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ABSTRACT

This document consists of test questions used in three state high schools teaching the new Matriculation pure mathematics course (approximately grade 12). This material was circulated to all schools teaching this course as a teacher resource. The questions are arranged in 14 papers of varying structure and length. Most questions are of the essay type. The topics covered include integral and differential calculus, algebraic and trigonometrical functions, the binomial expansion, linear algebra, fields and complex numbers. (MM)

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EDUCATION DEPARTMENT - VICTORIA

CURRICULUM AND RESEARCH BRANCH

CIRCULAR OF INFORMATION TO SECONDARY SCHOOLS

NO. M/6 - 7

MATRICULATION MATHEMATICS

PURE MATHEMATICS - TEST PAPERS

PURE MATHEMATICS (ALTERNATIVE SYLLABUS)

A meeting was held at University High School near the end of the term II the participants being teachers from State High Schools who were teaching the new Matriculation mathematics courses.

In the course of the meeting the scarcity of suitable test material was discussed.

It seemed that one way of providing useful material fairly quickly would be to publish in the form of a circular, test questions which have been set in schools taking the new courses.

This circular consists of test papers and questions from test papers for which the Branch is indebted to the following schools:

Camberwell H.S., Carey G.S., MacRobertson G.H.S.,
Maribyrnong H.S., M.L.C., Stawell H.S., St.Kevin's College,
University H.S., Waverley H.S., Wesley College, Sunbury H.S.

The papers have been checked for accuracy as far as time has permitted but teachers will be well aware how easily typing errors can be missed. Hence the questions should be read carefully and preferably worked out before being given to pupils.

It is hoped that these sets of questions will be of assistance to teachers taking the new courses this year and to teachers in the future.

PAPER 1

1. Obtain the derived function of sin2x, using the definition of a derived function.

2. Using the rules for differentiation find the derived functions of:

   (a) $\cos x + x.\sin x$ (b) $\arctan 4x$ (c) $\ln(\sec x)$ (d) $x^3.e^{3x}$

3. Obtain the critical points and use them to help sketch the graph of the function $f: R \to R$ where $f(x) = 4\sin x - 3\cos x$.

4. Obtain the terms as far as the term in x^3 in the expansion of $(3 - 2x)^{10}$

5. Find the constant term in the expansion of $(x - 2x^{-2})^9$.

6. Find the greatest term in the expansion of $(\frac{1}{4} + \frac{3}{4})^{20}$.

7. Using $\int_0^x \frac{dt}{1 + t}$ and $\int_0^x (1 + t)^{-1} dt$ find an expansion for

   $\ln(1 + x)$ which has the form $a + bx + cx^2 + dx^3 + ...$ where $a, b, c, d.........$ are constants.

8. Find an indefinite integral of:

   (i) $(x^3 + 2)^2$ (ii) $\frac{x}{(2x^2 - 3)^2}$ (iii) $\cos^2 2x$

9. Find the approximate value (to 3 significant figures) of:

$$\int_2^{12} \frac{2x + 5}{(2x - 3)(2x + 1)} .dx$$

11. Use a suitable substitution to evaluate:

$$\int_3^8 x\sqrt{1 + x}.dx$$

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1. (a) If $z = 2 - i$, show, on a single Argand diagram, the points represented by:-

$$z, \quad \bar{z}, \quad \frac{1}{z}, \quad z^2$$

(b) Find the six sixth roots of 1, and show that they include both the square roots and the cube roots of 1.

2. (a) Find an antiderivative of each of the following.

(i) $\tan^2 x \sec^2 x$,

(ii) $\dfrac{x^3}{x + 1}$,

(iii) $\sec^2 x - \tan^2 x$

(b) Evaluate the following:-

(i) $$\int_0^1 \frac{dx}{\sqrt{(4 - x^2)}}$$

(ii) $$\int_0^1 \frac{dx}{(x + 1)^2 \ (x + 2)}$$

(c) A metal sphere is dissolving in acid. It remains spherical and the rate at which it dissolves is proportional to the area of its surface. Prove that the radius decreases at a constant rate.

3. (a) If $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & 4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$

and $C = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$, show that $AB = AC$.

How does this result differ from what is true for algebra over the field R?

(b) If $X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $Y = \begin{bmatrix} 1 & 2 \end{bmatrix}$, find XY and the determinant of XY. Hence find the inverse of the matrix XY.

3. (c) If $M = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \\ 0 & -2 & 3 \end{bmatrix}$ $N = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $P = \begin{bmatrix} -3 \\ 3 \\ 8 \end{bmatrix}$

solve the system of linear equations:-

$$MN = P$$

4. (a) Find the largest subset, S, of R such that $f(x) = \sqrt{4 - x^2}$ defines a function $f : S \longrightarrow R$. With this domain, find the range of f, and sketch the graph of:-

$$\left\{ (x, y) : y = f(x) \right\}$$

(b) Given $f : R \longrightarrow R$ where $f(x) = \sin x$, find f^{-1}, the inverse of f, stating its domain and range. Find, also, the composite functions $f \circ f^{-1} \circ f$. Do they define the same identity function?

(c) Given two functions, f and g, such that:-

$f: \quad R \longrightarrow R,$ where $f(x) = 2x,$

$g: \quad R^+ \longrightarrow R,$ where $g(x) = \sqrt{x},$

find which of the composite functions $f \circ g$ and $g \circ f$ are defined, and state the domain and range of any such defined function.

5. Sketch the curve whose equation is:-

$$y = \frac{4x^2 + 5x + 1}{x^2 + 2x + 2},$$

locating any stationary points and asymptotes.

6. (a) By expanding both sides of the identity -

$$(1 + x)^{n + 2} = (1 + x)^n (1 + x)^2,$$

prove that:-

$$\binom{n+2}{r} = \binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2}$$

(b) If n is a positive integer, express $(1 + x)^n$ as a polynomial. Hence, deduce the polynomial expansion of $(1 - x)^n$, and use these two results to show that, when $n = 3$:-

(i) $(1 + x)^n + (1 - x)^n = 2(1 + 3x^2)$,

(ii) $(1 + x)^n - (1 - x)^n = 2x(3 + x^2)$

(c) In how many ways can eight boys be divided into two groups?

7. (a) Find $\{x: \quad x^3 + 3x^2 - x - 3 = 0, \quad x \in R\}$

(b) The following table gives the temperature θ° C of a body above the temperature of the atmosphere t minutes after it starts to cool:-

t	5	10	15	20	25
θ	63.5	50.5	40.1	31.8	25.3

Verify that the relationship between θ and t is of the form $\theta = a.10^{bt}$, where a, b, $\in$ R, and find the values of a and b.

(c) If $\{x: x^3 + ax^2 + bx + c = 0\} = \{x_1, x_2, x_3\}$, and $x_1 = x_2 + x_3$, prove that:-

$$a^3 - 4ab + 8c = 0.$$

8. (a) What fraction of the area of the region enclosed by the circle $x^2 + y^2 = 4$ lies outside the parabola $y^2 = 4(1 - x)$?

(b) A tank with plane sides has square, horizontal, cross sections whose sides vary in length from 2 feet at the base to 5 feet at the top; the height is 12 feet. Find, by integration, the volume of the tank.

Find the total mass of liquid in the tank, if the density decreases uniformly with height from 20lb./ft^3 at the bottom to 12lb./ft^3 at the top.

9. (a) The operation $\sim$ is defined on $J \setminus J^-$ by $x \sim y = |x - y|$

Give answers to the following questions, with a reason in each case:-

(i) Is the operation commutative?

(ii) Is it associative?

(iii) Has each element an inverse?

(b) Let $\oplus$ represent the binary operation "take the greater of" e.g., $3 \oplus 4 = 4$, and Θ represent "take the first mentioned of", e.g., $2 \Theta 3 = 2$. Do these operations, on the set $\{0, 1, 2, 3\}$, constitute a field?

10. (a) If x is any number greater than -1, prove that:-

$$(1 + x)^n \geqslant 1 + nx.$$

When is it true to say that -

$$(1 + x)^n = 1 + nx.$$

(b) Prove that, if $n \in N - \{1\}$:-

$$n^2 > n + 1.$$

Deduce that -

$$(n^n)^{(n^n)} < n^{(n^n)^n}$$

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PAPER 3

1. (a) Factorize completely $x^4 - 3x^2y^2 + y^4$

   (b) Express in partial fractions $\dfrac{1}{x^3 - 4x}$

   (c) If $z = x + iy$ and $\dfrac{z - 1}{z - i} = \dfrac{2}{3}$ find x and y.

   (d) Using only the formulae for modulus and phase of products, construct the argand point of i^2.

2. (a) Let $f(x) = x^2$ define a function on the closed interval $-2 \leq x \leq 8$, find (i) $f(4)$;

   (ii) $f(-3)$,

   (iii) $f(t - 3)$

   (b) Let $A = [-1, 1] = \{x: -1 \leq x \leq 1\}$ and the functions f, g, h of A into A be defined by-

   (i) $f(x) = x^2$,

   (ii) $g(x) = x^3$,

   (iii) $h(x) = \text{Sin } x$

   Which function, if any is onto?

   (c) Let the functions of f, g, h be defined by

   $f(x) = x^2$ where $0 \leq x \leq 1$

   $g(y) = y^2$ where $2 \leq y \leq 8$

   $h(z) = z^2$ where $z \in R$

   Which if any, of these functions are equal?

3. (a) Find from first principles the derivative of $\dfrac{1}{\sqrt{x}}$

   (b) Differentiate -

   $$\frac{1}{x - 1}, \quad \frac{x}{x - 1}, \quad \frac{x}{\sqrt{1 - x^2}}, \quad \cos^2 3x, \quad \left(\frac{1 + \sin x}{1 - \sin x}\right)^{\frac{1}{2}}$$

4. (a) Find the middle term in the expansion of $\left(\frac{a}{b} - \frac{b}{a}\right)^8$

(b) By considering the coefficient of x^r in the identity

$$(1 + x)^{m + n} = (1 + x)^m \; (1 + x)^n$$

Prove that :-

$$^{m+n}C_r = {}^nC_r + {}^mC_1 \; {}^nC_{r-1} + {}^mC_2 \; {}^nC_{r-2} + \ldots + {}^mC_{r-1} \; {}^nC_1 + {}^mC_r$$

5. (a) The graph of $f: R \longrightarrow R$ where $f(x) = x^4 + ax^3 + bx^2 + cx + d$ has turning points at $(0, 0)$ and $(1, 1,)$. Find a, b, c, d and sketch the graph.

(b) Sketch quickly the graph of -

$$y = \sqrt{(x - 2)^2} + \sqrt{(x + 2)^2}$$

6. (a) If $A = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 3 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

Calculate if possible -

AB, A + B, A(B + C), BA, BC' where C' is the transpose of C.

(b) If A and B are two 3 X 3 matrices explain why in general $(A + B)^2 \neq A^2 + 2AB + B^2$

(c) State the transpose rule for the product of two matrices and prove that if each of the matrices A, B and AB is symmetric, then $(A + B)^2 = A^2 + 2AB + B^2$.

7. (a) Show that if m is a positive integer which is <u>not</u> a multiple of 3

$$\left(\frac{-1 + i\sqrt{3}}{2}\right)^m + \left(\frac{-1 - i\sqrt{3}}{2}\right)^m = -1$$

(b) Find the solution set $\left\{x : x^6 + 2x^3 + 4 = 0\right\}$ and mark them on the argand diagram.

8. If $\left\{x: \quad x^3 - c_1x^2 + c_2x - c_3 = 0\right\} = \left\{x_1, x_2, x_3\right\}$

prove that $x_1 + x_2 + x_3 = c_1$

$$x_1{}^2 + x^2{}_2 + x_3{}^2 = c_1{}^2 - 2c_2$$

$$x_1{}^3 + x_2{}^3 + x_3{}^3 = c_1{}^3 - 3c_1c_2 + 3c_3$$

If $\longrightarrow$ $x_1 + x_2 + x_3 = -1$

$$x_2{}^2 + x^2{}_2 + x^2{}_3 = 5$$

$$x_1{}^3 + x_2{}^3 + x_3^3 = -7$$

find x_1, x_2, x_3.

9. If $C = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{pmatrix}$ find C^{-1}

Hence or otherwise solve the set of linear simultaneous equations

$$x + 2y + 3z = 2$$

$$x + 3y + 5z = 3$$

$$x + 5y + 12z = 7$$

If a student in his hurry wrote this down as $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 5 & 12 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}$

What can you say immediately about the inverse and why? And what can you infer from these equations?

10. If $f = \left\{(x, y); \quad x, y, \in R \text{ where } y = kx + \frac{1}{x}\right\}$

Prove that for $k < 0$, y may assume any value whereas for $k \geqslant 0$, y cannot assume certain values and find these values. On the same diagram sketch the graphs of the 3 cases $k = 1, k = 0, k = -1$, and show their asymptotes. Also state the values of m for which the line $y = mx$ intersects one and only one of the graphs.

In this paper, R stands for the set of real numbers; ph z stands for phase of z or angle of z or argument of z.

1. What is the largest subset X of R for which $F : X \longrightarrow R$ defines a function f having the rule $f(x) =$

   (a) $\dfrac{1}{1 + x^2}$

   (b) $\log_e(2x + 3)$

   (c) $2x \cos 2x$?

   Find the derivative and a set of antiderivatives of each of the functions, with such a domain.

2. If $z = 2 + i$, express as a complex number:
   z^{-1}, $\bar{z}$, $z + 5 - 2i$, z^2, $z^2 + iz + 5 - 2i$.
   Show the approximate position of each on a complex plane sketch.
   Calculate the absolute value of $z + 5 - 2i$ and of z^2.
   Express z and z^2 in polar form.
   Show these relations on complex plane sketches:
   $\left\{z : |z| = 5\right\}$ and $\left\{z : z^2 = 3 + 4i\right\}$

3. Given $f : R \longrightarrow R$ where $f(x) = \sin x$
   and $g : R^+ \longrightarrow R$ where $g(x) = \sqrt{x}$,
   determine, giving reasons, which of $\frac{g}{f}$, $f \circ g$, $g \circ f$, f^{-1}, g^{-1} are defined. For those which are, state the domain, range, and rule, and sketch their graphs.

4. a) Find the value of the greatest coefficient in the binomial expansion of $(1 + x)^{12}$. Give reasons.
   What is the sum of -

   $\binom{12}{1} + \binom{12}{2} + \binom{12}{3} + \ldots\ldots\ldots + \binom{12}{12}$? Give reasons.

5. a) Find, from first principles, the gradient of the graph of the function $f : R \longrightarrow R$ where $f(x) = \sin x$ at the point $(a, \sin a)$.

   b) Sketch the graphs of f, f^2, & three antiderivatives of f and indicate the connection between the values of each at the point where $x = \dfrac{\pi}{2}$.

   c) Calculate the volume enclosed by rotating the graph of $g : \left\{x : 0 \leq x \leq \pi\right\} \longrightarrow R$ where $g(x) = \sin x$ about the x axis.

6. a) If f is the function $f : R \longrightarrow R$, $f(x) = e^{\sin x}$,
find the sets $\{x : f(x) = 1\}$
and $\{x : f^1(x) = 0\}$

b) Find the set $\{x : x \in R \text{ and } \cos 3x = \cos 5x.\}$

7. <u>For the function f</u> : $X \longrightarrow R$ where $f(x) = \dfrac{(x-1)^3}{x^2}$ state the largest subset X of R for which f is defined.
Determine the values of x for which f(x) and f'(x) are zero, positive and negative. Sketch the graph of f, showing any straight line asymptotes and any turning points clearly.
For what values of x is f á decreasing function?

8. a) If r and α are given real numbers, list the elements of the set $\{z : z^3 = r(\cos\alpha + i \sin\alpha)\}$
and illustrate on a complex plane sketch if $r = 8$, $\alpha = \dfrac{3\pi}{4}$

b) Express as a product of linear factors over C :
$$x^3 + x^2 + 3x - 5$$
and find
$$\{x : x^3 + x^2 + 3x - 5 = 0, x \in C\}$$

c) The equation
$$2x^3 + bx^2 + cx + 8i = 0$$ has roots 2, i, and a.
Find b, c, and a.

9. a) Calculate:
$$\int_{-t}^{t} \frac{1}{\sqrt{1-x^2}} \, dx \text{ for } |t| < 1$$
and determine whether the result has a limit as $|t|$. approaches 1.
Give a geometrical interpretation of your result and the existence or otherwise of its limit.

b) For the function $f : R \longrightarrow R$ where $f(x) = x^2 \sin x$, show that $f(-x) = -f(x)$. Hence, or otherwise, find $\int_{-\pi}^{\pi} f$, giving a reason for your answer.

10. For any two elements a and b of the set J of integers

let $a + b = 4n + r$,

$ab = 4m + s$,

where n, m are integers, and r, s $\{0, 1, 2, 3\}$

Define $a \oplus b = r$

$a \otimes b = s$

Calculate $a \oplus b$ and $a \otimes b$ when

(i) $a = 10$, $b = 15$,

(ii) $a = 10$, $b = 10$

Establish which of the field laws are satisfied for J under $\oplus$ and $\otimes$ and which are not.

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PAPER 5

1. (a) If $Z = -1-i$ write down the modulus and phase of Z. Express in the form $x + iy$, where x and y are real numbers:

$$\text{(i)}\quad \bar{Z} \qquad \text{(ii)}\quad \frac{1}{Z} \qquad \text{(iii)}\quad (\bar{Z})^5$$

(b) Form the quadratic equation whose roots are

$$\cos\frac{\pi}{6} \pm i\sin\frac{\pi}{6}$$

2. Sketch the curve whose equation is $y = \dfrac{x^2 - 6x + 5}{(x+1)^2}$

locating any stationary points and asymptotes, and hence deduce the graph of $y = \dfrac{(x+1)^2}{x^2 - 6x + 5}$

3. (a) If $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 1 & -2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 0 & -1 \end{bmatrix}$ $\quad C = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 2 & 1 \end{bmatrix}$

$D = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

(i) Evaluate the product AB.

(ii) Find D^{-1}

(iii) Evaluate the determinant of C.

(iv) Find the transpose of C.

(v) Find the adjoint matrix of C.

(b) Find the solution set of the equations

$$-x + 2y - 3z = -8$$
$$2x - y + 4z = 17$$
$$3x + 4y + z = 22$$

by the method of inverting a matrix.

4. (a) Find the solution set for the equation

$$\sqrt{1-x} + \sqrt{4+x} = \sqrt{-4x-3}$$

(b) Prove $\binom{n}{k} = \binom{n}{n-k}$ and hence evaluate $\binom{30}{28}$

(c) Find the seventh term of the expansion $(x - \frac{1}{x^2})^9$

5. (a) Find all functions f such that $f'(x) = \frac{x^3 + 2x}{x^2}$

(b) Evaluate $\int_1^3 \frac{dx}{x^2 - 16}$

(c) Find the area bounded by the x-axis and the curve whose equation is

$$y = x^2 - 4$$

Find also the volume of the solid formed when this region is rotated about the x-axis.

6. (a) Simplify $\frac{\tan\theta \sec\theta}{1 + \tan^2\theta}$

(b) Prove the identity $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$

(c) Find the set $\{x: 0 \leq x \leq 2\pi \text{ and } 2\sin 3x \cos 2x - \sin 3x = 0\}$

7. (a) Let R* be the set $\{x: x \in R, x \geqslant 0\}$
If $f: R \to R$ is given by $f(x) = x^2 + 3$, and $g: R^* \to R$ is given by $g(x) = \sqrt{x}$ write down the ranges of f and g.
Find the composite functions $f \circ g$ and $g \circ f$, stating their domains and ranges. Calculate the value of $(g \circ f)'$ at $x = 1$.

(b) Find the largest subset S of R such that $f(x) = 1 + \sqrt{x}$ defines a function $f: S \to R$. With this domain find the range of f and sketch the graph of $\{(x,y): y = f(x)\}$
Find f^{-1}, the inverse of f, stating its domain and range.
Sketch (with the same axes as before) the graph of $\{(x,y): y = f^{-1}(x)\}$ and calculate where it intersects the first graph.

8. (a) Form a cubic equation whose roots are $2, 1 \pm 2i$

(b) Find $\sqrt{-7 + 24i}$

(c) Find x, y   R such that $\dfrac{1 + i}{2i} + \dfrac{2 - 3i}{5 + \frac{1}{i}} = x + iy$

9. (a) The equation $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{c} = 0$ occurs in electrical theory. Find by direct substitution two values of the constant λ, in order that the value $q = e^{\lambda t}$ should satisfy this equation. (L, R and C are constants and L, R, C $\in$ R.)

Determine the moduli and arguments of these values of λ, when $R < 2\sqrt{(L/c)}$, and plot the values of λ on an Argand diagram for $R = 4$, $L = 5$, $C = \frac{1}{2}$.

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PAPER 6

1. (a) Give the nth. term of the sequence:

$$\frac{1}{1.2}, \quad \frac{1}{2.3}, \quad \frac{1}{3.4}, \quad \frac{1}{4.5}, \quad \ldots\ldots\ldots$$

Show that the sequence may be written:

$$(1-\tfrac{1}{2}), \quad (\tfrac{1}{2}-\tfrac{1}{3}), \quad (\tfrac{1}{3}-\tfrac{1}{4}), \quad (\tfrac{1}{4}-\tfrac{1}{5}), \quad \ldots\ldots\ldots$$

Hence prove that the partial sum $Sn = 1 - \frac{1}{n+1}$

and that the sequence has a sum $S = 1$.

(b) Give 3 conditions for a function, f, to be continuous at $x = a$. Find if the functions defined by the following are continuous at the point given. If not, state which of the three conditions is not satisfied:

(i) $f(x) = \begin{cases} \dfrac{x^2 - 5x + 6}{x - 2} & \text{for all } x \neq 2; \quad \text{at } x = 2 \\ -1 & \text{for } x = 2 \end{cases}$

(ii) $f(x) = \dfrac{x+2}{x-2}$; at $x = 2$

(iii) $f(x) = \dfrac{x+2}{x-2}$; at $x = 3$

2. (a) For any function, f, give a definition of $D_x f$, explaining any symbols you use.

From your definition find $D_x f(x)$ where $f(x) = \sqrt{x+1}$

(b) Write down the rule which gives $D_x \left[f(x) \cdot g(x) \right]$

and use it to find a rule for $D_x \left[\dfrac{f(x)}{g(x)} \right]$

(c) Find $D_x f$ for the function, f, given by:

(i) $f(x) = (3x - 1)^4$

(ii) $f(x) = (x^2 - 1)\sqrt{x^2 + 1}$

(iii) $f(x) = \sqrt{\dfrac{1+x}{1-x}}$

3. Discuss the function $f : R \setminus \{1\} \longrightarrow R$ where $f(x) = \frac{x^2 + 3}{x - 1}$

noting

(i) where $f(0)$ and $f(x) = 0$

(ii) any limits to f, and the continuity of f

(iii) any stationary points.

Sketch the graph of the function, marking any features found in (i), (ii) or (iii).

4. AB, BC are two roads at right angles to one another. $\overline{AB} = 1$ mile and $\overline{BC} = 3$ miles. P is a point on BC. A man walks straight across country from A to P at 3 mph, and then along the road from P to C at 5 mph. Find the distance of P from B for the time taken for the whole journey to be a minimum.

5. (a) Obtain sufficient values of (r, θ) to sketch the graph of

$$\left\{(r, \theta) : r = 1 - \cos \theta\right\}$$

(b) Repeat part (a) for $r = \frac{1}{1 - \cos \theta}$. Obtain the cartesian form of the equation of this curve, and identify the curve.

6. (a) Find the derivative of sin 2x using the definition of a derivative.

(b) If $y = \frac{b}{\sin x}$ prove that -

$$D^2y. \sin x + 2Dy. \cos x - x = 0$$

(c) Obtain the stationary points of the functions, f_1 and f_2 where

$$f_1(x) = 2 \cos^2 x \text{ and } f_2(x) = \cos 2x.$$

Use these points to sketch the graphs of the functions.
What transformation will map f_1 onto f_2?
(For each function the domain is $0 \leq x \leq 2\pi$).

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PAPER 7

R stands for the set of real numbers. ph z stands for the phase of z or angle of z or argument of z. $\bar{a}$ means the conjugate of the complex number a.

1. (a) Evaluate correct to 3 decimal places $(1.01)^6$

   (b) Find the term independent of x in the expansion of $(x^2 - 2x^{-1})^{12}$

   (c) Find a value for $^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \dots\dots\dots\dots {}^nC_n$

2. (a) Express in partial fractions $\dfrac{x^2 - 9x - 6}{x^3 + x^2 - 6x}$

   (b) If $\{x: x^3 - 2x^2 + 3x - 4 = 0\} = \{x_1, x_2, x_3\}$

   find (1) $\sum 1/x_1x_2$

   (2) $\sum x_1{}^2$

   (3) $\sum x_1{}^3$

3. Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$ $P = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

   $Q = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ $V = \begin{bmatrix} 1 & 2 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

   (1) State which of the following expressions are meaningful:

   A + B, B + Q, AP, PQ, VA, A^2, BA.

   (2) Evaluate the meaningful expressions in (1).

   (3) Evaluate the determinant of PV.

   (4) Solve the system of linear equations AX = P.

4. Sketch the graph of $f : R \setminus \{3\} \rightarrow R$ and $f(x) = \dfrac{x^2 - x - 2}{x - 3}$

   Label the graph fully showing asymptotes and turning points.

5. (a) Find the set $\left\{x: \sqrt{x + 12} - \sqrt{8 - x} = 2 \quad x \in R\right\}$

(b) Find the cube roots of unity.

(c) If 1, w_1, w_2 are the cube roots of unity, prove $w_2 = \overline{w}_1 = w_1^{\,2}$

6. (a) Find the set $\left\{x: \; x \in R \text{ and } 2(1 - \cos^2 x) = 3\cos x\right\}$

(b) Find all functions f such that $f'(x) = (e^x + e^{-x})^2$ being sure to specify relevant domains.

(c) Find the set $\left\{p : y = e^{px} \text{ and } \frac{dy}{dx} + \frac{d^2y}{dx^2} - 6y = 0\right\}$

(d) The radius of a spherical balloon is increasing at the rate of 1 in/min. At what rate is the volume increasing when the radius is 5 in?

7. (a) Represent on separate diagrams (Argand) the following sets where z is a complex number.

(1) $\left\{z : \text{ph } z = \frac{\pi}{3}\right\}$

(2) $\left\{z : /z/ = 2\right\}$

(3) $\left\{z : -\pi/3 \leq \text{ph } z \leq \pi/3 \text{ and } 2 < /z/ < 3\right\}$

(b) If $d = -1 + i$ write down the modulus and phase of d and express in the form $x + iy$, each of $\overline{d}$, i/d, d^7 where $x, y \in R$.

8. (a) Find the measure of the area contained by the curves $y_1 = x^2$ & $y_2 = \sqrt{x}$.

(b) If the function f is given by $f(x) = \sin x$, $x \in R$ find (1) $\int_0^{\frac{\pi}{2}} f(x)\,dx$ and hence using the result of (1) find $\int_0^1 f^{-1}(y)\,dy$ where $y = \sin x$.

9. PQ is a variable focal chord of a parabola; TP is the tangent at P and TQ is parallel to the axis: show that the locus of the mid point of PT is the directrix.

10. (a) Find the largest subset S of R such that $f(x) = 2 + \sqrt{x}$ defines a function $f : S \longrightarrow R$. With this domain find the range of f and sketch the graph of $\{(x, y) : y = f(x).\}$

Find f^{-1}, the inverse of f, stating its domain and range.

Sketch (with the same axes as before) the graph of $\{(x, y) : y = f^{-1}(x)\}$ and calculate where it intersects the first graph.

(b) Let R* be the set $\{x : x \in R, x \geqslant 0\}$. If $f : R \longrightarrow R$ is given by $f(x) = x^2 + 5$, and $g : R^* \longrightarrow R$ is given by $g(x) = \sqrt{x}$, write down the ranges of f and g.

Find the composite functions fog and gof stating their domains and ranges. Calculate the value of $(g_o f)'$ at $x = 2$.

fog(x) means $f\{g(x)\}$

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In this paper -

R stands for the set of real numbers.

Ph Z stands for phase of Z or angle of Z or argument of Z

/Z/ stands for magnitude of Z or modulus of Z

In x stands for $\log_e x$.

1. (a) Find the set -

$$\left\{x : x \in R \text{ and } (\sin x + \cos x)^2 = 1 - \sin x\right\}$$

(b) Find the set -

$$\left\{m : y = e^{mx} \text{ and } \frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0\right\}$$

(c) Find all functions such that -

$$f^1(x) = x.\ \ln 2x, \text{ for } x > 0.$$

(d) The radius of a sphere is increasing at a constant rate P. Find the rate of increase of the volume of the sphere at the instant when the radius is r.

2. (a) Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ $P = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$Q = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ $V = \begin{bmatrix} 1 & 2 \end{bmatrix}$ $X = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$

(i) State which of the following expressions are meaningful:

A + B, B + Q, AP, PQ, VA, A^2, BA

(ii) Evaluate the meaningful expressions in (i).

(iii) Evaluate the determinant of PV.

(iv) Solve the system of linear equations AX = P.

(b) Write down the matrix that carries out the operation

(i) a rotation of θ where $\theta = \frac{\pi}{4}$

(ii) a reflection about the line $y = mx$ where $m = \tan\theta = \tan\frac{\pi}{6}$

(iii) a dilation of k where k = 2.

3\. (a) Sketch on separate Argand diagrams, the following sets (where z is a complex number):

(i) $\left\{ Z: \text{ph } Z = \frac{3\pi}{4} \right\}$

(ii) $(Z: /Z/ = 3)$

(iii) $\left\{ Z: -\frac{\pi}{3} < \text{ph } Z < \frac{\pi}{2} \text{ and } 1 \leqslant /Z/ \leqslant 3 \right\}$

(b) If $\alpha = 1 - i$, express in cartesian form.

(i) $\bar{\alpha}$

(ii) $\frac{i}{\alpha}$

(iii) α^5

(c) Express the cube roots of 1 in cartesian form.

(d) Solve $x^2 - 4x + 8 = 0$

4\. (a) Differentiate $y = \sqrt{x}$ with respect to x, from first principles.

(b) Differentiate with respect to x:

(i) $x. \text{arsin } x$

(ii) $\ln (\ln x)$

(c) A man in a rowing boat is 600 yards from the nearest point A, on a straight shore. He wishes to reach a point B, 1500 yards along the shore from A, in the shortest possible time. If he can row at 4 m.p.h., and run at 5 m.p.h., at what point on the shore should he land?

(d) At what point does the normal to $y = x^2$ at $x = 2$ cut the x - axis?

(e) Find the angle of intersection of the curves $y = \sin x$ and $y = \cos x$; in the first quadrant.

5. (a) Given that ${}^nC_r$, which is the number of ways of selecting r elements from a set of n elements, may also be written $\binom{n}{r}$

prove that $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$

(b) Using the expansion $(1 + x)^n$ and giving x an appropriate numerical value, prove that:

$$1 - 3.\ {}^nC_1 + 3^2.\ {}^nC_2 - 3^3.\ {}^nC_3 + \ldots\ldots\ldots\ldots + (-1)^n(3)^n\ {}^nC_n = (-2)^n$$

(c) Evaluate correct to 4 decimal places, making use of the Binomial Theorem:

$$(1.99)^5$$

(d) Find, using Newton's method, the first approximation to the root, which lies between 2 and 3, of the equation, $x^3 - 2x - 6 = 0$.

(e) Which term in the expansion of $(2x - 3y)^{12}$ has the greatest coefficient?

6. Sketch the curve whose equation is -

$$y = \frac{(x - 2)\ (x + 1)}{x\ (x - 1)}$$

locating any stationary points and asymptotes.

7. (a) Find the area enclosed between the curves

$$y = x^3 \text{ and } y = x.$$

(b) Show that the volume of a right circular cone of vertical height h and base radius r is given by :-

$$V = \frac{1}{3}\pi r^2 h.$$

(c) A serviette ring is made from a solid wooden sphere of radius r by removing all the wood inside a cylinder (of smaller radius) whose axis passes through the centre of the sphere. Measured parallel to its axis the finished ring is 2b units long. Show that, for constant b, the volume of wood forming the ring is the same, whatever the value of r, and is in fact

$$2\pi b^3 \text{ cubic units.}$$

(d) Evaluate each of the integrals.

(i) $$\int_{0}^{\frac{\pi}{4}} \sec^2 x \, (1 + \tan x) \, dx.$$

(ii) $$\int_{1}^{2} \frac{dx}{\sqrt{4 - x^2}}$$

8. (a) Express $\dfrac{5x + 8}{(2x-1)^2 \, (x^2+x + 1)}$ in partial fractions.

(b) By making use of partial fractions find -

(i) the sum to n-terms.

(ii) the sum to infinity of the sequence:

$$S = \left\{ \frac{1}{2.5}, \frac{1}{5.8}, \frac{1}{8.11} \ldots\ldots\ldots\ldots\ldots\ldots \right\}$$

(c) Given $\{x: x \in R \text{ and } 2x^3 - 21x^2 + 42x - 16 = 0\} = (x_1, x_2, x_3)$ and that (x_1, x_2, x_3) is a Geometric Sequence. Find x_1, x_2, x_3.

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PAPER 9

COMPULSORY: SECTION A

Answer briefly.

1. If $w = x + iy$ where $i^2 = -1$ then $w^{-1} =$

2. If $Z = a + ib$ then $Z - \overline{Z} =$

3. $(\exp iA)(\exp ib) =$

4. $/R \text{ cis } 2\theta/ =$

5. $(r \text{ cis } x)^k =$

6. $x^2 + 1 = 0$ where $x \in C$ has roots ----------

7. If $p(x)$ is a polynomial of degree n and $x \in C$ how many roots has $p(x) = 0$.

8. If $p(x) = 0$ has roots $-1, 3, 2 + i, 2 - i, 3 + i, 3 - i$ will the coefficients of all the terms in $p(x)$ be real?

9. Expand $(1 + x)^7 =$

10. If $/x/ < /$ the first linear approximation for $(1 - x)^p$ is - - - -

11. $\binom{n}{r} \div \binom{n+1}{r+1} =$

12. Express in partial fractions $\dfrac{2x}{x^2-1}$

13. If $y = f(x)$, $x, y \in R$ where $f(x)$ is a polynomial of degree $2n$ and the coefficient of x^{2n} is negative sketch the general shape of the graph of $y = f(x)$

14. If $y = \dfrac{x^2 + 1}{x^3 + 1}$ is sketched; for what values of x does a discontinuity occur.

15. If $y = \dfrac{kx^m}{qx^n}$ where $k, q \in R$ and $m < n$ investigate whether the graph of y has any horizontal asymptotes.

16. Write down -

(i) a typical 2 x 2 square matrix

(ii) a row matrix

17. If I (the unit matrix) is of order 3 then I =

18. If P = $\begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ then P^2 =

19. $\begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix}$ =

20. If X = $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$ then det A =

21. If B is a matrix then B^{-1} exists if and only if

22. If A = $\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix}$ then the cofactor matrix of A is

23. If det A $\neq$ 0 then $\frac{\text{adjoint } A}{\det A}$ =

24. If $y = x^2$ then $(y + \delta y) - y$ =

25. Exhibit graphically the statement "x is in a neighbourhood of 3 but not 3".

26. $\lim_{x \to 0} \frac{x^3 + 1}{x^5 - 11}$ =

27. If f(x) represents a constant function f for $x \in R$ then the limit of f(x) as x becomes large is

28. What continuity requirements must be satisfied if for a given function of -

(i) f'(x) exists

(ii) f"(x) exists

29. If $g(x) = k$ where $k \in R$ then $g'(x) =$

30. If $y = f(Z)$ and $z = g(x)$ then $\frac{dy}{dx} =$

31. If f is continuous on $a \leq x \leq b$ then -

$$\int_a^b f(x)\,dx \text{ is defined as}$$

32. $\int_a^b f + \int_b^a f =$

33. $\int_a^b f + \int_a^b f =$

34. $\int \left[D_x f(x)\right] dx =$

35. $\left[F(x)\right]_3^7 =$

36. Give illustrations of -

(i) a one-one mapping,

(ii) a many to one mapping,

(iii) a one to many mapping.

Which of these are functions?

37. If f and g are two functions defined on the same domain and range what can be said about f o g and g o f?

38. If $x^2 + px + 1 = 0$ has roots a and b express p in terms of a and b.

39. If p, q, r are roots of $ax^3 + bx^2 + cx + d = 0$ express c in terms of p, q, r.

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PAPER 9

SECTION B:

1. (a) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$ $C = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$

$D = \begin{bmatrix} 6 & 7 & 8 \\ 0 & 1 & 0 \end{bmatrix}$ $E = \begin{bmatrix} 9 \\ 8 \\ 7 \end{bmatrix}$ $F = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

evaluate the following where possible -

(1) $A + D$ (2) C^{-1}

(3) F^T (4) $B + E$

(5) BA (6) DE

(7) $C-B$

Where no answer is possible give reasons why not.

(b) If X and Y are n x n matrices, does $x + y = y + x$ always?

(c) (1) The product matrix PQ exists if ....

(2) If PQ exists, does QP also necessarily exist?

(d) If A is an orthogonal 2 x 2 matrix, prove that -

(1) $A^T = A^{-1}$

(2) Det A = 1.

(e) If A is a square matrix and A^{-1} exists prove that -

$$A \times A^{-1} = A^{-1} \times A$$

(f) Use the property proved in (e) to evaluate x, y and z for

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -3 \\ 3 & 2 & 5 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}$$

2. (a) If $p(x) = 0$ is a 5th degree equation with 5 unequal roots a, b, c, d, e and the coefficient of x^5 is 1 prove that -

(1) the coefficient of x^4 is proportional to the sum of the 5 roots.

(2) find the coefficients of x^2 and x^0 in terms of the roots.

(b) $P(x) = 0$ is a polynomial equation of degree 4 such that it has 4 real roots a, b, c and d. If $a = 1$, and $\frac{b}{a} = \frac{c}{b} = \frac{d}{c} = \frac{3}{1}$ form the equation using the symmetric relations between the roots and check that c is a root using the Remainder Theorem.

(c) If $x^3 + 9x^2 - 39x - 36 = 0$, has roots p, q and r and $p + q = 3$ use the properties of the roots to find the solution set of the equations.

(d) If $f(x) = 0$ is a polynomial equation of degree n with 2 equal roots of value p prove that $f'(x) = 0$ when $x = p$.

3. (a) If $c = a + ib$, $w = p + iq$, $z = x + iy$, where a, b, p, q, x and y are real and $i^2 = -1$, complete the following -

(1) $w = z = c$ if.....

(2) $c + w =$

(3) $/z/ =$

(4) $w \times z =$

(5) $/\bar{c} + c/ =$

(6) the conjugate of $(z - \bar{z}) =$

(b) State the fundamental theorem of algebra.

(c) If $x \in C$; $x^n + x^{n-1} + \ldots\ldots\ldots\ldots + 1 = 0$ will have ___?___ roots.

(d) Find $X = \left\{ x;\ x^4 + 8x^3 - x - 8 = 0 \text{ and } x \in C \right\}$ if $\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) \in X$

(e) If $z \in C$, illustrate where possible these sets of points; with Argand diagrams.

(1) $\left\{z;\ /z/ \leq a, \text{ phase of } z = -\frac{\pi}{3}\right\}$

(2) $(z;\ \bar{z} = z,\ /z/ \leq 5)$

(3) $(z;\ /z/ = -10,\ 0 \leq \arg z \leq 1)$

The -ve is intentional.

4. (a) If $(x+y)^n = \binom{n}{o} x^n\ y^o + \ldots\ldots + \binom{n}{r} x^{n-r}\ y^r \ldots\ldots + \binom{n}{n} x^o\ y^n$ is true for $n = k$ where n and k are natural numbers prove it is true for $n = k + 1$.

(b) Find the coefficient of x^o in the expansion of $(x^{\frac{1}{2}} - x^{-\frac{1}{2}})^7$

(c) (1) show that if h is small $(1 + h)^n \simeq 1 + nh$.

(2) use this to find two successive approximations to the root of $x^3 - 3x^2 + 1 = 0$ which lies between 0 and 1.

(d) Prove that $\binom{K}{3} + \binom{K}{2} = \binom{K+1}{3}$

5. (a) Give the first and second derived functions of the functions represented by -

(1) $y = \operatorname{artan} kx$

(2) $y = \log_{10} 10^x$

(3) $y = \cot(\exp 3x)$

(b) If $y = \sin kx + \cos(\pi - kx)$ and $y - \frac{d^2y}{dx^2} = \sqrt{4k^2 - \left(\frac{dy}{dx}\right)^2}$,find k.

(c) A cube of side x inches is expanding in volume at the rate of "a" cubic inches per second. Find the rate at which the length of the side is increasing when $x = 2a$.

6. (a) State which of these are functions. If they are not a function give a sufficient reason in each case.

(1) $X = \{ (a, b, c), (d, e, f); (g, h, i) (j, k, 1) \}$

(2) $V = \{ (0, 1), (1, 2), (2, 3) (3, 2), (4, 1), (5, 0) \}$

(3) $F = \{ (x, y);\ y = /x/,\ x \in R,\ y \in R \}$

(4) $G = \{ (x, y);\ y = \sqrt{x};\ x, y \in J^+ \cup \{o\} \}$

(5) $P = \{ (x, y);\ y = k,\ x \in R,\ y \in N \}$

(6) $Q = \{ (x, y);\ /y/ = x,\ x \in J,\ y \in J \}$

(b) If $x \in (-2, -1, 0, 1, 2,)$ and $y = f(x) = x^3 + 1$,

(1) Sketch the graph of $y = x^3 + 1$ on the specified domain.

(2) What is the domain?

(3) What is the range of y?

(4) What type of mapping is this?

(c) If $f: R \rightarrow R$ and $f(x) = 5 - 7x$
$g: R \rightarrow R$ and $g(x) = 3 - \sqrt{x^2}$

(1) define the range of each function

(2) define $g(f(x))$ and $f(g(x))$ and state the domain and range of each composite function.

(3) Is gof equal to fog for any domain and range?

7. (a) If $f(x) = \text{ar} \sin x$, prove that $f'(x) = \dfrac{1}{\sqrt{1 - x^2}}$ for certain domains of f and f'. Mention these domains explicitly.

(b) Assuming that product and function of a function rules are already proved, show that if

$$F(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$$

$$D_x F(x) = \frac{g(x)\, f^1(x) - f(x)\, g^1(x)}{g(x)^2}$$

Use this rule to find $\dfrac{Dx\ (\text{ar} \tan 2x)}{1 + x^2}$

(c) Use algebraic and calculus methods to sketch the curve of the relation

$$\left\{ (x, y): \ y = \frac{x^2 + x - 2}{x^3} \right\}$$

On the curve emphasise all the important characteristics of this relation.

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PAPER 10

The universe of discourse for this paper is

{real numbers}.

1. (a) Simplify: $\dfrac{\sin\theta + \sin(\theta+\phi) + \sin(\theta + 2\phi)}{\cos\theta + \cos(\theta + \phi) + \cos(\theta + 2\phi)}$

(b) Solve for x over the domain $-2\pi \leqslant x \leqslant 2\pi$:

$$\sin 2x \cot x - \sin^2 x = \tfrac{1}{2}$$

(c) Express $\sqrt{3}\cos 2x + \sin 2x$ in the form $C \sin (bx + c)$ where $0 < c < \pi/2$. Hence state the maximum value of $\sqrt{3}\cos 2x + \sin 2x$, and all values of x in the interval $[-\pi, \pi]$ for which this maximum value occurs.

Write down the co-ordinates of the turning-points of the graph of

$$\left\{ (x, y) : y = \frac{1}{\sqrt{3}\cos 2x + \sin 2x}, \ -\pi < x < \pi \right\}$$

2. (a) The Cartesian equation of a circle is $x^2 + y^2 - 4x + 2y - 4 = 0$ Find the equation of the chord which has (3, 1) as its mid-point.

(b) Find the equation to the locus of a point P such that the areas of the triangles PAB, PCD are equal, if A, B, C, D are the points (0, 1), (0, 7), (3, 0), (5, 0) respectively.

3. (a) A locus is specified by the parametric equations

$$x = 4\cos t + 1$$
$$y = 4\sin t - 5$$

Eliminate the parameter t, and hence sketch the locus.

(b) Sketch the graphs of -

(i) $\left\{ (x, y) : y = \dfrac{2x^2 + x - 1}{x^2 - x + 1} \right\}$.

(ii) $\left\{ (x, y) : y = x^2 (x^2 - 9) \right\}$

4. (a) Write down, and simplify, the term independent of x in the expansion of

$$\left(3x^2 - \frac{1}{2x}\right)^9$$

(b) If a and b are the values of the second and third terms respectively in the expansion of $(1 + x)^n$, prove that

$$n = \frac{a^2}{a^2 - 2b}, \qquad x = \frac{a^2 - 2b}{a}.$$

(c) Find the coefficient of x^5 in the expansion of $(2 - 3x + x^2)^6$.

5. (a) Let f, g, h be functions with domain R, such that

$$f(x) = 2x^2 + 3, \quad g(x) = x^5 + 7, \quad h(x) = \cos x.$$

Give the rule which specifies

(i) $g \circ h$,

(ii) $(f \circ g) \circ h$,

(iii) $f \circ g^{-1}$

(iv) $h \circ h$.

(b) Show that $f : R \setminus \{1\} \to R$ where $f(x) = \dfrac{1}{1 - x}$

$g : R \setminus \{0\} \to R$ where $g(x) = \dfrac{x - 1}{x}$

are an inverse pair of functions, over a certain domain which is to be specified.

6. (a) Differentiate from first principles:

(i) $\frac{1}{2} \cos 3x$, (ii) $\sqrt{2x + 3}$

(b) Differentiate with respect to x:

(i) $\dfrac{1 - \sqrt{x}}{1 + \sqrt{x}}$

(ii) $(x^2 - x)^6$

(iii) $\sin^2 \dfrac{1}{x}$

(iv) $\sin x . \sin 3x$

(c) Find the value of $\dfrac{d}{dx}\left[\cos\left(\frac{1}{2}x + \frac{\pi}{3}\right)\right]$ when $x = 0$.

7. (a) Show that the curves whose Cartesian equations are

$$y = 5x^4 - x^3 + 2x + 29 \text{ and } y = (1 + 3x)(1 - x + 4x^2)$$

have a common tangent at the point (2, 105).

(b) If $f = \left\{(x, y) : y = x^3 - 6x, /x/ \leq 10\right\}$ find the greatest and least values of $f(x)$.

8. (a) Find $\frac{dy}{dx}$ when $y^3 - 3x^2y + 2x^3 = 0$.

(b) If $x^2 - y^2 = 1$, express $\frac{d^2y}{dx^2}$ in terms of y.

(c) If $x = 3t + 1$, $y = t^2 + t$, find $\frac{d^2y}{dx^2}$.

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Section One.

1. (a) Find in the form a + ib where a and b are real, the square roots of i.

(b) Let $1, w, w_1$, be the three cube roots of one. Show that $w_1 = \bar{w} = w^2$; and simplify the expressions;

$1 + w + w^2$, $(1 + w)(1 + w^2)$, $(1 + w)(1 + 2w)(1 + 3w)(1 + 5w)$.

2. (a) If $z_1 = 2(-1 + i\sqrt{3})$ express z_1 in polar form and hence find

$$\{z: \ z^2 = z_1\}$$

(b) Find in the form x + iy, the complex number z such that

$$\frac{3z + 1}{z - 2} = \frac{4 + 3i}{-1 + i}$$

(c) The points A, B, C, are such that B is the midpoint of AC. If A, B, C, represent, on the Argand diagram, the complex numbers α, β, γ what is the relationship between α, β and γ.

Section Two.

3. For each of the following sets state which of the field laws with respect to matrix addition and multiplication are satisfied:

(i) the set of all matrices of the form $\begin{bmatrix} k & 0 \\ 0 & n \end{bmatrix}$ where k and n are real.

(ii) the set of all matrices of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ where a, b, c are real non-zero.

4. (a) Verify the associative law, $[AB]\,C = A\,[BC]$, and the distributive law $A(B + C) = AB + AC$, in the case where -

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and } C = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

(b) If A and B are respectively the matrices

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

find their product AB and hence obtain the matrix A^2. Find the value of α for which $A^3 = 1$, the unit matrix, and for this value of α, interpret geometrically the transformation represented by A.

5. For what real value of α is the solution set of the following equations non-empty;

$$x - y + 2z = 1$$
$$3x + y - 6z = 4$$
$$5x - y - 2z = \alpha$$

Find the solution set in this case.

Section Three.

6. (a) Find the largest subset S of R such that -

$$f(x) = \log_e (x^2 - 1)$$

defines a function $f: S \rightarrow R$.

(b) Given $g(x) = x^2 - 3x$, find two of the largest possible subsets S of R such that $g: S \longrightarrow R$ has an iverse function. State the domains and ranges of the inverse functions.

7. Let the functions, f_1, f_2, f_3, and f_4 be defined with domain R by: $f_1(x) = \sin x$, $f_2(x) = /x/ + 1$, $f_3(x) = \begin{cases} 0, & x \leqslant 0 \\ 1, & x > 0 \end{cases}$

$f_4(x) = (1 + x^2)^{-1}$.

Find (i) the ranges of these functions;

(ii) the derivatives of f_2 and f_3, stating their domains;

(iii) the derivative of the composite function $f_1 o f_4$;

(iv) the two composite functions that can be formed with f_1 and f_2. Sketch the graphs of the composite functions in (iv). Show also that, if $a < 0$ and $x > a$, $\int_x^a f_3(t)dt = \frac{1}{2}(x + /x/)$

8. Let the functions f_1, f_2, f_3, and f_4 be defined with domain R and by; $f_1(x) = 2^x$, $f_2(x) = -2^x$, $f_3(x) = 2^{-x}$, $f_4(x) = 2^{-|x|}$
Sketch on the same axes, the graphs of the functions, stating their respective ranges.
State, giving brief reasons, which of the following assertions about $p = 2^{1/100}$ is true, where p is real:

$$p < 0, \quad 0 < p < 1, \quad 1 < p < 2, \quad p > 2$$

9. A conic has polar equation $r = (1-e\cos\theta)^{-1}$. Sketch (using separate diagrams) the graph of the conic in the cases $e = \frac{1}{3}, 1, 4$. Cartesian axes are taken with the focus as origin. Find the cartesian equation of the conic in the case $e = 1$ and hence obtain the cartesian equation of the tangent to the conic at the point where it is cut by the ray $\{[r, \theta] : r \geqslant 0, \ \theta = \frac{1}{2}\pi\}$.

Section Four.

10. Three points A, C, B lie on the base of a semi-circle of radius length r; O is the centre of the base and A and B are equidistant from O. If D and C are points on the curved part of the semi-circle and such that ABCD is a rectangle of greater area than that of all other rectangles inscribed in the same way, find the lengths of AB and BC.

11. Find the area of the region enclosed by the graph of the function defined by -

$$f(x) = \frac{x + 5}{(x + 1)\ (x + 3)},$$

the ordinates $x = 1$ and $x = 5$, and the x-axis.

12. (a) Find the derivatives (stating their domains) of;

(i) $f: R \longrightarrow R$, where $f(x) = e^{2x}\cos x$.

(ii) $g: R^{+} \rightarrow R$, where $g(x) = \frac{1}{x}\log_e x$

(b) Find real numbers x and y such that

$$(2 - i)x + (1 + 3i)y + 2 = 0.$$

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1. (a) If $f(x) = \frac{a^x - a^{-x}}{a^x + a^{-x}}$, prove that $f(x + y) = \frac{f(x) + f(y)}{1 + f(x)f(y)}$

(b) Express in partial fractions:

$$\frac{1 + x^2}{(1 + x)\ (1 + x^3)}$$

(c) If $\sec A - \tan A = x$, prove that $\frac{1 - x}{1 + x} = \tan \frac{A}{2}$

(d) Solve for θ :

$$3 \tan^3 \theta - 3 \tan^2 \theta = \tan \theta - 1.$$

2. If the coefficients of x^{r-1}, x^r, x^{r+1} in the binomial expansion of $(1 + x)^n$ are in arithmetic progression, prove that $n^2 - n(4r + 1) + 4r^2 - 2 = 0$.
Find the three consecutive coefficients of the expansion $(1 + x)^{14}$ which form an arithmetic progression.

3. (a) (i) If $y = e^x \sin x$, show that $\frac{dy}{dx} = \sqrt{2}\ e^x \sin(x + \frac{\pi}{4})$ and $\frac{d^2y}{dx^2} = 2\ e^x \sin\ (x + \frac{\pi}{2})$

(ii) Find the derivative of:

$$\sin(\ln 2x) + e^{2x} \tan 3x + 7^x$$

(b) Find the area enclosed between the curves $y = 6x^2 - 5x$, $y = -4x^2 + 5x$, and show that the chord joining the points of intersection divides this area in the ratio 3 : 2.

4. Calculate the volume generated, when the area bounded by the curve $y = x^4$ and the line $y = x$ is rotated

(i) about the x - axis;

(ii) about the y - axis.

5. (a) For the equation : $3x^3 - 2x^2 - 4x - 5 = 0$, find

(i) the sum of the squares of the roots,

(ii) the sum of the reciprocals of the roots,

(iii) the sum of the cubes of the roots.

(b) If α, β are the roots of the quadratic equation $ax^2 + bx + c = 0$, obtain the equation whose roots are $\frac{1}{\alpha^3}$ and $\frac{1}{\beta^3}$.

6. (a) Evaluate the following integrals correct to 1 dec. place.

(i) $\int_0^4 \frac{x\,dx}{\sqrt{x^2+1}}$

(ii) $\int_0^4 \frac{x\,dx}{x^2+1}$

(iii) $\int_0^4 \frac{dx}{x^2+1}$

(b) Integrate the following :

(i) $\sin 2x \cos 3x$,

(ii) $x^2 \ln x$

(iii) $\sin^4 x$

7. (a) Sketch the conic : $\{(x,y) : y^2 + 4x - 12y + 4 = 0.\}$

(b) An ellipse and a hyperbola have the same foci $(6,0)$ and $(-6, 0)$; the eccentricities are $\frac{1}{2}$ and 2 respectively. Find their equations and sketch the graphs.

(c) Prove that the triangle formed by the asymptotes and any tangent to the hyperibola $\frac{x^2}{4} - \frac{y^2}{2} = 1$, is of constant area.

8. Through the vertex O of a parabola chords OP and OQ are drawn at right angles to one another. Prove that for all positions of P, P Q cuts the axis of the parabola in a fixed point K.

Show also that the locus of the midpoint of P Q is a parabola, and find its vertex and the length of the latus rectum.

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PAPER 13

1. (a) $G = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 2 \\ 3 & 7 & 5 \end{pmatrix}$ $\qquad H = \begin{pmatrix} -4 & -1 & 2 \\ 1 & -1 & 0 \\ 1 & 2 & -1 \end{pmatrix}$

Find GH.

Can you deduce HG without further working? Why?

(b) Find the column vector whose image is $\binom{4}{29}$ under the transformation represented by the matrix $\begin{pmatrix} 2 & 5 \\ 3 & -4 \end{pmatrix}$.

(c) What matrix represents a rotation through angle α about O?

Compute the product

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

(d) Find an approximate value for $\cos \frac{5\pi}{12}$ without using trigonometric tables, but using $\sqrt{2} \approx 1.414$ and $\sqrt{3} \approx 1.732$.

(e) Give an example of two matrices A and B for which AB = BA.

(f) Write the matrix algebra equation which is equivalent to $i^2 = -1$.

(g) What matrix represents the complex number $3 - 2i$?

2. (a) Find the solution set $\{x_1, x_2, x_3\} = \{x : x^3 - 5x^2 - 2x + 24 = 0\}$ given that $x_1 x_2 = 12$.

3. Sketch (in your script book, not on graph paper) the graph of $f : R \setminus \{3\} \rightarrow R$ and $f(x) = \dfrac{x^2 - x - 2}{x - 3}$ after first finding any asymptotes, then the limitations on the range and hence the critical points.

4. Solve the equations over the field indicated -

(a) $2x^2 + 3ix + 4 = 0$ Over C

(b) $\sqrt{2x+1} - 2\sqrt{x-3} = \sqrt{2x-7}$ Over R.

(c) $\left.\begin{array}{l} 3x^2 - 4xy + 3y^2 = -8 \\ 5x^2 + 7xy + 5y^2 = 14 \end{array}\right\}$ Over R and over C.

5. Find an approximate solution, correct to three significant figures, of $x = 10^{-x^2}$ by first sketching (in your script book) the loci $\{(x,y) : y = -x^2\}$ and $\{(x,y) : y = \log_{10}x\}$, and then drawing a careful graph (on graph paper) to a sufficiently large scale.

6. (a) Show that the set of numbers of the form $a + bi$, where $i^2 = -1$, is closed under the operations of addition and multiplication.

(b) Put $(4 - 4\sqrt{3}.i)$ in its simplest polar form.

(c) Express $[5(\cos 13^\circ + i.\sin 13^\circ)] . [3(\cos 32^\circ + i.\sin 32^\circ)]$ in rectangular form.

(d) Find the square roots of $8(1 + \sqrt{3}.i)$.

7. (a) Find the largest subset S of R such that $f(x) = 2 + \sqrt{x}$ defines a function $f : S \to R$. With this domain find the range of f and sketch the graph of $\{(x,y) : y = f(x)\}$

Find f^{-1}, the inverse of f, stating its domain and range. Sketch (with the same axes as before) the graph of $\{(x,y) : y = f^{-1}(x)\}$ and calculate where it intersects the first graph.

(b) Let R^* be the set $\{x : x \in R, x \geqslant 0\}$. If $f : R \to R^*$ is given by $f(x) = x^2 + 5$, and $g : R^* \to R$ is given by $g(x) = \sqrt{x}$, write down the ranges of f and g.

Find the composite functions $f \circ g$ and $g \circ f$, stating their domains and ranges.

Calculate the value of $(g \circ f)'$ at $x = 2$.

$[f \circ g\,(x)$ means $f(g(x))]$.

8. (a) Find (i) $D_x \frac{x^2 - 1}{2x + 3}$

(ii) $D_x \sqrt{2x + 1}\ (x - 2)^3$

(iii) $D_x \sin \left[(2x + 5)^2\right]$

(iv) $D_x(x.\text{arsin } x + \sqrt{1 - x^2})$

(b) Find $D_x y$ given that $y = 2^x$.

(c) Find the set

$$\left\{m : y = e^{mx} \text{ and } \frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0\right\}$$

(d) Find (i) $\int (y^2 + y - 2)^5(2y + 1)\ dy$

(ii) $\int_1^2 \frac{x + 2}{x^2 + 4x + 1}\ dx.$

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PAPER 14

1. (a) Find real numbers x & y; $(2-i)x + (1 + 3i)y + 2 = 0$.

(b) If $Z = x + iy$ express $\frac{Z-1}{Z+1}$ in theform $a + ib$.

(i) If this quotient is real, show that $y = 0$.

(ii) If it is a pure imaginary quantity, prove that $x^2 + y^2 = 1$.

(iii) Show that $z\bar{z}$ and $z + \bar{z}$ are both real.

(c) If $Z_1 = 2(-1 + \sqrt{3}i)$, express Z, in polar form and hence find $(z : z^2 = z_1)$.

(d) The points A,B,C are such that B is the mid-point of AC. If A,B,C, represent, on the Argand diagram, the complex numbers α, β, γ what is the relationship between α, β, γ ?

2. (a) Express as a single complex number, $\left[\frac{\sin\theta + i\cos\theta}{\cos\theta + i\sin\theta}\right]^6$

(b) Using the usual formula for the solution of a quadratic equation, solve $Z^2 + 2iZ - 2 = 0$. Express the roots in the form $(x + iy)$ and find the modulus and phase of each.

3. (a) Let $A = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$ $B = \begin{bmatrix} 3 & -12 \\ -2 & 8 \end{bmatrix}$ $P = \begin{bmatrix} 1 & 2 \end{bmatrix}$

$Q = \begin{bmatrix} 4 & -3 & 0 \\ -1 & 2 & 1 \end{bmatrix}$

(i) State which of the following expressions are meaningful.

A + B, A + Q, QP, AB, P^2, AQ.

(ii) Evaluate the meaningful expressions in (i).

(iii) Evaluate, where possible, the determinants of the resulting expressions in (ii).

(b) Find the square roots of the second order identity matrix,

4. For what real value of α is the solution set of the following equations non-empty.

$$\begin{aligned} x - y + 2z &= 1, \\ 3x + y - 6z &= 4, \\ 5x - y - 2z &= \alpha \end{aligned}$$

Find the solution set in this case.

5. (a) Prove that $\frac{\sin x}{x}$ tends to a limit as x tends to zero.

(b) Find the derivative of cos 2x with respect to x, from first principles.

(c) Use this derivative to deduce the derivative of $\cos^2 x$.

6. (a) Differentiate the following functions with respect to x:

(i) $\sin^2 x$

(ii) $\sin (x^2)$

(iii) $\ln \left(\frac{1 - \cos x}{\sin x}\right)$

(iv) $\sqrt{\frac{1 - x}{1 + x}}$

(b) If $V = \frac{4}{3} \pi r^3$ and $S = 4 \pi r^2$, find $\frac{dS}{dV}$.

7. A warehouse has the form of portion of a long circular cylinder of radius 16 feet with axis horizontal. A vertical section is a segment of a circle, the highest point being 20 feet above the horizontal floor. A tank in the shape of a right circular cylinder with base on the floor, is to be built in the warehouse.

Find the height of the tank in order that its volume should be a maximum.

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