Box's method of approximation for the null distributions of likelihood criteria is described. It simplifies the formulas, describes a method of obtaining $f$, $\phi$, and $\rho$ directly from given values, and provides two illustrations of the method. (DJ)
A NOTE ON BOX’S GENERAL METHOD OF APPROXIMATION FOR
THE NULL DISTRIBUTIONS OF LIKELIHOOD CRITERIA

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by

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1. Introduction

Many multivariate test statistics (such as the likelihood ratio test statistic for MANOVA and the Bartlett modification of the likelihood ratio test statistic for testing the equality of covariance matrices) have null distributions whose $h$-th moment $M_h$ is of the form

$$M_h = K \left( \prod_{j=1}^{J} \frac{y_j}{(y_j)^{x_j}} \right)^h \left( \prod_{i=1}^{I} \frac{x_i}{\Gamma(x_i(1+h) + \xi_i)} \right) \left( \prod_{j=1}^{J} \frac{y_j}{\Gamma(y_j(1+h) + \eta_j)} \right), \quad h = 0, 1, 2, \ldots,$$

where $K$ is a constant (such that $M_0 = 1$), the $x$'s and $y$'s are positive numbers, and

$$\frac{\sum_{i=1}^{I} x_i}{J} = \frac{\sum_{j=1}^{J} y_j}{J}.$$

Such statistics have a range of variation from 0 to 1, so that the moments $M_h$, $h = 0, 1, 2, \ldots$, determine the null distribution.

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For any statistic \( W, 0 \leq W \leq 1 \), whose moments are of the form (1.1), Box (1949) has proposed an asymptotic expansion for the cumulative distribution function (c.d.f.). This expansion provides an accurate method for determining the critical constants defining rejection regions for the multivariate tests mentioned above. One form in which this expansion is often given [see Anderson (1958; p. 207)] is the following:

\[
(1.3) \quad P(-2 \log W \leq t) = (1 - \phi)P(X_1^2 \leq pt) + \phi P(X_2^2 \leq pt) + R(\delta)
\]

Here, \( \delta = \sum_{i=1}^{I} x_i = \sum_{j=1}^{J} y_j \),

\[
(1.4) \quad f = -2[\sum_{i=1}^{I} x_i \xi_i - \sum_{j=1}^{J} \eta_j - \frac{1}{2} (I - J)]
\]

\[
(1.5) \quad \phi = \frac{1}{-6\rho} \left[ \sum_{i=1}^{I} \frac{x_i^2}{2} B_3((1 - \rho)x_i + \xi_i) - \sum_{j=1}^{J} \frac{y_j^2}{2} B_3((1 - \rho)y_j + \eta_j) \right]
\]

\( \rho \) is the solution of the equation

\[
(1.6) \quad \sum_{i=1}^{I} x_i^{-1} B_2((1 - \rho)x_i + \xi_i) - \sum_{j=1}^{J} y_j^{-1} B_2((1 - \rho)y_j + \eta_j) = 0
\]

and \( R(\delta) \) is a remainder term. [Note: \( \rho \) is chosen so that Anderson's \( \omega_1 \) is 0.] Also \( B_1(u), B_2(u), \) and \( B_3(u) \) are the Bernoulli polynomials of degree 1, 2, and 3 defined by:

\[
(1.7) \quad B_1(u) = u - \frac{1}{2} \quad , \quad B_2(u) = u^2 - u + \frac{1}{6} \quad , \quad B_3(u) = u^3 - \frac{3}{2} u^2 + \frac{1}{2} u
\]
If \( \lim_{\delta \to \infty} \delta^{-1} x_i > 0 \) and \( \lim_{\delta \to \infty} (1 - \rho)x_i \) exists, \( i = 1,2,\ldots,I \), and if \( \lim_{\delta \to \infty} \delta^{-1} y_j > 0 \) and \( \lim_{\delta \to \infty} (1 - \rho)y_j \) exists, \( j = 1,2,\ldots,J \), then

\[
\text{Anderson (1958, pp. 205-207) sketches an argument which shows that the remainder term } R(\delta) \text{ in (1.3) is } O(\delta^{-3}) \text{ as } \delta \to \infty. \text{ In the case of the likelihood ratio test for MANOVA, Anderson (1958) gives examples showing the high accuracy provided by the approximation (1.3).}
\]

Determination of the constants \( \phi \) and \( \rho \) defined by (1.5) and (1.6), respectively, is usually a very cumbersome task. In Section 2, we show how this task can be considerably simplified by taking advantage of certain relationships among the Bernoulli polynomials \( B_k(u) \), \( k = 1,2,3 \). As a result, we obtain convenient formulas for \( f \), \( \rho \), and \( \phi \) for the important special case (which includes the null distributions of the likelihood ratio test statistic for MANOVA and for various special cases of the likelihood ratio test statistic for independence of sets of variates) in which \( T = J \), the \( x_i \)'s and \( y_j \)'s are all equal, \( \xi_i \) is linear in \( i \), and \( \eta_j \) is linear in \( j \).

Let \( W_1 \) and \( W_2 \) be two statistically independent random variables, where each \( W_i \) has range of variation from 0 to 1 and has moments of the form (1.1). It follows that an asymptotic expansion (1.3) for the c.d.f. of \( W_i \) can be given, \( i = 1,2 \). Let \( f_i \), \( \rho_i \), and \( \phi_i \) be the constants in the asymptotic expansion (1.3) for the c.d.f. of \( W_i \), \( i = 1,2 \), and suppose that the values of \( f_1 \), \( f_2 \), \( \rho_1 \), \( \rho_2 \), \( \phi_1 \), and \( \phi_2 \) are known to us. Let \( W = W_1W_2 \). It can easily be shown that \( 0 \leq W \leq 1 \) and that the moments of \( W \) have the form (1.1). Thus, one can obtain an
asymptotic expansion (1.3) for the c.d.f. of \( W \). Although the constants \( f, \rho, \) and \( \phi \) in this expansion can be obtained ab initio by calculating the moments of \( W \) and then using formulas (1.4) through (1.6), or the simplifications of these formulas given in Section 2, it would be more convenient to obtain the constants \( f, \rho, \) and \( \phi \) directly from the given values of \( f_1, f_2, \rho_1, \rho_2, \phi_1, \) and \( \phi_2 \). Formulas for doing this are given in Section 3. Section 4 provides two illustrations of the method.

2. Simplification of the Formulas for \( \rho \) and \( \phi \)

In this section it is shown that if the \( h \)-th moment of the random variable \( W, \ 0 \leq W \leq 1 \), is given by (1.1), then the constants \( \rho \) and \( \phi \) in Equation (1.3) can be obtained as follows:

\[
\rho = 1 - \frac{1}{f} \left[ \sum_{i=1}^{I} x_i^{-1} B_2(\xi_i) - \sum_{j=1}^{J} y_j^{-1} B_2(\eta_j) \right],
\]

and

\[
\phi = -\frac{1}{6\rho^2} \left[ \sum_{i=1}^{I} x_i^{-2} B_3(\xi_i) - \sum_{j=1}^{J} y_j^{-2} B_3(\eta_j) + \frac{3}{2} (1 - \rho)^2 f \right],
\]

where \( f \) is given by (1.4).

To verify (2.1), note from (1.7) that

\[
B_2(w + v) = B_2(v) + 2wB_1(v) + w^2,
\]

and hence
Note that

\[
\begin{align*}
(2.4) \quad \sum_{i=1}^{L} u_i^{-1} B_2((1 - \rho)u_i + v_i) \\
= \sum_{i=1}^{L} u_i^{-1} B_2(v_i) + 2(1 - \rho) \sum_{i=1}^{L} B_1(v_i) + (1 - \rho)^2 \sum_{i=1}^{L} u_i.
\end{align*}
\]

Note that

\[
(2.5) \quad \sum_{i=1}^{I} B_1(\xi_i) - \sum_{j=1}^{J} B_1(\eta_j) = \sum_{i=1}^{I} \xi_i - \frac{1}{2} I - \sum_{j=1}^{J} \eta_j + \frac{1}{2} J
\]

\[= - \frac{1}{2} f.
\]

Since \( \sum_{i=1}^{I} x_i = \sum_{j=1}^{J} y_j \), it follows from (1.6), (2.4), and (2.5) that \( \rho \) is the solution of

\[
(2.6) \quad \sum_{i=1}^{I} x_i^{-1} B_2(\xi_i) - \sum_{j=1}^{J} y_j^{-1} B_2(\eta_j) - (1 - \rho)f = 0,
\]

which is equivalent to (2.1).

To verify (2.2), note from (1.7) that

\[
(2.7) \quad B_3(w + \psi) = B_3(v) + 3wB_2(v) + 3w^2B_1(v) + w^3,
\]

and hence

\[
(2.8) \quad \sum_{i=1}^{L} u_i^{-2} B_3((1 - \rho)u_i + v_i)
\]

\[= \sum_{i=1}^{L} u_i^{-2} B_3(v_i) + 3(1 - \rho) \sum_{i=1}^{L} u_i^{-1} B_2(v_i)
\]

\[+ 3(1 - \rho)^2 \sum_{i=1}^{L} B_1(v_i) + (1 - \rho)^3 \sum_{i=1}^{L} u_i.
\]
Equation (2.2) now follows from (1.5), (2.1), (2.5), and the fact that

\[
\sum_{i=1}^{I} x_i = \sum_{j=1}^{J} y_j.
\]

If the \( h \)-th moment of \( W \) has the form (1.1) with \( I = J = L \), \( x_1 = x_2 = \ldots = x_I = y_1 = y_2 = \ldots = y_J = z \), \( \xi_i = a + bi \), \( i = 1, 2, \ldots, I \), and \( \eta_j = c + dj \), \( j = 1, 2, \ldots, J \), then from (1.4),

\[
(2.9) \quad f = -2 \left[ \sum_{i=1}^{L} (a + bi) - \sum_{j=1}^{L} (c + dj) \right]
\]

\[
= -2L[(a - c) + \frac{1}{2}(b - d)(L + 1)].
\]

Also, from (2.1), (2.9), and (1.7),

\[
\rho = 1 - \frac{1}{L^2} \left[ \sum_{i=1}^{L} (a + bi)^2 - \sum_{i=1}^{L} (a + bi) - \sum_{j=1}^{L} (c + dj)^2 + \sum_{j=1}^{L} (c + dj) \right]
\]

\[
= 1 - \frac{L}{L^2} \left[ (a^2 - c^2) + (ab - cd)(L + 1) + \frac{1}{6} (b^2 - d^2)(L + 1)(2L + 1) + \frac{1}{21} \right].
\]

Finally, note from (1.7) that

\[
B_3(u) = u^3 - \frac{3}{2} B_2(u) - B_1(u) - \frac{1}{4}.
\]

From this fact, (2.1), (2.2), and (2.5), it follows that
By a direct evaluation we have that

\[ \sum_{i=1}^{L} [(a + bi)^3 - (c + di)^3] \]

(2.11) 

\[ + \frac{3}{2} (1 - \rho) \frac{z^2}{\rho} f. \]

(2.12) 

Thus, (2.11) and (2.12) together give us a formula for computing $\phi$.

3. Approximation for the Distribution of the Product of Independent Statistics Whose Moments Are of the Form (1.1)

Let the independent random variables $W_g$, $0 \leq W_g \leq 1$, be independent, with $h$-th moment of the form

\[ E(W_g)^h = K_g \left( \prod_{j=1}^{J_g} (y_{gj})^{x_{gj}} \right)^h \left( \prod_{i=1}^{I_g} \Gamma(x_{gi} (1 + h) + \xi_{gi}) \right), \quad h = 0, 1, 2, \ldots, \]

(3.1)

where $K_g$ is a constant (such that $E W_g^0 = 1$), and

\[ \Sigma x_{gi} = \Sigma y_{gi}, \]

for $g = 1, 2$. Let
\( W = W_1 W_2 \).

It then follows that \( E(W)^n = E(W_1)^n E(W_2)^n \), and hence the moments of \( W \) are of the form (1.1) with \( I = I_1 + I_2, \ J = J_1 + J_2, \)

\[
\begin{align*}
x_i &= \begin{cases} x_{1i} & \text{if } i = 1, 2, \ldots, I_1, \\ x_{2i-I_1} & \text{if } i = I_1 + 1, I_2, \ldots, I. \end{cases} \\
t_i &= \begin{cases} t_{1i} & \text{if } i = 1, 2, \ldots, I_1, \\ t_{2i-I_1} & \text{if } i = I_1 + 1, I_2 + 1, \ldots, I. \end{cases} \\
y_j &= \begin{cases} y_{ij} & \text{if } j = 1, 2, \ldots, J_1, \\ y_{2j-J_1} & \text{if } j = J_1 + 1, J_1 + 2, \ldots, J. \end{cases} \\
\end{align*}
\]

Note also that \( \sum_{i=1}^{I} x_i = \sum_{j=1}^{J} y_j = \delta. \)

Since the moments of \( W_1, W_2, \) and \( W \) are all of the form (1.1), it follows from the results of Sections 1 and 2 that the c.d.f.'s of these three variables can be expanded in the form (1.3). The coefficients \( f_g, \rho_g, \phi_g \) for the expansion of the c.d.f. of \( W_g, \ g = 1, 2, \) and the coefficients \( f, \rho, \phi, \) for the expansion of the c.d.f. of \( W \) are expressible by means of equations (2.5), (2.1), and (2.2). From these expressions and (3.3), it can be shown by some straightforward algebra that

\[
\begin{align*}
f &= f_1 + f_2, \\
\rho &= \frac{f_1 \rho_1 + f_2 \rho_2}{f}, \\
\end{align*}
\]

and
Equations (3.4), (3.5), and (3.6) thus provide a way to compute the coefficients $f$, $\rho$, and $\phi$ in the expansion (1.3) of the c.d.f. of

$W = W_1 W_2$ in terms of the coefficients $f_g$, $\rho_g$, and $\phi_g$ in the expansion (1.3) of the c.d.f. of $W_g$, $g = 1, 2$. We may generalize these results by induction and obtain the following:

**Theorem 1.** Let the statistically independent variables $W_g$, $0 < W_g < 1$, have moments of the form (3.1), $g = 1, 2, \ldots, G$. Let $f_g$, $\rho_g$, $\phi_g$,

and $\delta_g = \sum_{i=1}^{I_g} x_i$, $\gamma_g = \sum_{j=1}^{J_g} y_{gj}$ be the constants in the Box expansion (1.3) of the c.d.f. of $W_g$, $g = 1, 2, \ldots, G$. Finally, let

\[(3.7) \quad W = \prod_{g=1}^{G} W_g.\]

Then

\[(3.8) \quad P(-2 \log W \leq t) = (1 - \phi)P(\chi_f^2 \leq pt) + \phi P(\chi_f^2 \leq pt) + R(\delta),\]

where $\delta = \sum_{g=1}^{G} \delta_g$, $f = \sum_{g=1}^{G} f_g$, $\rho = \frac{1}{\phi} \sum_{g=1}^{G} \rho_g$, $\phi$, and

\[\phi = \frac{1}{\rho^2} \sum_{g=1}^{G} \rho_g^2 + \frac{1}{4\rho^2} \sum_{f g h} (\rho_g - \rho_h)^2.\]
Two applications of Theorem 1 to multivariate testing problems are given in Section 4. Note that if in the Box expansions for $W_1, W_2, \ldots, W_g$, each $R_g(\delta_i)$ is $O(\delta_i^{-3})$ as $\delta_i \to \infty$, and if $\delta_1, \delta_2, \ldots, \delta_g$ are all asymptotically of the same order of magnitude (i.e., $\lim_{\delta_i \to \infty} \frac{\delta_i \delta_h^{-1}}{\delta_h} > 0$ as $\delta_i, \delta_h \to \infty$ for all $\delta_i \neq \delta_h$), then $R(\delta_i)$ in (3.8) is $O(\delta_i^{-3})$. In practical use of Theorem 1, the $\delta_i$’s will usually be asymptotically of the same order of magnitude. If the $\delta_i$’s are not asymptotically of the same order of magnitude, Equation (3.8) is formally correct (when $\delta$, $f$, $\rho$, and $\sigma$ are defined as in Theorem 1), but the order of magnitude of the remainder term $R(\delta_i)$ in $\delta_i$ must be separately investigated.

4. Applications to Multivariate Hypothesis Testing Problems

Suppose we are interested in testing whether either the mean vectors and/or covariance matrices of $k$ multivariate normal populations are identical. Suppose that an observation (p dimensional row vector) $x^{(i)}$ from the $i$-th population has a p-variate normal distribution with mean vector $\mu^{(i)}$ and covariance matrix $\Sigma^{(i)}$, $i = 1, 2, \ldots, k$. Let $x^{(i)}$ be partitioned as $(x_1^{(i)}, x_2^{(i)})$, where $x_1^{(i)}$ is 1 x q, and let

$$
(4.1) \quad \mu^{(i)} = (\mu_1^{(i)}, \mu_2^{(i)}) , \quad \Sigma^{(i)} = \begin{pmatrix} \Sigma_{11}^{(i)} & \Sigma_{12}^{(i)} \\ \Sigma_{21}^{(i)} & \Sigma_{22}^{(i)} \end{pmatrix},
$$

be correspondingly partitioned, $i = 1, 2, \ldots, k$. Suppose that we observe $N_i$ observations from the $i$-th population, $i = 1, 2, \ldots, k$.

We consider two tests of hypotheses. The first test compares the null hypothesis.
(4.2) \( H_{mvc} : \mu(1) = \mu(2) = \ldots = \mu(k) \), \( \Sigma(1) = \Sigma(2) = \ldots = \Sigma(k) \),

against general alternatives. In the second test, we compare the null hypothesis \( H_{mvc} \) against the alternative:

(4.3) \( H_{m'vc} : \mu_1(1) = \mu_1(2) = \ldots = \mu_1(k) \), \( \Sigma_{11} = \Sigma_{11} = \ldots = \Sigma_{11} \).

Let \( \bar{x}(i) = (\bar{x}_1(i), \bar{x}_2(i)) \) be the sample mean vector and let

\[
V(i) = \begin{pmatrix} V_{11}^{(i)} & V_{12}^{(i)} \\ V_{21}^{(i)} & V_{22}^{(i)} \end{pmatrix}
\]

be the sample cross-product matrix from the \( i \) -th population, \( i = 1, 2, \ldots, k \).

Then \( (\bar{x}(1), \bar{x}(2), \ldots, \bar{x}(k), V(1), V(2), \ldots, V(k)) \) is a sufficient statistic for both hypothesis testing problems. Let \( N = \sum_{i=1}^{k} \frac{N_i}{N} \),

\[
\bar{x} = (\bar{x}_1, \bar{x}_2) = \frac{1}{N} \sum_{i=1}^{k} N_i (\bar{x}_1(i), \bar{x}_2(i))
\]

and

\[
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \sum_{i=1}^{k} N_i (\bar{x}(i) - \bar{x}) (\bar{x}(i) - \bar{x})
\]

4.1 Test of \( H_{mvc} \) Versus General Alternatives

Anderson (1958) suggests testing \( H_{mvc} \) against general alternatives by means of the test statistic
\[ W = \left( \prod_{i=1}^{k} \left( \frac{1}{n_i} \right)^{\frac{1}{2}} \right)^{\frac{1}{2n}} \frac{\sum_{i=1}^{k} \frac{1}{n} \left( \sum_{i=1}^{k} V(i) \right)^{\frac{1}{2n}}}{\left( \frac{1}{n} \sum_{i=1}^{k} V(i) + A \right)^{\frac{1}{2n}}} = W_1 W_2, \]

where \( n_i = N_i - 1 \), \( i = 1, 2, \ldots, k \), and \( n = \sum_{i=1}^{k} n_i \). The statistic \( W_1 \) is the likelihood ratio test statistic for testing \( \Sigma^{(1)} = \Sigma^{(2)} = \ldots = \Sigma^{(k)} \) against general alternatives, but modified [along lines suggested by Bartlett (1937)] by everywhere replacing the sample sizes \( N_i \) by the degrees of freedom, \( n_i \), of \( V(i) \). The statistic \( W_2 \) is the \((n/N)\) -th power of the likelihood ratio test statistic for MANOVA.

As Anderson (1958) shows, the statistics \( W_1 \) and \( W_2 \) are independent when \( H_{mvc} \) holds, and also

\[ E(W_1)^h = K_1 \left( \frac{\prod_{i=1}^{k} p \prod_{r=1}^{p} \Gamma(\frac{1}{2} n_i (1 + h) + \frac{1}{2} (1 - s))}{\prod_{t=1}^{p} \prod_{s=1}^{p} \Gamma(\frac{1}{2} n (1 + h) + \frac{1}{2} (1 - t))} \right). \]

Thus, the moments of \( W_1 \) under \( H_{mvc} \) are of the form (1.1), and we can use the Box expansion (1.3) for the c.d.f. of \( W \) when \( N_1, N_2, \ldots, N_k \) are large and of the same order of magnitude. Although the constants \( \tau_1, \rho_1 \), and \( \phi_1 \) in the expansion (1.3) of the c.d.f. of \( W_1 \) can be obtained directly from (4.5) by use of the methods of Section 2, Anderson (1958; p. 255) has already shown that
\[ f_1 = \frac{1}{2} (k - 1)p(p + 1), \]

\[ \rho_1 = 1 - \left( \sum_{i=1}^{k} \frac{1}{n_i} - \frac{1}{n} \right) \frac{2p^2 + 3p - 1}{6(p + 1)(k - 1)}. \]

(4.6)

\[ \phi_1 = \frac{p(p + 1)[(p - 1)(p + 2)\left( \sum_{i=1}^{k} \left( \frac{1}{n_i} \right)^2 - \left( \frac{1}{n} \right)^2 \right) - 6(k - 1)(1 - \rho_1)^2]}{48\rho_1^2}. \]

[In comparing (4.6) with Anderson's results it should be noted that his \( q \) is our \( k \).

Anderson (1958; p. 207) gives the \( h \)-th moments of \( \lambda = (W_2)^{N/h} \) under \( H_{mv} \) as

(4.7)

\[ E(\lambda)^h = K_2 \sum_{s=1}^{p} \prod_{t=1}^{n} \frac{\Gamma\left( \frac{1}{2} n(1 + h) + \frac{k}{2} - \frac{1}{2} s \right)}{\Gamma\left( \frac{1}{2} n(1 + h) - \frac{1}{2} t \right)}, \]

where (4.7) holds for all real \( h \) for which the gamma functions exist.

Hence,

(4.3)

\[ E(W_2)^h = K_2 \sum_{s=1}^{p} \prod_{t=1}^{n} \frac{\Gamma\left( \frac{1}{2} n(1 + h) + \frac{1}{2} - \frac{1}{2} s \right)}{\Gamma\left( \frac{1}{2} n(1 + h) + \frac{k}{2} - \frac{1}{2} t \right)}. \]

Thus, applying (2.9), (2.10), (2.11), and (2.12), with \( L = p, \ z = n/2, \ a = 1/2, \ b = -(1/2), \ c = k/2, \ d = -(1/2) \), we find that
\[ P(-2 \log W_2 \leq t) = (1 - \phi_2)P(X^2_{\nu_2} \leq \rho_2 t) + \phi_2 P(X^2_{\nu_2} + 4 \leq \rho_2 t) + O(n^{-3}) \]

where

\[ f_2 = \rho(k - 1) \]

\[ \rho_2 = 1 - \frac{p - k + 2}{2n} \]

\[ \phi_2 = \frac{\rho(k - 1)[\rho^2 + (k - 1)^2 - 5]}{48n^2(\rho_2)^2} \]

To obtain an asymptotic expansion for the c.d.f. of the test statistic \( W \) under \( H_{\text{mv}} \), Anderson (1958; p. 255) goes back to the \( h \)-th moments of \( W \) and applies the Box expansion method ab initio.

However, we already have the constants \( f_1, f_2, \rho_1, \rho_2, \phi_1, \phi_2 \) from the asymptotic expansions of the c.d.f.'s of \( W_1 \) and \( W_2 \). Using Theorem 1 of Section 3, we thus conclude that the constants \( f, \rho, \phi \) in the Box expansion (1.3) of the c.d.f. of \( W \) are given by

\[ f = f_1 + f_2 = \frac{1}{2} (k - 1)p(p + 3) \]

\[ \rho = \frac{f_2\rho_1 + f_1\rho_2}{f} = 1 - \sum_{i=1}^{n} \frac{1}{n} - \frac{1}{n} \left( \frac{2\rho^2 + 3p - 1}{6(p + 3)(k - 1)} - \frac{p - k + 2}{n(p + 3)} \right) \]

and
\[
\phi = \frac{\rho_1^2 + \rho_2^2}{\rho^2} + \frac{f_1f_2}{4p} (\rho_1 - \rho_2)^2
\]

\[
(4.12) = \frac{p}{288p} \left[ 6 \left( \prod_{i=1}^{k} \frac{1}{n_i} \right) \right. \\
\left. \left( \frac{1}{n} \right)^2 \right] - \frac{1}{n^2} \left( p^2 - 1 \right) \left( p + 2 \right) \\
- \frac{1}{n^2} \left[ \frac{12(k - 1)}{n^2} (-2k^2 + 7k + 3pk - 2p^2 - 6p - 4) \right]
\]

4.2 Test of \( H_{mv} \) Versus \( H_{m'v} \)

Gleser and Olkin (1972) show that the likelihood ratio test statistic
\[
\lambda_{mvc,m'vc},
\]
for testing \( H_{mvc} \) against the alternative \( H_{m'vc} \), is

\[
(4.13) \quad \lambda_{mvc,m'vc} = \frac{\prod_{i=1}^{k} \frac{1}{N_i} V(i)_{22}^{1} \prod_{i=1}^{k} \frac{1}{N_i} \left( \frac{1}{N_1} \left( \frac{1}{N_i} \left( \sum_{i=1}^{k} V(i)_{11} + A_{11} \right) \right) \right)^{1/2N} \\
\times \left( \frac{1}{N_1} \left( \sum_{i=1}^{k} V(i) + A \right) \right)^{1/2N}
\]

where

\[
(4.14) \quad V(i)_{22} = V(i)_{22} - V(i)_{11} V(i)_{12}^{-1} V(i)_{12}, \quad i = 1, 2, \ldots, k
\]

However, instead of \( \lambda_{mvc,m'vc} \), let us modify the statistic by replacing \( N_1 \) by \( n_i = N_1 - 1, \quad i = 1, 2, \ldots, k \), and \( N \) by \( n = \sum_{i=1}^{k} n_i \), everywhere in (4.13). [There is more than one way to modify the likelihood ratio test statistic along the lines suggested by Bartlett (1937). One way is given
here; another, and possibly preferable, way is considered in Gleser and Olkin (1972). The resulting statistic is

\[ U_2 = \frac{\left( \prod_{i=1}^{k} \frac{1}{n_i} \frac{v(i)^{\frac{1}{2}n_i}}{\sum_{i=1}^{k} \frac{1}{n} \left( \frac{1}{n} \sum_{i=1}^{k} v(i) + A_{11} \right)^{\frac{1}{2}n}} \right)}{\left( \frac{1}{n} \left( \frac{k}{n} \sum_{i=1}^{k} v(i) + A_{11} \right)^{\frac{1}{2}n} \right)} . \]

To obtain the c.d.f. of \( U_2 \) let \( U_1 \) be a similar modification of the likelihood ratio test statistic for testing hypothesis \( H_{m'vc} \) against general alternatives. Gleser and Olkin (1972) have derived the likelihood ratio test statistic. From their result, we find that

\[ U_1 = \frac{\left( \prod_{i=1}^{k} \frac{1}{n_i} \frac{v(i)^{\frac{1}{2}n_i}}{\sum_{i=1}^{k} \frac{1}{n} \left( \frac{1}{n} \sum_{i=1}^{k} v(i) + A_{11} \right)^{\frac{1}{2}n}} \right)}{\left( \frac{1}{n} \left( \frac{k}{n} \sum_{i=1}^{k} v(i) + A_{11} \right)^{\frac{1}{2}n} \right)} . \]

Comparing the statistic \( W \) defined in (4.4) with \( U_1 U_2 \), and recalling that \( |v(i)| = |v_{11}| |v_{22.1}| \) for \( i = 1, 2, \ldots, k \), we see that

\[ W = U_1 U_2 . \]

Since under \( H_{m'vc} \), the statistics \( \tilde{x}_2^{(i)} - \frac{\tilde{x}_1^{(i)} v(i)}{12} \), \( v_{11} \), \( v_{22.1} \), \( i = 1, 2, \ldots, k \), \( \tilde{x}_1^{(i)} \), and \( \frac{k}{n} \sum_{i=1}^{k} v(i) + A_{11} \) are complete and sufficient, and since the distribution of \( U_1 \) is the same for all values of the parameters \( \mu(1), \mu(2), \ldots, \mu(k) \), \( \Sigma(1), \Sigma(2), \ldots, \Sigma(k) \) obeying \( H_{m'vc} \), it follows from a theorem of Ba,a (1955) that \( U_1 \) and \( U_2 \) are statistically independent under \( H_{m'vc} \) (and thus under \( H_{mvc} \)).
Note that $U_1$ has the same form as $W$, except that $U_1$ is a function only of $x^{(1)}(j), j = 1,2,\ldots,N; i = 1,2,\ldots,k$. That is, $U_1$ is a $q$-dimensional version of $W$. The moments of $W$ under $H_{mvc}$ are known [they equal the product of (4.5) and (4.8)]. The moments of $U_1$ under $H_{mvc}$ can be obtained from the formula for the moments of $W$ by everywhere replacing $p$ by $q$. The moments of both $U_1$ and $W$ under $H_{mvc}$ are thus of the form (1.1). Since $U_1$ and $U_2$ are independent under $H_{mvc}$, and since $W = U_1 U_2$, the $h$-th moment of $U_2$ under $H_{mvc}$ equals the $h$-th moment of $W$ under $H_{mvc}$ divided by the $h$-th moment of $U_1$ under $H_{mvc}$. Thus the moments of $U_2$ under $H_{mvc}$ have the form (1.1).

From the preceding discussion and the results of Sections 1-3 (particularly Theorem 1), it follows that the c.d.f.’s of $U_1$, $U_2$, and $W$ all have asymptotic expansions of the form (1.3). Let $f$, $\rho$, and $\phi$ be the coefficients in the expansion (1.3) for the c.d.f. of $W$; these constants are given by Equations (4.10), (4.11), and (4.12), respectively. Let $f^*_g$, $\rho^*_g$, and $\phi^*_g$ be the coefficients in the expansion for the c.d.f. of $U_g$, $g = 1,2$. Since $U_1$ is a $q$-dimensional version of $W$, the coefficients $f^*_1$, $\rho^*_1$, and $\phi^*_1$ can be obtained by substituting $q$ for $p$ in the formulas (4.10), (4.11), and (4.12) respectively. Finally, from Theorem 1 we know that

\begin{align*}
f &= f^*_1 + f^*_2, \\
\rho &= \frac{f_1^* \rho^*_1 + f_2^* \rho^*_2}{f}, \\
\phi &= \frac{(\rho_1^*)^2 \phi_1^* + (\rho_2^*)^2 \phi_2^*}{\rho^2} + \frac{f_1^* f_2^*}{4f} (\rho_1^* - \rho_2^*)^2,
\end{align*}

(4.18)
Solving for $f_2^*$, $\rho_2^*$, and $\phi_2^*$ in (4.18) yields

$$(4.19) \quad f_2^* = f - f_1^* = \frac{1}{2} (k - 1)(p - q)(p + q + 3),$$

$$\rho_2^* = \frac{f_p - f_1^*}{f_2^*} = 1 - \left[ \sum_{i=1}^{k} \frac{1}{n_i} - \frac{1}{n} \right] \left[ \frac{2p^2 + 2pq + 2q^2 + 3p + 3q - 1}{6(p + q + 3)(k - 1)} \right]$$

$$(4.20) \quad \phi_2^* = \frac{\rho_2^* - (\rho_1^*)^2}{(\rho_1^*)^2} = \frac{f_1^* f_2^* (\rho_1^* - \rho_2^*)^2}{4f}$$

$$= \frac{1}{238(\rho_1^*)^2} \left[ \sum_{i=1}^{k} \frac{1}{(n_1)^2} - \frac{1}{n^2} \right] \left[ (p^2 - 1)(p^2 + 2p) - (q^2 - 1)(q^2 + 2q) \right]$$

$$- \left[ \frac{(12)(k - 1)(p - q)}{n^2} \right] \left[ 3(p + q)(k - 2) - 2(p^2 + pq + q^2) \right]$$

$$- 2k^2 + 7k - 4 \right] - 72f_2^*(1 - \rho_2^*)^2.$$
**References**


