = .05)$

| MODEL | | DOUBLE EXPONENTIAL | | | | | NORMAL | | | | | LAMBDA | | | | | UNIFORM | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| m,n | T' \ Δ | 0 | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 3 | 4 |
| | t | 44 | 150 | 479 | 792 | 918 | 48 | 125 | 419 | 773 | 941 | 52 | 122 | 398 | 765 | 947 | 54 | 125 | 385 | 795 | 960 |
| | U | 44 | 170 | 465 | 751 | 875 | 46 | 128 | 399 | 741 | 916 | 43 | 118 | 378 | 731 | 934 | 47 | 110 | 350 | 710 | 940 |
| | r | 61 | 131 | 321 | 579 | 717 | 45 | 96 | 224 | 470 | 683 | 57 | 80 | 205 | 448 | 658 | 53 | 69 | 170 | 388 | 626 |
| m=5 | A,B | 45 | 158 | 436 | 720 | 862 | 45 | 124 | 366 | 714 | 894 | 45 | 111 | 348 | 730 | 927 | 45 | 106 | 359 | 731 | 940 |
| n=5 | A+B | 48 | 165 | 465 | 748 | 871 | 45 | 133 | 387 | 742 | 915 | 46 | 117 | 359 | 735 | 926 | 44 | 117 | 358 | 716 | 944 |
| | t | 52 | 169 | 493 | 822 | 950 | 56 | 163 | 455 | 832 | 970 | 54 | 165 | 443 | 834 | 972 | 55 | 163 | 437 | 829 | 977 |
| | U | 48 | 183 | 527 | 844 | 958 | 47 | 147 | 405 | 786 | 956 | 48 | 132 | 372 | 760 | 950 | 48 | 139 | 374 | 735 | 933 |
| | r | 51 | 184 | 498 | 774 | 928 | 50 | 132 | 290 | 570 | 827 | 55 | 104 | 255 | 507 | 761 | 54 | 99 | 195 | 390 | 645 |
| m=10 | A,B | 55 | 114 | 316 | 567 | 733 | 55 | 119 | 357 | 666 | 876 | 55 | 124 | 413 | 771 | 945 | 55 | 190 | 575 | 899 | 981 |
| n=10 | A+B | 53 | 134 | 363 | 630 | 802 | 51 | 142 | 397 | 705 | 898 | 53 | 141 | 412 | 783 | 950 | 50 | 188 | 559 | 892 | 979 |
| | t | 31 | 183 | 540 | 823 | 958 | 34 | 178 | 504 | 826 | 967 | 37 | 176 | 491 | 825 | 968 | 41 | 174 | 496 | 818 | 974 |
| | U | 40 | 228 | 634 | 891 | 983 | 40 | 173 | 462 | 801 | 960 | 40 | 155 | 434 | 781 | 946 | 40 | 160 | 443 | 759 | 939 |
| | r | 40 | 224 | 557 | 856 | 959 | 37 | 130 | 309 | 609 | 839 | 40 | 95 | 245 | 522 | 765 | 39 | 84 | 200 | 404 | 629 |
| m=20 | A,B | 49 | 75 | 119 | 310 | 509 | 49 | 97 | 291 | 515 | 736 | 49 | 127 | 410 | 733 | 930 | 49 | 329 | 830 | 969 | 998 |
| n=20 | A+B | 42 | 85 | 237 | 373 | 584 | 43 | 105 | 335 | 572 | 800 | 43 | 132 | 446 | 758 | 944 | 44 | 313 | 814 | 972 | 1000 |
| | t | 44 | 187 | 483 | 799 | 943 | 45 | 169 | 434 | 787 | 959 | 47 | 157 | 433 | 789 | 963 | 49 | 157 | 427 | 787 | 967 |
| | U | 36 | 202 | 497 | 811 | 930 | 35 | 151 | 420 | 762 | 943 | 41 | 153 | 391 | 738 | 939 | 40 | 154 | 390 | 707 | 935 |
| | r | 56 | 153 | 384 | 656 | 799 | 38 | 112 | 271 | 470 | 725 | 43 | 81 | 220 | 423 | 709 | 47 | 90 | 192 | 355 | 673 |
| m=10 | A,B | 51 | 136 | 356 | 624 | 802 | 51 | 127 | 362 | 674 | 903 | 58 | 142 | 379 | 728 | 945 | 48 | 182 | 465 | 798 | 958 |
| n=5 | A+B | 49 | 160 | 402 | 680 | 849 | 50 | 138 | 376 | 722 | 909 | 54 | 136 | 374 | 730 | 937 | 51 | 155 | 419 | 774 | 954 |
| | t | 36 | 191 | 514 | 812 | 964 | 37 | 178 | 472 | 809 | 972 | 45 | 175 | 460 | 802 | 975 | 46 | 173 | 442 | 803 | 980 |
| | U | 38 | 202 | 538 | 807 | 941 | 38 | 159 | 412 | 751 | 958 | 38 | 151 | 372 | 727 | 957 | 38 | 149 | 366 | 690 | 938 |
| | r | 52 | 204 | 482 | 676 | 840 | 53 | 132 | 332 | 571 | 812 | 55 | 114 | 281 | 523 | 791 | 48 | 86 | 221 | 466 | 782 |
| m=20 | A,B | 50 | 100 | 238 | 433 | 662 | 50 | 139 | 292 | 542 | 795 | 50 | 171 | 364 | 632 | 875 | 50 | 227 | 540 | 765 | 931 |
| n=5 | A+B | 44 | 157 | 309 | 546 | 747 | 43 | 161 | 350 | 652 | 885 | 46 | 173 | 405 | 726 | 954 | 46 | 209 | 515 | 839 | 989 |



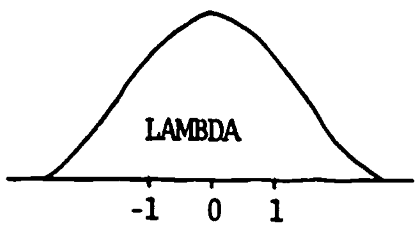

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-1/2x^2}$$

$$-\infty < x < \infty$$




$$f(x) = \frac{1}{2} e^{-|x|}$$

$$-\infty < x < \infty$$

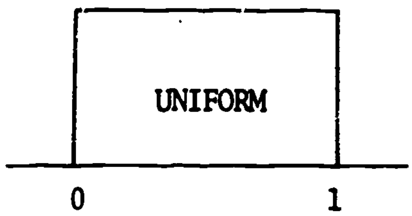



(inverse of the CDF)

$$x = F^{-1} = \frac{u^{.35} - (1-\mu)^{.35}}{.385}$$

where $\mu$ is uniform $0 \leq \mu \leq 1$

$$|x| < 2.6$$




$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

FIGURE 1

PROBABILITY DISTRIBUTIONS SAMPLED

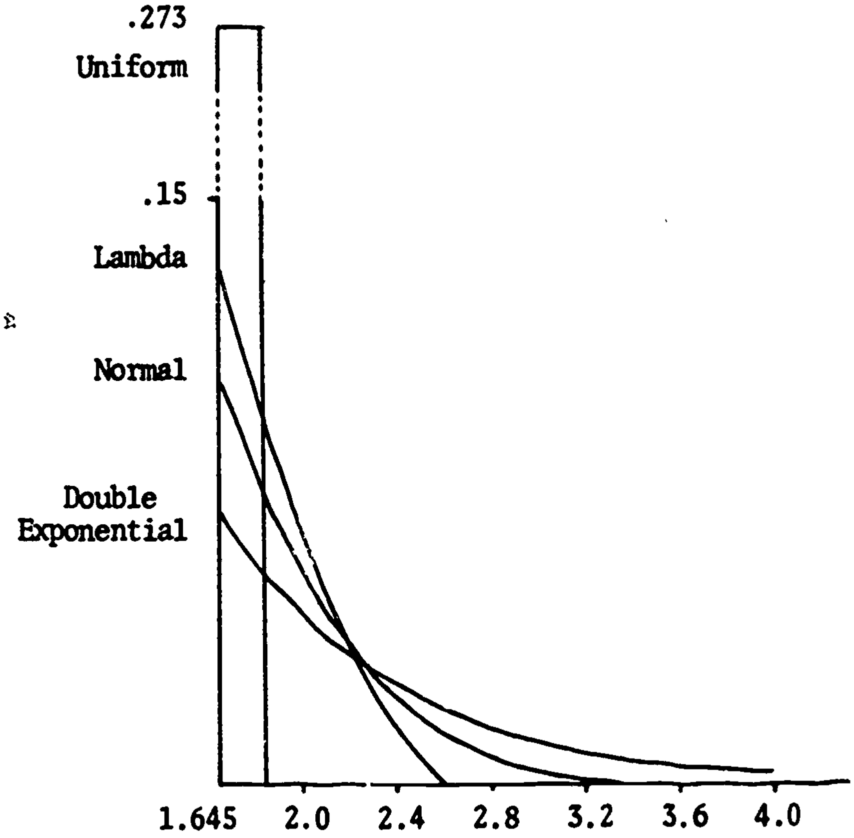
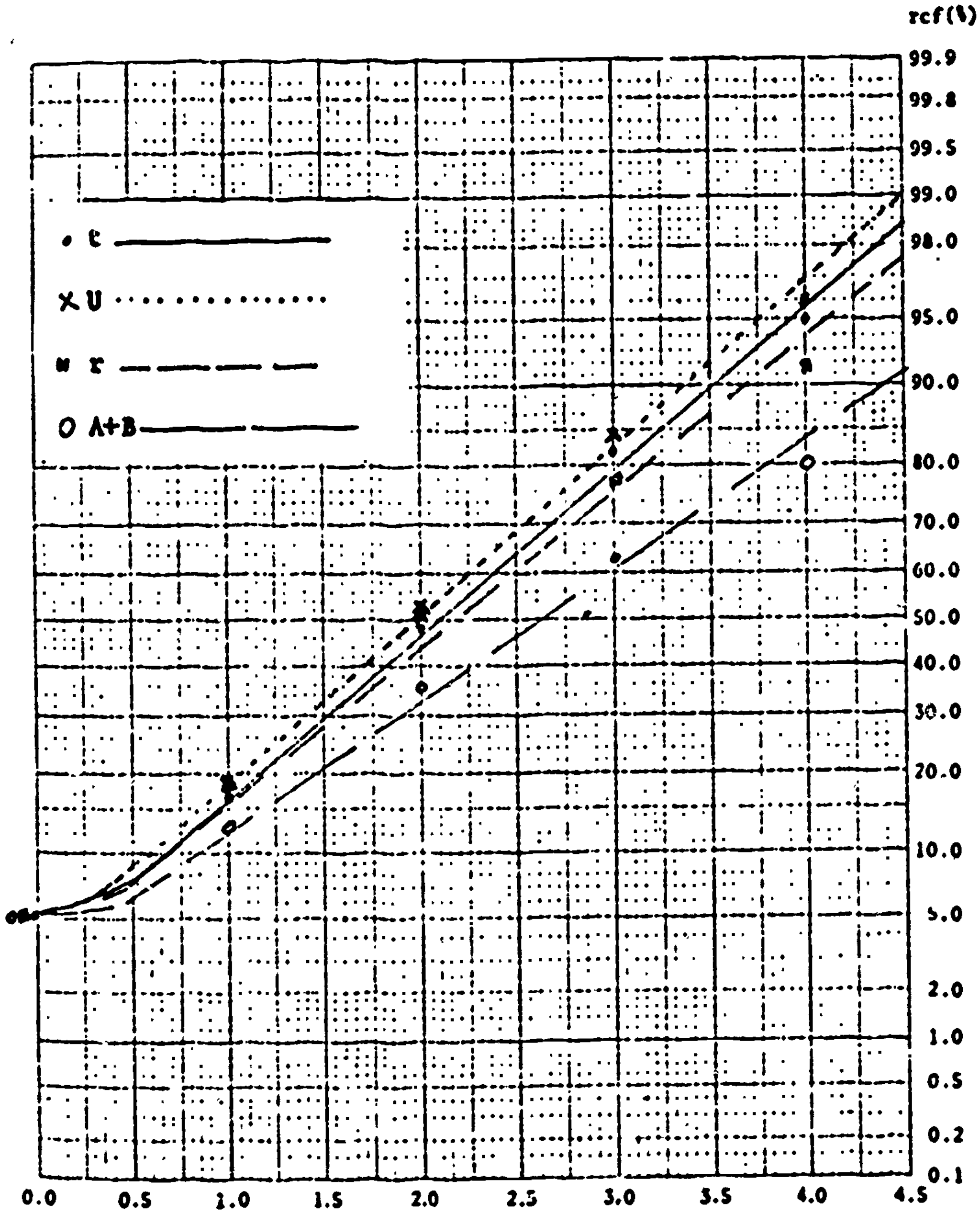



FIGURE 2

UPPER 5% TAILS OF DISTRIBUTIONS SAMPLED

FIGURE 3

Empirical Power Values and Smoothed Power Curves
for t, U, r, and A+B for Double Exponential Distributions
$m = n = 10,\ \alpha = .05$




$\Delta = (\mu_y - \mu_x)\cdot(\sigma_x^2/m + \sigma_y^2/n)^{-\frac{1}{2}}$

# FIGURE 4

Empirical Power Values and Smoothed Power Curves for t, r, U, and A+B for Lambda Distributions
$m = n = 20$, $\alpha = .05$

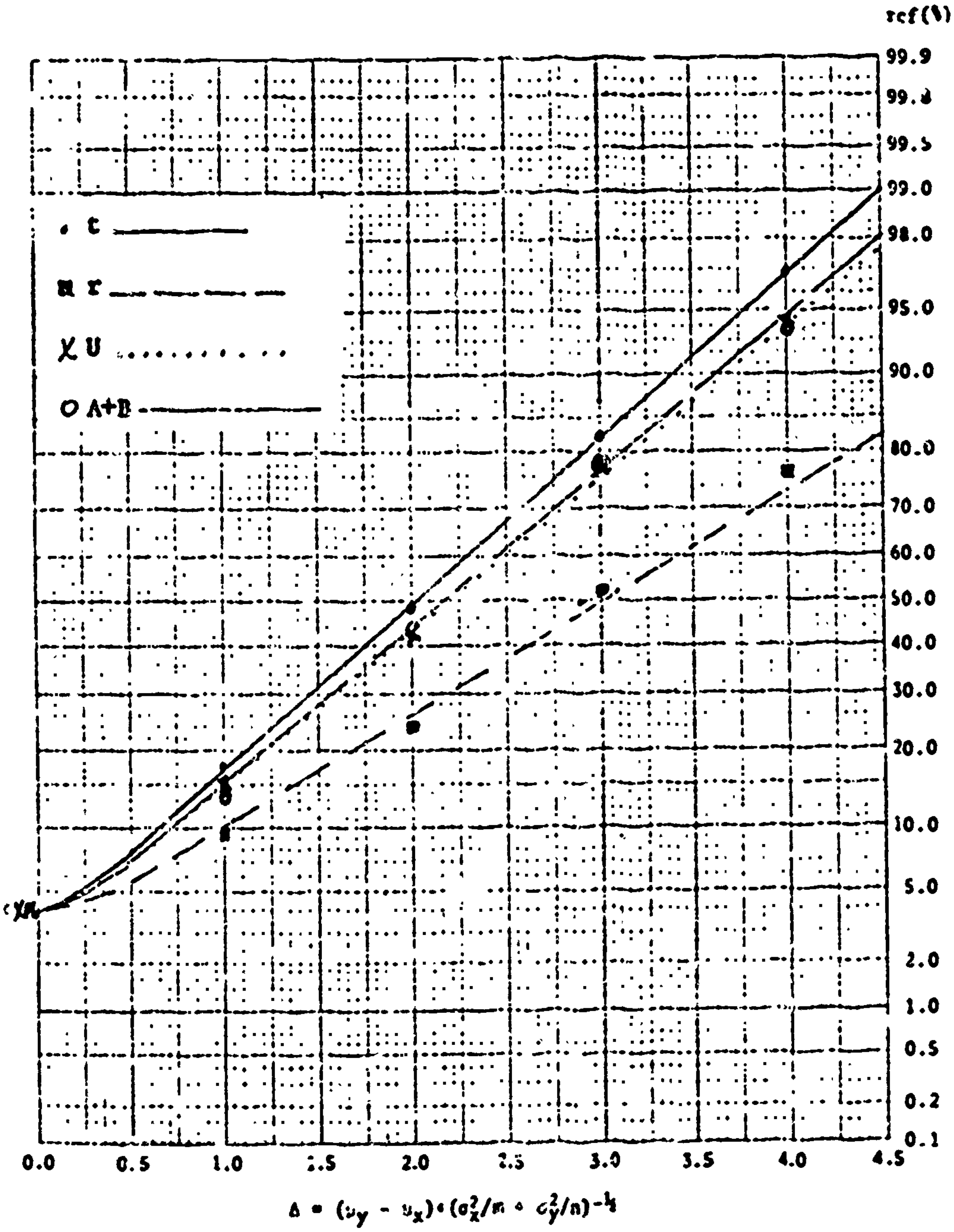
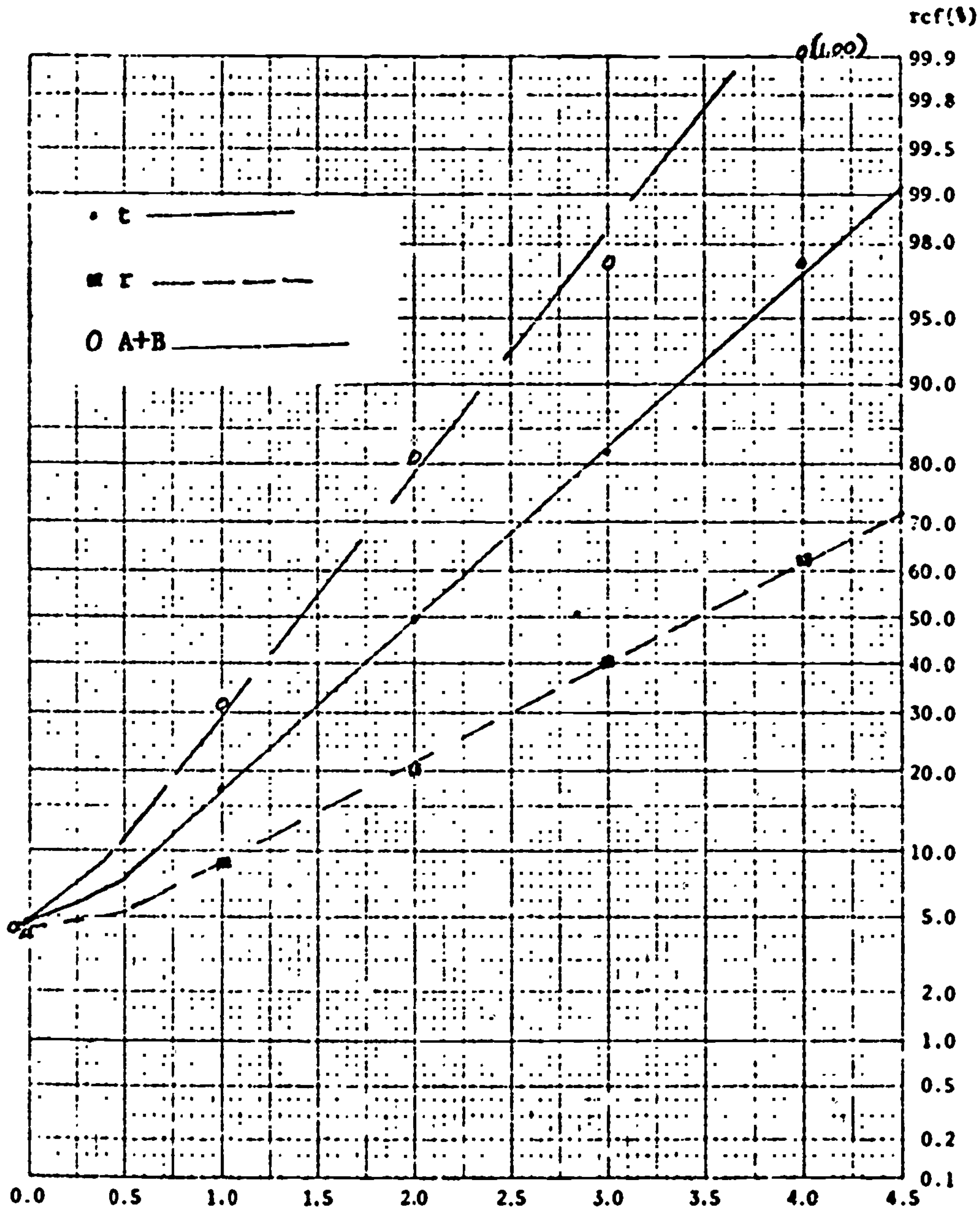


FIGURE 8

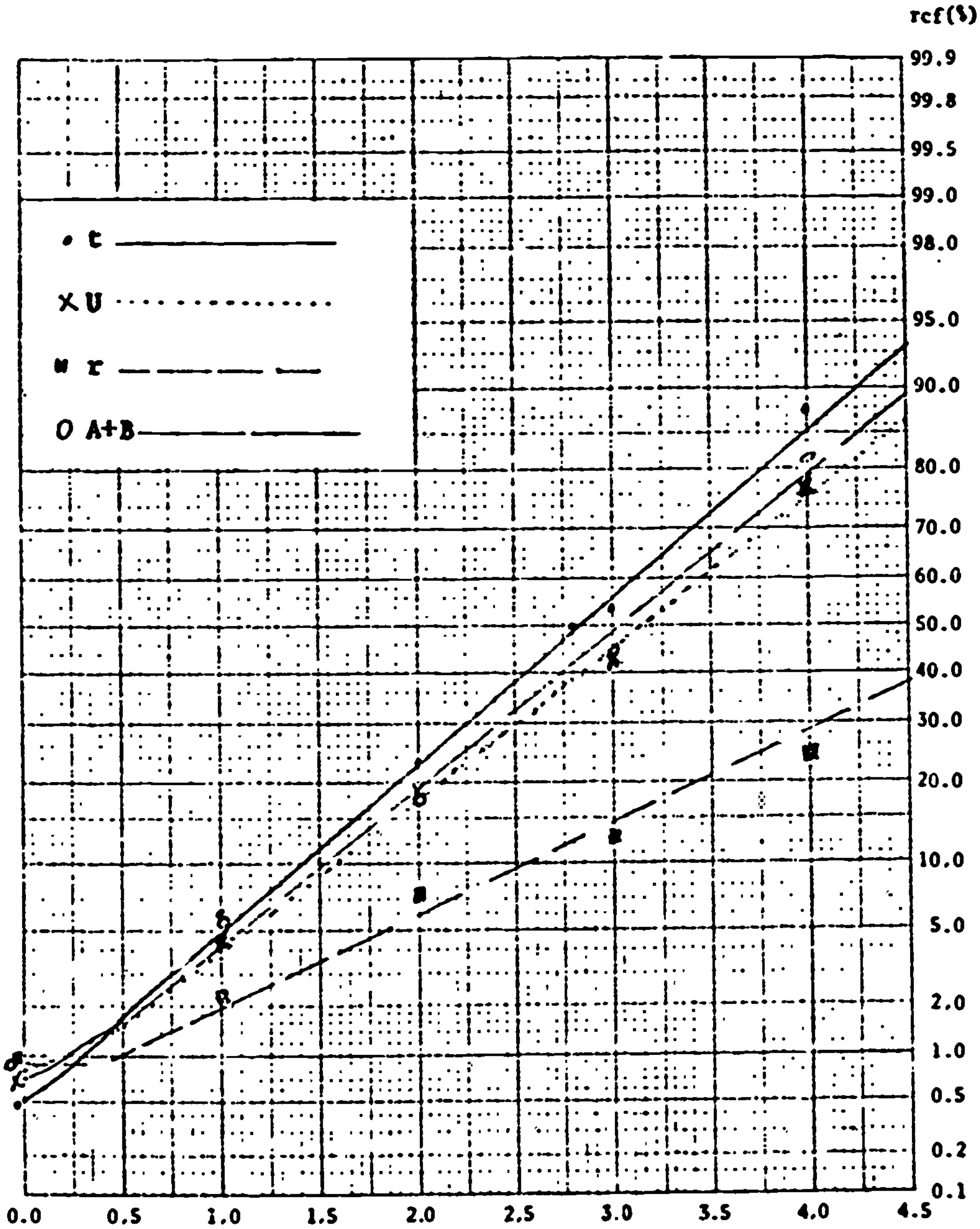
Empirical Power Values and Smoothed Power Curves
for t, r, and A+B for Uniform Distributions
$m = n = 20$, $\alpha = .05$




$\delta = (\mu_y - \mu_x)\cdot(\sigma_x^2/m + \sigma_y^2/n)^{-\frac{1}{2}}$

FIGURE 6

Empirical Power Values and Smoothed Power Curves
for t, U, r, and A+B for Uniform Distributions
$m = 20,\ n = 5,\ \alpha = .01$




$\Delta = (\mu_y - \mu_x)\cdot(\sigma_x^2/m + \sigma_y^2/n)^{-1/2}$

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