# DOCUMENT RESUME

| | | |
|---|---|---|
| ED 069 729 | | TM 002 181 |
| AUTHOR | Olkin, Ingram | |
| TITLE | Testing and Estimation for Structures which are Circularly Symmetric in Blocks. | |
| INSTITUTION | Educational Testing Service, Princeton, N.J. | |
| SPONS AGENCY | National Science Foundation, Washington, D.C. | |
| REPORT NO | ETS-RB-72-41 | |
| PUB DATE | Aug 72 | |
| NOTE | 21p.; Draft | |
| EDRS PRICE | MF-$0.65 HC-$3.29 | |
| DESCRIPTORS | *Models; *Structural Analysis; Testing | |


ABSTRACT

In this report, the circularly symmetric model is extended to the point where the symmetries are exhibited in blocks. In addition, it is shown how maximum likelihood estimators (MLEs) and likelihood ratio tests (LRTs) can be obtained. The circularly symmetric model is reviewed and it is shown how MLEs and LRTs can be obtained by reducing the model to a canonical form. The various ways of generating the extended model (blocked circularity) are discussed. It is noted that a reduction to a canonical form is possible for the block circular case. Hypotheses representing block versions of sphericity, intraclass correlation, circular symmetry, and a general matrix are presented. Also, shown are likelihood ratio tests and their approximate null distribution for testing symmetric structures and tests for means given that the covariance matrix is circular. (JS)

RB-72-41

RESEARCH BULLETIN

# TESTING AND ESTIMATION FOR STRUCTURES WHICH ARE CIRCULARLY SYMMETRIC IN BLOCKS

Ingram Olkin

Stanford University
and Educational Testing Service



Educational Testing Service
Princeton, New Jersey
August 1972

# TESTING AND ESTIMATION FOR STRUCTURES WHICH ARE CIRCULARLY SYMMETRIC IN BLOCKS[1]

Ingram Olkin

Stanford University and Educational Testing Service

## 1. Introduction

There has been considerable study of models in which the observations arise in a structured form. These structures may yield covariance matrices which exhibit special patterns or symmetries. An early example of patterned covariance structure is the intraclass correlation model (in which all the variances and all the covariances are homogeneous) considered by Wilks (1946). Another example is the spherical model in which all the variances are equal and the covariances are zero. A wide class of structured models, called radex models, was introduced by Guttman (1954). In these models, test scores are generated from components which may be viewed as having a special geometrical structure, and hence the more recent name simplex models. Although Guttman (1954, 1957) provided examples of data which approximated simplex structure, there was little work in developing estimators of the parameters or in designing tests of hypotheses. The paper of Wilks (1946) was concerned with inference for the particular intraclass correlation model. This model has now been studied in some detail in various contexts. Similarly, tests for sphericity also have been studied in detail (for references, see Gleser, 1966).

The first general study of some simplex models was that of Mukherjee (1966), in which solutions of the maximum likelihood equations are discussed. In a subsequent paper, Mukherjee (1970) was able to obtain explicit maximum likelihood estimators for a certain class of simplex models.

---

[1] Supported in part by the National Science Foundation Grant 32326X at Stanford University and by Educational Testing Service.

Another general paper which has relevance to some of the Guttman's simplex models is that of Anderson (1970). The method of analysis considered by both Anderson (1970) and Mukherjee (1970) is based on covariance matrices $\Sigma$ which may have a representation of the form

$$(1.1) \quad \Sigma = \Theta_1 A_1 + \dots + \Theta_m A_m \quad ,$$

where the matrices $A_j$ are known and the $\Theta$'s are unknown. Many patterned covariance matrices may be expressed in the form of (1.1). Jöreskog (1970) provides a general discussion of the solution of the maximum likelihood equations for simplex models.

Although general methods were considered in these papers, greater depth can be achieved in the study of some particular patterns. One such pattern, called the circumplex by Guttman, was considered by Olkin and Press (1969), in which an extensive study was made of a hierarchy of models. In addition to studying the covariance structure, patterns among the means are also considered. Olkin and Press provide not only the likelihood ratio tests of various hypotheses, but also different approximations to the null and non-null distributions. It is of interest to note that the genesis of their study is a physical model in which observations are made at the vertices of a regular polygon. Because of stationarity [see Olkin and Press (1969)], a circularly symmetric model is generated, which is identical to that of the circumplex.

In the present paper we extend the circularly symmetric model to the case where the symmetries are exhibited in blocks, and show how maximum likelihood estimators (MLE) and likelihood ratio tests (LRT) can be obtained.

## 2. Preliminaries on the Circularly Symmetric Model

Before providing an extension of the model, we first review the circularly symmetric model, and show how the reduction to a canonical form enables us to obtain easily the MLE and LRT. In both Guttman's circumplex model and in that considered by Olkin and Press (1969), we have random variables $x_1,\ldots,x_k$ with means $\mu_1,\ldots,\mu_k$ and covariance matrix $\Sigma_c$, which is a circular symmetric matrix. A circular symmetric matrix $A_c$ is given by

$$(2.1)\qquad A_c = \begin{bmatrix} a_1 & a_2 & \cdots & a_r \\ a_r & a_1 & \cdots & a_{r-1} \\ & & \cdots & \\ & & \cdots & \\ a_2 & a_3 & \cdots & a_1 \end{bmatrix} \quad , \text{ where } a_j = a_{r-j+2} \quad , \quad j = 2,\ldots,r \quad .$$

Every symmetric matrix has a representation

$$(2.2)\qquad A_c = \Gamma' D_\tau \Gamma \quad ,$$

where $\tau_1,\ldots,\tau_r$ are the eigenvalues of $A_c$, $D_\tau = \mathrm{diag}(\tau_1,\ldots,\tau_r)$, and $\Gamma = (\gamma_{jk})$ is orthogonal.

A key point in the development is the fact that if $A_c$ is circularly symmetric, then

$$(2.3)\qquad \tau_j = \tau_{r-j+2} \quad , \quad j = 2,\ldots,r \quad ;$$

furthermore, the elements $\gamma_{jk}$ are given by

$$(2.4)\qquad \gamma_{jk} = r^{-1/2}\{\sin 2\pi r^{-1}(j-1)(k-1) + \cos 2\pi r^{-1}(j-1)(k-1)\} \quad ,$$

which are independent of the elements of $A_c$.

Another way to express (2.1) [see Wise (1955)] is

$$(2.5) \qquad A_c = a_1W_0 + a_2W_1 + \cdots + a_rW_{r-1} \quad ,$$

where

$$(2.6) \qquad W_0 = I \quad , \qquad W_j = \begin{pmatrix} 0 & I_{r-j} \\ I_j & 0 \end{pmatrix} \quad , \qquad j = 1,\ldots,r-1 \quad .$$

Remark

Since $a_j = a_{r-j+2}$, we may combine terms having the same coefficients to yield terms $W_j + W_{r-j+2}$. But $W_{r-j+2} = W_j'$, so that $W_j + W_{r-j+2}$ is symmetric. It is easily verified that $W_j = W_1^j$, so that all matrices $(W_j + W_j')$ $j = 0,\ldots,r-1$ are commutative, and hence may be diagonalized by the same orthogonal matrix. This fact will be used later.

Suppose we have a sample of size $N$, $(x_{1\alpha},\ldots,x_{p\alpha})$, $\alpha = 1,\ldots,N$, from a normal population with mean vector $\mu$ and covariance matrix $\Sigma$. By sufficiency we may consider the mean vector $\bar{x}$, which has a $\mathcal{N}(\mu,\Sigma/N)$ distribution, and the cross product matrix $S$, which has a Wishart distribution, $\mathcal{W}(\Sigma;p,n)$, $n = N - 1$, with density function

$$p(S) = c(p,n)|S|^{(n-p-1)/2} \; |\Sigma|^{-(n/2)} \exp[-\tfrac{1}{2} \operatorname{tr} \Sigma^{-1}S] \quad ,$$

where

$$c(p,n) = 2^{-pn/2}\pi^{-p(p-1)/4}[\prod_1^p \Gamma(\tfrac{1}{2}(n-i+1))]^{-1} \quad .$$

Now transform $\bar{x}$ and $S$ by

$$(2.7) \qquad y = \sqrt{N}\,\bar{x}\Gamma' \quad , \qquad V = \Gamma S\Gamma' \quad ,$$

so that

$$(2.8) \qquad y \sim \mathcal{N}(\nu,\bar{\Sigma}) \quad , \qquad V \sim \mathcal{W}(\bar{\Sigma};p,n) \quad ,$$

where $\nu = \sqrt{N}\,\mu\Gamma$ and $\bar{\Sigma} = \Gamma\Sigma\Gamma'$ . If $\Sigma$ is circular, then $\bar{\Sigma}$ is diagonal, and estimators of the parameters are readily available.

In the following extension, we make use of these ideas to afford an analogous simplification for the extended problem.

## 3. Block Circularity

The extended model may be generated in various ways. In terms of the physical model mentioned in Section 1, we again have a point source located at the geocenter of a regular polygon of $p$ sides, from which a signal is transmitted. Identical signal receivers are positioned at the $p$ vertices. However, now the signal received at the $i$ -th vertex is characterized by $k$ components, and is denoted by $x_i = (x_{i1},\ldots,x_{ik})$ . The main assumption is that the covariance matrices depend only on the number of vertices separating the two receivers, so that

$$(3.1) \qquad \mathrm{Cov}(x_i,x_{i+m}) = \Sigma_m = \Sigma_{p-m} \quad , \qquad m = 0,\ldots,p \quad ,$$

where each $\Sigma_m$ is a $k \times k$ matrix. Thus, for example, if $p = 4$ and 5, we obtain

$$(3.2) \qquad \begin{bmatrix} \Sigma_0 & \Sigma_1 & \Sigma_2 & \Sigma_1 \\ \Sigma_1 & \Sigma_0 & \Sigma_1 & \Sigma_2 \\ \Sigma_2 & \Sigma_1 & \Sigma_0 & \Sigma_1 \\ \Sigma_1 & \Sigma_2 & \Sigma_1 & \Sigma_0 \end{bmatrix} \quad , \qquad \begin{bmatrix} \Sigma_0 & \Sigma_1 & \Sigma_2 & \Sigma_2 & \Sigma_1 \\ \Sigma_1 & \Sigma_0 & \Sigma_1 & \Sigma_2 & \Sigma_2 \\ \Sigma_2 & \Sigma_1 & \Sigma_0 & \Sigma_1 & \Sigma_2 \\ \Sigma_2 & \Sigma_2 & \Sigma_1 & \Sigma_0 & \Sigma_1 \\ \Sigma_1 & \Sigma_2 & \Sigma_2 & \Sigma_1 & \Sigma_0 \end{bmatrix} \quad .$$

In terms of Guttman's circumplex model, vectors of scores are generated from a structured model as follows. For simplicity, consider the special case in which there are five tests $t_1$, $t_2$, $t_3$, $t_4$, $t_5$ made up from 5 components $c_1$, $c_2$, $c_3$, $c_4$, $c_5$, where each $c_i$ is a $k$-dimensional vector:

$$
\begin{aligned}
t_1 &= c_1 + c_2 + c_3 \quad , \\
t_2 &= \phantom{c_1 + {}} c_2 + c_3 + c_4 \quad , \\
t_3 &= \phantom{c_1 + c_2 + {}} c_3 + c_4 + c_5 \quad , \\
t_4 &= c_1 \phantom{{} + c_2 + c_3} + c_4 + c_5 \quad , \\
t_5 &= c_1 + c_2 \phantom{{} + c_3 + c_4} + c_5 \quad .
\end{aligned}
$$

If $c_1$, $c_2$, $c_3$, $c_4$, $c_5$ are on a circle [Guttman (1954) provides a rationale for this], so that

$$\mathrm{Cov}(c_i, c_{i+k}) = \Delta_k \quad , \quad \text{for } i = 1,2,\ldots,5 \quad ,$$

with $\Delta_1 = \Delta_4$, $\Delta_2 = \Delta_3$ [because the "distance" from $c_1$ to $c_2$ is that of $c_1$ to $c_5$, and the "distance" from $c_1$ to $c_3$ is that of $c_1$ to $c_4$ ], then we obtain (3.2) for $p = 5$ as the covariance matrix.

## 4. Reduction to Canonical Form

The critical question at this point is whether a reduction to a canonical form is possible for the block circular case. Although not obvious, by using Kronecker products it becomes straightforward to see that such a reduction is possible, indeed.

First we note several facts concerning the Kronecker product

$$A \otimes B \equiv (a_{ij}B) \quad .$$

If $A : m \times n$ , $B : p \times q$ , then $A \otimes B$ is an $mp \times qn$ matrix. We would like to generate block circular matrices as in (3.2). To do this, the matrix we use (2.5) and form

$$(4.1) \qquad \Sigma = (W_0 \otimes \Sigma_0) + (W_1 \otimes \Sigma_1) + \cdots + W_{p-1} \otimes \Sigma_{p-1} \quad ,$$

where the matrices $W_j$ are defined in (2.5), and $\Sigma_j = \Sigma_{p-j}$ , $j = 1,\ldots,p-1$ . For example, when $p = 4$ , we obtain

$$\Sigma = \begin{pmatrix} \Sigma_0 & 0 & 0 & 0 \\ 0 & \Sigma_0 & 0 & 0 \\ 0 & 0 & \Sigma_0 & 0 \\ 0 & 0 & 0 & \Sigma_0 \end{pmatrix} + \begin{pmatrix} 0 & \Sigma_1 & 0 & 0 \\ 0 & 0 & \Sigma_1 & 0 \\ 0 & 0 & 0 & \Sigma_1 \\ \Sigma_1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \Sigma_2 & 0 \\ 0 & 0 & 0 & \Sigma_2 \\ \Sigma_2 & 0 & 0 & 0 \\ 0 & \Sigma_2 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \Sigma_1 \\ \Sigma_1 & 0 & 0 & 0 \\ 0 & \Sigma_1 & 0 & 0 \\ 0 & 0 & \Sigma_1 & 0 \end{pmatrix}$$

$$= \begin{bmatrix} \Sigma_0 & \Sigma_1 & \Sigma_2 & \Sigma_1 \\ \Sigma_1 & \Sigma_0 & \Sigma_1 & \Sigma_2 \\ \Sigma_2 & \Sigma_1 & \Sigma_0 & \Sigma_1 \\ \Sigma_1 & \Sigma_2 & \Sigma_1 & \Sigma_0 \end{bmatrix} \quad .$$

Next we need several well-known facts concerning Kronecker products:

$$(4.2) \qquad (A_1 \otimes B_1)(A_2 \otimes B_2) = A_1A_2 \otimes B_1B_2 \quad ,$$

$$(4.3) \qquad A \otimes (B + C) = (A \otimes B) + (A \otimes C) \quad ,$$

$$(4.4) \qquad (A + B) \otimes C = (A \otimes C) + (B \otimes C) \quad .$$

Applying these facts to (4.1), we obtain

$$(4.5) \qquad (\Gamma \otimes I)\Sigma(\Gamma' \otimes I)$$

$$= (\Gamma W_0 \Gamma' \otimes \Sigma_0) + (\Gamma W_1 \Gamma' \otimes \Sigma_1) + \cdots + (\Gamma W_{p-1} \Gamma \otimes \Sigma_{p-1}) \quad .$$

Recall that $W_0 = I$, $W_j = W_1^j$, $j = 1,\ldots,p-1$. Consequently, if $\Gamma$ diagonalizes $W_1 + W_{p-1} = W_1 + W_1'$, i.e., $\Gamma(W_1 + W_1')\Gamma' = \text{diag}(\epsilon_1,\ldots,\epsilon_k)$, then $\Gamma(W_j + W_j')\Gamma' = D_\epsilon^j$. But the matrix $\Gamma$ defined by (2.3) is exactly that orthogonal matrix which diagonalizes $W_j + W_j'$, and the diagonal elements $\epsilon_j$ are the $p$ roots of unity. Thus

$$(4.6) \qquad (\Gamma \otimes I)\Sigma(\Gamma' \otimes I) = \text{Diag}(\psi_1,\psi_2,\ldots,\psi_p) \equiv D_\psi \quad ,$$

where the matrices $\psi_j$ are positive definite and satisfy

$$(4.7) \qquad \psi_j = \psi_{p-j+2} \; , \qquad j = 2,\ldots,p \quad .$$

As in the case when the blocks are single elements, we may now use (2.6) and (2.7) as our starting point, noting that we have $pk$ variates instead of $p$.

Remark: If we wish to recapture estimates of $\Sigma_j$, we may do so from (4.6), namely,

$$(4.8) \qquad \Sigma = (\Gamma' \otimes I)\, D_\psi(\Gamma \otimes I) \quad .$$

Indeed (4.8) yields simple linear equations of the form

$$(4.9) \qquad \Sigma_\alpha = a_{\alpha 1}\psi_1 + \cdots + a_{\alpha p}\psi_p \quad , \qquad \alpha = 1,\ldots,p \quad ,$$

where the coefficients $a_{ij}$ are functions of the $\gamma_{ij}$.

## 5. Hypotheses for Symmetric Structures in the Covariance Matrix and the Likelihood Functions

The following hypotheses represent block versions of (1) sphericity, (2) intraclass correlation, (3) circular symmetry, and (4) a general matrix:

(5.1) $$H_1 : \Sigma = \mathrm{Diag}(\Sigma_0, \ldots, \Sigma_0) \quad ,$$

$$H_2 : \Sigma = \begin{bmatrix} \Sigma_0 & \Sigma_1 & \cdots & \Sigma_1 \\ \Sigma_1 & \Sigma_0 & \cdots & \Sigma_1 \\ & & \cdots & \\ \Sigma_1 & \Sigma_1 & \cdots & \Sigma_0 \end{bmatrix} \quad ,$$

$$H_3 : \Sigma = \Sigma_c \quad ,$$

$$H_4 : \Sigma > 0 \quad .$$

In terms of the canonical representation, we may now test hypotheses such as

(a) sphericity versus intraclass correlation,

(b) sphericity versus circular symmetry,

(c) intraclass correlation versus circular symmetry,

(d) circular symmetry versus general structure.

Because of the canonical form (4.6), the parameter space for each of the hypotheses $H_1 - H_4$ becomes

$$(5.2)\qquad \omega_1 = \{\psi\colon \psi_1 = \cdots = \psi_p > 0 \text{ , given } \psi_j = \psi_{p-j+2} \text{ , } j = 2,\ldots,p\} \quad ,$$

$$\omega_2 = \{\psi\colon \psi_1 > 0 \text{ , } \psi_2 = \cdots = \psi_p > 0 \text{ , given } \psi_j = \psi_{p-j+2} \text{ , } j = 2,\ldots,p\} \quad ,$$

$$\omega_3 = \{\psi\colon \psi_1 > 0 \text{ , } \psi_2 = \psi_{p-j+2} > 0 \text{ , } j = 2,\ldots,p\} \quad ,$$

$$\omega_4 = \{\Sigma\colon \Sigma > 0\} \quad .$$

Following the procedure of Olkin and Press (1969), we may obtain the maxima of the likelihood functions $L(y,V)$ over the regions $\omega_i$ and $\omega_j$, and thereby generate the likelihood ratio tests (LRT) for hypothesis $H_i$ versus $H_j$ by

$$(5.3)\qquad \lambda_{ij} = \frac{\sup_{\omega_i} L(y,V)}{\sup_{\omega_j} L(y,V)} \quad .$$

Because the circular symmetric model is equivalent to the condition $\psi_2 = \psi_p$ , $\psi_3 = \psi_{p-1}$ , etc., it is clear that we will want to pool the covariance matrices $V_{22}$ with $V_{pp}$ , $V_{33}$ with $V_{p-1,p-1}$ , etc. in estimating the common $\psi_2$ , $\psi_3$ , etc. Thus, it will simplify our notation if we write

$$(5.4)\qquad (V_1,\ldots,V_{m+1}) = (V_{11}, V_{22} + V_{pp}, \ldots, V_{mm} + V_{m+2,m+2}, V_{m+1,m+1}) \quad ,$$

when $p = 2m$ is even, and

$$(5.5)\qquad (V_1,\ldots,V_{m+1}) = (V_{11}, V_{22} + V_{pp}, \ldots, V_{m+1,m+1} + V_{m+2,m+2}) \quad ,$$

when $p = 2m + 1$ is odd. For later use we define

$$(5.6) \qquad V_j = V_{p-j+2} \quad , \quad j = 2,\ldots,p \quad .$$

Since the mean vector $y$ and the $V_j$'s are independently distributed, when $\Sigma$ is circular, we have as a canonical model:

Mean Vector

$$(5.7) \qquad y_1 \sim \mathcal{N}(\nu_1,\psi_1) \, , \quad y_j \sim \mathcal{N}(\nu_j,\psi_j) \, , \quad \psi_j = \psi_{p-j+2} \, , \quad j = 2,\ldots,p \quad .$$

Covariance Matrices

$$(5.8) \qquad V_1 \sim \mathcal{W}(\psi_1;k,n) \, , \quad V_j \sim \mathcal{W}(\psi_j;k,2n) \, , \quad j = 2,\ldots,m \, ,$$

$$V_{m+1} \sim \mathcal{W}(\psi_{m+1};k,n) \, , \quad \text{when} \quad p = 2m \quad .$$

$$(5.9) \qquad V_1 \sim \mathcal{W}(\psi_1;k,n) \, , \quad V_j \sim \mathcal{W}(\psi_j;k,2n) \, , \quad j = 2,\ldots,m + 1$$

when $p = 2m + 1$ .

The maxima of the likelihoods may now be obtained in a straightforward manner, and to a certain extent, the results parallel those in Olkin and Press (1969). The results are based on the assumption that the mean vectors $\nu_1,\ldots,\nu_p$ are unknown. A slight modification in the development yields analogous results when the means are known.

We now list the maxima of the likelihood function for the various models using (5.7) - (5.9) as our starting point. In each case the maximum involves a common term

$$(5.10) \qquad c(V) = c(pk,n)(2\pi)^{-pk/2}|V|^{(n-pk-1)/2}\, e^{-pkN/2} \quad .$$

Spherical Model

$$(5.11) \qquad \sup_{\omega_1} L(y,V) = \frac{c(V)\ (pN)^{pkN/2}}{|\sum_1^p V_{ii}|^{pN/2}} .$$

Intraclass Correlation Model

$$(5.12) \qquad \sup_{\omega_2} L(y,V) = \frac{c(V)\ [N^p(p-1)^{p-1}]^{kN/2}}{|V_{11}|^{N/2}|\sum_2^p V_{ii}|^{(p-1)N/2}} .$$

Circular Model

$$(5.13) \qquad \sup_{\omega_3} L(y,V) = \frac{c(V)\ N^{pkN/2}}{|V|^{N/2}} .$$

General Case

$$(5.14) \qquad \sup_{\omega_4} L(y,V) = \frac{c(V)\ N^{pkN/2}}{|V|^{N/2}} .$$

For each of the hypotheses considered when the hypothesis is true, the LRT is distributed as a product of independent beta variates, so that the procedure of Box (1949) may be used to obtain an approximation for the null distribution. We here present an approximation to $O(N^{-2})$. Because some of the hypotheses are closely allied to testing for the equality of covariance matrices, the Bartlett modification may be preferable to the LRT [see Anderson (1958) for details concerning this test]. Also, because some of the hypotheses are nested, we may use the procedure of Gleser and Olkin (1972) to provide an easier evaluation of the needed constants.

## 6. Likelihood Ratio Tests and Their Approximate Null Distribution for Testing Symmetric Structures

Using the results (5.11)-(5.13), we may readily form the LRT for various hypotheses. In some instances the results for $p = 2m$ or $p = 2m + 1$ differ; whenever possible, we combine our results for even and odd $p$ by using the parameter $m$.

### 6.1 Test for Sphericity Versus Circularity

From (5.11) and (5.13), the LRT, $\lambda_{13}$, is given by

$$(6.1) \qquad \lambda_{13}^{2/N} = \frac{p^{pk} \prod_{1}^{p} |V_j|}{2^{2k(p-m-1)} \; |\sum_{1}^{p} V_{ii}|^p} \quad .$$

The modified Bartlett statistic, $L_{13}$, is a simple function of $\lambda_{13}$, namely, $L_{13} = \lambda_{13}^{n/N}$. Using the result of Anderson (1958, p. 254) we obtain the approximate null distribution:

$$(6.2) \qquad P\{-\rho \log L_{13} \leq z\} \cong P\{X_f^2 \leq z\} + O(n^{-2}) \quad ,$$

where

$$f = \frac{1}{2} mk(k + 1) \quad ,$$

$$\rho = 1 - \frac{1}{n} \frac{[p(3m + 3 - p) - 2]\,(2k^2 + 3k - 1)}{12mp(k + 1)} \quad .$$

## 6.2 Test for Intraclass Correlation Model Versus Circularity

From (5.12) and (5.13), the LRT, $\lambda_{23}$ , is given by

$$(6.3) \qquad \lambda_{23}^{2/N} = \frac{(p-1)^{(p-1)k} \prod_{2}^{p} |V_j|}{2^{2k(p-m-1)} \left| \sum_{2}^{p} V_{ii} \right|^{p-1}} \quad .$$

As in Section 6.1, the modified Bartlett statistic, $L_{23}$ , is given by $L_{23} = \lambda_{23}^{n/N}$ . Similarly,

$$(6.4) \qquad P\{-\rho \log L_{23} \le z\} \cong P\{\chi_f^2 \le z\} + O(n^{-2}) \quad ,$$

where

$$f = \frac{1}{2} mk(k+1) \quad ,$$

$$\rho = 1 - \frac{1}{n} \frac{(3mp - p^2 - 3m + p - 3)(2k^2 + 3k - 1)}{12(p-1)m(k+1)} \quad .$$

## 6.3 Tests for Circular Versus General Structure

From (5.13) and (5.14), the LRT is given by

$$(6.5) \qquad \lambda_{34}^{2/N} = \frac{2^{2k(p-m-1)} |V|}{\prod^{p} |V_j|} \quad .$$

In order to show that this statistic is distributed as a product of independent beta variates, note that

$$\frac{|V|}{\prod_1^p |V_j|} = \frac{|V|}{\prod_1^p |V_{ii}|} \frac{|V_{22}|\ |V_{pp}|}{|V_{22} + V_{pp}|} \cdots \frac{|V_{mm}|\ |V_{m+2,m+2}|}{|V_{mm} + V_{m+2,m+2}|} ,$$

if $p = 2m$ , and

$$\frac{|V|}{\prod_1^p |V_j|} = \frac{|V|}{\prod_1^p |V_{ii}|} \frac{|V_{22}|\ |V_{pp}|}{|V_{22} + V_{pp}|} \cdots \frac{|V_{m+1,m+1}|\ |V_{m+2,n+2}|}{|V_{m+1,m+1} + V_{m+2,m+2}|} ,$$

if $p = 2m + 1$ . Under the null hypothesis, these statistics are independently distributed. Furthermore, each term is known to be distributed as a product of independent beta variables, Anderson (1958, Chapters 9, [illegible] Consequently, we may use the following result of Gleser and Olkin (1972): If $Z = \prod_1^G Z_i$ , and appropriate regularity conditions prevail, then

$$P\{-2 \log Z \leq z\} \cong P\{\chi_f^2 < \rho z\} ,$$

where $f = \sum_1^G f_i$ , $\rho = \sum_1^G f_i \rho_i / f$ , and the $f_i$ and $\rho_i$ are obtained by applying the Box procedure to each $Z_i$ .

In the present case, we may let $Z_1 = |V| / \prod_1^p |V_{ii}|$ . This is the test statistic for testing for independence in a covariance matrix [see Anderson (1958, p. 233)]. Here $f_1 = \frac{1}{2} k^2 p(p - 1)$ , $\rho_1 = 1 - [2k(p + 1) + 9]/6N$ .

The remaining statistics $Z_j$ are of the form $|V_{11}|\,|V_{22}| / |V_{11} + V_{22}|$ , which is the test statistic for testing for the equality of two covariance matrices [see Anderson (1958, p. 255)]. For each such test, we obtain

$$f_j = \frac{1}{2}k(k+1) \quad , \quad \rho_j = 1 - \frac{2k^2 + 3k - 1}{12N(k+1)} \quad .$$

Note that $p - m - 1$ is equal to $m - 1$ when $p = 2m$, and is equal to $m$ when $p = 2m + 1$. Thus, in either case of $p$ even or odd, we obtain

$$(6.6) \quad f = \frac{1}{2}k^2 p(p-1) + \frac{1}{2}(p-m-1)k(k+1) \quad ,$$

$$\rho = 1 - \frac{k}{24fN}\{2kp(p-1)(2kp + 2k + q) + (p-m-1)(2k^2 + 3k - 1)\} \quad .$$

The final approximation to the null distribution is then given by

$$P\{-\rho \log \lambda_{34} \leqq z\} = P\{\chi_f^2 \leq z\} + O(N^{-2}) \quad ,$$

where $f$ and $\rho$ are given by (6.6).

## 7. Tests for Means Given That the Covariance Matrix Is Circular

When the population covariance matrix has no special structure, and we wish to test that the mean vector is zero, the appropriate test is Hotelling's $T^2$. However, when we know that there is a circular structure, we can take advantage of this information in constructing a test.

From the canonical form (5.7)-(5.9), we see that under $H: \nu = 0$, we should estimate $\psi_1$ by $V_{11} + y_1'y_1$, and we should estimate $\psi_j$ by

$$V_{jj} + V_{p-j+2,p-j+2} + y_j'y_j + y_{p-j+2}'y_{p+j+2} \quad ,$$

$j = 2,\ldots,p$. Thus the LRT statistic is given by

$$(7.1) \qquad \lambda^{2/N} = \prod_{1}^{p} \frac{|V_j|}{|V_j + W_j|} \quad ,$$

where when $p = 2m + 1$

$$(7.2) \qquad Z_1 = y_1'y_1 \quad , \quad Z_j = y_j'y_j + y_{p-j+2}'y_{p-j+2} \quad , \quad j = 2,\ldots,m+1 \; ,$$

and when $p = 2m$ ,

$$(7.3) \qquad Z_1 = y_1'y_1 \quad , \quad Z_j = y_j'y_j + y_{p-j+2}' \quad , \quad Z_{m+1} = y_{m+1}'y_{m+1} \quad ,$$

$j = 2,\ldots,m$ . Each component of the product in (7.1) is distributed as a $U$ -statistic [see Anderson (1958, p. 193)]. Thus, we may again use the Lemma of Gleser and Olkin (1972) to obtain an approximation to the null distribution of the LRT. To do this we need to know the degrees of freedom $f$ , and the value of $\rho$ . For ratios $|V_j|/|V_j + Z_j|$ , which do not involve pooling, e.g., $j = 1$ , we have

$$f = k \quad , \quad \rho = 1 - \frac{k^2}{(k+1)n} \quad ;$$

for terms which involve pooling, e.g., $j = 2,\ldots,m$ , we have

$$f = 2k \quad , \quad \rho = 1 - \frac{k-1}{2n} \quad .$$

Consequently, the overall value of $f$ is

$$f = pk \quad , \quad \rho = 1 - \frac{[k^3 + (k^2 - 1)(p - m - 1)]}{2f(k+1)n} \quad .$$

Remark: The methods outlined lend themselves to the development of other tests concerning the means. For example, Olkin and Press (1969) consider the test that the means are equal when there is circular symmetry. This model may be extended to test that the mean vectors are equal in blocks, when the covariance matrix is circularly symmetric in blocks. Similarly, we may simultaneously test for the equality of the mean vectors and circular symmetry. The key point in the development of such tests is to start with the canonical form (5.7)-(5.9), from which the LRT may be readily obtained.

References

Anderson, T. W. (1958). An Introduction to Multivariate Statistical Analysis. Wiley, New York.

Anderson, T. W. (1970). Estimation of covariance matrices which are linear combinations or whose inverses are linear combinations of given matrices, pp. 1-24 in Essays in Probability and Statistics, University of North Carolina Press, Chapel Hill.

Box, G. E. P. (1949). A general distribution theory for a class of likelihood criteria. Biometrika 36, 317-346.

Gleser, L. (1966). A note on the sphericity test. Ann. Math. Statist. 37, 464-467.

Gleser, L. J. and Olkin, I. (1972). A note on Box's general method of approximation for the null distributions of likelihood criteria. Submitted for publication.

Guttman, L. (1954). A new approach to factor analysis: the radex, pp. 258-348, 430-433 in Mathematical Thinking in the Social Sciences (ed. by P. F. Lazarsfeld), The Free Press, Glencoe, Ill.

Guttman, L. (1957). Empirical verification of the radex structure of mental abilities and personality traits. Educ. Psychol. Meas. 17, 391-407.

Jöreskog, K. G. (1970). Estimation and testing of simplex models. British J. Math. Statist. Psychol. 23, 121-145.

Mukherjee, B. N. (1966). Derivation of likelihood ratio tests for the Guttman quasi-simplex covariance structures. Psychometrika 31, 97-123.

Mukherjee, B. N. (1970). Likelihood ratio tests of statistical hypotheses associated with patterned covariance matrices in psychology. British J. Math. Statist. Psychol. 23, 89-120.

Olkin, I. and Press, S. J. (1969). Testing and estimation for a circular stationary model. Ann. Math. Statist. 40, 1358-1373.

Wilks, S. S. (1946). Sample criteria for testing equality of means, equality of variances, and equality of covariances in a normal multivariate distribution. Ann. Math. Statist. 17, 257-281.

Wise, J. (1955). The autocorrelation functions and the spectral density function. Biometrika 42, 151-159.