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ABSTRACT

This paper examines the research and theories of Piaget and Bruner and some implications for education, particularly as applied to mathematics education in the elementary grades. Piaget's theories are divided into the general development and conceptual thinking and mathematical concepts in children. Experimental evidence is cited and a short section on structural materials for the mathematics classroom follows. A brief summary ties the two men's beliefs together with respect to readiness, curriculum, and structural materials. (JM)

SOME IMPLICATIONS OF THE THEORIES OF JEAN PIAGET AND J. S. BRUNER FOR EDUCATION

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INTRODUCTION

Current statements on education indicate that one of its primary purposes is to teach children how to think. Thinking is a very complex process, and until recently, has been little understood. This deficiency is now being remedied by the work of a number of investigators -- the most notable of them being Jean Piaget in Switzerland, and J. S. Bruner in the United States.

These two men have directed themselves to similar questions from divergent points of view. Piaget's major work has been concerned with the growth of intelligence (i.e., logical thinking) from infancy to adulthood, and his developmental studies of children's thinking are beginning to have considerable impact on educational planning. Bruner, on the other hand, has directed a programme of research into concept attainment and problem-solving in adults. More recently, he has turned his attention to the applicability of his findings to children. The present paper is an examination of the investigations of these two men, and some of the implicationsof their theories for education.

PIAGET

I THEORY

In a voluminous collection of studies, Jean Piaget, a Swiss psychologist, has undertaken the tremendous task of exploring the mind of the child "as Galileo explored the sky". The following theoretical formulation is the fruit of his investigation.

Piaget (1950) begins with the biological concept of adaptation, which is seen as the interplay of two processes, which he calls "assimilation" and "accommodation". Assimilation occurs when the child acts on an environmental object according to previous experience with some similar object and imposes some of his own conceptions on it. Objects are subservient to his needs -- a ball exists only to be handled, grasped, etc. In accommodation, new activities are incorporated into the child's repertoire in response to the demands of the environment. For example, the child learns to crawl towards an object that he desires. The development of intelligence involves a progressively more complex balancing or "equilibration" of these two processes.

Piaget argues that intellectual activity begins with physical actions upon the environment. These actions become internalized with the result that a mental structure* is formed. The child is able to perform certain "mental operations" which are the symbolic version of his earlier actions or operations on the physical environment; e.g.,

* The term "structure" refers here to a system of mental activities, which operate according to definite laws analogous to those of mathematics and logic.

classification (gathering), analysis (pulling apart), serialization (arranging in series).*

This structure can be described as "a semi-mobile equilibrium" capable of understanding the following logical and psychological operations:

(a) Combinativity or Closure: Two classes can be combined into a third; e.g., in mathamatics, $a + b = c$.

(b) Reversibility: Each operation implies a converse; e.g., in mathematics, $a \times b = b \times a$.

(c) Associativity: Different operations can achieve the same result; e.g., in mathamatics, $a + (b + c) = (a + b) + c$.

(d) Identity: Any operation combined with its converse annuls the operation; e.g., in mathematics, $a + (-a) = 0$.

According to Piaget, development of mental structures proceeds in four major stages. The first is described as the sensori-motor period, and develops from birth to about 2 years. This consists of simple innate reflex actions which are eventually modified by learning. These motor activities are not yet capable of being internalized into mental representations. Definition is in terms of action; e.g., "a hole is to dig". By the end of this period, however, the child has come to appreciate the permanence and stability of objects, including those which exist outside his immediate perceptual field. The child is capable of reversibility and associativity of <u>action</u>, but not of thought. For example, the child realizes that an object which has passed temporarily out of his visual field, still exists. The second major stage -- the pre-operational level -- may be subdivided into two phases. In the first or "transductive" phase, between 2 and 4 years, the development of symbolic

* See Lawrence (1963) for a further description.

function occurs; i.e., symbolic play, internalized imitation, and language. Actions can now be internalized to some degree through symbolization; i.e., language allows for action through internal images, which are relatively autonomous. The child can use symbolic play to assimilate reality to his own egocentric interests, and his intellectual functioning operates midway between the generality of a true concept and the particular instances of which it is composed. Past and future events can now be conceived. However, reasoning with these "pre-concepts" is irreversible, lacks generality, and is tied to action.

In the "intuitive" phase (4 to 7 years), the beginnings of proper thought groupings are apparent. The child accepts the constancy of a relation between one group of objects, but not another. He understands that operations are reversible, but this understanding is highly unstable and subject to perceptual rearrangement. He finds it almost impossible to "decentre" his perceptions from a single attribute of an object, although movement from two successive, and often contradictory, judgments to two simultaneous centrings has begun.* He cannot understand that the number in a row of counters does not change when the row is more or less spread out. There is a limited amount of conceptual activity, but this is still tied to perceptual appearances.

The period of "concrete operations", from 7 to 11 years of age, occurs when certain basic concepts are acquired and organized into qualitatively new <u>stable</u> structures. For the first time, the child is capable of the logical operations <u>on objects</u> of conservation, reversibility,

* In his earlier writings, Piaget described this narrowness of the field of attention as "egocentrism" (inability to distinguish between the self and the object) rather than "centring", and the development from it to logical thought ("decentred") was attributed to the progressive socialization of children's thinking. The child learns to view the world from a point of view other than his own.

associativity, and identity. Three simultaneous advances occur at about age 7to8: (1) The child formulates the concept of a logical class as the complete internalization of action groupings of objects. (2) Asymmetrical relations are now developed as the internalization of the action of ordering. For example, the child accepts the relativity of "left" and "right". (3) Number concepts can be understood as the "logical multiplication" of classification and ordering. All these diverse developments, however, are the expression of one basic change. Whereas thinking at the intuitive level resembled perception in its centring on the object as seen from a single point of view, thought at the stage of concrete operations has become "decentred". It can deal effectively with rearrangements of the objects, at least at an active level, because it has achieved an equilibrium of the assimilation of objects to the child's action and the accommodation of concepts ("schemata") to modifications of environmental objects. A tightly grouped collection of objects no longer seems "larger" when it is spread out into a line.

Concrete operations still have their limitations, however, since the concepts are still not generalized to all situations. For example, conservation of weight and volume follows that of number.* They are fragmentary, and have yet to be combined into an organized, structured whole.

It is only in the final, or "formal operational" level, that the emancipation from perception and action is complete. At about the age of 11 or 12, the capacity for formal "abstract" thought begins to mature. The adolescent is capable of thinking beyond the present, of

* For a summary of the experiments on these and other operations, see Inhelder and Matalon (1960, Pp. 445-448).

formulating hypotheses and theories about "the possible", of reasoning on the basis of purely formal assumptions (true or false) -- in short, of reflective thought. He can now make "logical experiments", not merely factual ones. The logic of propositions permits him to test the validity of statements by reference to their pure logicality, rather than to their correspondence with the real empirical world; i.e., to their form rather than their content. The hypothetico-deductive methods of science, with their rigorous deductions and systematic testing of variables can now be utilized. Bruner (1962c) has reported an illustrative experiment by Huttenlocher on the attainment of formal thinking. In a complex problem-solving situation, only subjects of 12 years of age or more successfully solved _all_ the problems requiring extended "logical experiments".

How has this change occurred? Piaget suggests that here we are dealing with a second-order system of operations upon _operations_, rather than upon objects or events. The operations of the concrete level -- classes and relations, asymmetry and symmetry, logical addition and multiplication -- have become organized into a structure with rules of its own. It is, in fact, a system of logical relations among propositions, whose logicality is testable in terms of the implications and contradictions involved in it. New operational schemata appear, permitting the adolescent to comprehend the various results of formal analysis: proportionalities, probabilities, permutations, combinations, reciprocals, and other types of logical and mathematical transformations.

To summarize then, we can see the exciting saga of mental development unfold as follows: The sensori-motor level combines primitive perceptual experiences, and allows the child to deal with objects in terms of his own action. The development of language permits imitative, imaginal, and symbolic activity. Concrete operations give the child a representation of empirical reality, but it is not until the stage of formal operations that the world of the possible is opened up to the individual. Mathematics and logic, in the abstract sense, are first available at this stage. The adolescent may not know the verbally stated laws of logic and mathematics, but he can <u>think</u> in accordance with them, surely a much more valuable achievement.

The sequence of development, then, consists of a continuing process of progressively more complex "schemata". The latter may be described as modes of action which are capable of conservation, generalization, and combination into higher order schemata. The culminating point comes with the complete reversibility of thought processes, at the formal level. Thought has become mobile, flexible, and free.

II NUMBER CONCEPTS

The discussion up to this point has been concerned with the general part of Piaget's work -- the development of conceptual thinking in childhood, from birth to intellectual maturity. One of the most exciting areas of research in this field has been a topic of obvious interest to educators; i.e., the growth of basic number concepts. Piaget's studies (1952)* have brought into serious question the whole problem of how, when, and what to teach in mathematics. This next section will outline some of the experiments relevant to this question.

In their 1961 report, the Mathematical Association stated the following aims for the teaching of mathematics:

> ...children, developing at their own individual rates, learn through their active response to the experiences that come to them; through constructive play, experiment and discussion, children become aware of relationships and developmental structures which are mathematical in form and are in fact the only sound basis of mathematical techniques. The aim of primary teaching...is the laying of this foundation of mathematical thinking about the numerical and spatial aspects of the objects and activities which children of this age encounter. (Pp.v-vi of the Preamble)

What would lead the group to such a decision? Why are educators, in fact, so concerned about the fact that many children "cannot" seem to learn to handle mathematics adequately? To answer these questions, one must turn to Piaget's studies of children's mathematical concepts.

Piaget begins with two basic assumptions: (1) that ability to count is quite different from true number understanding; (2) that the concept of number is developed concurrently with the growth of logic.

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* Summaries may be found in Isaacs (1960, 1961), Lawrence (1954).

His experiments on the growth of number concepts take the following form: Using children of ages 4 to 12, the investigator interviews each child individually, and asks him to predict the outcome of certain manipulations with simple objects, such as beads, dolls, poker chips, glasses, etc. The first test concerns the conservation of continuous quantities. Two identical vessels, A_1 and A_2, are filled to the same level with a coloured liquid. Half of the contents of A_2 are then poured into a vessel B_1 and the other half into a vessel B_2 (both the same size as A_2). The child is then asked if the quantity of liquid in A_1, and that in B_1 and B_2 together, remain the same. But the young child of 4 to 5 years has no conception of the constancy of quantity. For him, the quantities are unequal because they <u>seem</u> to change -- they get "taller" or "wider". At age 6, there is a transitional period towards conservation, but a conservation concept which is unstable and subject to the vagaries of perceptual change. It is not until the child reaches about age 7 to 8 that the assumption of constancy appears. The child at this age is able to coordinate the dimensions of height and width (compensating an increase in one with a decrease in the other) and reach the concept of constant quantity. The concrete operation has appeared. The process has become "reversible"; i.e., the child can imagine the water as being poured back into the original beaker without changing.*

One might expect that discontinuous quantity would be much easier to handle. If the child is asked to count out an equal number of beads into two beakers, one tall and thin, the other short and wide, he should surely, by adult standards, understand that the quantities are

* Recent evidence from Bruner's studies of children's thinking (cited in 1962c) has indicated that covering the beakers (removing the perceptual cues) lowers the age of attainment of the concept to 4 to 5 years.

equal. But here again one finds the same stages and pace of development -- no understanding of constancy of quantity, transitional understanding based on perceptual correspondence, and finally full attainment of the concept, independent of perceptual distortions.

It is generally assumed that a child who can count correctly has a reasonable conception of what number means. But it is here, in the area of basic number concepts, that educators will be most surprised. From Piaget's experiments, we learn that the logical principle of conservation of quantity is a pre-condition for the concept of number. The child is required to match objects, and thus establish a one-to-one correspondence between eggs and egg-cups, flowers and vases, glasses and bottles, etc. The child aged 4 to 5 merely creates a second row of approximately equal length. In the transitional period, the objects are correctly matched. However, if the experimenter rearranges the objects in one row, so that they are more of less spread out, the child's judgment about the second group is altered. Only when the conservation of number is grasped at age 7 to 8, can equivalence be established. Counting aloud is of no assistance when perception dominates thinking. It is only a verbal exercise, and of little effect in bringing the child to an appreciation of the constancy of a number despite the rearrangement of its parts. When the child comprehends that an increase in the length of the row is proportional to the length of interval between objects, he can disengage his concepts from perceptual experience, and begin to understand number.

Piaget describes an important group of experiments dealing with the concept of seriation, or ordinal correspondence. Children are asked to match each of ten dolls of graduated sizes with ten sticks similarly graduated, to construct a series with rods of varying lengths, to match one series with another, insert odd pieces into already constructed series, and so on. The same course of development is seen to pertain in these tasks. The child moves from a rough perceptual approximation, through correct analysis which depends on perceptual correspondence, to completely autonomous and quantitative ordinal thought (with constant unit values).

An essential aspect of numerical thinking is the problem of class inclusion. To attain any sort of stable number concept, the child must realize that there are more elements in the whole class B than in the sub-class A included in it (where $B = A + A^1$). Given a box of wooden beads, of which a few are white and the rest brown, the child cannot tell if there are more wooden or brown beads. The idea of the stability of part-whole relationsand the concept of classification in general, are unavailable to him until age 7to 8years. Nor does he recognize that equal groups of objects in dissimilar arrangements, can be split up into a number of equal groups.

What do the results of these tests represent? To answer this question, it is necessary to realize that Piaget's conception of number involves a synthesis of two <u>logical</u> principles: classification (the basis of cardination), and seriation (the basis of ordination). Number is the system of grouping which simultaneously involves classification based on similarity, and seriation based on cumulative difference. For example, the number "5" has two basic properties. Its cardinal or classificatory property -- its "five-ness" -- consists in the fact that it is a class of events which includes the sub-class of 4, just as 6 includes 5. Its ordinal function is contained in the fact that it occurs in the series of events between 4 and 6. If it is 5th, then there must be 4 objects preceding it -- i.e., there is an interdependency of ordination and cardination. It is the task of the child who tries to deal with number, to coordinate these two aspects. But herein the difficulty lies. For the young child, the simultaneous entertaining of two dimensions of experience is unlikely at the ages of 4to7. The child at the stage of "global comparisons" (4to5), has no concept of equivalence because he is unable to "decentre" his perception from a single aspect to more than one attribute at a time. In the "intuitive" stage, he is in the transitional phase of conceptual development. He can coordinate the high-wide dimensions of the beaker, or match red to blue poker chips, as long as the objects do not depart too far from the world of appearances. Destroy perceptual correspondence, and you destroy what was only a very unstable concept in the first place. But when the full development of "concrete operations" has occurred through the attainment of "decentring", the child is able to attain a systematic, persistent, and necessary concept of equivalence, of conservation of quantity, of reversibility in classification and seriation — in short, of number.

III EXPERIMENTAL EVIDENCE

The fascinating, if somewhat startling discoveries by Piaget and his co-workers about children's number concepts have not gone unchallenged. Although his more recent theory and empirical investigations have been much more precise and objective than the earlier work, Piaget is still often accused of an "unscientific" approach, as the result of his failure to specify the sorts of samples he uses, the mental ages of his subjects, the precise age criteria for each stage of development, etc. Nevertheless, there is a small, but rapidly developing body of experimental studies which is attempting to remedy these deficiencies.

One of the first of these efforts was a study by Estes (1956). Using 52 children who ranged in age from 4 to 6 years, she presented four Piaget-type number situations to them. Contrary to Piaget's results, Estes found that children who could count could do so correctly even when the objects were rearranged, making no errors attributable to perceptual "centring". In addition, the subjects did not confuse apparent increases in number with true increases, another major hypothesis of the Geneva group.

Nevertheless, the reader must not immediately conclude that Piaget's results are worthless. In another study on a sample of some 250 Kindergarten and Primary school children, Dodwell (1960) found substantial support for the proposition that there are three stages in the development of number concepts -- the global, the intuitive, and the concrete operational. Although there were variations between children at any given age level, and within a given child for various types of testing situation, the evidence certainly offers considerable corroboration to the basic structure of Piaget's theory and experimental findings. In addition, Dodwell points out the necessity of considering mental age as an important variable.

This last factor is shown to be of some relevance in a study by Carpenter (1955). She used a number of Piaget tests, including the water-beaker technique for testing conservation of continuous quantities. Her results indicate a substantially higher correlation between total score on the tests and mental age, than between total score and chronological age. The sequence of development, however, is confirmed. Moreover, this study also showed that performance can be improved by presenting the test material in more concrete and familiar situations (Lunzer, 1955). Thus, the age levels obtained by investigators who control for mental age may be too high in the absolute sense, though accurate with respect to <u>sequence</u> of development.* A well-controlled study by Hood (1962) used some 126 normal children and 40 sub-normals. His findings indicate that mental stature is a much more relevant factor with respect to the development of number concepts than is chronological age in normal children. The pattern was similar for retarded children, but the stages occurred at a much slower rate of development.

It is fairly clear to most investigators in this area that Piaget's age ranges may have to be taken "with a grain of salt". Piaget himself has admitted that the <u>sequence</u> is the important finding. Not only is the factor of mental age confounding the issue, but environmental background is also relevant. For example, Dodwell (1961) has found some

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* Similar results have been obtained by Elkind (1961).

small differences in number understanding in favour of children of higher socio-economic status, and from urban as opposed to rural areas. Harker (1960) reports that previous experience with number tests has a facilitating effect on performance. Price-Williams (1961) found a similar development of the concepts of continuous and discontinuous quantity among illiterate West African bush children, though age levels were difficult to establish.

A critical study has been performed by Eileen Churchill (1958 a and b), showing the effect of environmental enrichment on the development of number concepts. Two groups, each consisting of eight children aged five, were matched for their understanding of number concepts. One of these groups was then given a special programme of number experiences over a period of four weeks. These sessions were devoted to guided play situations in which the children were given the opportunity for seriation, matching, ordering, sharing, comparing, grouping, etc. They were encouraged to discover for themselves the invariance of numerical relations. At the end of the period both groups were given another battery of "tests". The experimental group were found to have made a highly significant improvement when compared with the control group, as judged by the number of questions which were answered at an "operational" level. Their relative superiority was maintained on retesting three months later.

The sequence of stages described by Piaget was borne out in this study, and the majority of Churchill's five-year olds showed the same lack of conservation as Piaget's own younger subjects. On the other hand, the experiment strongly suggests that environment, and especially concrete number experiences, can do a great deal to accelerate the developmental progress of children in their understanding of number:

> ...where the children's environment is carefully planned to involve experience of these relationships at a concrete level through play activities and stories which interest them, and where the teacher shares them with the children, helping those who are ready to make explicit those relationships already known implicitly and acted upon, children can be helped in their development towards an appreciation of the meaning of number. (Churchill, 1961, P. 96.)

Inhelder (1953) has also supported this position concerning the importance of a salutary environment.

The Churchill experiment is of considerable importance to education. Piaget's critics have often complained that his emphasis on inward maturation and inward growth leaves no room for the effects of a stimulating environment. While this is a partial misunderstanding of his theory, the difficulty could be resolved easily by the realization that Piaget is referring to a continuous interaction between the child and his environment. If the teacher provides a setting which offers the opportunity for activity in certain general directions, and guidance towards these goals, then the course of mental development can be hastened, the learning immeasurably enriched, and the foundations for true understanding firmly laid.

The work of Piaget on number concepts is now recognized as being of vital importance to teachers of mathematics. In the past, educators have been tempted to accept the child's ability to count as evidence of number understanding. But Piaget's work has stimulated more precise investigations of this topic. For example, Williams (1958) tested groups of children on ordination, cardination, and arithmetic achievement. He found little relationship between the ability to count, or even to compute sums, and the understanding of number concepts. Those children who had achieved such understanding, however, were able to appreciate such sophisticated mathematical notions as the complementary nature of addition and subtraction. There is apparently a difference between the verbal activity of counting, and the mental one of quantification. Churchill's (1958) results emphasize even more the necessity for caution in teaching elementary arithmetic before the underlying mental structures have developed. Her conclusion is that development may even be reversed or at least substantially retarded, by premature training in mechanical sums before understanding of the conservation of number <u>even when these computations are disguised as play situations</u>. Another study of Saad (1960) points out the wideness of the gap between arithmetical skill and understanding.

IV GENERAL IMPLICATIONS

What are the implications of these findings for education? With respect to the teaching of arithmetic, a recent conference concluded

> ...not that we should begin to teach number at this or that age, but that the teaching of number should be geared to the understanding which has been achieved so far, in order to facilitate progress to a higher level. (In Churchill, 1961, P. 6)

"Number readiness" may be an important factor to be assessed before formal instruction proceeds, and Williams, Dodwell and Churchill have provided us with useful testing instruments using Piaget's experimental techniques for this task.* But the critical question is how to assist the child's growth to this level of understanding. This brings educators back to the old question of the "mechanical vs. meaningful" controversy over arithmetic teaching. While teachers are generally agreed that all teaching should stress meaning, the place of mechanical skills in arithmetic is debatable, as is the question of exactly how to make primary arithmetic experiences meaningful.

Churchill (1961) has provided us with the beginning of an answer to this question. Taking as her starting point Piaget's idea of thought as the internalization of action, she points out that:

> Man felt his way into it [the language of number] through his own bodily manipulations of his environment. Children do likewise, and must be given the opportunity as well as the time to do this. (P. 43)

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* See a discussion of this question in Crawford (1960).

The mathematician Dienes (1959) has provided three basic hypotheses for anyone who attempts to formulate a programme of mathematics teaching which would take Piaget's findings into account:

> (1) Visual, tactile and muscular images must be formed to create perceptual equivalents of a concept. From the common essence of these will be abstracted the conceptual structure.
>
> (2) The higher the level of generality at which a concept is formed, the wider its fields of possible applications.
>
> (3) A concept involving variables is best understood in its full generality if the variables contained in it are made to vary. (Pp. 16-17)

The specific relevance of these propositions to Dienes' own structural material will be amplified further below. But the point to be made here is Dienes' agreement with the notion that a variety of <u>action</u> experiences (tactual, kinesthetic) with different materials in different situations is perhaps the best way to lay the foundation for full concrete operational, and ultimately, formal mathematical understanding. Children begin to develop their ideas of number from groupings in their environment. They learn to discriminate qualities of objects, to sort them into classes, and to label these classes. Such action is the necessary prior condition for number learning. As Inhelder (1962) has noted, "the development of knowledge seems to be the result of a process of elaboration that is based essentially on the <u>activity</u> of the child". (P. 20) Formal education must simply amplify and extend these concrete experiences. If children are permitted <u>active</u>, <u>manipulative</u> experiences and self-determined exploration in <u>concrete</u> situations, then teachers will be fulfilling their valuable role of "planned intervention in the maturational process"; i.e., of helping the child to build up a better "model" of the world which relies on the understanding of the relational nature of number concepts and not on the rote memorization of rules. But the final "leap" will come from the child himself. The Mathematical Association Report (1961) cited above, reiterates what by now must be an oft-repeated theme, but one that bears further repetition:

> Understanding is the first aim; familiarity through many and rich experiences is the second; memorization to the point of automatic response, if it is to have any usefulness, must wait for the accomplishment of the other two.* (P. 10)

In one of his relatively rare excursions into educational philosophy, Piaget (1951) has summed up his own attitude to the problem of instruction. His version of the United Nation's "universal right to education" is as follows:

> During his formative period everyone has the right to an educational environment which will permit him to fashion in their completed form these indispensable instruments of adaptation, the logical faculties. (P. 73)

He directs his attention to the teaching of elementary mathematics, noting that the apparent failure to grasp the most basic concepts is not due to the lack of any special aptitude, but rather to affective blocking. If mathematics is nothing but a form of simple logical principles, then mathematical failure would seem to imply an overall defect of reasoning. To deny this conclusion is to suggest the possibility that the problem may lie in the area of the approach to teaching.

He draws the following implications from his own experimental studies on children's concepts of number, space, time, etc.:

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* One might even question the need for any training in computation at all in the age of the high speed electronic calculator. One mathematician espousing this view defined true mathematics as "the art of not-calculating".

(1) "Every normal pupil is capable of sound mathematical reasoning if his own initiative is brought into play." (P. 95)

(2) Concerning the teaching of geometry, Piaget points out that..."the child's ideas are at first much less influenced by the obvious metrical relationships involved than is generally imagined. On the contrary, they proceed from that kind of relationship which the mathematicians call 'topological' and they arrive only much later at the stage of Euclidean geometry (a fact which is of the greatest interest from the standpoint of modern mathematics)." The curriculum now presents geometry according to the axiomatic order, despite the fact that both the child and the race developed from topological to projective and Euclidean concepts. (P. 97)

(3) "The real cause of the failure of formal education must be sought primarily in the fact that it begins with language (accompanied by illustrations and fictitious or narrated action) instead of beginning with real practical action. The preparation for subsequent mathematical teaching should begin in the home by a series of manipulations involving logical and numerical relationships, the idea of length, area, etc., and this kind of practical activity should be developed and amplified in a systematic fashion throughout the whole course of primary education, gradually developing at the beginning of secondary education into elementary physical and mechanical experiments." (P. 98)

Too often in the past, the teaching of arithmetic in primary grades has been based on the premature assumption of number understanding. Piaget's work shows us that arithmetic teaching should take the following order, and not the reverse, as has so often been the case:

(1) Some things are stable and invariable;

(2) These things can be grouped, and groups have the qualities of "one-ness", "two-ness", etc.;

(3) Number properties have an order of size;

(4) These quantities are represented by certain symbols in our culture.

V STRUCTURAL MATERIALS

It may be seen from the preceding discussion that Piaget's discoveries have made imperative a revaluation of the standard conceptions of mathematics teaching, especially at the primary level. Churchill (1961) has pointed out that early work must be chiefly concerned with active, manipulative experiences with number concepts. With the current interest in the use of structural materials for the teaching of mathematics, it is tempting to attempt to analyze some of these methods in the light of Piaget's theory. But such an analysis is worthy of an entire discussion of its own, and therefore only a few brief comments will be advanced.

Williams (1961) has described in some detail the principal types of structural material, or "concrete analogues" of mathematical concepts. A summary of their contents appears in the following table (Williams, 1961, Pp. 120-121):

COMPARATIVE SUMMARY OF DEVICES USED IN DIFFERENT SYSTEMS*

	Shaw	Stern	Unifix	Avon	Cuisenaire	Montessori	Dienes	Lowenfeld	Bass	Arnold
Abacus.						Value indicated by colour.				Value shown by size.
Number Lengths.	Cylindrical vertically-structured.	Wooden blocks.	Vertically or horizontally structured. Components attachable to one another.	Flat pieces 2 units wide, bearing one dot per unit.	Units not marked on rods, which have colour values.	Large wooden blocks and small bars of beads.	Lengths not provided but can be constructed with M.A.B. pieces.	Lengths not provided but can be constructed.		Measurement stair of flat numbered pieces.
Containing and measuring devices.	Base with two rows of 10 holes divided by groove.	Trays holding squares of numbers from 1-10 and 20.	Container divided into five 20-unit channels.						Counterplay board.	
Number-stairs.		Number-stair holder.	Number-stair holder. Number boats.							
Number tracks.	Bases can be juxtaposed.	1-100 track with units marked.	1-20 track with 1-10 track joined.	Number channel with tens marked.						
Positional notation.	Tens and units card.	Dual board holding 10 tens.	Holds only 5 tens.	Holds only 9 tens.			Teacher constructs own.			
Charts, etc., for measuring.	Multiplication Chart and 100-Chart.	Multiplication Machine.				Multiplication board. Addition strip board.			Convecta board.	
Number Patterns.		Pattern Boards with pits for cubes.		Patterns on Number-pieces.		Patterns formed out of counters.				
Pegboards.	Pegs structurable vertically.					Pegs have colour value. Division Board.	For use in A.E.M.			
Fraction devices.		Fraction plates and frames.				Insets, skittles, and device for decimals.		Pieces showing *shapes* of fractions.	Convecta fraction board.	Keighley fraction board.
Balances.							For use in A.E.M.			
Devices illustrating powers.		Grooved wooden blocks to base 10.				Joined beads to base 10.	Grooved wooden blocks to various bases.			
Some devices for illustrating algebra and geometry.					Geoboards.	A variety of devices including flat shapes.	A.E.M. apparatus including triangular and rectangular slabs of wood.	Shapes, areas and volumes demonstrable.		

* NOTE: (1) Systems often make provision for operations, without using specially constructed devices, so this table does not reflect comparative comprehensiveness.

(2) "Unifix" refers to a *set of materials* rather than a "system," and "Arnold" to the *supplier* of certain devices.

Churchill (1961) presents a discussion of the relative merits of various sorts of structural material. She points out that the Cuisenaire, Stern, and Montessori material all provide reasonably accurate and useful models of the number system for early learning of these concepts. However, she notes that "unless there is a good deal of direction from the teacher they do not achieve their purpose as tools of number education. Moreover, because the range of shapes and sizes is limited their possibilities for constructive activities are exhausted pretty quickly" (P. 98). Structural materials may be useful but "until more sustained experimental work has been undertaken and reported upon, it would be premature to form conclusions about their value" (P. 98).

Dienes' notions about structural material are based on what he calls the "variability principle". If the attainment of a concept depends on the ability to abstract common elements from a variety of situations, then the greater the variety of such experiences for each concept (within limits), the easier the acquisition of it. Thus he suggests a variety of different structures on which mathematically equivalent tasks can be performed. From these varied situations, the child is better able to generalize to the overall mathematical concept, and to realize its applicability to all situations. Churchill draws a similar conclusion about the importance of variety of materials in the transition from the concrete to the abstract. She suggests that a successful programme will use any or all of: counters, beads, abacus, Stern and Cuisenaire material, Dienes' multi-base blocks, a ladder, etc. In addition, different materials may be more or less appropriate for different children, ages, and stages.

Presenting the child with groups of objects of varying sizes and patterns, as in the Montessori material, provides some measure of visual and kinesthetic supplement to verbal statements about numbers, but this method operates on the assumption that cardinality precedes ordinality. Stern's technique allows for both ordinality and cardinality, as do the Cuisenaire rods. Dienes' multi-base blocks have the additional advantage of diversity, and thus the apparatus is useful for the discovery of the concepts of place value and base. Dienes also suggests the use of the balance beam for the teaching of linear equations.

Two major issues arise here: concrete vs. abstract material, and structural vs. natural material. The use of concrete material through active manipulation has its dangers, as well as its advantages. Difficulty can occur in transferring from the specific concrete instance to the formal concept; i.e., a problem of "weaning" may occur.

Experimental evidence on the efficacy of Cuisenaire rods, for example, is mixed. The Vanccuver Study (Ellis, 1962) reports generally beneficial results but it is difficult to tell the extent to which these effects can be attributed to the material itself, or to the overall "modern arithmetic" programme which they use, and the enthusiasm of teachers and administrators in the Vancouver system about the new approach. Moreover, there was no difference in performance on problem-type questions, only on Cuisenaire ones. A study in Manitoba schools (Lucow, 1962) reports superior performance in multiplication and division for pupils using Cuisenaire rods in Grade 3, but the study is confounded by the previous experience of some of the children with the material before the study began, and by the presence of over-age children and repeaters in the non-Cuisenaire classes. Moveover, the study indicated a much greater effect in rural Manitoba schools than with an urban population, thus suggesting that the use of the rods with city children may be superfluous. As Lucow states, "there is some doubt of its general superiority over current methods of instruction" (P. 23). Lucow concludes that the method is a valuable one, and worthy of attention but that "children should be taught by whatever method they respond to best. No teacher should limit herself to one method of instruction in the face of the abundant individual differences in children" (P. 25). Passy (1963) reports that children taught with Cuisenaire do significantly worse on a standardized arithmetic achievement test. Piaget (cited in Churchill, 1960) reports a study of some children who learned arithmetic by the Cuisenaire method; they were successful at number tests using similar material, but were quite lost when tested on the same concepts using different structural devices.

Allied to the question of active participation is the matter of its directness. There is some limited evidence (Williams, 1958) that watching the teacher demonstrate a principle, and verbally anticipating it, may produce as effective learning in some children as actual performance of the activity.

It has even been suggested that concrete material and specific images may be necessary to some degree, or in some situations, at all stages of learning to think (Bruner, 1962). The question is how to strike the balance between the two extremes.

With respect to the problem of structural vs. natural materials, it can probably be safely stated that the former are to be preferred for the teaching of any mathematical concepts beyond the most elementary properties of the number system.* Nevertheless, none of the present systems is without its own mathematical limitations, especially with regard to

* This conclusion is contrary to that of Mathematical Association, who believe that artificiality in dealing with number merely lengthens the "weaning" process.

some of the "New Mathematics". In addition, different methods may be effective with different children. Perhaps some children use more visual images in their thinking; others, auditory. In still others, proprioceptive feedback is the source of information about the environment. The teacher may have to provide a varied programme; perhaps no one method will suffice.

There is a further danger in the use of structural materials. Crawford (1960) has reported that gadgetry is sometimes used as an aid to increased computational skill and speed, rather than for a heightened understanding of basic concepts. The teacher must carefully guide children in the use of such materials to prevent the occurrence of this practice. Some of the more general implications of these devices will be presented below in Section VII of this paper.

BRUNER

VI THEORY

In "The Psychology of Thinking", Thomson (1959) draws a distinction between two basic thought processes. The first is "concept formation -- the acquisition of new basic concepts -- [which] appears to take place in children up to the age of about fifteen years". (P. 69) This is the ground that Piaget has covered. But there is a second aspect to thinking -- concept attainment -- which is a matter of "modifying and adapting one's existing concepts to new uses rather than of forming completely original concepts". (Thomson, 1959, P. 69) For an explanation of this type of cognitive activity, we must turn now to the work of J. S. Bruner.

Bruner's major work on concept attainment appeared in "A Study of Thinking" (1956), the report of a group of detailed experiments with adult human subjects. These individuals were shown a series of 81 cards, each with a different combination of figure, border, and colour. The subjects' task was to sort these cards in such a way as to illustrate realization of the correct principle. The detailed results will not be reported here, but a brief summary of Bruner's theoretical interpretation of them is in order.

Bruner begins with the assumption that "virtually all cognitive activity involves and is dependent on the process of categorizing". (Bruner, 1956, P. 246) All of our interaction with the environment consists of dealing with <u>classes</u> of events.* We build up a system of

* In another paper, Bruner (1957) extends the omnipotence of the categorization notion to include perception as well as thought.

"categories" or concepts through learning, and these help us to handle the multitude of stimuli which are constantly impinging upon us. Each category involves a group of objects or events which are treated by the individual "as if" they were equivalent, whether or not this is objectively the case. Thus, cognitive activity depends on the prior placing of each event in the right category. Classification is useful in that it permits us to decrease the complexity of the environment, and thereby to identify new events easily and efficiently.

The experiments on concept attainment -- on the search for an identifying principle of which several events are examples -- outline the mechanisms by which an individual seeks and tests out the usefulness of environmental clues. The process is somewhat as follows:*

(1) A number of objects exists which exemplifies a class. These objects can be described in terms of a number of qualities or dimensions, each of which is ordered along a range of values. For example, the objects might vary with respect to colour, the possible colours being red, blue, and green. (2) For each instance encountered by the individual, a tentative prediction or decision about the nature of the concept is made. (3) Validation of the prediction comes when other environmental information arrives to confirm, invalidate, or leave indeterminate the correctness of the prediction. (4) The validation procedure (decision and test) provides information about which attributes to note. (5) A sequence of validations exists en route to a concept. This may be termed a "strategy", or purposive pattern of behaviour. The choice of strategy is determined by three main objectives -- to maximize the information

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* This sequence of events applies as well to a real-life situation as to a laboratory experiment.

gained in each validation; to reduce mental strain by operating within the limits of one's own capacity (different strategies impose varying demands on the individual); and to regulate one's behaviour according to the risks, penalties and rewards, time limits, etc. of the situation (i.e., the "payoff matrix"). Bruner's experiments provide a systematic investigation of these three objectives.

Bruner found that his subjects tend to use a variety of strategies, selecting particular ones to fit the type of task. In general, however, an individual usually chooses a specific approach, and maintains it throughout the course of the experiment. The most frequently used method is the "focusing" technique, where the subject selects a particular attribute of the objects and explores it fully and systematically until it is conclusively demonstrated to be wrong, before moving to another feature. These subjects work rather cautiously, using the most probable clues, and avoiding clues derived from negative instances.

In a later paper (1958), Bruner discusses the implications of the "coding system" for thinking; i.e., a complex system of related conceptual categories. Once such a system is constructed, an individual can handle each environmental situation by placing it into the appropriate category of the "generic coding system" (system of classes or categories). He can then "read off" any additional relevant information about its unobserved properties, previously learned as being germane to this particular category of events.

Bruner outlines the conditions under which a complex coding system is acquired. A moderate degree of motivation is necessary for the development of such a system. With repetitive attention to, and mastery of, specific details of a situation, "generic codes" can be developed at increasing levels of generality. But simple repetition is insufficient. The learner must have experience with a variety of dissimilar instances of the same concept, as well as examples of what it is not, in order to draw the appropriate conclusions.*

Two major advantages accrue from the acquisition of generic coding systems. These extend the use of concepts to organizing information and manipulating environmental facts. Sheer brute learning of details would soon burden the mind with an overwhelming array of unrelated items of information. There is a maximum capacity for handling such items at any one moment, which Miller establishes at about seven items. (1956) Thus it is necessary to regroup (recode) the events in the environment into categories for easy handling and storing. We must, as Miller has stated, increase the capacity of these categories; we must fill them not with dross but with gold. The coding system must be formed during the process of learning about the world so as to make possible later generalization to new situations. It is to this process of formation that we must now turn.

The preceding discussion has outlined Bruner's earlier investigations into the thinking processes of adults. More recently, however, Bruner has turned his attention to the work of Piaget, and in his current experiments, is attempting a synthesis of these two streams of research. In a recent lecture (1962c), he has outlined the characteristics of the "active", "iconic", and "symbolic" modes of representation of the world in the mind.**

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* Compare Dienes' (1959) conclusions on structural materials for the teaching of mathematics.

** These correspond roughly with Piaget's "pre-operational", "concrete operational", and "formal" levels of thinking.

At the active level of representation, objects exist for the individual only in terms of their physical presence and the actions associated with them. As with brain-damaged adults, the definition of an object can only be determined by actual physical manipulation of it. Bruner cites an illustrative experiment by Emerson, in which children were asked to reproduce the position of a ring attached to one board on a second board. The second one could be placed in various spatial relationships with respect to the first; i.e., parallel, at right angles, etc. For younger children, performance is satisfactory only if the two boards are adjacent. The more they have to change their own bodily orientation, the more difficult the task is for them. Thinking is tied to action.

At the iconic level, thinking proceeds with the aid of perceptual imagery. However, perception is highly subject to fluctuation due to motives or attitudes, and can even inhibit more advanced forms of thought. The symbolic level, on the other hand, permits the abstract formal reasoning which Piaget has described. Bruner (1962c) reports an experiment by Potter which illuminates this distinction. Children were presented with colour photographs of varying degrees of focus, and asked to make guesses about their content. Up to the age of 7 to 8 years, their guesses are random, related to the child's own point of view, and highly concrete. Older children formulate and test hypotheses about the pictures in a highly systematic and consistent fashion. Inferences are made much more on the basis of the probability of the occurence of the event. The internalization of action, and the symbolic use of language have made possible formal logical induction and deduction.

Bruner also cautions us against the rigid acceptance of the concept of stages. Concerning both Piaget's and his own investigations, he notes that the various levels of thought process are not completely discrete psychological stages. They may well be chronological, but are by no means exclusive, since each depends to some degree on those which precede it. All levels persist to some degree, and may interact in a complex fashion in adult thinking. The symbolic level is more powerful and economical, perhaps, but the iconic and even the active methods may be brought to bear on a particularly thorny problem. We often find it useful in solving a mathematical problem, for example, to "draw a diagram" (iconic) or even to "construct a model" (iconic and active). In fact, the latter device is a powerful tool of reasoning whereby the scientist advances his symbolic analysis of the world. And the history of the use of "teaching aids" in modern education is another manifestation of this phenomenon.*

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* See Section VII for a description of Bruner's use of these principles in teaching quadratic equations to eight-year olds.

VII IMPLICATIONS

Theoretical considerations of this kind are of obvious import to education, and some of the implications have already been examined at the Woods Hole Conference. Bruner's report on this conference (1962b) begins with the importance of teaching according to the structure of a subject. As his own work on cognitive processes has shown, the learning of a basic concept (i.e., an underlying principle or general category) alone leads to the nonspecific transfer so necessary for the building up of an organized view of the world. A curriculum must be designed in such a way as to embody the basic structural principles inherent in the subject matter. Just as the scientist or philosopher at the frontier of knowledge seeks to discover the basic concepts which will unify, organize and permit manipulation of vast bodies of knowledge, so the pupil must be intreduced to a curriculum which is organized around these principles. Introduced to them, he can easily enough find, predict, or regenerate the other details.

But surely basic concepts are far too abstract for a young pupil to comprehend! It is Bruner's contention that "any subject can be taught effectively in some intellectually honest form to any child at any stage of development". (1962b, P. 33) And thus we are intreduced to the idea of the "episodic" or "spiral curriculum", where the basic concepts of mathematics, science, the arts, and the humanities are introduced in a concrete way into the curriculum at the earliest levels, with later studies developing out of, and yet dependent upon, the former ones.

In the spiral curriculum,

> ...ideas are presented in homologue form, returned to later with more precision and power, and further developed

and expanded until, in the end, a student has a sense of mastery over at least some body of knowledge. (1960, P. 617)

Bruner is not arguing for the "central theme" or "project" method of teaching. Rather he believes that "the object of learning is to gain facts in a connected context that permits the facts to be used generatively".* (1959, P. 189) The criterion for introducing any subject into primary grades is whether it gives delight, and whether, in its final form, it will make a significant contribution to the individual's cognitive life by providing the basis for generalization to new situations.

The manner of presentation of these concepts will depend, of course, on the level of intellectual maturation of the pupil. At each age, a child has a particular way of viewing the world, and the fundamental concepts must be translated (<u>not</u> reduced) into a form commensurate with the level of development. This is not an easy task by any means. The teacher can only present the basic structural concepts of a subject if he or she understands them well. The problem of embodiment of these concepts in the forms that a child understands is secondary to this first requirement. Curriculum planners will have to take into account the work of Piaget on children's intellectual development. When the world is defined in terms of action-linked concepts, then the child must encounter knowledge through his own actions. When he has constructed a system of concrete operations to deal with reality, the teacher must present her material in terms of specific, concrete examples. When the formal stage is reached, however, theory is not only a possible, but a necessary, means of codification of experience and extending the use of mind.

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* Compare the idea advanced at the meeting of the Ontario Association for Curriculum Development ~~concerning the~~ "idea of sets as a unifying principle in all branches of mathematics". (1962, P. 19)

Nevertheless, none of these levels is completely discrete or independent. Each must be approached from the foundation of the last, so that continuity is maintained (but not, however, at the expense of depth of treatment).

It would seem that an impossible task now faces the already overburdened teacher. To present the totality of knowledge in terms of a few structural principles would seem to be an unwieldy task at best. But Bruner draws attention to a most potent source of assistance; namely, the child himself.

The formation or attainment of concepts in any field can never be a passive process, and to permit it to become so is to do a disservice to the child and to the aims of education. The most effective means of developing a complex coding system is by allowing the child to discover these basic concepts for himself. To present a "predigested" version of knowledge, a tidy catalogue of facts and figures, is the most effective way of preventing true learning and growth. Children must be directed towards the investigation of the unknown, as much as instructed in the accumulated known.

This principle can be applied in any area of curriculum -- from mathematics to literature, from geography to physics. With respect to the first of these, mathematics teaching, Bruner has argued elsewhere (1960, 1963) that the teacher of mathematics can accomplish a great deal in helping pupils towards the discovery of mathematical ideas for themselves. This may be by the provision of various "model devices" or structural materials, such as those of Cuisenaire, Stern, or Dienes (see above), by permitting small group discussion, or by other means. Bruner (1962b) cites evidence from a study in which a group of fifth-grade pupils wrote their own geography of the North Central United

States by inferring from information about various configurational and economic features of the region to the location of the major cities. Another group of eight-year olds discovered the basic principles of the mathematical square and its relevance to the solution of quadratic equations (symbolic), by means of manipulative activity (active and iconic) with blocks of wood (X by X, X by 1, and 1 by 1 units of size).

Learning by discovery has a number of advantages to commend it:

(1) It increases intellectual potency by leading the learner to use a "cumulative constructionist", rather than an "episodic empiricist" strategy. His hypotheses will be systematic and connected; they will attempt to locate the relevant constraints or attributes of the situation. His organized approach will permit him less cognitive strain, and thus more persistence of effort. In short, he is more likely to attain Piaget's "formal thought level" in the fullest sense.

(2) Discovery is self-rewarding, and thus the extrinsic system of rewards and punishments in the school can be supplanted by the intrinsic reward of intellectual mastery. Success and failure in cognitive efforts now act as a source of information, not as reward and punishment.

(3) The pupil "discovers how to discover". The heuristics of inquiry, the intuitive sense of the rightness or wrongness of an approach -- both of these can be developed in an atmosphere of discovery.

(4) Discovery of a mediating principle to connect unrelated items of information is the most effective method for ensuring that the material will be <u>remembered</u>. (Bruner, 1961)

But one cannot wait patiently for discovery to emerge. The teacher's function must become much more that of the guide, the source of encouragement, and the planner for the child's self-activated discovery. According to Bruner (1962a), the good teacher will learn to practise the art of "intellectual temptation". The encouragement of discovery may very well foster the development of intuition -- the mainspring of creativity, and perhaps the most neglected aspect of the curriculum. The opportunity for discovery, coupled with the teacher's assistance in the understanding of the basic structural relations of a topic, can prepare the ground for those creative leaps which exist at the frontiers of knowledge, and which should exist at the frontiers of the child's developing mind.

In summary, Bruner presents five axioms for the guidance of curriculum planners:

(1) "Get it right." Teach fundamental concepts honestly without distortion.

(2) Translate it into more concrete terms.

(3) Let the child discover for himself, while providing sufficient structure in the situation to guide his efforts in the right directions.

(4) Be diversified in aim, technique and criterion.

(5) Devise an atmosphere of learning which permits understanding, coding, generalization, and not the passive storing of knowledge. (1963, Pp. 24-25)

If educators take seriously these axioms, they can design curricula which will encourage the appreciation of intrinsic values in learning. Students should be able to say, "it matters not *what* we have learned. What we can do with what we have learned: this is the issue". (1959, P. 192) When these criteria are vested in the curricula of public education systems, society will judge students not on the comprehensiveness of their factual accumulation, but rather on the quality of their thinking.

CONCLUSIONS

A very brief overview of this area of this research reveals three major areas for consideration by education. The first, and most obvious, is the old and thorny question of "readiness". The experiments of Piaget and Bruner (and their students) have suggested that there are certain maturational stages or levels of thought process. Surely this would imply that we must wait before introducing certain types of concepts into the curriculum? In fact, the first inference that one might draw from these investigations is that schools should pre-test all children when they enter the primary grades to see what stage of thinking they have attained. This practice would be an essential prerequisite to any kind of streaming programme, or even to intellectual groupings within classes.

A note of caution should, however, be sounded at this juncture with respect to curriculum planning. Piaget's research into intellectual development in children could easily be interpreted as specifying the exact ages when certain concepts should be introduced. However, Bruner's investigations into self-development as the guiding principle of education must be taken into account. When Bruner talks about "learning by discovery", he is implying exactly this kind of consideration. Teachers, practising the art of "intellectual temptation", will need curricula which create the conditions of readiness. Bruner has provided education with a powerful tool for this task in the notion of the "spiral curriculum". The basic structural concepts of each subject will be translated into the form appropriate to the level of development of each child. The mathematical ideas of sets, of combinativity and associativity of numbers can be introduced early into the school programme by means of concrete, manipulative activities, with plenty of opportunity given for self-directed exploration. With teacher guidance and amplification of those concepts which they have inferred from their explorations, children can be led upward and onward through the spiral to advanced mathematical thinking at a later age. Presented in <u>this</u> way, mathematics can never become a meaningless manipulation of symbols, but rather a potent tool for thinking.

In connection with this kind of educational programme, the use of structural materials will have to be carefully evaluated. In Section V of this paper, a brief discussion of some of the potentialities of such devices for mathematics teaching* was presented. It only remains here to point out one further caution. It is quite common for the proponents of one or the other of these materials to insist on its virtues, its "phenomenal" results to the exclusion of all others. The purpose of this paper is <u>not</u> to advocate the use of any one of them. Perhaps all of them have value; perhaps none of those yet developed have value. It will require a systematic and patient programme of carefully designed research to answer such questions as these. More critical, however, will be the use of a variety of structural and natural materials on a trial basis, by individual teachers in their classrooms to see how the <u>child</u> uses them to "leap the barrier from learning to thinking", (Bruner, 1959, P. 192)

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* Note also the implications of "concrete analogues" for other subjects, such as physics, geography, chemistry, etc.

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