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ABSTRACT

This booklet contains suggestions for developing six of the twenty topics recommended for students in Secondary Forms III and IV in the schools of Victoria, Australia. These suggestions are intended for students for whom the standard course is unsuitable because of subject time allotments, ability levels, or vocational or educational needs. The six topics are Hire Purchase; Income Tax; Topology; Curve Stitching; Probability; and Relations, Loci and Graphs. Objectives of the modified course, and a bibliography of reference materials, are also included. (MM)

EDUCATION DEPARTMENT OF VICTORIA

MODIFIED MATHEMATICS

FORMS III AND IV

SUGGESTED TOPICS FOR STUDY

in

SECONDARY SCHOOLS

MODIFIED MATHEMATICS
Forms III and IV

EDUCATION DEPARTMENT OF VICTORIA

1970

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CONTENTS

INTRODUCTION

These topics are presented in an endeavor to cater for the mathematical needs of pupils in Forms III and IV, for whom the suggested V.U.S.E.B. courses may not be considered entirely suitable because of subject time allotments, ability levels, or vocational or educational needs. These pupils might be found in commercial classes and other similar groups.

The course as presented is the result of deliberations of the Secondary Mathematics Committee Working Group on Modified Mathematics Courses—Forms III and IV.

The word " course " as used in this document does not refer to a prescribed course which teachers are committed to use in its published form. It is to be considered as a series of suggestions for starting-points which should be of help to teachers in constructing courses appropriate to their own particular situation.

THE NATURE OF THE PROBLEM OF MATHEMATICS COURSE DEVELOPMENT

Before 1967 the V.U.S.E.B. or its predecessors prescribed courses of study for Forms IV, V, and VI. With the abolition of the Intermediate Certificate after 1967, the development of courses of study for Forms I–IV has become the responsibility of individual schools. The courses for Forms V and VI are still prescribed and published by V.U.S.E.B. annually. The V.U.S.E.B. has agreed to continue the publication of courses of study for the first four years of the secondary school. These latter courses are not prescriptive but are prepared for the guidance of teachers by the various Standing Committees of the V.U.S.E.B. Schools are free to alter courses to suit particular classes of pupils who either do not have the full time allotment for mathematics or are unable to cope with the courses published in *V.U.S.E.B. Courses of Study, Forms I–IV*. The present publication is concerned with courses for those children noted above, many of whom have had as many as eight or nine years of lack of success in mathematics or other subjects.

A Working Group for Modified Mathematics Courses, Forms III and IV, was set up in 1967 to consider such a course. The course contains topics which are—

(i) mathematically significant ;

(ii) seen to be so by children undertaking the course ;

(iii) within the capabilities of the children.

The members of the Working Group made suggestions for the development of this course from their experience as practising classroom teachers, and from topics which they had found to be successful in that context.

Acknowledgement is made of the assistance of the following Working Group members : Mrs. L. Green (Waverley High School) ; Mrs. J. Maddox (Niddric High School) ; and Messrs. R. C. Barrington (Blackburn High School), M. Blakey (Highett High School), R. H. Cowban (Curriculum and Research Branch ; Convener), P. L. Galbraith (Secondary Teachers' College), S. Hosking (Kingswood College ; replaced by B. Phillips, Kingswood College, 1969), R. L. Jones (Glenroy High School), C. M. Powers (Curriculum and Research Branch), L. C. Press (Chadstone High School), W. Stanford (Brunswick Girls' High School), and G. R. Willis (Reservoir High School).

Acknowledgement is also made of the assistance of the Technical Schools Standing Committee on Mathematics in the preparation of some of the material contained in this publication.

SUGGESTED TOPICS FOR A MODIFIED COURSE

Many aspects of a modified course were considered by the Working Group. Among these aspects were the objectives of such a course, suitable topics, reference books, and working papers on certain topics.

In this section, objectives, topics, and reference material are considered. The working papers are presented in the following section.

Objectives of a Modified Course

(a) *Ultimate Objectives*

The objectives of the "Alternative Mathematics" courses as set down in the *V.U.S.E.B. Courses of Study, Forms I–IV*, September 1967 (pp. 182–183), were agreed to as suitable for any modified course, and in particular the objective—

> "To develop desirable *attitudes*, *interests*, and *appreciations*, e.g. senses of curiosity, wonder, and beauty; attitudes of independence and reasonable self-confidence; a permanent interest in mathematics; an appreciation of our cultural heritage."

(b) *Immediate Objectives*

(i) An appreciation of the space in which we live, and of spatial relationships.
(ii) A knowledge and appreciation of some of the major themes and concepts of mathematics.
(iii) An understanding of the role of mathematics in society—past, present, and future.
(iv) The development of methods of investigation and means of expressing results of such investigations.

The objectives of a modified course could be stated as the inculcation of a love of mathematics, together with an appreciation of mathematics as a tool which contributes to the development of confidence and clarity of thought.

Topics Suggested for a Modified Course

Modified courses in mathematics at Forms III–IV levels have been used in a number of schools, and a list of topics used is given, together with suitable references.

This list is not prescriptive. Schools could make a selection from the topics listed below; nevertheless, the list is not exhaustive, and teachers may well choose other topics which achieve the purposes of a modified course.

The Working Group recommends that the topics be listed in two separate categories. Those called "core topics" should be taken by all classes; those called "elective topics" are those whose selection depends more on the personal tastes and the experience of teachers. The limit to which each topic is treated depends upon interests and abilities of teachers and classes and should become clear to individual teachers as the year develops.

Core Topics

1. Sets, Venn diagrams.
2. Statistics.
3. Graphs—algebraic and statistical.
4. Simple interest, growth and decay patterns.
5. Income tax.
6. Use of tables.

Elective Topics

7. Binary arithmetic, codes and matrices, punch cards.
8. Adding machines.
9. Tiling problems, mensuration.
10. Topology.
11. Groups—informal treatment.
12. Curve stitching.
13. Construction of polyhedra, 3D models.
14. Trigonometry.
15. Household arithmetic.
16. Number patterns.
17. Probability—selections and arrangements.
18. Slide rules.
19. Hire purchase.
20. Logarithms.

REFERENCES

In each case below, the name of a book is followed by numbers which refer to the topics noted earlier for which the book is considered to be suitable. It is not intended that this list be thought of as complete.

FLETCHER T. J. (ed.) : *Some Lessons in Mathematics* (C.U.P., 1964)
(Topics 1, 3, 7, 8, 10, 11, 12, 18)

LEWIS, W. D. : *Teaching School Mathematics with the Desk Calculator*
(Topic 8)

LOCKWOOD, E. H. : *A Book of Curves* (C.U.P.)
(Topics 3, 12)

HUME, B. : *An Introduction to Probability and Statistics* (Univ. of W.A.)
(Topics 2, 3, 17)

BUDDEN, F. J. : *An Introduction to Number Scales and Computers* (Longmans)
(Topics 7, 8)

THOMAS, J. G. : *Topics in Modern Mathematics* (Blackie, 1965)
(Topics 1, 2, 3, 7)

CASE, D. H. : *Modern Mathematical Topics* (Mills and Boon, 1966)
(Topics 1, 2, 7, 16, 17)

CADWELL, J. H. : *Topics in Recreational Mathematics* (C.U.P., 1966)
(Topics 9, 10, 11, 13, 16, 17)

JOHNSON, D. A. : *Paper Folding for the Mathematics Class* (N.C.T.M.)
(Topics 12, 13)

CRADDY, O. : *Topics in Mathematics* (Batsford, 1967)
(Topics 3, 5, 13, 15, 16)

EDWARDS, A. : *Mathematics Modern Style*, Book I (Oliver and Boyd, 1968)
(Topics 2, 9, 10, 13, 15, 16, 18, 19, 20)

EVENSON, A. B. : *Modern Mathematics* (Gage, 1962)
(Topics 2, 3, 4, 5, 13, 15, 16—Teachers' reference)

HAMILL, C., and MARSHALL, J. J. : *Modern Mathematics*, Books I, II (Hulton, 1967)
(Topics : Book I—1, 3, 7 ; Book II—3, 7, 10, 11)

KANNEGIESSER, H. J., and MIRSKY, M. H. : *Business Mathematics* (Macmillan, 1969)
(Topics 2, 3, 4, 5, 6, 9)

Exploring Mathematics on Your Own Series (John Murray)

" Calculating Devices "	(Topics 8, 18)
" Short Cuts in Calculating "	(Topic 8)
" Sets, Sentences, and Operations "	(Topic 1)
" The World of Statistics "	(Topics 2, 3)
" Topology, the Rubber Sheet Geometry "	(Topic 10)
" Number Patterns "	(Topic 16)
" Understanding Numeration Systems "	(Topic 7)
" Graphs "	(Topic 3)
" Curves "	(Topics 3, 12)
" Probability and Chance "	(Topic 17)
" Finite Mathematical Systems "	(Topic 11)
" Logic and Reasoning in Mathematics "	(Topic 7)
" The Theorem of Pythagoras "	(Topic 14)
" The World of Measurement "	(Topic 9)
" Computer Programming "	(Topic 7)

Topics in Modern Mathematics Series (Ginn)

" Digital Computers and Related Mathematics "	(Topic 7)

Topics from Mathematics Series (C.U.P.)

" Computers "	(Topic 7)
" Statistics "	(Topics 2, 3)
" Circles "	(Topics 12, 13)
" Solid Models "	(Topics 12, 13)
" Cubes "	(Topic 13)
" Tessellations "	(Topic 12)

Midlands Mathematics Experiment (Harrap, 1968)

Volume 1, Part A	(Topics 1, 9)
Volume 1, Part B	(Topics 7, 16, 17, 18, 20)
" Excursions from Mathematics "	(Topics 2, 3, 9, 10)

Houghton Mifflin Mathematics Enrichment Series

" Stereograms "	(Topics 9, 12)
" Mosaics "	(Topics 12, 13)

Exploring Mathematics Series (Grant)

" Introducing Topology "	(Topic 10)
" Introducing Sets "	(Topic 1)
" Introducing Polyhedra "	(Topic 13)
" Calculating Devices "	(Topics 8, 18)
" Curves "	(Topics 3, 12)
" Number Patterns "	(Topic 16)
" Ciphers "	(Topic 7)
" Great Mathematicians "	(Topics 3, 6, 17, 20)
" Statistics "	(Topic 2)
" Histograms "	(Topics 2, 3)
" Simple Line Graphs "	(Topics 2, 3)

Modern Mathematics for Achievement, Book 4. (Second Course)
" Learning about Geometry " (Houghton Mifflin) (Topic 13)

School Mathematics Project (C.U.P.)

S.M.P. Book A	(Topics 3, 13, 14, 16)
S.M.P. Book B	(Topics 2, 3, 7, 9, 10, 16)
S.M.P. Book C	(Topics 2, 3, 7, 9, 13, 18)

Contemporary School Mathematics (St. Dunstans)

" Sets and Logic I "	(Topics 1, 7)
" Sets and Logic II "	(Topics 1, 7)
" Computers I "	(Topic 7)
" Computers II "	(Topic 7)
" Matrices I "	(Topic 7)
" Matrices II "	(Topic 7)
" Probability and Statistics "	(Topics 2, 3, 17)
" Shape, Size, and Place "	(Topics 3, 10, 11)
" Geometry and Logic "	(Topic 9)
" Mathematics I "	(Topics 1, 2, 3, 7, 8, 9, 13, 16)
" Exercises I "	(Topics 1, 7, 10)

SUGGESTIONS FOR THE DEVELOPMENT OF TOPICS

During the meetings of the Working Group a great deal of time was taken up in the presentation and discussion of working papers on some of the topics noted earlier.

These working papers are presented as a possible guide for teachers. They are not to be thought of as presenting prescribed or ideal methods, but rather as suggestions based upon successful use in various schools.

The topics covered by working papers are :

1. Hire Purchase
2. Income Tax
3. Topology
4. Curve Stitching
5. Probability
6. Relations, Loci, and Graphs

HIRE PURCHASE

Hire purchase has become an accepted method of purchasing goods in our community, affecting many households directly or indirectly. It is within the capacity of pupils in Forms III and IV to understand how hire purchase works and to appreciate some of the advantages and the disadvantages of the system. The following topics lead up to the calculation of the approximate effective rate of interest being paid under hire purchase and similar loans.

A. Simple Interest

Use of the formula $I = \frac{PRT}{100}$, where

I represents the interest,
P represents the principal borrowed,
R represents the rate per cent per annum,
T represents the time in years.

Transformation of the formula to make R, P, or T the subject.

Examples should include cases where T has fractional values, e.g. $T = \frac{1}{12}$.

Note : To simplify $\frac{100 \times 20}{750 \times \frac{1}{12}}$ we may multiply by $\frac{12}{12}$, although most pupils follow a rule

Examples

1. Make R the subject of the formula $I = \frac{PRT}{100}$, and find R when—

(i)	(ii)	(iii)
$I = 20$	$I = 35$	$I = 2\frac{1}{2}$
$P = 200$	$P = 280$	$P = 90$
$T = 1$	$T = 1$	$T = \frac{1}{12}$

2. Find the rate per cent per annum if it costs \$10 to borrow \$500 for one month ($\frac{1}{12}$ year).

3. Find the rate of interest per annum if it costs \$2 to borrow \$150 for one month.

4. In a hire purchase transaction the hirer borrows the equivalent of \$1,000 for one month. He is charged \$12 for terms. What is the effective rate of interest ?

5. Repeat example 4, where \$20 is charged for borrowing the equivalent of \$720 for one month.

B. Summation of Series

1. A series such as
$120 + 110 + 100 + 90 + 80 + 70 + 60 + 50 + 40 + 30 + 20 + 10$ (12 terms)
may be summed by arranging the terms in pairs :
$(120 + 10) + (110 + 20) + (100 + 30) + \ldots$ etc. (6 pairs)
$= 6 \times 130$
$= 780.$

2. $900 + 800 + 700 + 600 + 500 + 400 + 300 + 200 + 100,$
which has an odd number of terms may be written as
$900 + 800 + 700 + 600 + 500 + 400 + 300 + 200 + 100 + 0$
to give an even number of terms, and the sum is then obtained as in (1) above.

Examples

By arranging the terms of these series in pairs, find the sums of the following :

(i) $40 + 35 + 30 + 25 + 20 + 15 + 10 + 5.$

(ii) $240 + 220 + 200 + 180 + \ldots + 20$ (12 terms).

(iii) $10 + 20 + 30 + 40 + 50 + \ldots + 120.$

(iv) $10 + 20 + 30 + 40 + 50 + \ldots + 110.$

(v) $720 + 700 + 680 + \ldots + 20.$

(vi) $7 + 14 + 21 + \ldots + 70.$

(vii) $5 + 10 + 15 + 20 + \ldots$ to 20 terms.

C. The Formula $S = \frac{n}{2}(a + l)$

where S is the sum of n terms,
n is the number of terms,
a is the first term,
l is the last term.

This formula arises from the method used in B above, and is particularly useful when n is large, e.g. to sum $720 + 700 + 680 + 660 + \ldots$ to 36 terms. It is applicable to this type of series (arithmetic series).

Exercises

1. Use the formula $S = \frac{n}{2}(a + l)$ to find the sums in the exercises under Section B.

2. Similarly, find the sum of $2 + 4 + 6 + 8 + \ldots$ to 20 terms.

3. In a raffle 100 tickets are sold, the first for 1 cent, the second [illegible] the third for 3 cents, and so on until the last is sold for \$1.00. If the prizes and the [illegible] together cost \$10, calculate the profit made.

D. Equivalent Loans

The idea that a loan paid off in regular monthly instalments can be expressed as the equivalent of a loan made for one month is a vital concept illustrated by a simple example :

A borrows \$30 from B and pays it back at \$10 per month. Show that this is equivalent to borrowing \$60 for one month.

Observe that A borrows \$30 for one month,
then A borrows \$20 for one month,
then A borrows \$10 for one month.

This is equivalent to A borrowing \$(30 + 20 + 10) for one month, i.e. \$60 for one month.

Exercises

1. A man borrows \$100 for one month, \$80 for one month, \$60 for one month, \$40 for one month, and \$20 for one month. Show that this is equivalent to borrowing \$300 for one month.

2. In buying a TV set on hire purchase a man agrees to pay off \$200 over 10 months in equal instalments.

(i) How much is each instalment ?

(ii) How much does he owe during the 1st month ?
How much does he owe during the 2nd month ?
How much does he owe during the 3rd month ?
How much does he owe during the 4th month ?

(iii) By finding the sum of the amounts owing in each month, express the amounts owing over the period as an equivalent amount owing for one month.

3. A man buys a car on hire purchase agreeing to pay off \$720 over 3 years in equal monthly instalments.

(i) How much is each instalment ?

(ii) Write down some of the terms of the series showing the amounts owing during the first month, the second month, the third month, and so on.

(iii) Express the amounts owing over the period as an equivalent amount owing for one month.

(iv) Express this as the equivalent of an amount owing for one year.

4. (*a*) A man invests \$50 per year in an insurance policy. He continues this for 25 years.

(i) How much was invested in the first year ?

(ii) How much altogether would the insurance company hold in the second year ?

(iii) How much altogether would the insurance company hold during the third year, the fourth year, and so on ?

(*b*) Express these investments as the equivalent of an amount invested for one year.

(*c*) Calculate the simple interest the money would earn at 5 per cent per annum.

5. In the following hire purchase transaction express the loans involved over the period as an equivalent of a loan for one month.

(*a*) \$360 payable in 12 equal monthly instalments.

(*b*) \$300 payable in 12 equal monthly instalments.

(*c*) \$480 payable in 24 equal monthly instalments.

(*d*) \$500 payable in 24 equal monthly instalments.

(*e*) \$1200 payable in 36 equal monthly instalments.

(*f*) \$52 payable in 52 equal weekly instalments.

6. A bank offers to lend money at a fixed rate of interest, the money to be repaid in equal monthly instalments.

(i) If a man borrows \$960 over 24 months, how much does he have to repay each month ?

(ii) Write out the series of the amounts owing during the first month, the second month, the third month, and so on to the twenty-fourth month.

(iii) Show that the man has borrowed from the bank the equivalent of \$12,000 for one month.

(iv) Does the man ever hold \$12,000 of the bank's money ?

E. Flat Interest Rates and Effective Interest Rates

1. In hire purchase transactions the term "flat" rate of interest refers to the *rate per cent per annum* charged on the *full amount* borrowed for the *full term* of the contract. For example, if the hirer borrows \$100 to be repaid in 10 equal monthly instalments at 8 per cent flat rate of interest, then the interest is calculated on \$100 for 10 months ($\frac{5}{6}$ year).

$$\text{Since } I = \frac{PRT}{100}$$

$$I = \frac{100 \times 8 \times \frac{5}{6}}{100}$$

$$= 6\tfrac{2}{3}.$$

Interest $\simeq$ \$6 67.

2. Since the loan is paid off in instalments, the full amount (\$100) is not in fact borrowed for the full time (10 months). The *effective* rate of interest is calculated having regard for the principle that interest should not be reckoned on an amount repaid by instalment, since it is no longer owing for the remainder of the term of the contract. In the example used above it may be shown that the hirer borrowed the equivalent of \$550 for 1 *month*, being charged \$6$\frac{2}{3}$ (approx).

The effective rate of interest is calculated as follows :—

$$R = \frac{100\,I}{P\,T}$$

$$\left.\begin{array}{l} I = 6\tfrac{2}{3} \\ P = 550 \\ T = \tfrac{1}{12} \end{array}\right\} \Rightarrow \qquad R = \frac{100 \times 6\tfrac{2}{3}}{550 \times \tfrac{1}{12}} = \frac{100 \times \tfrac{20}{3}}{550 \times \tfrac{1}{12}}$$

$$= 14\tfrac{6}{11}$$

Effective rate of interest per cent per annum $\simeq$ 14.55.

Examples

1. Mr A borrows $1,200 from a bank at a flat rate of 6 per cent per annum, capital plus interest to be repaid in 12 equal monthly instalments.

(i) How much interest would Mr A have to pay ?

(ii) How much would Mr A have to pay back to the bank altogether ?

(iii) If the amount in (ii) is paid in 12 equal monthly instalments, how much would each instalment be ?

(iv) Since the actual amount borrowed from the bank was $1,200, how much of this principal is paid off each month in equal instalments ?

(v) Write down the amount of principal owing during the first month, the second month, the third month, and so on. Hence express the loan from the bank as the equivalent of a loan for one month only.

(vi) Show that the *effective* rate of interest in borrowing $7,800 for one month at a cost of $72 is just over 11 per cent per annum.

2. Repeat Question 1, where $1,200 is borrowed at 6 per cent flat rate of interest over a 2-year period, repayments being made in equal monthly instalments.

F. Hire Purchase Problems

The Hire Purchase Act requires that the hirer be given information specifying cash price, terms price, charges for terms, term of repayment.

Some discussion of the consequences of failing to meet instalments and the repossession conditions will assist pupils to appreciate the topic.

Exercises

1. A housewife obtained a washing machine on hire purchase on the following terms :

Cash price of machine	$240
Terms price	$260
Charges for terms	$20
Deposit	$60

Balance due, $200 payable in 20 equal monthly instalments of $10 per month.

(*a*) Write down the amount owing for the first month, the second month, the third month, and so on to the twentieth month.

Note that these amounts include the interest.

(*b*) Considering the cash price of the machine was $240 and the deposit paid was $60, the actual amount borrowed from the hire purchase company was $180. In paying off $180 over 20 months write down—

(i) the amount to be paid off each month ;

(ii) the amount owing for the first month, the second month, the third month, and so on to the twentieth month.

(*c*) Express the amounts borrowed from the hire purchase company in (*b*) above as an equivalent amount borrowed for one month.

(*d*) Calculate the approximate effective rate of interest *per cent per annum*.

2. **Study the following table for hire purchase commitments:**

	(i) Radio	(ii) Car	(iii) Refrigerator
Cash price ..	\$65	\$2,200	\$220
Terms price ..	\$70	\$2,400	\$250
Charges for terms	\$5	\$200	\$30
Deposit	\$15	\$600	\$40
Balance due ..	\$55	\$1,800	\$210
Instalments ..	10 monthly of \$5.50 each	20 monthly of \$90 each	20 monthly of \$10.50 each
Approximate answers ..	10.9% p.a.	14.3% p.a.	19% p.a.

(*a*) Determine the amount owing, including interest, in the first month, the second month, the third month, and so on.

(*b*) Determine the actual principal (exclude interest) borrowed in the first month.

(*c*) Determine the amount of principal (exclude interest) paid off in each equal instalment.

(*d*) Write out the series of the amounts of principal (exclude interest) owing during the first month, the second month, the third month, and so on.

(*e*) Express these as the equivalent of a loan for one month.

(*f*) Calculate the approximate effective rate of interest.

Worked Example

A person purchases a car for \$2,000 and is allowed \$650 trade-in on his old car. He takes \$450 of the trade-in in cash and the remaining \$200 is used in part payment on the new car.

The monthly repayments extend over 36 months, and the interest is charged at a rate of 10 per cent on the initial cost (flat rate). In addition the buyer has accessories fitted to the car before purchase. The cost of these accessories is \$150 and is added to the purchase price. A handling charge of \$210 is also added to the purchase price and includes insurance, registration, and so on.

Find

(i) the interest payable,

(ii) the total amount payable,

(iii) the monthly repayments,

(iv) the amount of principal repaid each month,

(v) the effective rate of interest.

Cash price of car	\$2,000
Cash price of accessories	\$ 150
Handling charge	\$ 210
	\$2,360
Deposit (trade-in)	\$ 200
Balance due	\$2,160
Charges for terms (interest)	$\$(\frac{10}{100} \times 2{,}160 \times \frac{3}{1})$

(i) Interest payable $= \$\ 648$

Total balance due $= \$(2{,}160 + 648)$

$= \$2{,}808$

(ii) Total amount payable $= \$2{,}808$

(iii) Monthly repayments (i.e. amount of each instalment) $= \$\left(\frac{2{,}808}{36}\right) = \$78.$

Actual principal borrowed $= \$2{,}160$

(iv) Principal repaid each month $= \$\left(\frac{2{,}160}{36}\right) = \$60.$

Principal amounts owing $= \$(2{,}160 + 2{,}100 + \ldots + 60)$

$$S = \frac{n}{2}(a + l)$$

$$= \frac{36}{2}(2{,}160 + 60)$$

$$= 18 \times 2{,}220$$

$$= \$39{,}960$$

and

$$R = \frac{100\,I}{P\,T}$$

$$= \frac{100 \times 648}{39{,}960 \times \frac{1}{12}} \times \frac{12}{12}$$

$$= \frac{2{,}160}{111}$$

$$\simeq 19.5$$

(v) So effective rate of interest is approximately 19.5 per cent per annum.

Note : It may be of interest to teachers and pupils to work this problem through with more of the trade-in value being used as deposit on the new car.

INCOME TAX

A. Justification for Inclusion of This Topic in a Modified Course

1. Utilitarian justification. Many students following modified courses at this level will be taxpayers within two years.

2. Familiarity with tabular methods of recording information (e.g. insurance tables, train time-tables).

3. Practice at filling in tables.

4. Increased knowledge of operation of governments and government departments.

5. Graphical interpretation of progressive tax.

6. Applications of the idea of a fraction in a practical context.

B. Part of Year Best Suited to Topic

June and/or July. Duration—till interest lags.

C. Suggested Development

1. Descriptive—approach either by discussion or by assignment. Points that can be covered are :

(*a*) Need for government to control economy to some extent at least (Compare *laissez-faire* governments of the 19th century in the U.K. and the U.S.A.).

(*b*) Taxation is not a modern invention. Primitive tax was a percentage of harvest. This still applies in some underdeveloped countries.

(*c*) Modern methods of taxation in Australia—income tax, customs and excise duty, sales tax, pay-roll tax, estate duty, gift duty, others. Merits of all or some—especially income tax as compared with sales tax.

(*d*) Short history of taxation in Australia.

(*e*) Students could imagine that they lived in a country in which no tax was paid. Ask the class :

If you were Prime Minister for one year, what laws would you make ?

Are there any countries in the world where income tax is not paid ?

What is a duty-free port ?

(*f*) Governments sometimes tax imports for reasons other than for the raising of money. Some imported goods, for example, cars, are taxed to protect local products.

2. Examination of tax systems with progressively increasing rates :

(*a*) From a study of a current " S Form ", students calculate percentage tax paid for selected incomes to $32,000.

Income ($)	0	400	500	32,000
Tax	0	0	10	16,000
% tax	0	0	2	50

(Approximate tax for ease of calculation.)

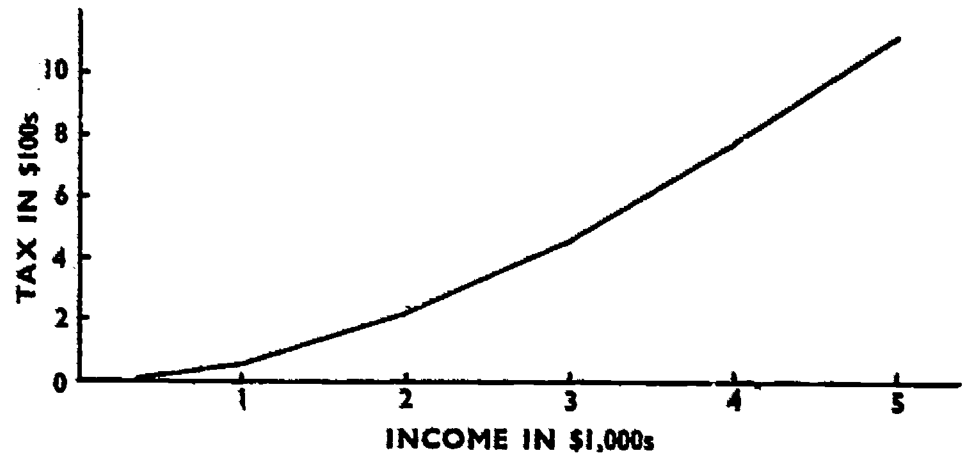
(*b*) Graphs (pictorial representations) of tax systems.

(i) The tax system of Australia (System A).

(ii) Another system (System B).

System A : Graph 1

Income ($) ..	0	416	500	1,000	2,000	3,000	4,000	5,000
Tax ($)	0	0	10	53	218	463	771	1,140

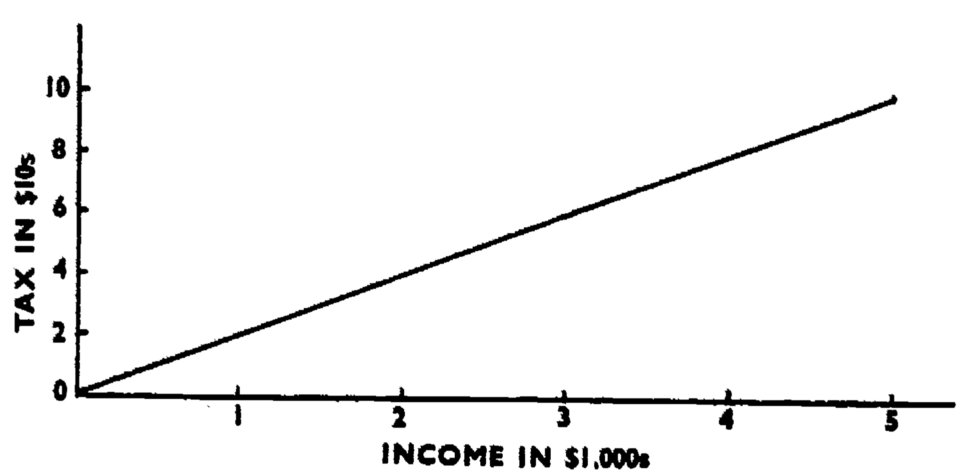


Incomes and tax up to $32,000 can be examined. What shape is the graph up to $32,000 ? Why ? What shape is it over $32,000 with 2½% levy ? What shape is it over $32,000 without 2½% levy ?

System B : Graph 2

Income ($)	0	500	1,000	2,000	3,000	4,000	5,000
Tax ($)	0	10	20	40	60	80	100

(This can be compared with exponential growth curve with more able children.)



Students should note the different shapes of graphs. They should realise that with a progressive tax a person pays a proportionally greater income tax on a high income than he does on a low income, hence the graph is shaped the way it is. The increase is always increasing.

With a system such as B, more income earned means more tax is paid, but the rate of increase is always constant.

3. Exercises in Income Tax :

(*a*) It is advisable to do exercises on actual S Forms.

(*b*) Suggested items (on the S Form) to be involved in problems—1, 8, 11, 12, 15, 17, 18, 19, 20, 21, 22, 25, 26, 27.

With items 11, 12, and 15, exercises should be given when a dependent is not "wholly dependent" and actual deduction has to be calculated.

(*c*) Most Form III–IV arithmetic (and modern mathematics) text books and the Taxation Department Information Circular have suitable problems on finding gross income, total deduction, taxable income, income tax payable, and tax refund or tax payable. However, students are often more interested in the imaginery income tax return of the teacher, their parents, and friends and classmates who have part-time employment.

(*d*) Where possible, actual group certificates and tax stamps should be shown. (Students in part-time employment are often willing to exhibit their group certificates.)

(*e*) Some students enjoy typing tax returns.

4. References :

(*a*) Descriptive—

(i) *Information on Taxation for Use in Victorian Schools* (1968) (One copy was sent to each Victorian secondary school during 1968. This replaced the previous Information on Taxation, VJ 187/1 1965).

(ii) DROHAN and DAY : *Australian Economic Framework.*

(iii) NANKERVIS : *Descriptive Economics.*

(*b*) Exercises—

(iv) *Information on Taxation for Use in Victorian Schools* (1968). (See (i) above.)

(v) HEFFERNAN : *Elements of Arithmetic*, Parts III and IV.

(vi) ADAMSON and TURNER : *Third and Fourth Year Arithmetic.*

(vii) BAKER, MOORE, and PITHOUSE : *Mathematics for Form III.*

(viii) RICHARDSON, et al : *Integrated Mathematics*, Book 3.

TOPOLOGY

The study of topology for the purpose of this topic is concerned with the properties of a given figure which remain unchanged during a topological transformation. Such a transformation occurs when one geometrical figure is mapped into another in such a way that—

(*a*) For each " point " of one figure there is a corresponding " point " on the other figure. Some mathematicians maintain that a point, as an abstract mathematical thing, cannot be moved. In the present treatment we use the word " point " informally, e.g. as the moving " point " which is the mid point of a sliding ladder.

(*b*) If we move a point on one figure towards another point on the same figure until the distance between them approaches zero, the distance between corresponding points on the other figure will also approach zero.

Hence this is the kind of alteration that can take place by str:tching and distorting a figure as if it had been drawn on a rubber sheet. Measures of distance, angles, curvature, or area are not preserved in these transformations.

Children can be led to an appreciation of such transformations by a study of particular problems associated with such concepts. Intuitive topological ideas develop very early in children, according to Piaget. Work on topology can extend children's thinking to reason in relation to many situations for which Euclidean geometry is inappropriate.

References

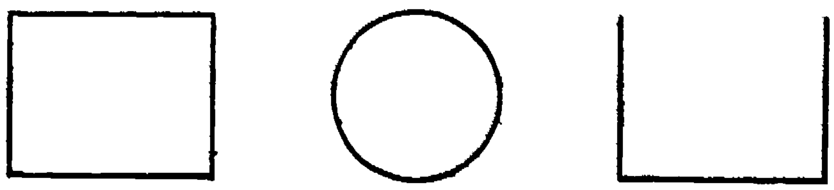
JOHNSON, D. A., and GLENN, W. H. : *Topology, the Rubber-sheet Geometry* (" Exploring Mathematics on Your Own Series "), John Murray.

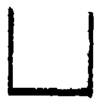
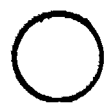
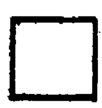
FRENCH, P. : *Introducing Topology* (" Exploring Mathematics Series "), Grant, 1964.

School Mathematics Project, Book 2, C.U.P., 1966.

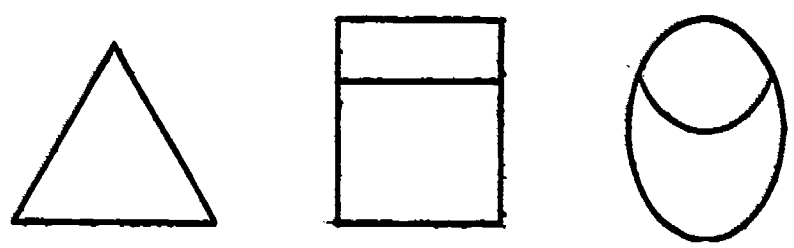
1. Topological Transformations

(*a*) Discussion and demonstration of shapes drawn on sheets of rubber and whether these shapes can be changed into each other without " welding " or " tearing " or similar distortion. Demonstrating this.

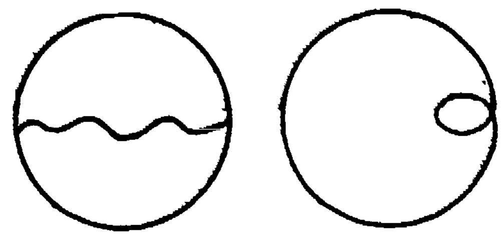


Why can't  be changed into  whereas  can ?

(*b*) Which of the following curves can be " changed " into one of the other two ?

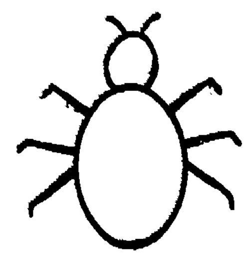


(c) Class perform many similar determinations to establish equivalent topological figures, e.g.



(d) More complicated examples.

(e) What different figures can the beetle become ?



Class can discover similar examples.

(f) Discussion of simple closed curves. Which of the following are such ?

(i) An equilateral triangle.

(ii) A letter L.

(iii) The letter P, etc.

2. Networks

(a) An appropriate introduction to this topic could be through the marking of basket-ball courts. A discussion of the difficulties involved in the use of many of the marking machines could involve the difficulty of stopping and starting the " lines " (formation of thicker areas, lines overlapping, etc.). The cost of the material is unimportant compared with these difficulties, and so the exact length of line is not the important thing.

" Could one traverse the lines for this court in one " sweep " or would one have to use more than one " sweep " ?" This is an important question.

(b) Many other networks could be examined for traversability.

(c) The children could make up traversable networks and see if other children could find the appropriate route. In constructing these, children should begin to form generalisations as to characteristics of networks which are traversable.

(d) Classification of networks in terms of traversability could be made and recorded on a table such as the following :

Number of even vertices	Number of odd vertices	Whether traversable

(e) Situations in which closed networks appear to be always traversable could be established and tested.

3. Map Coloring

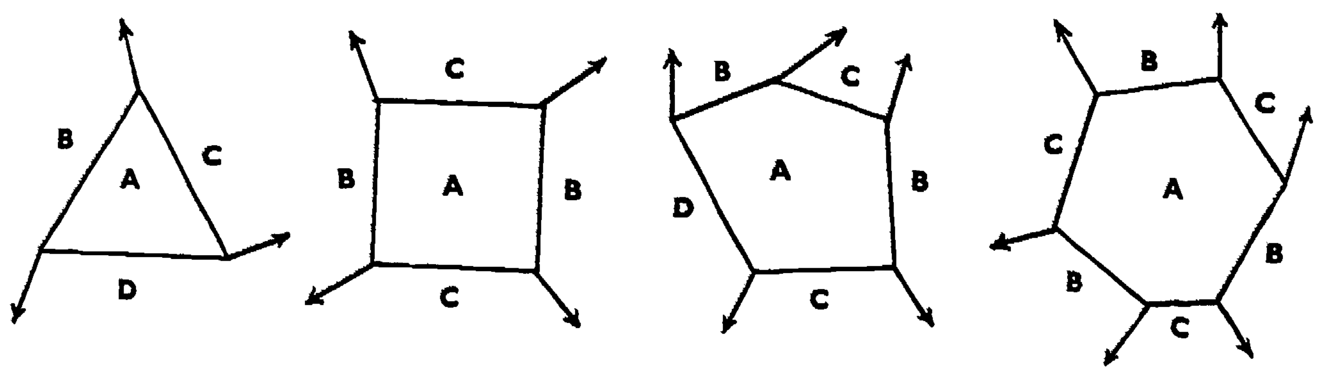
(a) Development, through questioning, of the idea that maps are not colored to establish characteristic colors for countries or to make them attractive, but to facilitate recognition of the boundaries of each country.

(*b*) The question arises as to how many colors are needed to achieve this with a complex map. Discussion of how a printer achieves a finished color plate and the relation between the number of colors used and the expense make the finding of a minimal number of colors of interest. For the child, a more practical question of whether he can color any map with blue biro, black lead, and red biro is equally motivating.

(*c*) Assignments relating to the coloring of maps, graded for complexity, should lead to the discovery that no more than four colors ever seem to be needed.

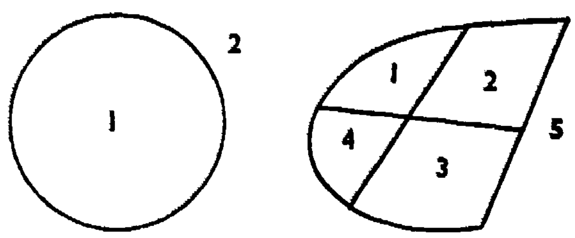
(*d*) Can the children make up a map in which more than this number of colors will be needed ?

(*e*) Use of polygons and surrounding regions should establish some kind of logical certainty about the previous result.

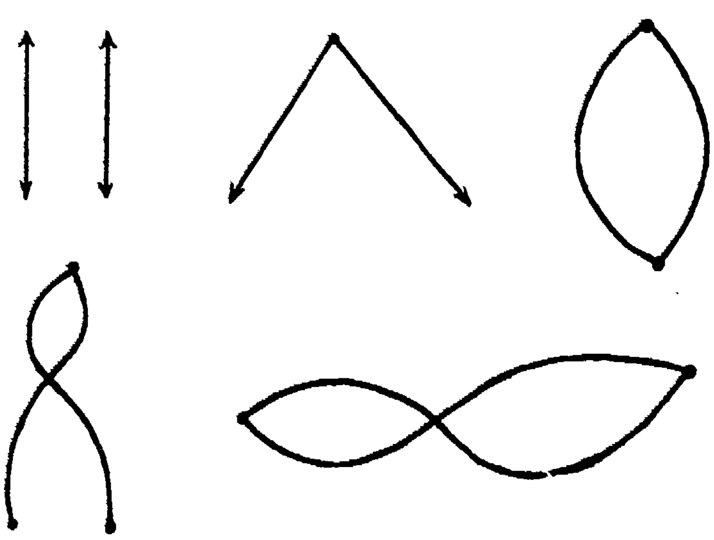


4. Closed Curves and Regions

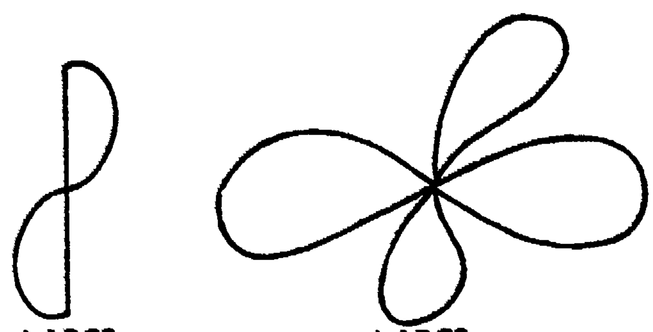
(*a*) Development of the idea that lines partition a plane into regions.



(*b*) Do two lines always partition the plane into the same number of regions ? What are the possibilities for two rays ? For two line intervals ? For other combinations of elements.



(*c*) Appropriateness of using the word "arc" to signify parts of lines or curves between vertices (nodes, points of intersection).



Can you construct diagrams which show four arcs dividing the plane into two regions? Into four regions? Into six regions?

5. Surfaces

(*a*) Can a piece of paper be manipulated to have only one surface?

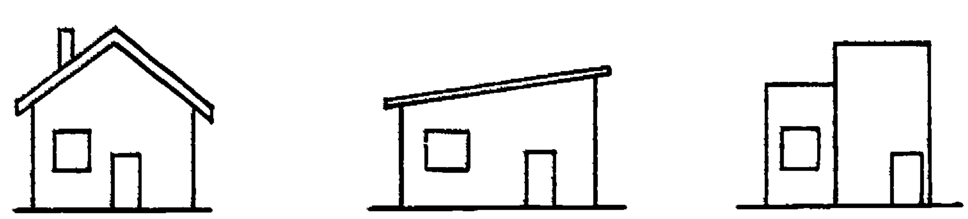
(*b*) The division of strips of papers into differing types of strips by cutting along lines parallel to the edges after formation of a Möebius strip.

(*c*) Study of the effect of twisting the strip more than once before joining the ends. Generalisations associated with this.

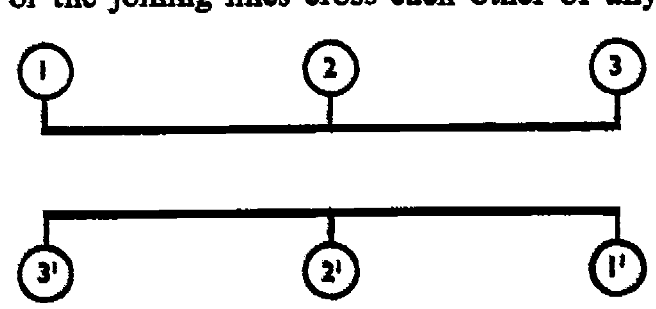
(*d*) Classification of figures with a single closed surface into singly connected figures, doubly connected figures, and so on.

6. Puzzles

(*a*) Is it possible for each house to be connected to gas, water, and electricity supplies in such a way that no pipes or wires cross on the map?



(*b*) Can corresponding numbers (1 and 1^1; 2 and 2^1; 3 and 3^1) be connected in such a way that none of the joining lines cross each other or any of the existing lines?



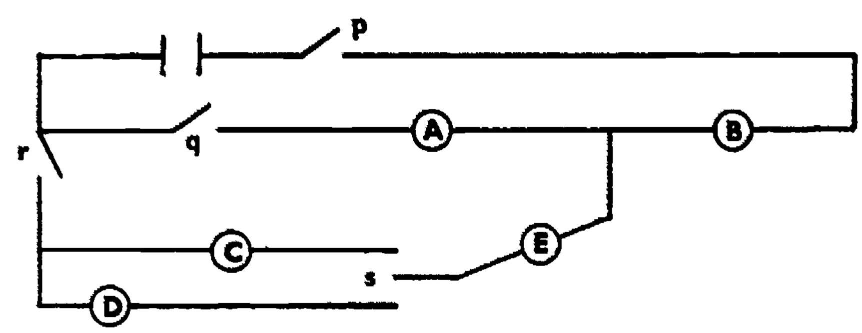
The two problems above can be discussed as relevant to the design of printed circuits.

(*c*) **A, B, C, D, and E represent lamps ; p, q, and r are switches ; and s is a two-way switch.**

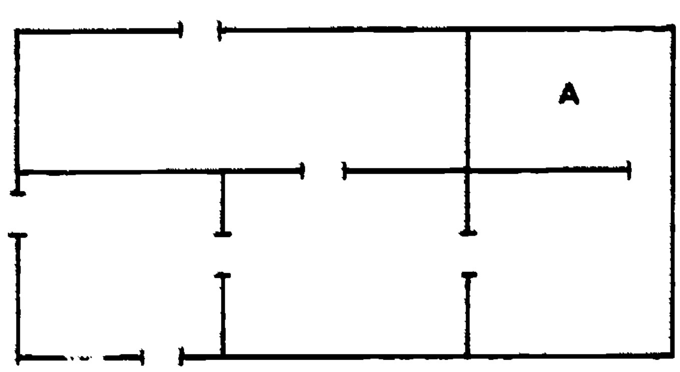
Write down, if possible, an arrangement of switches under which—

(i) A and B are on ; C, D, and E are off.

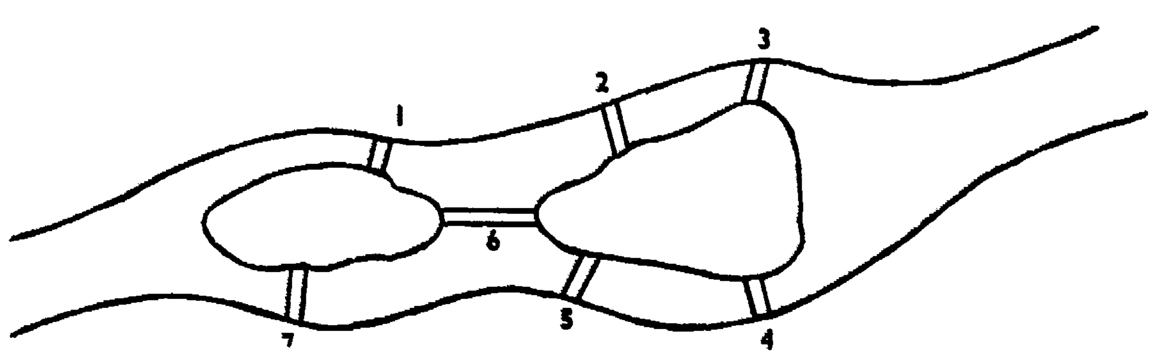
(ii) C, E, and B are on ; A and D are off.



(*d*) The figure shows a plan of a house. Can you start at A and walk through every door of the house exactly once ? Can you start anywhere else and successfully go through each door only once ?



(*e*) The town of Koenigsberg is situated near the mouth of a river; part of it is built on two islands. The islands and the mainland are connected by seven bridges. Is it possible to go for a walk and return to the starting-point having crossed each bridge only once ?



(*f*) Tie a piece of string to each of your wrists. Tie a second piece of string to each of the wrists of a partner in such a way that the second string loops the first. Can you separate yourself from your partner without cutting the string, untying the knots, or taking the string off your wrists ?

CURVE STITCHING

As a guide to teachers, the approximate number of periods spent on this topic would be about four to eight. Most of the actual curve stitching is done at home by the students. It is envisaged that class time will be spent in discussing techniques, designs, the finished designs, and the use and the application of the students' curves. In the early stages it might be necessary to actually use class time to demonstrate to the students or, perhaps, to let the students do some of the work at school.

A curve may be regarded as the path traced out by a " point " which is moving continuously with regard to position and direction. For present purposes we are going to regard a curve as the envelope of its tangents. By tangents, we mean a series of lines, all drawn to satisfy some given condition. The envelope curve will be the curve touched by each tangent.

To illustrate the above, take two rays at right angles to each other and mark fifteen points, each the same distance apart, on each ray as shown in Figure 1.

Using a ruler and pencil proceed as follows :

(*a*) Join the first point on the vertical line to the last point on the horizontal line.

(*b*) Join the second point on the vertical line to the second-last point on the horizontal line.

(*c*) Continue until the last point on the vertical line has been joined to the first point on the horizontal line. See Figure 2.

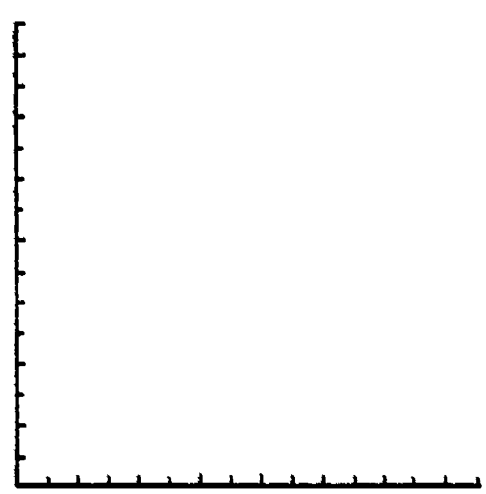


FIGURE 1

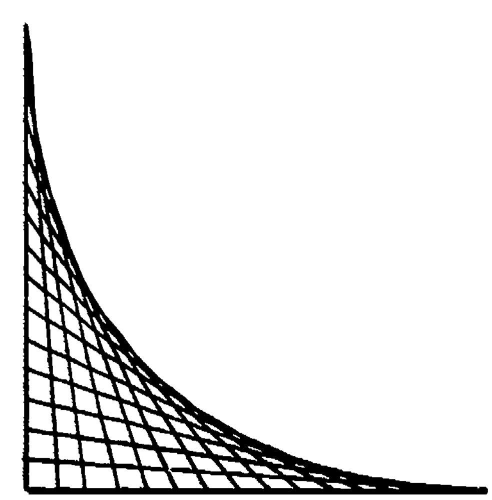


FIGURE 2

All the straight lines drawn are tangents to the curve formed, since they all touch the curve. The curve formed in this way is an *envelope* of the family of tangets. The above method is all that is needed as an introduction to the exciting and fascinating study of curves.

Materials Needed

Needle, cotton, and surface board.

A board well suited for this type of work is 4-sheet surface board. This surface board (size 20 in. by 25 in.) is available in many different colors.

For the first experiment, white cotton on black surface board is suggested. One sheet of surface board set out into quarters is an ideal size to produce a first, beautiful design.

Hints on Procedure

1. Rule initial lines lightly in pencil.
2. Carefully mark the position of each point.
3. Pierce each point in the surface board with a needle before commencing sewing. Pierce the holes from the surface, not from the underside, as white flecks often appear (especially when using black cardboard) which detract from the beauty of the finished design.

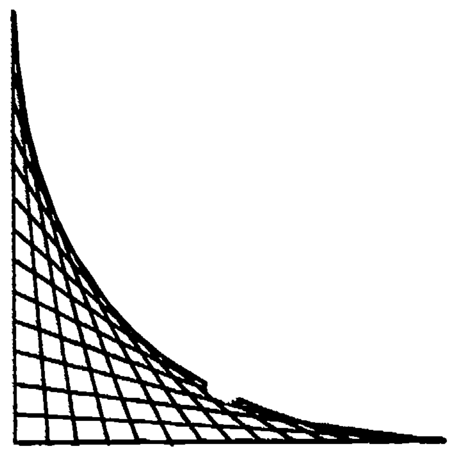


FIGURE 3

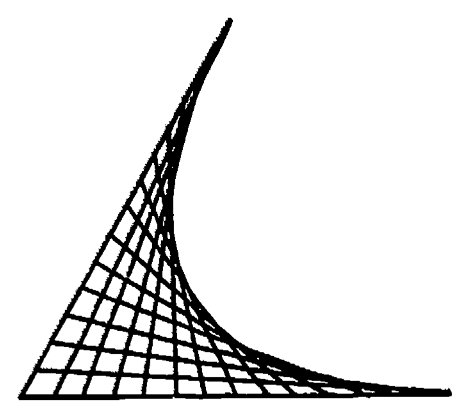


FIGURE 4

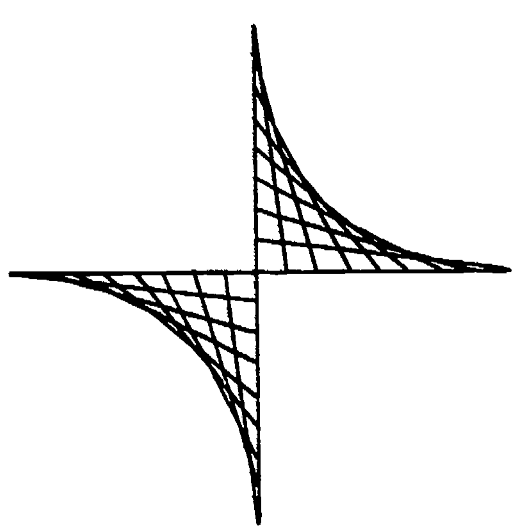


FIGURE 5

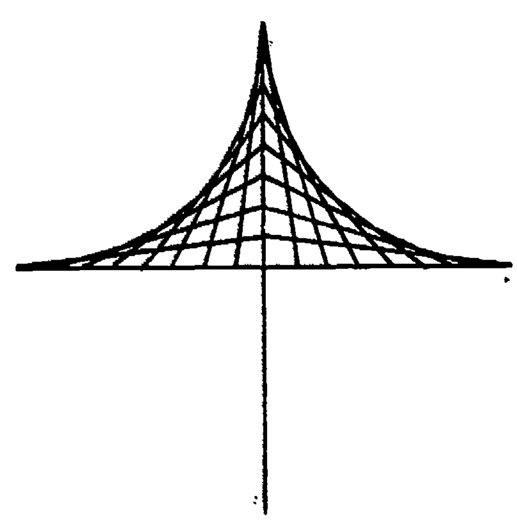


FIGURE 6

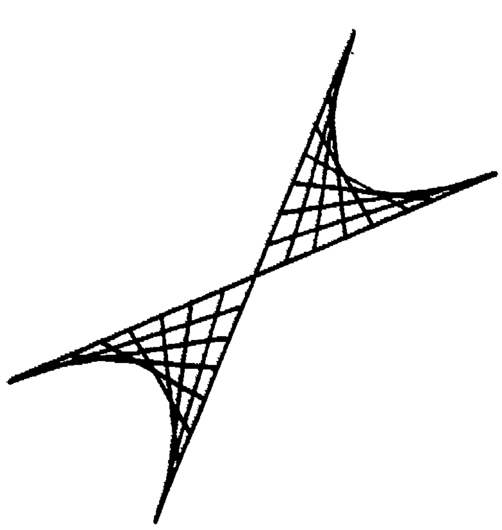


FIGURE 7

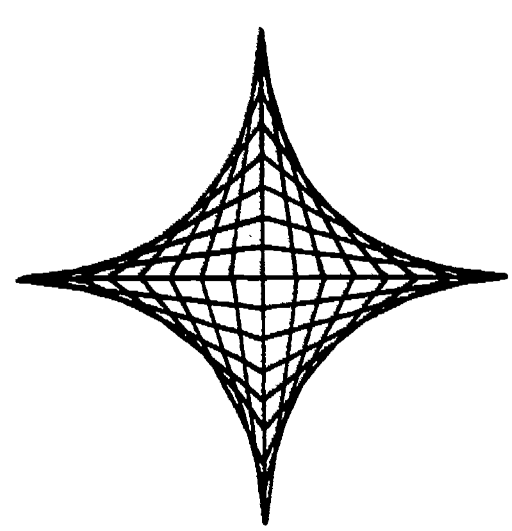


FIGURE 8

4. Cover the completed design with either clear plastic or self-adhesive clear plastic to form a permanent and protective cover.

5. Minimise the amount of cotton used by avoiding long stitches on the back of the design. After finishing one stitch on the front of the design the needle should, in nearly all cases, be passed through the next adjacent hole.

Allow approximately 4 weeks for the following assignments to be completed.

Assignment 1

Construct and present four completed cotton designs :

1. A simple design, white cotton on black card.
2. A more complex design, still in black and white.
3. A simple design in multi-color variegated embroidery cotton. This produces most pleasing designs.
4. A more complex multi-colored design, mounted and covered.

Assignment 2

Complete a curve stitching design and present mounted and covered (if possible). This design should be done on a full sheet of surface board 20 in. by 25 in.

Assignment 3

Using 4-sheet surface board (for best results), 10 in. by 12 in., construct the curve stitching design illustrated opposite. ("Slipping ladder" curves, with axes at right angles or inclined. See Figures 3, 4, 5, 6, 7, 8.)

Assignment 4

1. Can you find a name, or names, for any of the illustrations shown in Assignment 3 (Figures 3 to 8) ?

2. What general effect takes place when the angle between the two axes is decreased ? Example, Figures 3 and 4.

3. In Figure 3, what would be the locus of the mid-point of each line ?

4. What is the name given when two curves meet as illustrated in Figure 8 ?

Assignment 5 : *The Cardioid*

The cardioid (more commonly known as a heart shape) may be constructed as follows :

Step 1. Draw a circle and mark a point on the circumference.

Step 2. Starting at the point marked on the circumference, mark off further points on the circumference at intervals of 5° in a clockwise direction.

Step 3. Number each point starting with 0 as illustrated in Figure 9.

Step 4. Join the points as follows :—1 to 2, 2 to 4, 3 to 6, 4 to 8, x to $2x$, and so on until 71 is joined to 142.

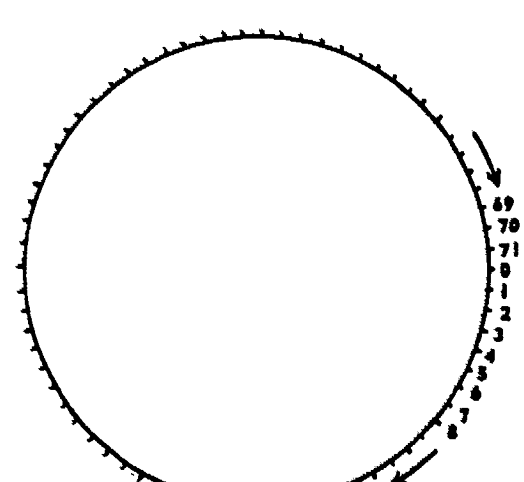


FIGURE 9

Assignment 6 : ***The Deltoid***

The deltoid is constructed as follows :

Step 1. Draw two concentric circles with their radii in the ratio 1 : 3 and mark in one diameter.

Step 2. Starting with one extremity of a diameter of the inner circle, mark off points around the circle in an anticlockwise direction and number them 0, 1, 2,... 71, i.e. the points are 5 degrees apart.

Step 3. Starting at the other extremity of the same diameter of the inner circle, and moving in a clockwise direction number points at intervals of 10 degrees, also starting with 0 and continuing to 71 (i.e. twice around the circle).

The construction so far would appear as in Figure 10.

Step 4. Join points with the same number and extend to the outer circle.

N.B. A more complete diagram of the construction of the deltoid is found in BLAKEY, M. : *Modified Mathematics* (Cassell, 1968), page 81.

Note—When constructing a deltoid in cotton, the above construction would be done lightly in pencil except for Step 4 where points on the outer circle are marked with numbers as illustrated in Figure 11. A large circular protractor is most convenient in this construction.

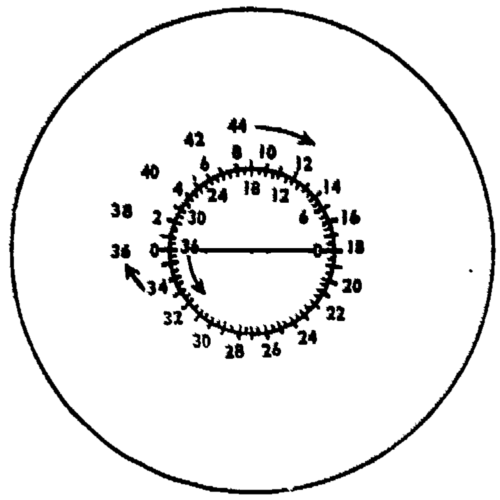


FIGURE 10

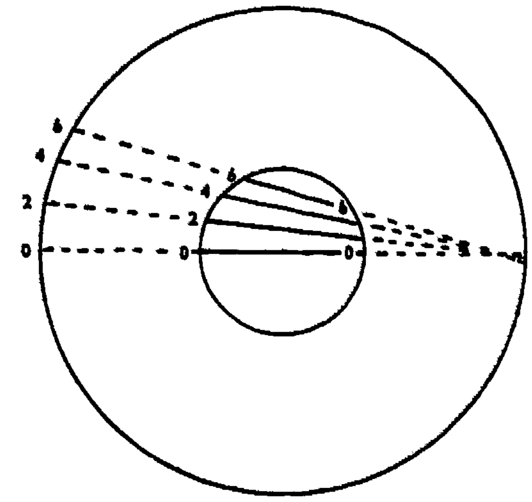


FIGURE 11

Several points are very close together. When stitching this could cause difficulty, so it would be a good idea to place cellotape (on the back of the board) over these holes.

Assignment 7

1. Construct a cardioid with pencil, ruler, protractor, and compass.
2. Construct a cardioid in cotton.

Assignment 8

1. Construct a deltoid using, pencil, compass, ruler, and protractor.
2. Construct a deltoid in cotton.

Assignment **9 (for advanced students)**

1. Construct a double cardioid.
2. Construct a triple cardioid.
3. Construct a triple cardioid together with a deltoid.

N.B. Each of these should be drawn on paper first.

Reference

BLAKEY, M. : *Modified Mathematics* (Cassell, 1968).

PROBABILITY

References

1. *Exploring Mathematics on Your Own Series—Probability and Chance* (John Murray).
2. A. J. SHERLOCK : *Contemporary School Mathematics—Probability and Statistics* (Edward Arnold, 1964).
3. BERYL HUME : *An Introduction to Probability and Statistics*, University of W.A. (Longmans, 1966).

Reasons for Including the Topic

1. The growing use of statistical methods in the world demands that every school-leaver should have some knowledge of sampling techniques.

2. The modern trend towards statistical description in terms of averages requires a knowledge of probability for the meaningful interpretation of broad statements expressed numerically.

3. The topic is interesting, with many applications in games.

4. The topic is appropriate to the third-form level, where experiments can be used to develop intuitive ideas and general interest in mathematics.

5. The topic provides a good example of what is meant by a "mathematical model"—its strengths and its weaknesses. The ease with which predictions from the model may be checked make the topic valuable for this reason alone.

Suggested Development

1. Randomness.
2. Order from disorder—relative frequency and stability.
3. Properties of relative frequency.
4. Predicting frequencies—an introduction to probability.
5. The mathematical model—calculating probabilities.
6. Special events.

Some Experiments

Experiment 1 : *Randomness*

To develop intuitive ideas about randomness—the unsystematic and unpredictable order in which the possible *outcomes* of an experiment occur.

The experiment consists of reading the last digit of the first 100 numbers from a table of figures (logarithms, square roots, random numbers, or the like), and recording the results in a *frequency table*. As each digit occurs, a stroke is placed in a tally column in the row beginning with that digit. Each fifth stroke is drawn horizontally to group the strokes into 5s for easy counting.

Last Digit	Tally	Frequency
0	~~1111~~ 1	6
1	1111	4
2	~~1111~~ 111	8
.	.	.
.	.	.
9	111	3

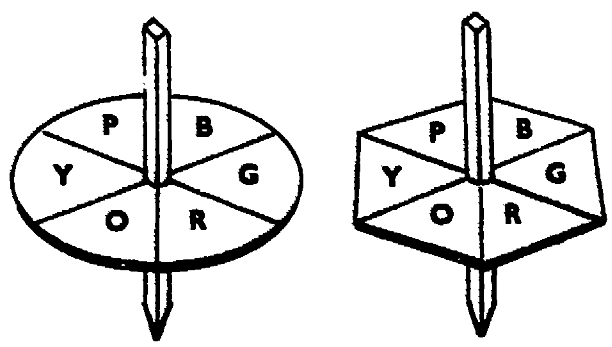
The concept of randomness is illustrated by the way in which the tally marks "jump" around as they are recorded—it is impossible to predict where the next mark will appear.

Other similar experiments can be performed, e.g.

(*a*) rolling a die,

(*b*) using a *random selector disc*.

Random selector discs are simply cut out of cardboard and pierced centrally with a sharpened match. Differently colored regions of the disc provide different outcomes when the disc is spun. A useful variation is to trim the cardboard to a regular hexagon, (or octagon etc.) as this produces greater discrimination and speed. These discs are easy to make, more readily available than other aids, and less noisy to use than dice or playing cards.



Another interesting exercise would be to allow students to work out a distribution of letters of the alphabet in a number of passages of literature of varying lengths. From these distributions a possible layout of a typewriter keyboard could be prepared for comparison to an actual keyboard.

Experiment 2 : *Relative Frequency*

In this experiment, the students discover that in a random experiment the outcome in any one trial is unpredictable, but in a very large number of trials some pattern does emerge.

(i) **The students construct a disc that is half black.**
(ii) The disc is spun 500 times, the results being tallied as in Experiment 1.
(iii) After 5, 10, 20, 50, 100, 200, 300, 400, 500 spins, the *relative frequency* (proportion of trials in which black arose) is calculated.
(iv) The relative frequency (f) is plotted against the number of trials.
(v) The shape of the graph is analysed, and the different graphs obtained by the groups of students are compared.

Total Number of Spins	Cumulative Frequency of Black		Relative Frequency of Black
5	111	3	$\frac{3}{5} = 0.6$
10	~~1111~~	8	$\frac{8}{10} = 0.8$
20	1111	12	$\frac{12}{20} = 0.6$
.		.	.
.		.	.
.		.	.

The observations by the students could include :

(*a*) In any number of trials, the relative frequency (f) is such that $0 \leqslant f \leqslant 1$ (a consideration of extreme results for small numbers will show the possible extremes).

(*b*) For a large number of trials, the graph tends to level out at a value of 0.5, which approximates to the proportion of black on the disc. Any divergence from this value provides the substance of valuable class discussion. This might concern imperfections, which result in systematic bias.

Thus there is an "ideal" value, equal to the proportion of black on the disc, which the observed relative frequency is "trying" to equal (but chance variations will affect the result). This "ideal" value we can term the *probability* of black.

Experiment 3 : *Prediction*

In this experiment, the students list their hypotheses about proportion (probability) and the observed relative frequency.

1. Each group of students constructs a spinner in which a different fraction is colored black—for example, using a hexagon shape, the fractions $\frac{0}{6}$, $\frac{1}{6}$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{4}{6}$ and $\frac{5}{6}$ are possible.

2. The predicted relative frequency is calculated from the proportion of black on the spinner and entered in a table.

3. Each disc is spun 200 times and the observed relative frequency is entered in a table.

4. The predicted values are compared with the observed values.

Proportion of Black	Predicted Relative Frequency	Frequency of Black (200 spins)	Observed Relative Frequency
0	0		
$\frac{1}{6}$	$\frac{1}{6}$		
$\frac{1}{3}$	$\frac{1}{3}$		
$\frac{1}{2}$	$\frac{1}{2}$		
$\frac{2}{3}$	$\frac{2}{3}$		
$\frac{5}{6}$	$\frac{5}{6}$		
1	1		

The students will be able to see that estimates of the relative frequency can be made. Examples could be given which reverse this procedure—estimating the nature of the disc from a set of observed frequencies.

Experiment 4 : *Equally Likely Outcomes*

In this experiment, the students investigate equally likely outcomes and find that the sum of the relative frequencies is always unity.

1. Each group colors each segment of a hexagonal spinner a different color.

2. The frequency with which each color turns up in 200 spins is recorded.

3. The relative frequency of each color is calculated for each group.

4. The frequencies for the whole class are added into a class table, and the relative frequencies calculated, together with the sum of these.

The students will be able to see that if there are n equally likely outcomes of an experiment, the relative frequency of each is approximately $\frac{1}{n}$. In addition, the sum of the relative frequencies is 1.

The Mathematical Model

We now wish to construct a mathematical model which closely approximates to our observations in each experiment.

1. Any experiment has a number of possible outcomes. We refer to the set of these possible outcomes as the *outcome space*. The elements of the set (i.e. the outcomes) are called *sample points*.

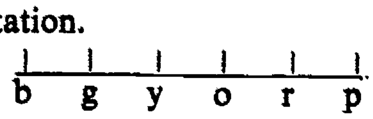
The outcome space (or universal set) S may be specified :

(i) As a listed set, e.g. S = {blue, green, yellow, orange, red, pink}.

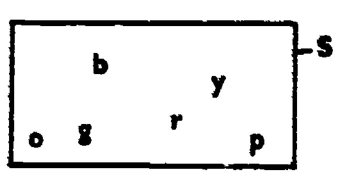
(ii) As a described set.

(iii) In set builder notation.

(iv) As a graph, e.g.



(v) As a Venn diagram, e.g.

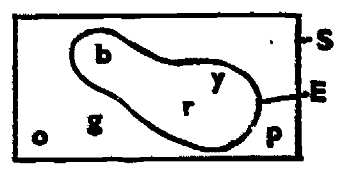


2. To complete our model we assign to each outcome a number which is the best estimate of the proportion of times that outcome is expected to occur. This is the probability of that outcome. If each of n outcomes is equally likely, then the probability of each is $\frac{1}{n}$.

3. Thus our mathematical model is a set of ordered pairs (s, p) in which $s \in S$ and $p \in R$ (where R is the set of real numbers), such that $0 \leqslant p \leqslant 1$ and $\Sigma p = 1$. This set could be represented as follows :

$$\begin{array}{cccccc|c} b & g & y & o & r & p & (s) \\ | & | & | & | & | & | & \\ \hline \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & (p) \end{array}$$

4. An *event* is described as a set of outcomes (i.e. when any one of this set of outcomes occurs in a trial, we say the event has occurred). Alternatively *an event is a subset of the outcome space.*



Example. Suppose we wish to find the probability of a primary color turning up on our spinner.

$$E = \{\text{blue, yellow, red}\}$$

Then the probability of an event E, Pr (E), is given by the sum of the probabilities of the outcomes contained in E.

$$\begin{aligned} \text{Above, Pr (E)} &= \text{Pr (b)} + \text{Pr (y)} + \text{Pr (r)} \\ &= \tfrac{1}{6} + \tfrac{1}{6} + \tfrac{1}{6} \\ &= \tfrac{1}{2}. \end{aligned}$$

This result can be verified experimentally or from the results of Experiment 3.

If the outcomes are equally likely, the probability can be found simply by counting the elements of E and S.

$$\text{Pr (E)} = \frac{n(E)}{n(S)}.$$

Examples

1. Draw Venn diagrams illustrating (*a*) an impossible event ; (*b*) a certain event.

2. Examples on the calculation of probabilities, with verifying examples, are given in HUME : p. 127 ; p. 129, Ex. 1–5 ; p. 132, Ex. 7–8.

3. *Sample spaces :* See HUME : p. 135, Ex. 1–5, 7–10.

4. *Assigning probabilities :* See HUME : p. 139, Ex. 7–12.

RELATIONS, LOCI, AND GRAPHS

The approach described here was designed first to arouse interest where it has flagged, and then to extend this interest and study mathematical topics.

The order summarised here presents a logical development in which figures are established geometrically first, then examined analytically. However, with a little ingenuity, any section may be used by itself.

The approach is also suitable for a " full mathematics " group with, of course, suitable variations in the time spent on constructional work and the depth to which abstract concepts are considered.

Summary of Sections

1. Arousing interest in sets of points with a new constructional exercise.

2. The idea of a geometrical figure as a set of points, a relation, or a locus. The need for a reference system.

3. Geometrical relations or loci with defining conditions involving length of line intervals.

   Two important loci, the sphere and the infinitely long cylinder.

   Corresponding *plane* loci, the circle, and the parallel line-pair.

   Other *plane* loci using—

   (*a*) Finite sets of points—two numerical conditions.

   (*b*) Infinite sets of points—families of circles and line pairs in plotting loci with geometrical relationships such as PB = 2PA as given conditions. Definition of parabola, ellipse, and hyperbola as loci.

4. Cartesian relations in the plane, with specifications involving length and direction of two intervals.

   The Cartesian framework and notations.

   The parabola and the straight line (and perhaps other curves) as Cartesian relations.

5. Polar relations. Specification of direction, and of the distance along it. Some ideas to try out.

SECTION 1

AROUSING INTEREST IN SETS OF POINTS

Select one of the following drawing exercises. Clear instructions are essential and individual assistance may be needed at first. Although exercise 1.3 is simpler, exercise 1.1 probably captures more interest. Choosing different exercises for different classes and years also provides greater interest.

Exercises

1.1 Draw a line L down the c .tre of your page and mark a point C about $\frac{1}{2}$ in. from L. (These given points and lines could be emphasised by using a blue biro.) See Figure 1.1 (*a*). Prepare a strip of cardboard with a straight edge $\overline{xy}$ approximately 10 in. long (paper, wood, or even the edge of a book will do quite well). Mark (with blue biro) a particle of the strip at about the centre of $\overline{xy}$; name it z. Place a piece of plasticine on $\overline{xy}$ about 2 in. from z ; name it k (a piece of adhesive tape or even a red mark will do). See Figure 1.1 (*b*).

Now choose a point Q on L near the top of the page, place and hold z there while moving the strip around until $\overleftrightarrow{xy}$ passes through C. With a red biro, mark the position P_1, which particle k occupies *on the paper*. See Figure 1.1 (*c*).

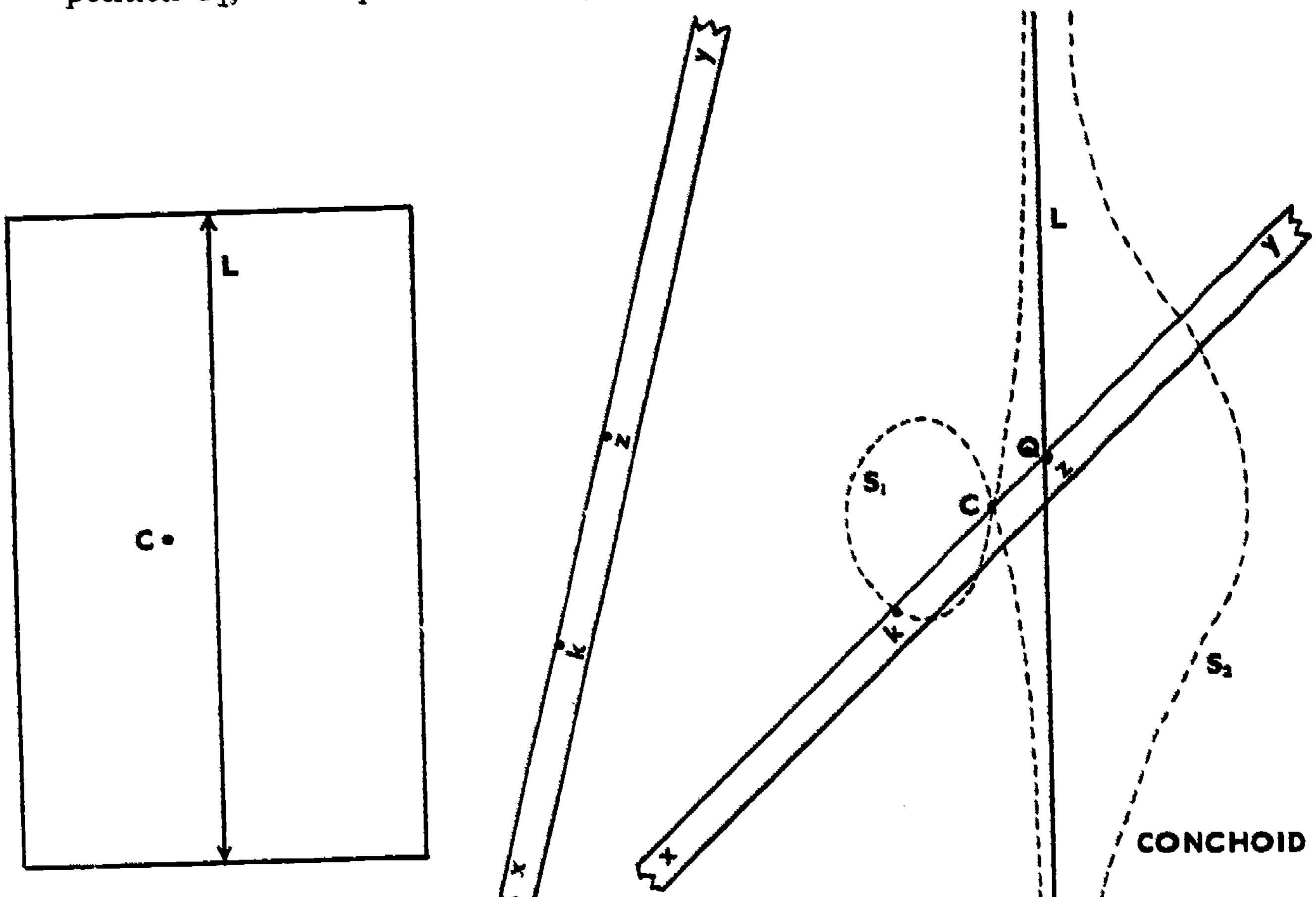


FIGURE 1.1 (a) FIGURE 1.1 (b) FIGURE 1.1 (c)

Then choose another point on L a little lower down and repeat to find the position P_2, which position particle k *now* occupies on the paper. Move z along L in easy stages (" blue on blue, red on red " helps in the instructions) marking the positions on the paper taken by particle k at each stage ; name them P_3, P_4 ... Mark many such points. Can you suggest a way of showing *all* such points without spending the rest of your life constructing ? Test that two of the points you have guessed *do* satisfy the above instructions.

This set of points S_1 forms part of a curve called a *conchoid* (dotted line in the sketch). Another section S_2 may be constructed by turning $\overline{xy}$ through 180° when z reaches the bottom edge of the page, enabling points on the other side of L to be plotted.

The whole conchoid may be imagined if you think of $\overline{xy}$, the page, and L becoming infinitely large.

Extensions

Individuals could try another exercise with C at a different distance from L ; others might try k at a different distance from z. Compare the results informally.

1.2 Draw a circle L (approximate radius length 1½ in.) with centre O (use blue biro). Mark a point C of the circle (with blue).

Now follow the same instructions as in Exercise 1.1 : " Prepare a strip of cardboard....satisfy the above instructions." Note that the particle k moves twice around the circle before the pattern repeats.

This set of points forms a curve called a *limaçon*. You may have met this figure before if you have been adventurous with circle patterns.

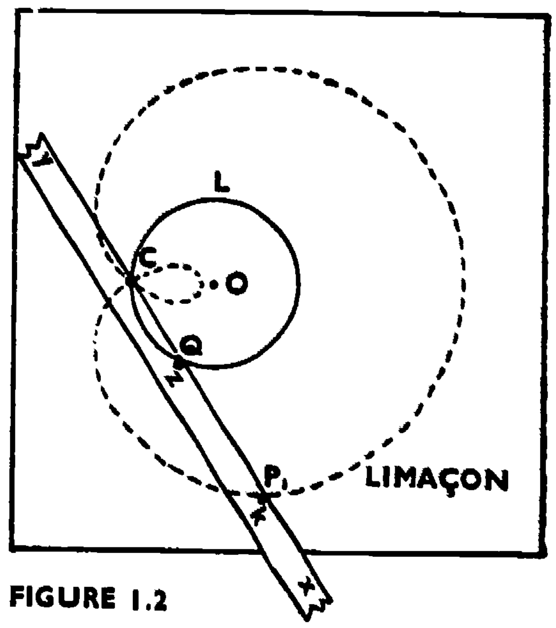


FIGURE 1.2

Extensions

The size of the circle L could be altered.

The distance zk could be changed.

The point C could be outside the circle or inside the circle.

Discuss the possibility of C not being in the plane of the paper.

For a limaçon without a loop, use radius length ½ in., zk = 2 in. ; for a special limaçon (also called a cardioid) use radius length 1 in., zk = 2 in.

1.3 Name one corner of a fresh page BOA, using blue biro as in Figure 1.3 (*a*).

Prepare a strip of cardboard with a straight edge $\overline{xy}$ approximately 6 in. long. Divide $\overline{xy}$ into three approximately equal parts and place a piece of plasticine or red adhesive tape on the particle at the point of division nearer to x ; name it k, as in Figure 1.3 (*b*). Now place the strip with x on $\overrightarrow{OA}$ and y on $\overrightarrow{OB}$. With a red biro mark the position P_1, which particle k occupies on the paper. See Figure 1.3 (*c*). Move the strip about, keeping x on $\overrightarrow{OA}$ and y on $\overrightarrow{OB}$, marking other positions P_2, P_3, P_4, . . . of the particle k. (You can imagine k an ant on a falling ladder xy if you wish !)

This set of points is part of an ellipse. A complete one can be obtained by using intersecting perpendicular lines, $\overleftrightarrow{AOA'}$, $\overleftrightarrow{BOB'}$, in the centre of the page, and a smaller strip, perhaps 3 in. long. See Figure 1.3 (*d*).

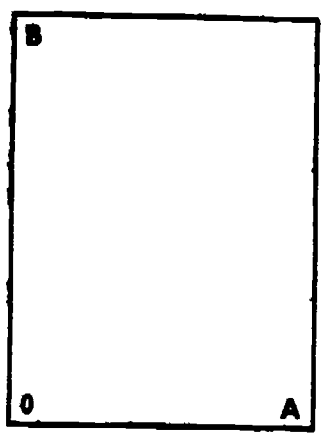


FIGURE 1.3 (a)

FIGURE 1.3 (b)

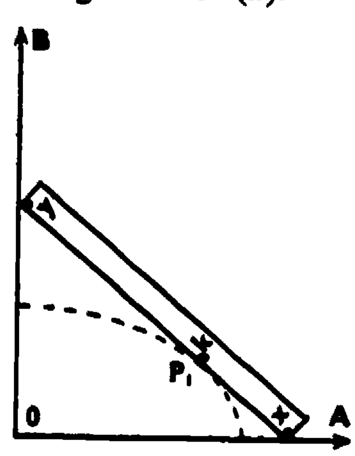


FIGURE 1.3 (c)

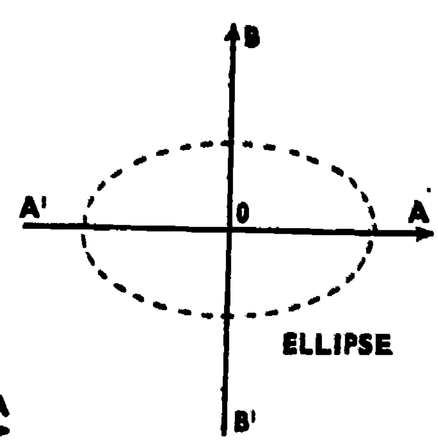


FIGURE 1.3 (d)

Extensions

A strip of a different length may be used.

The distance of k from x may be altered.

The lines $\overleftrightarrow{AA'}$ and $\overleftrightarrow{BB'}$ might be non-perpendicular.

Discuss the possibilities if the lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$ do not intersect, i.e. the lines are parallel or they are skew lines.

1.4 Name one corner of a fresh page AOB (use blue biro). Prepare a strip of cardboard with a straight edge $\overline{xy}$ approximately 6 in. long. Color all particles on the edge $\overline{xy}$ with red.

Now place the strip with x on $\overrightarrow{OA}$ and y on $\overrightarrow{OB}$. Mark the positions now occupied *on the page* of all the particles making up $\overline{xy}$ (it will actually be a line interval, but a series of dots shows more clearly that it is also a set of points). Repeat for many positions of $\overline{xy}$.

Show, by shading, where all the points are expected to lie. See Figure 1.4 (*a*).

This set of points, S, is part of an *astroidal region* (see Note 2 below). A complete region can be obtained by using intersecting perpendicular lines, $\overleftrightarrow{AOA'}$, $\overleftrightarrow{BOB'}$, in the centre of the page, and a smaller strip about 3 in. long. See Figure 1.4 (*b*).

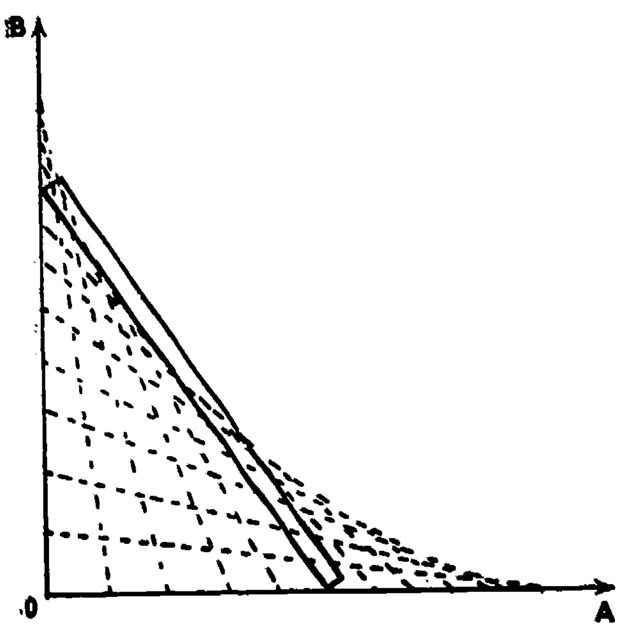


FIGURE 1.4 (a)

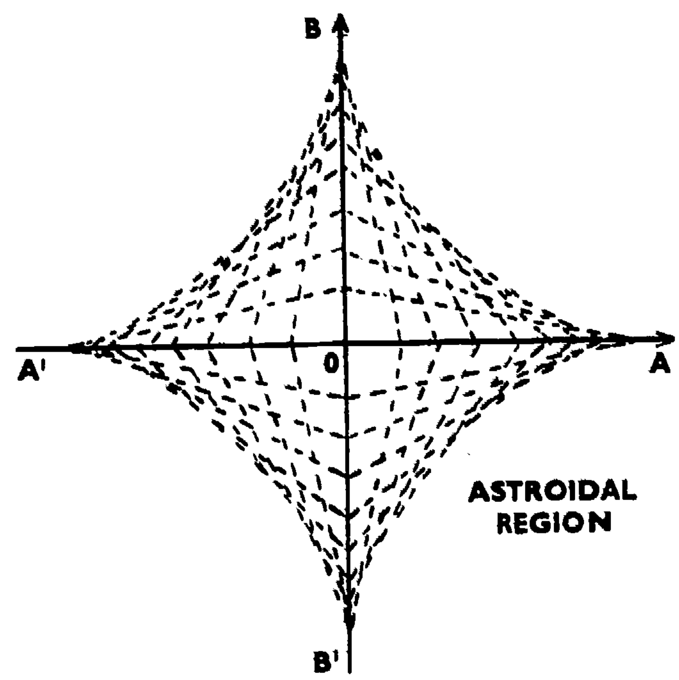


FIGURE 1.4 (b)

Extensions

A strip of a different length may be used.

The lines $\overleftrightarrow{AA'}$ and $\overleftrightarrow{BB'}$ may be at a different angle.

Discuss, or even construct, a model with dowel and string, when $\overleftrightarrow{AA'}$ and $\overleftrightarrow{BB'}$ are skew lines.

Try replacing AOB by a circle. Thin sticks of equal length (e.g. matches) may be glued into position in any of these models.

Notes

1. **The term " moving point " will not be used here. A point has no " thickness " ;** it has only position, and just one position to each reference system. Instead we shall think of a tiny part or particle (alternatives are a very small body, a pin-point body, or a sizeless body—corresponding to a weightless rod in mechanics). This gives a more concrete approach, particularly at this level, and " the less the size the better " in plotting suggests informally the limiting concepts involved.

2. Geometrical " solids " and plane closed geometrical figures which are made up of the interior *and* the boundary points will be referred to here as " regions ".

Examples

A pyramid will consist of the sets of points forming its plane surfaces.

A pyramidal region consists of the sets of points forming the surfaces and the interior.

A circle will be the set of boundary points. (The word " circumference " will be used to correspond to perimeter, a length measurement.)

A circular region or disc will refer to the boundary and the interior points enclosed by a circle.

A sphere will be the set of points forming the curved surface of a spherical region.

SECTION 2

GEOMETRICAL FIGURES AS SETS OF POINTS

(Relations ; Loci)

Reference System

Consider the sets of points in Section 1. Notice the need when stating position to say to what body or system you are referring. (See Note 1 above regarding " moving point ".)

For example, in Section 1, Exercise 1.1, if referring to the strip of cardboard, particle k has just one position ; however, if you are referring to the page, the positions of k form a conchoid curve.

The instructions must say clearly (although informally) to what system to refer. By convention, if no reference system is stated, reference to the earth in our " space " is assumed.

Geometrical Figures

Examine objects that can be seen and/or handled, e.g. a box, a water pipe, the building called the " Myer Music Bowl ". Consider the set of points that describe the positions of particles of material which form the objects. Introduce the term geometrical figure (" figure " if context is clear) as the set of boundary points which form a shape such as a pyramid, a square, or a sphere (see Note 2 at the end of Section 1), the reference system here being a part of the figure. Notice that if the reference system is our earth, different sets of points may, in fact, describe *congruent* geometrical figures.

Defining Conditions

Some geometrical figures are well defined by an expression such as, " the set of points for which . . . " where the expression quoted states the instructions or the conditions (including the reference system) under which the points belonging to the figure may be collected.

Examples

1. The sphere, centre O, radius length 2 in., is clearly defined by the statement. " the set of points at a given distance 2 in. from a given point O ", or in the symbols, $\{$points $P : OP = 2$ in.$\}$. The reference system is the point O.

2. The circle, centre O, radius length 2 in., is defined by including an extra condition that the set of points be confined to a plane containing O.

3. For the astroidal region S in Exercise 1.4 (*b*), the defining conditions are the instructions given there ; or they may be stated, " each point is a position of particle k situated on straight edge $\overline{xy}$ when x lies on the given line $\overleftrightarrow{AA'}$ and y on the given line $\overleftrightarrow{BB'}$. In symbols, $S = \{P : P \in \overline{xy}, x \in \overleftrightarrow{AA'}, y \in \overleftrightarrow{BB'}\}$. Reference system consists of lines $\overleftrightarrow{AA'}$ and $\overleftrightarrow{BB'}$.

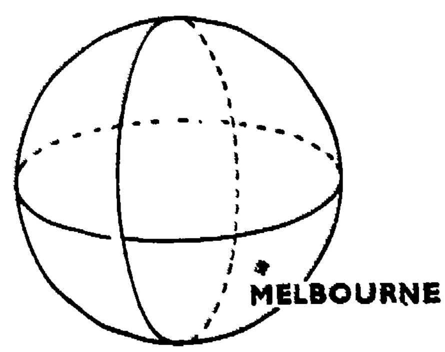
4. The defining conditions for the limaçon of Exercise 1.2 are lengthy. The condition that points are confined to the plane of the paper need not be stated as the choice of C in the same plane as L, since the reference system automatically does so.

Terms

(*a*) *Relation :* These conditions *relate* the position of each point to the reference system previously decided upon. The above expression, " the set of points for which .... ", may be called a *relation.* Sometimes we say the conditions locate the position of each point.

(*b*) *Locus :* This may be demonstrated with a globe or a model of the earth (assumed spherical), noting—

(i) that the surface consists of a set of points ;
(ii) that Melbourne is located " here, at this point " ;
(iii) what instructions are required when constructing a model of the locations or positions of the set of points?



The term *locus* as " the set of all points (locations) satisfying certain given conditions " is another name for the expression above— " the set of points for which .... ". Notice that, in collecting a set of points, we need to check (although often informally) that we have all the points which belong, and that there are no extra points which do not belong.

Use

Many fascinating figures can now be drawn or constructed using a relation or locus definition. Select one only, as there are others more important in Sections 3, 4, and 5. Ask the students for informal statements of the reference system being used, introduce the names of the geometrical figures for interest, and notice that the instructions saying how to construct are generally lengthy and can seldom be shortened.

In Sections 3, 4, and 5 we study ways to write simpler conditions and also study the loci or relations which are certain geometrical figures.

Exercises

2.1 Rolling disc exercises, i.e. plotting the set of points in a plane taken by a particle fixed on a disc which rolls according to given instructions.

Draw a line L on the page (use blue biro). Prepare a cardboard disc, marking (in red) a particle k on the boundary. Place the disc on the right-hand side of and just touching L at Q_1. Mark position P_1 (in red) which k now occupies on the paper. Roll the disc along L a little, but *without any slipping*, and plot the new position, P_2, of k.

Repeat for many such positions of the disc. (k might be regarded as a spot of mud on a bicycle wheel as the bicycle travels along a straight road L.) Show {points P on the paper such that the conditions given are satisfied}. The curve is called a cycloid. See Figure 2.1 (*a*). (L could be drawn down the crease of a double page, the disc being $\frac{3}{4}$ in. radius length. An old toothed wheel makes a good disc. Later, a small hole at k helps.)

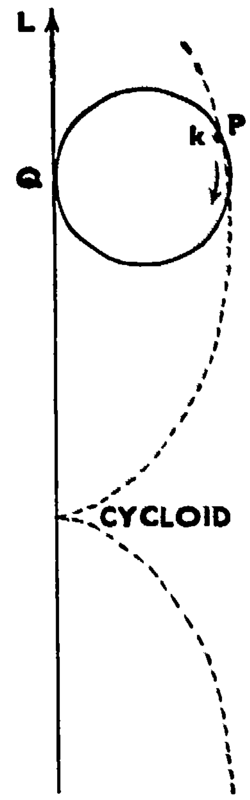


FIGURE 2.1 (a)

Extensions

(*a*) Replace the given line by a circle L. Roll the disc without slipping as follows :

(i) On the outside of L. The curve is an *epicycloid*. See Figure 2.1 (*b*).

(ii) On the inside of L. The curve is a *hypocycloid*, See Figure 2.1 (*c*).

NOTE.—The rolling circle radius length is $\frac{2}{5}$ that of the fixed circle.

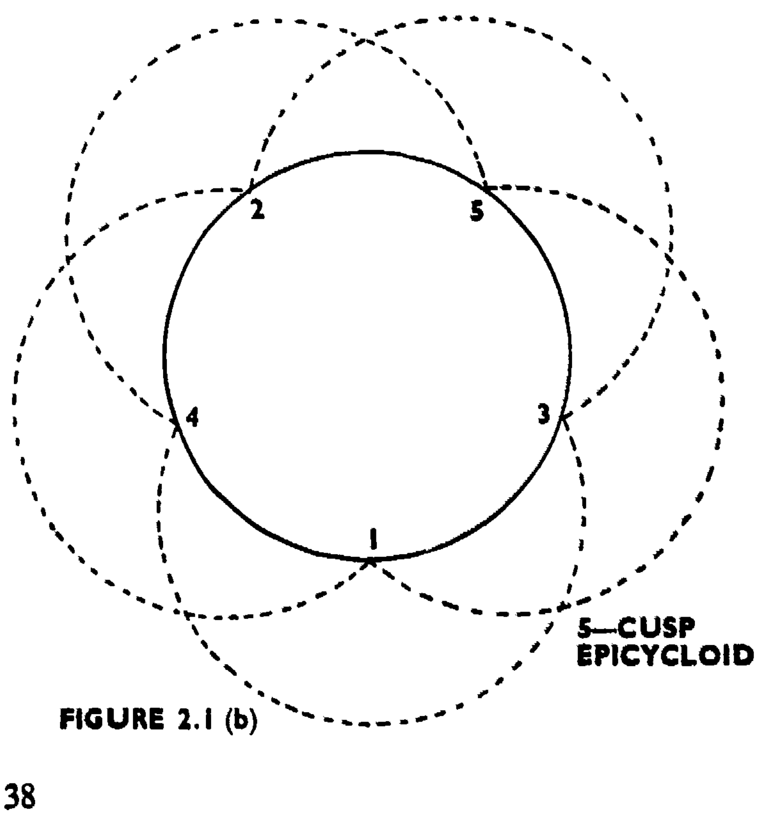
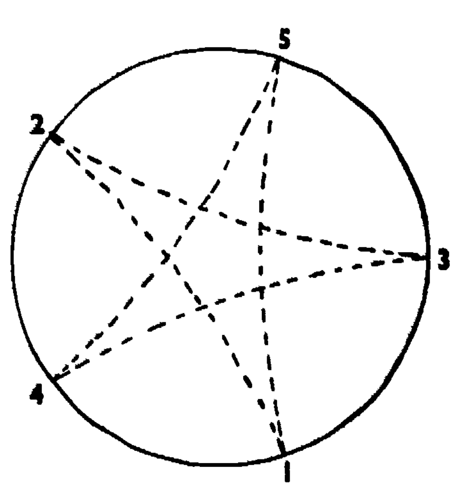


FIGURE 2.1 (b)



HYPOCYCLOID

FIGURE 2.1 (c)

The " Spirograph " toy uses exactly this principle to construct a hypocycloid ; teeth on the disc and on L prevent slipping.

After constructing as above, students can see the toy as a device for constructing geometrical loci.

(*b*) Replace the given line by any curve L.

(*c*) Make k an interior particle of the disc in any of the above exercises.

2.2 Families of circles exercises, i.e. plotting the set of all points belonging to all circles which satisfy certain given conditions.

Draw a circle C, radius length 1½ in., and a line L passing through its centre. Choose a point Q of the circle C. Construct a circle with centre Q which just touches line L at one point.

Collect all of the points belonging to all such circles and you have a *nephroidal region*. See Figure 2.2 (*a*).

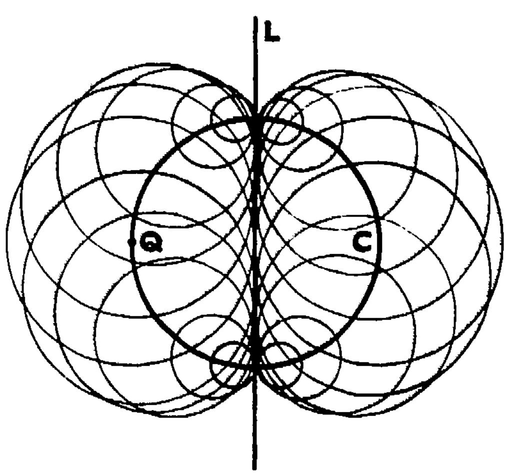
Extensions

(*a*) Choose L not through the centre of circle C.

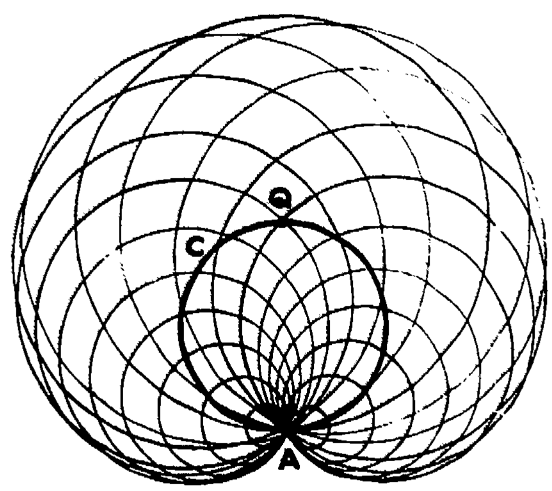
(*b*) Given a circle C, and a point A of C, construct the family of circles with centre Q on C and radius length equal to QA (*cardioid region*). See Figure 2.2 (*b*).

Extend to A not a point of C (*limaçon region*).

Note on Envelopes : A rather more sophisticated concept may be observed here if desired. Each of the boundary curves, the nephroid, the cardioid, and the limaçon is called the envelope of the family of circles. It is the curve to which every circle of the family is tangential.



NEPHROIDAL REGION
FIGURE 2.2 (a)



CARDIOID REGION
FIGURE 2.2 (b)

2.3 Families of line interval exercises, i.e. plotting the set of all points belonging to all line intervals satisfying certain given conditions.

Prepare a cardboard disc, radius length 1 in., by marking the circumference into arcs of equal length. This may be done by assuming that congruent arcs result from congruent angles at the centre ; or that congruent chords cut off congruent arcs. However the following method does measure and divide the circumference more directly and has links with other topics.

Place the disc on a ruler as in Figure 2.3 (*a*). Label the point of the disc in contact with zero on the ruler as 0.

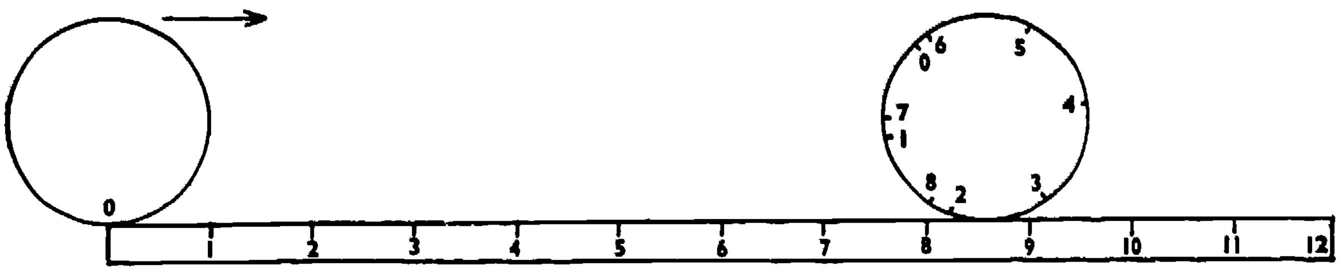
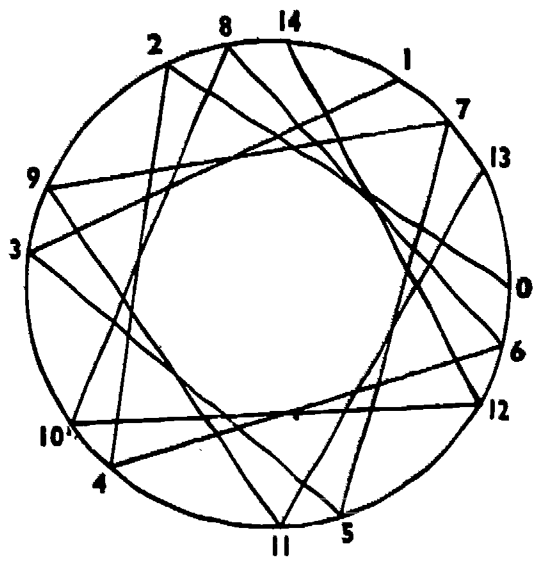


FIGURE 2.3 (a)

Roll the disc without slipping along the ruler marking points at 1 in., 2 in., 3 in., 4 in., 5 in., 6 in. Continue without pausing on through 0 to 7 in., 8 in., ...

Now draw the straight line interval joining 0 and 2. Then join 1 to 3, 2 to 4, 3 to 5, and in general x to x + 2. The set of points belonging to this family of intervals forms an *annular region.*

" Stop when you can guess the final shape," brings out the infinite nature of the set of intervals.



Not completed: more chords are needed before guessing the locus.

FIGURE 2.3 (b)

Extensions

(*a*) Find {points belonging to line intervals joining x to x + 5}.

(*b*) Students may make the intervals with large stitches in colored thread or use light rods, wire, or even match-sticks.

(*c*) Replace the circle by any closed polygon or curve.

(*d*) Replace the circle by a pair of line intervals which—

(i) intersect—Figure 2.3 (*c*) ;

(ii) are skew—Figure 2.3 (*d*).

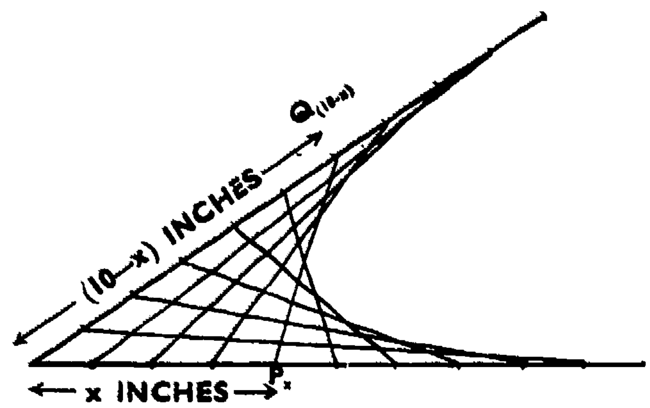
Devise methods of numbering and joining.

Notes

1. Students can incidentally get a feeling for (and perhaps discuss) the value of π and for some of the properties of chords of equal (and unequal) magnitude.

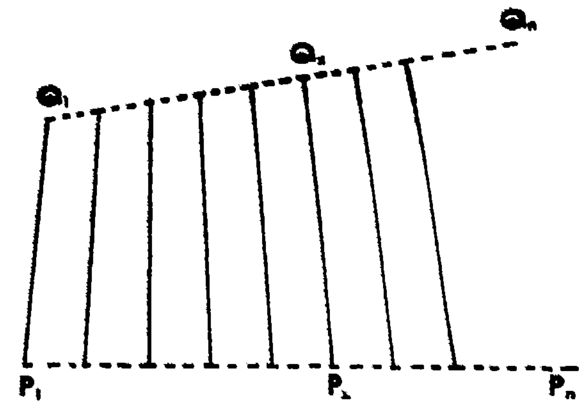
2. Use can be made of the principle of (*d*) (ii) above in building construction. Straight beams are generally stronger and probably cheaper to work with than curved ones, yet a curved surface can be made as in (*d*) (ii) ; the structure can be covered with mesh and other suitable material on to which concrete may be poured.

3. Comparison with exercise 1.4, where congruent line intervals were used instead of congruent arcs, may be made.



P_x JOINED TO $Q_{(10-x)}$

FIGURE 2.3 (c)



$Q_1 Q_n$ AND $P_1 P_n$ ARE NOT COPLANAR

FIGURE 2.3 (d)

SECTION 3

GEOMETRICAL RELATIONS

These relations use geometrical concepts, such as distance or length of an interval, in the defining conditions of a set of points.

The term locus is more traditionally used than relation here.

Very simple defining conditions use distance from a point (Examples 1 and 2 of Section 2) and from a line. Discuss briefly the shortest and other distances between two points, and between a point and a line. Make clear our convention that "distance" means "shortest or perpendicular distance" unless otherwise stated.

Start by introducing in a practical way two important loci in space :

A = {points that are a given distance from a given point O}.

B = {points that are a given (perpendicular) distance from a given line L}.

For example, the surface of the earth S (assumed spherical) is the set of points P which are 3,960 miles from the centre O of the earth ; more shortly :

S = { P : OP = 3,960 miles, O a given point}.

Discuss the set of points T forming the whole of the earth, and W (or T′) the set of points which do not belong to it.

T = { P : OP $\leqslant$ 3,960 miles, O given}.

W = { P : OP $>$ 3,960 miles, O given}.

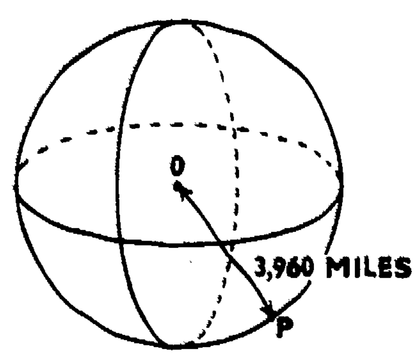


FIGURE 3.1

Another interesting starting-point is to consider distances from a piece of radio-active material.

Now imagine a straight telephone cable of the future linking Melbourne on earth with "Lunbon" on the moon. Suppose it needs to be enclosed in a protective casing. See Figure 3.2 (*a*). Demonstrate with a glass tube and a firm wire. Describe the set of points C forming the outer surface of the casing :

(i) as the surface of a cylinder of great length ;

(ii) as the set of points that are the same (perpendicular) distance (say ½ in.) from the wire.

Let $(P, \overleftrightarrow{ML})$ be the (perpendicular) distance of any point P from the line $\overleftrightarrow{ML}$ of which the wire is part.

Then $D = \{P : (P, \overleftrightarrow{ML}) = \frac{1}{2} \text{ in.}, \overleftrightarrow{ML} \text{ given}\}$.

Extend to $E = \{P : (P, \overleftrightarrow{ML}) = \frac{1}{2} \text{ in.}, \overleftrightarrow{ML} \text{ given}\}$ by imagining a wire and a cylinder, never ending or "of infinite length". See Figure 3.2 (*b*).

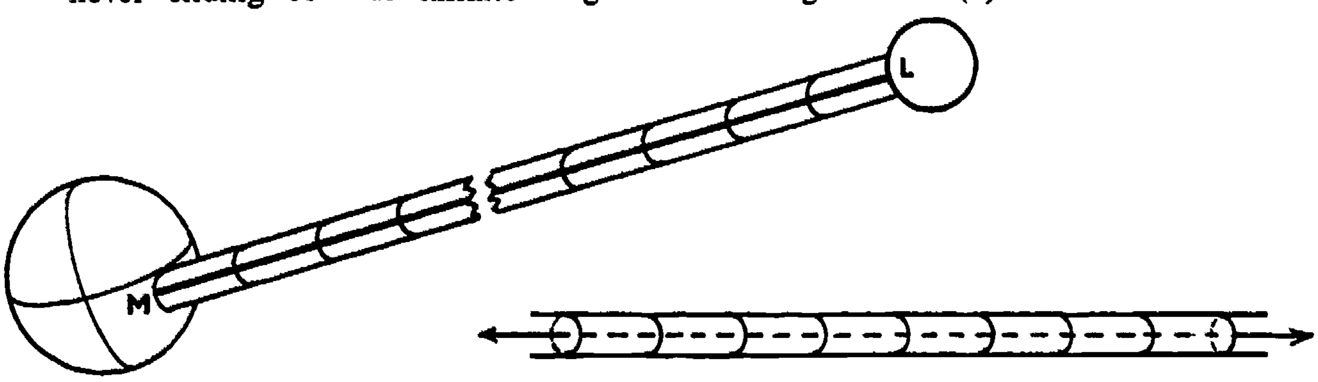


FIGURE 3.2 (a) FIGURE 3.2 (b)

Discuss also the infinite cylindrical region :

$$F = \{P : (P, \overleftrightarrow{ML}) \leqslant \tfrac{1}{2} \text{ in.}, \overleftrightarrow{ML} \text{ given}\},$$

and the region exterior to the cylinder :

$$G \text{ (or } F') = \{P : (P, \overleftrightarrow{ML}) > \tfrac{1}{2} \text{ in.}\}.$$

Other starting-points include discussions of distance from a red-hot poker, of straight pipe-lines, and of cylinders of various kinds.

Construction of these and other loci in space are possible with building materials such as metal, wood, plastic, and rubber, However, we shall concentrate on sets of points in a plane in the following discussion.

Note : We shall use OP to indicate the distance between two points O and P ; d(O,P) as the measure of this distance in a unit already stated ; $(P, \overleftrightarrow{AB})$ to indicate the perpendicular distance from P to $\overleftrightarrow{AB}$; and $p(P, \overleftrightarrow{AB})$ as the measure of $(P, \overleftrightarrow{AB})$ in the above unit.

[An alternative notation is Pb for $(P,\overleftrightarrow{AB})$, where b is used instead of $\overleftrightarrow{AB}$.]

Plane Loci

If we confine our points to the plane of the page, many curves may be drawn. The two important plane loci may be considered as plane sections through the given point or the line of the sphere and the cylinder above respectively ; alternatively, they may be plotted according to the defining conditions.

Thus :

(1) Given a plane A and a point O on it, the set of points at a given distance k units from O defines a circle C. See Figure 3.3.

$C = \{P : OP = k \text{ units}, O, P \in A\}$.

O and A define the reference system given.

O is called the centre of the circle.

The set of points is easily plotted with a pair of compasses or a drawing pin and a piece of string.

(2) Given a plane A and a line $\overleftrightarrow{BC}$ in it, the set of points at a given (perpendicular) distance k units from $\overleftrightarrow{BC}$ defines a pair of straight lines parallel to BC. See Figure 3.4.

$D = \{P : (P, \overleftrightarrow{BC}) = k \text{ units}, P, \overleftrightarrow{BC} \in A\}$.

$\overleftrightarrow{BC}$ and A define the given reference system.

$\overleftrightarrow{BC}$ is often called the axis.
The points can be plotted with a set square (a torn-off corner of a page will do).

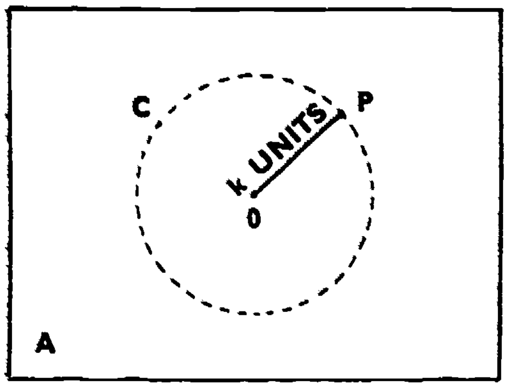


FIGURE 3.3

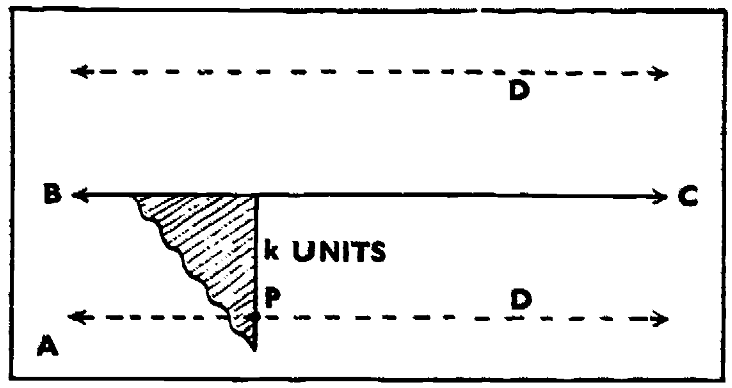


FIGURE 3.4

Use

We now use these two loci to find further loci.

(*a*) *Finite Sets*

The defining condition uses two or more specific distances. Motivate with ideas such as a pirate treasure verse, a clandestine lovers' meeting, a car trial secret destination.

EXAMPLE 1 : Reference to two given points.

Long John has half of a treasure map. Thus Long John will need to dig everywhere around this circle. See Figure 3.5 (*a*).

Along comes Pedro with the other half of the map. Now John and Pedro need only dig in two places. See Figure 3.5 (*b*).

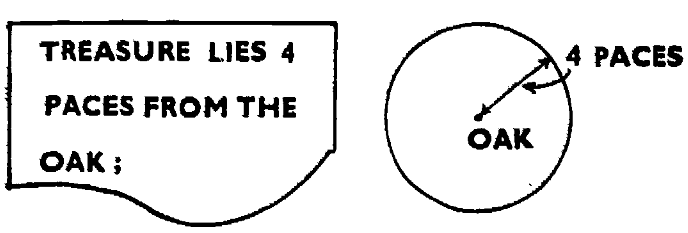


FIGURE 3.5 (a)

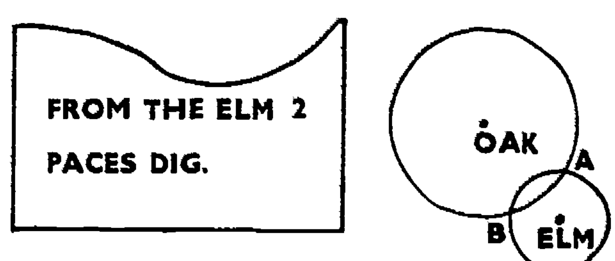


FIGURE 3.5 (b)

The locus with the two defining conditions OP = 4 units, EP = 2 units consists of a set of two points A and B.

Notice that a false note might be made ; if O and E were far apart, the circles might not intersect. What if OE is 6 paces ?

(*Note*—Keep to words until students themselves demand shorthand. Even then express in words in discussion.)

EXAMPLE 2. Reference to two given axes, not **necessarily at right angles to one another.**

"The oak and elm a line do make,
Four paces from it you do take ;
Look now at the fallen birch,
Two paces therefrom do you search,"

The locus with the two defining conditions, i.e.

$\{P : (P, \overleftrightarrow{OE}) = 4$ paces, $(P, \overleftrightarrow{BB'}) = 2$ paces$\}$, consists of the 4 points marked by crosses in Figure 3.6.

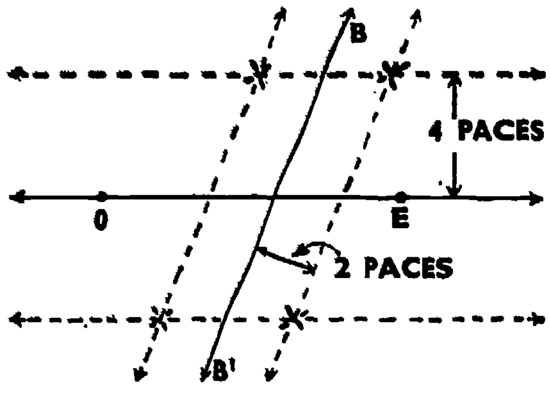


FIGURE 3.6

A useful device for parallel lines is to use a blue-lined exercise page with a sheet of air-mail paper on top, with lines at the required angle. Take care, however, that all the lines on the two pages are the same distance apart. Geometrical figures will be distorted if different scales are used.

EXAMPLE 3. Reference to one point and one axis.

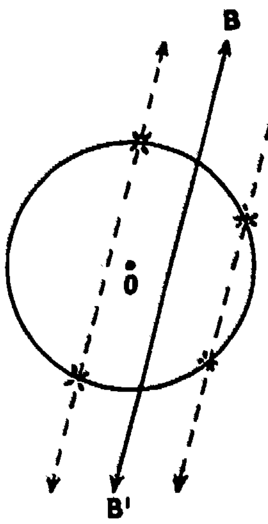


FIGURE 3.7 (a)

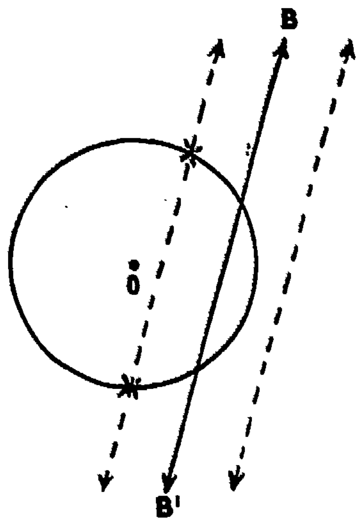


FIGURE 3.7 (b)

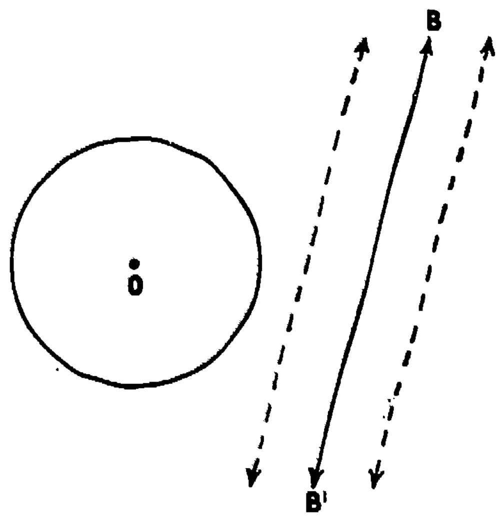


FIGURE 3.7 (c)

$\{P :\ OP = 4$ paces, $(P, \overleftrightarrow{BB'}) = 2$ paces, O and BB′ given$\}$ is shown by the set of crosses for three different positions of the given centre O and the axis $\overleftrightarrow{BB'}$ in Figure 3.7. Can you find positions which result in (*a*) three points, (*b*) one point ?
(Spend a little more time on this example because the definitions of conic sections depend on it.)

(*b*) *Infinite Sets*

The defining condition is a relationship between the distances of a point from two or more given points or axes, such as (i) $OP = 2PE$; (ii) $(P, \overleftrightarrow{OE}) = 2(P, \overleftrightarrow{BB'})$; (iii) $OP = 2\ (P, \overleftrightarrow{BB'})$.

Use words frequently at first ; e.g. in (iii) the distance of P from O is to be twice the distance of P from $\overleftrightarrow{BB'}$, and given O and $\overleftrightarrow{BB'}$ as illustrated, the construction in Example 3 above (Figure 3.7 (*a*)) provides four points of the set since $OP = 4$ paces and $(P, \overleftrightarrow{BB'}) = 2$ paces. (Shown by light dots in Figure 3.8.)

One idea is to extend the problem on the pretence of not knowing what size the unit pace was. Lead slowly to families of circles and line pairs, and plot the four (or less) points determined by each circle and its corresponding line pair for different pace lengths. (Figure 3.8.)

The idea of choosing a positive *number* t is also helpful. Construct the line pair t units from BB′; then calculate 2t units and construct a circle, centre O, radius length 2t units. Mark the intersections, which belong to the locus, in red.

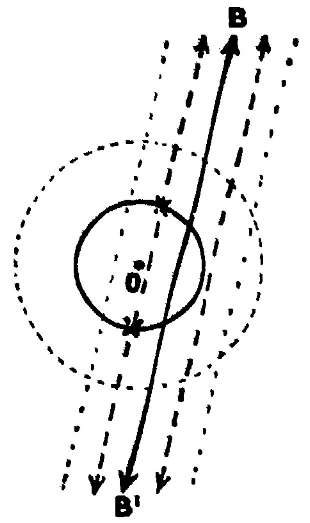


FIGURE 3.8

Plot many points, then join by a smooth curve as a guess at the complete set. Test some of the extra points to see if they do satisfy the defining condition. This converse aspect is particularly important near unusual " bumps ", near end points (if there are any), and where plotted points are far apart.

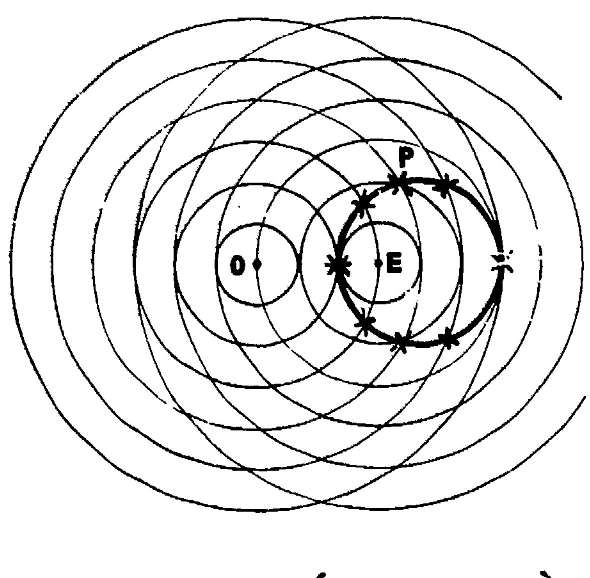
The loci which result are sufficiently varied to provoke discussion. Figures 3.9 (*a*), 3.10 (*a*), and 3.11 (*a*) illustrate these.

Other defining conditions are :

$OP = PE$; $(P, \overleftrightarrow{OE}) = (P, \overleftrightarrow{BB'})$; $OP = (P, \overleftrightarrow{BB'})$. (Line, perpendicular line pair, parabola.) See Figures 3.9 (*b*), 3.10 (*b*), 3.11 (*b*).

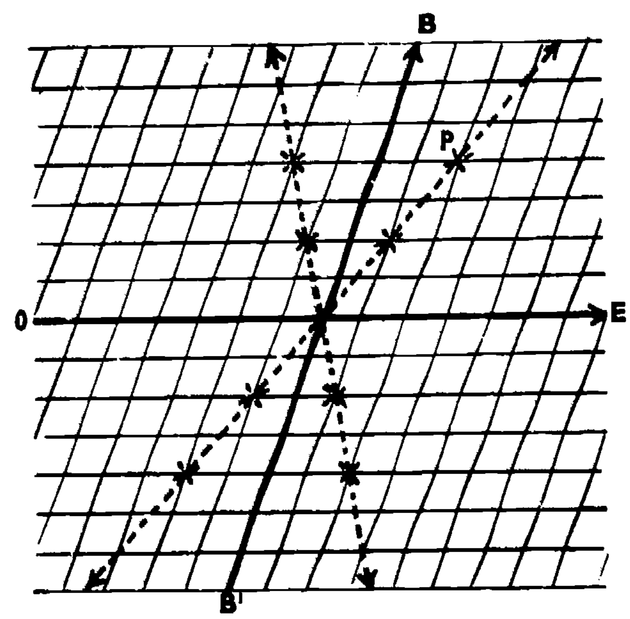
$OP = \frac{1}{2} PE$; $(P, \overleftrightarrow{OE}) = \frac{1}{2} (P, \overleftrightarrow{BB'})$; $OP = \frac{1}{2} (P, \overleftrightarrow{BB'})$. (Circle, line pair, ellipse.) See Figures 3.9 (*c*), 3.10 (*c*), 3.11 (*c*).

In all these cases O, E, B, B′ are given.



THE CIRCLE IS $\{P : OP = 2\, PE\}$

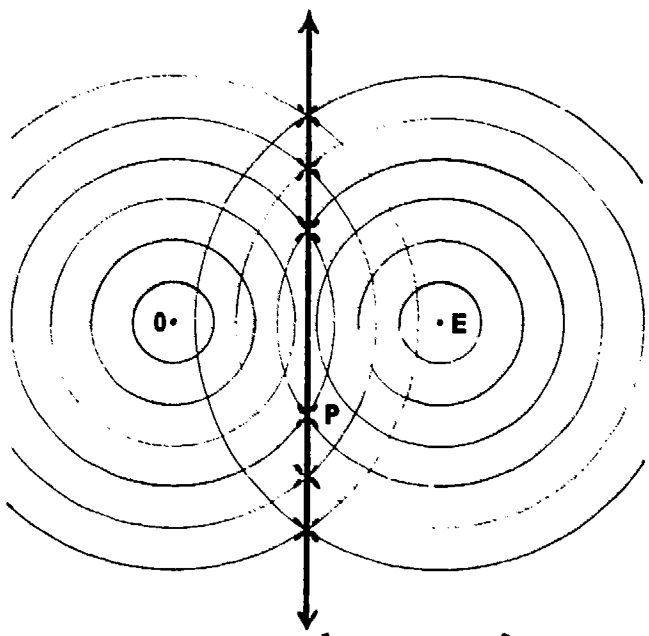
FIGURE 3.9 (a)



THE DOTTED LINE PAIR IS $\{P : (P, \overleftrightarrow{OE}) = 2\,(P, \overleftrightarrow{BB'})\}$

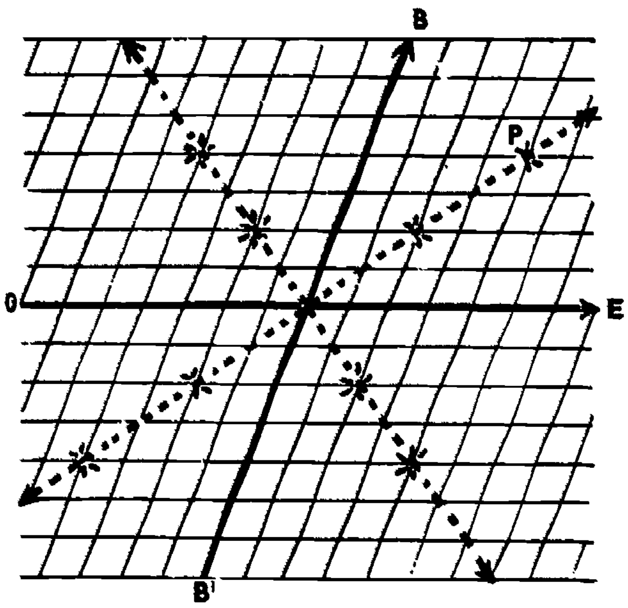
FIGURE 3.10 (a)

After students have constructed their own families of circles and line pairs once, it is a good idea to provide graph paper which can readily be made by using a stencil. Roneoing on both sides of the duplicating paper is quite satisfactory. Include some examples with line pairs at right angles. A $\frac{1}{4}$ in. grid is useful for this purpose.



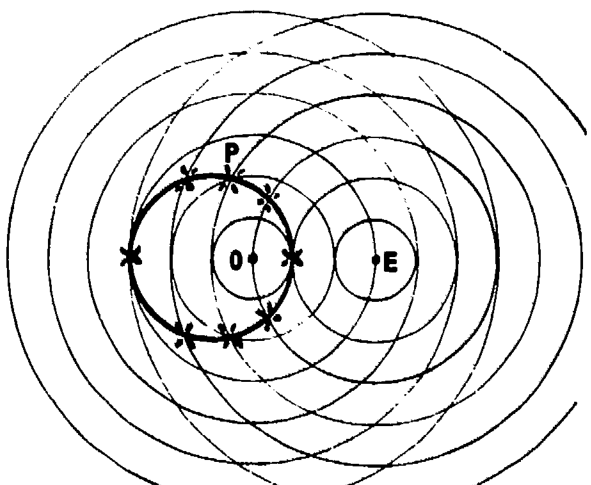
THE LINE IS $\{P : OP = PE\}$

FIGURE 3.9 (b)



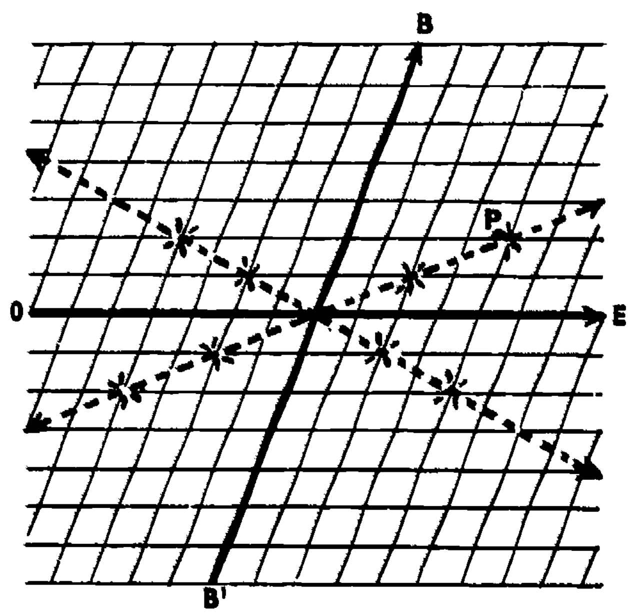
THE DOTTED LINE PAIR IS $\{P : (P, \overleftrightarrow{OE}) = (P, \overleftrightarrow{BB'})\}$

FIGURE 3.10 (b)



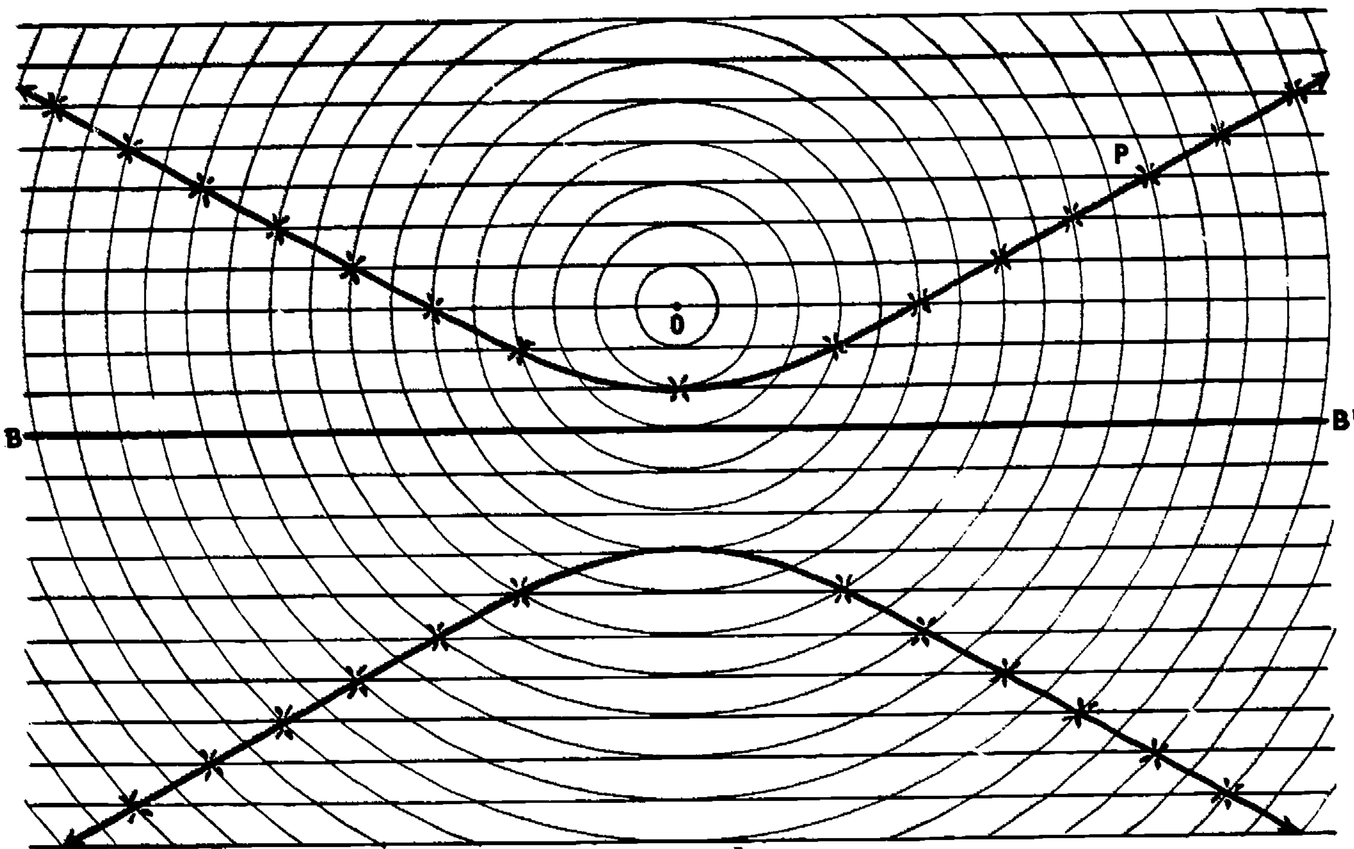
THE CIRCLE IS $\{P : OP = \frac{1}{2} PE\}$

FIGURE 3.9 (c)



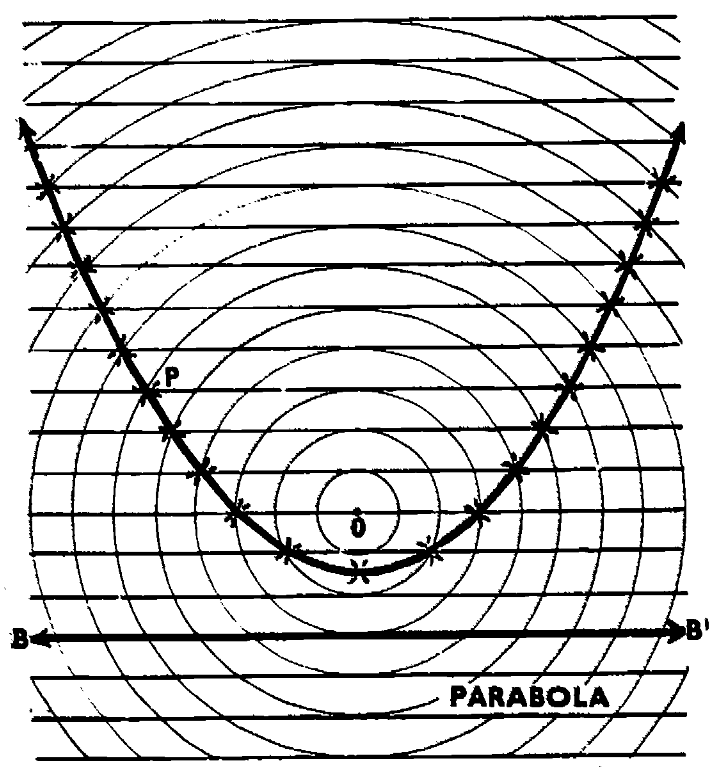
THE DOTTED LINE PAIR IS $\{P : (P, \overleftrightarrow{OE}) = \frac{1}{2} (P, \overleftrightarrow{BB'})\}$

FIGURE 3.10 (c)



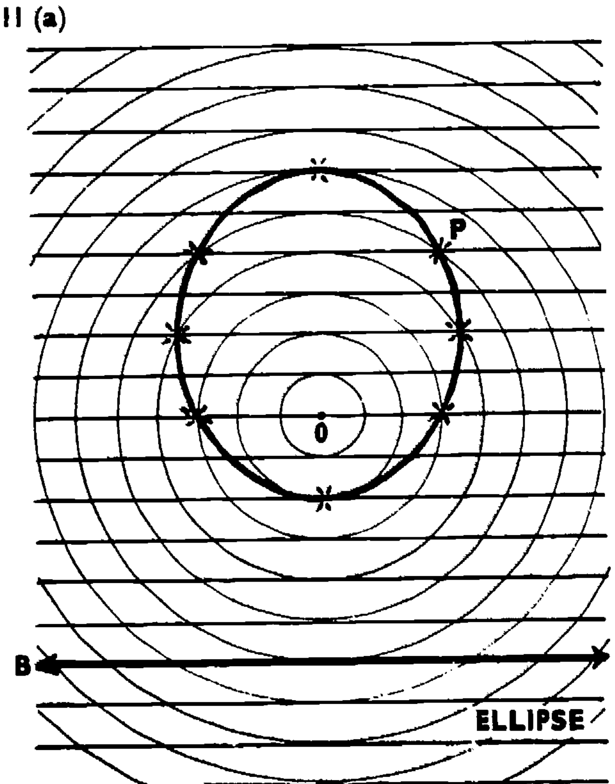
THE CURVE IS $\left\{P : OP = 2\,(P, \overleftrightarrow{BB'})\right\}$ HYPERBOLA

FIGURE 3.11 (a)



THE CURVE IS $\left\{P : OP = (P, \overleftrightarrow{BB'})\right\}$

FIGURE 3.11 (b)



THE CURVE IS $\left\{P : OP = \frac{1}{2}\,(P, \overleftrightarrow{BB'})\right\}$

(Note: In this figure the distance $(0, \overleftrightarrow{BB'})$ has been doubled to give a larger ellipse.)

FIGURE 3.11 (c)

Extensions

Vary the relative positions of reference points and lines.

Vary the relationship, e.g. $2\ OP = 3\ (P, \overleftrightarrow{BB'})$. (Approximately a regular hyperbola.)

For future reference it is important to construct several parabolas. The same stencil may be used ; by a suitable choice of reference line the distance between it and the reference point can be altered. Cut out templates of some of the parabolas (see Section 4).

Other examples to try if desired (and if time permits) are :

$\{P : AP + PB = 3 \text{ units}\}$. (Ellipse.)

$\{P : AP - PE = 2 \text{ units}\} \cup \{P : PE - AP = 2 \text{ units}\}$. (Hyperbola.)

$$\{P : d(A,P) = \frac{1}{d(P, B)}\}.$$

Selection

Time and interest are important. Selection is possible and should be made carefully. For example, it may be felt necessary to limit locus work to the parabola. In this case consider only Example 3, with 4 paces from O and 4 paces from the birch line—

$\{P : OP = (P, \overleftrightarrow{BB'})\}$ and adjust the detailed descriptions.

Perhaps you may prefer the general set-up as described here, but consider the ellipse (a closed curve) which is a simpler concept. In Examples 1, 2, and 3 make adjustments to 2 paces from O, 6 paces from E, ... (this results in a better shaped ellipse) ; then try 4 paces and 2 paces.

SECTION 4

CARTESIAN RELATIONS IN THE PLANE: GRAPHS

Here the defining conditions are concerned with the number of units in two particular directions (directed intervals or vectors) and make use of negative numbers to describe distances in the opposite directions.

Pinpointing

Discuss the possibility of a one-point locus in Example 2 of Section 3.

(*a*) At least one other distance condition is needed. Perhaps :

"Now note five paces from the elm,
From these your reward will surely stem."

See Figure 4.1 (*a*).

(*b*) Directions could be given, e.g. "On the same sides as the birch top and the elm."

See Figure 4.1 (*b*).

(*c*) The conditions could state simply : "Walk this distance in this direction, then that distance in that direction." For example :

"At yon trunk of silver birch a start do make,
From oak toward elm four paces take,
Now proceed two paces parallel to the fallen birch
From root towards top, then search."

See Figure 4.1 (*c*).

Watch that the measurements are made along the direction, i.e. along a ray. Compare with (*b*) above and note that this is not the same treasure map.

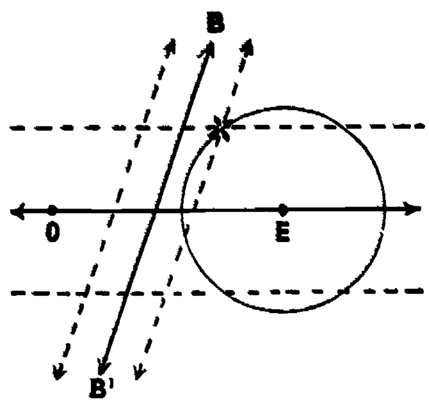


FIGURE 4.1 (a)

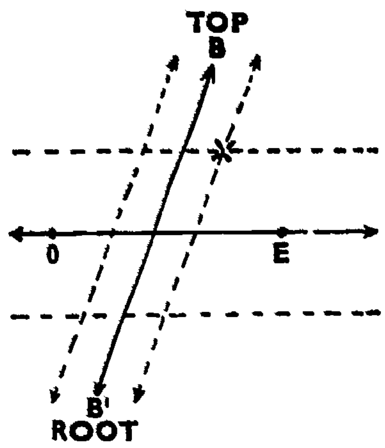


FIGURE 4.1 (b)

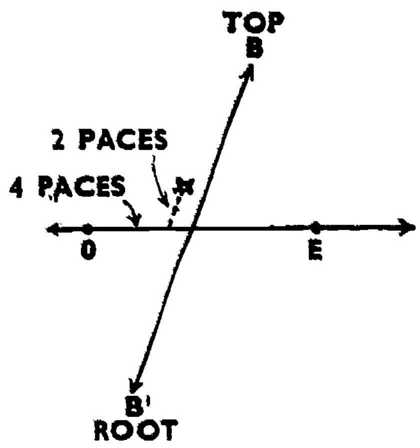


FIGURE 4.1 (c)

Directed Intervals and the "GO" Method

Suppose we are given a ray $\overrightarrow{AB}$ and a starting-point O. See Figure 4.2 (*a*). We shall call a distance of 3 units in the direction $\overrightarrow{AB}$ a directed interval. This means that we think of a tiny particle starting at O, going 3 units to a point P along a ray $\overrightarrow{OC}$ parallel to $\overrightarrow{AB}$ and in the same direction as it. See Figure 4.2 (*b*).

Also $(-\overrightarrow{AB})$ will mean a ray in the opposite direction to $\overrightarrow{AB}$, i.e. ray $\overrightarrow{BA}$. See Figure 4.2 (*c*).

Directed numbers will be used thus : (-4) units in $\overrightarrow{AB}$ direction will mean a distance of 4 units in the $(-\overrightarrow{AB})$ or $\overrightarrow{BA}$ direction. For example, to get to Q go from O (-4) units in $\overrightarrow{AB}$ direction. See Figure 4.2 (*d*).

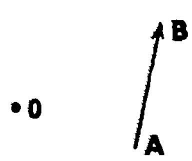


FIGURE 4.2 (a)

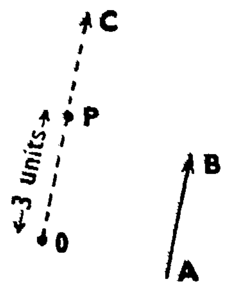


FIGURE 4.2 (b)

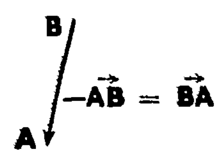


FIGURE 4.2 (c)

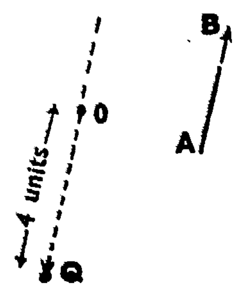


FIGURE 4.2 (d)

Cartesian Reference System

If we wish the particle to be able to go from O to all points in the plane, we could consider all directions from O and how to specify them. See Figure 4.3 (*a*). We do this in Section 5.

However, the particle can go from O to all points in the plane along paths using two non-parallel directions. For example, a path from O to P using only given directions $\overrightarrow{AB}$ and $\overrightarrow{CD}$ may be found. See Figure 4.3 (*b*). Draw line $\overleftrightarrow{OL}$ parallel to $\overleftrightarrow{CD}$, and $\overleftrightarrow{PQ}$ parallel to $\overleftrightarrow{AB}$. Let $\overleftrightarrow{OL}$ and $\overleftrightarrow{PQ}$ intersect in M. Then the path $\overline{OM}, \overline{MP}$ is a suitable one. Perhaps the students could find another suitable path.

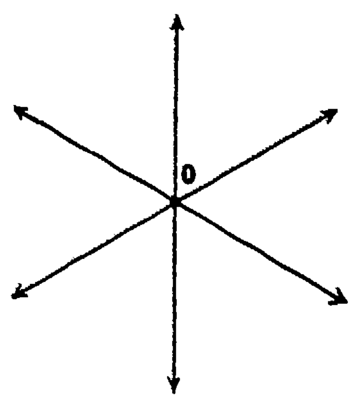


FIGURE 4.3 (a)

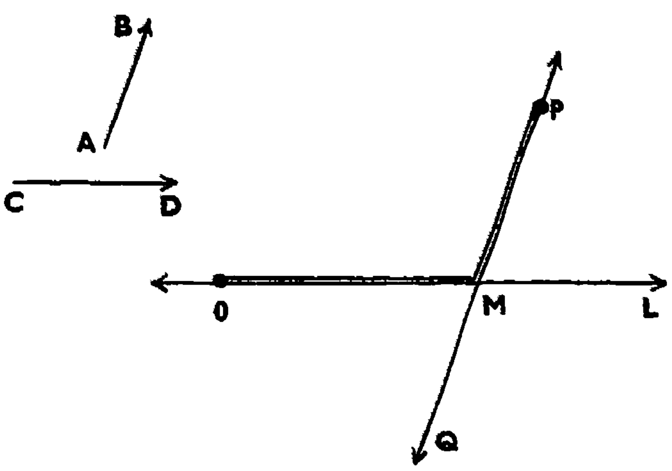


FIGURE 4.3 (b)

Hence we introduce the Cartesian reference system consisting of a starting-point and two non-parallel directions. Give students practice at finding points as meeting places by approximate construction methods. For example, issue students with a duplicated sheet (without grid lines of any kind) showing two directions $\overrightarrow{AB}$ and $\overrightarrow{CD}$ and having a starting-point O, several other points, and, perhaps, a figure marked on it. See Figure 4.3 (*c*).

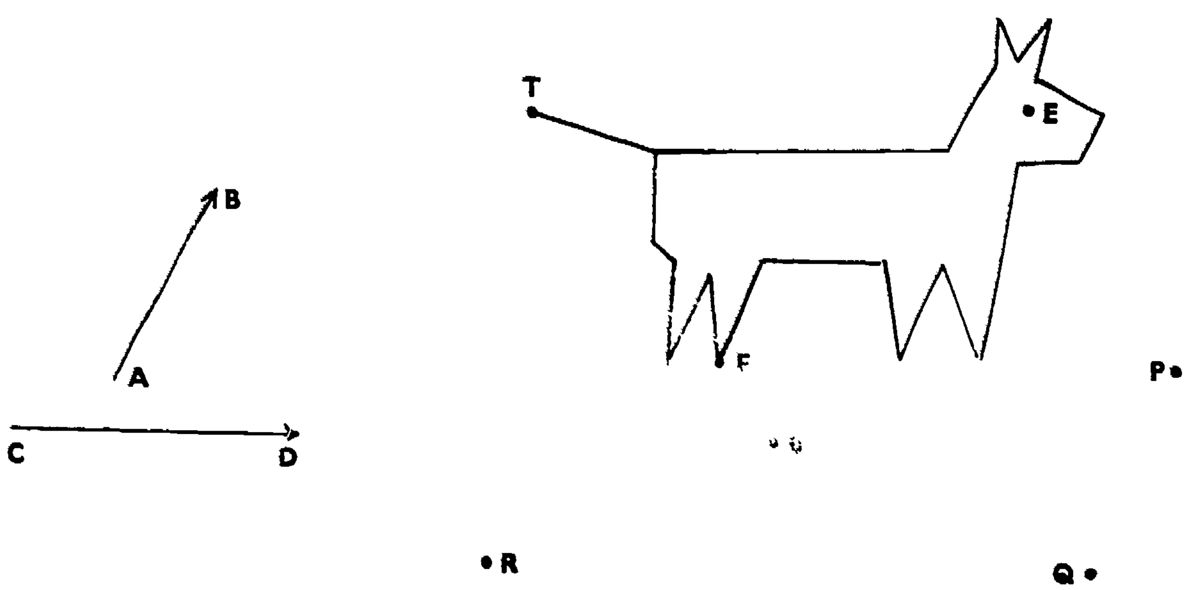


FIGURE 4.3 (c)

Ask students to *sketch* parallels and make approximate measurements to—

(i) show a meeting place M described by : " Go from O 5 in. in the $\overrightarrow{AB}$ direction, then (− 4) in. in the $\overrightarrow{CD}$ direction ;

(ii) describe with similar words the positions of P, Q, R, T, F, and E ;

(iii) practise putting further points on the page by " going ".

Let students observe that the same point results no matter which instruction is given first. Suggest a particular order be observed to speed up the grasping of instructions

Grid

Have students observe, after plotting several points, that it would be helpful to have a grid of lines parallel to the two given directions to *go* along, as in Figure 4.4. The air-mail paper system suggested in Section 3, Example 2, may be used, but care should be taken to measure "distance along" in contrast to "(perpendicular) distance from". A point such as N, 5 units in the $\overrightarrow{CD}$ direction and (− 4) units in the $\overrightarrow{AB}$ direction from O, is then quickly placed in position in either of two ways. One is indicated by the dotted lines in Figure 4.4.

FIGURE 4.4

Give students practice at plotting points, using the intersection method, with their own grid or with ¼ in. duplicated grid. Include one rectangular system.

Plot also, perhaps, the set of points for which the number of units in the $\overrightarrow{AB}$ direction is twice the number of units in the $\overrightarrow{CD}$ direction.

Rectangular Cartesian System

Compare "distance along" and "distance from", e.g. refer to the position of the treasure in Figures 4.1 (*b*) and 4.1 (*c*). Discuss the possibility of measurements coinciding, leading to the special case of two perpendicular directions as being very useful,

Discuss street-directories, map references, etc. A ¼ in. duplicated rectangular grid is useful now. Later, graph paper is a useful device for in-between measurements.

The starting-point O is traditionally called the *origin*. The *rays* through O, parallel to the given directions, are usually named $\overrightarrow{Ox}$, $\overrightarrow{Oy}$, and placed as shown in Figure 4.5 (*a*).

The *lines* $\overleftrightarrow{Ox}$, $\overleftrightarrow{Oy}$ are called the *x-axis* and the *y-axis* respectively.

Numbers are placed on the x-axis to show the number of units along the $\overrightarrow{Ox}$ (often shortened to x) direction, and similarly on the y-axis. See Figure 4.5 (*b*).

For this purpose *a scale* may need to be given, e.g. 1 in. to 2 of the units in use. The *same scale should be used on both axes* so that geometrical figures will not be distorted ("dilated" from an axis, actually). Properties are pictured truthfully, shapes, areas, congruent intervals, and gradient and angle of slope of a straight line are as seen. Later, if different scales are needed so that equations may be solved, students are aware of the distortion. Different scales, as used in column graphs and statistics, do not usually refer to the same physical quantity.

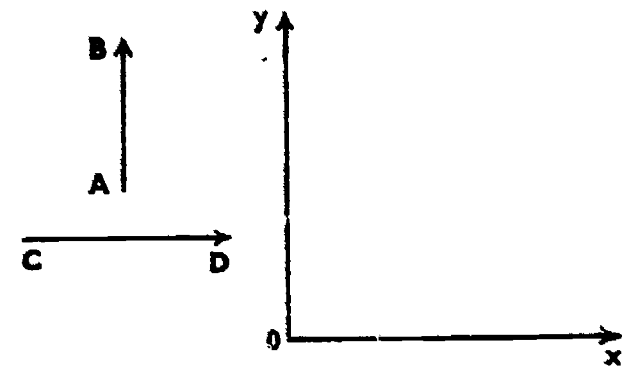


FIGURE 4.5 (a)

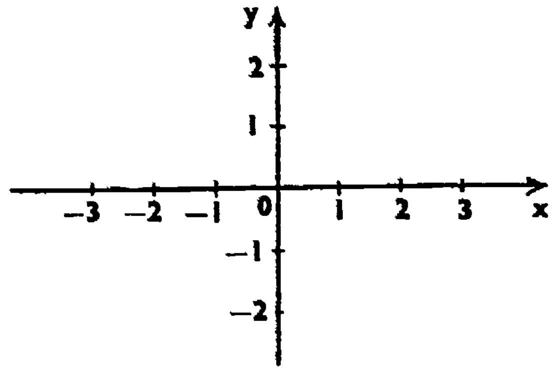


FIGURE 4.5 (b)

At this level the origin can well be placed at about the middle of the page, unless otherwise directed, and a suitable scale indicated.

Cartesian Notation

Introduce shorthand slowly :

(i) x-co-ordinate as shorthand for "the number of units to go in the $\overrightarrow{Ox}$ direction", and y-co-ordinate for "the number of units to go in the $\overrightarrow{Oy}$ direction". Plot several points using this language. For example,

plot P which has an x-co-ordinate of 2 and a y-co-ordinate of 5 ;
plot Q which has an x-co-ordinate of 2 and a y-co-ordinate twice as large ;
plot another point R which has a y-co-ordinate twice as large as its x-co-ordinate ;
plot others.

(ii) $x = 2$, $y = 5$ as further shorthand, and some plotting using this shorthand ; leading to several points for which y is three more than x.

(iii) Ordered pair notation, $(x, y) = (2, 5)$, noting carefully that we all agree to observe a rule that the x co-ordinate is to be written (and read) first. Plot some given ordered pairs. Read off ordered pairs for points on figures recorded on a grid, for example, as in Figure 4.6.

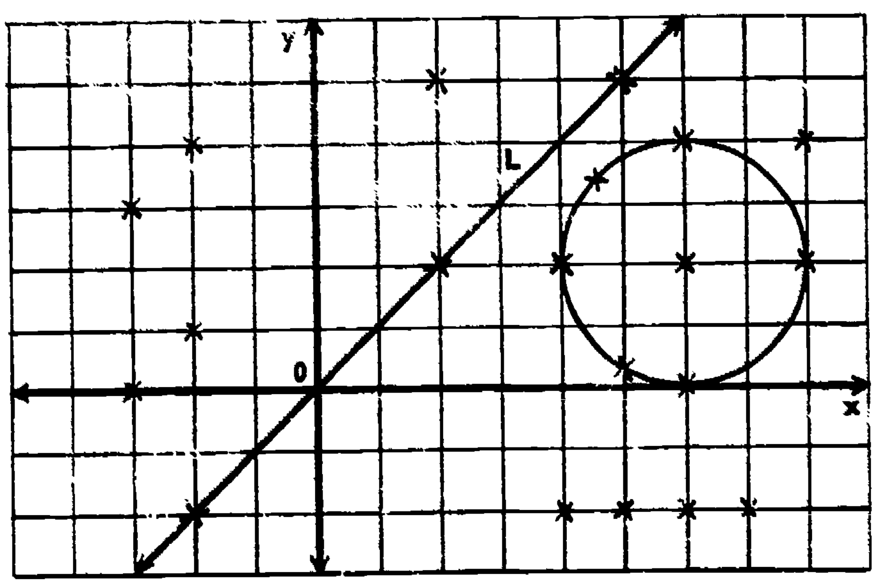


FIGURE 4.6

Cartesian Relations

Observe and discuss the fact that every point on L, see Figure 4.6 above, appears to have its x-co-ordinate the same number as its y-co-ordinate.

Test other points on L. Make a conclusion : " It appears that the line L is the set of points for which the x-co-ordinate is equal to the y-co-ordinate.". In symbols :

$$L = \{\text{points } (x, y) \text{ such that } x = y\},$$
$$\text{or } L = \{(x, y) : x = y\}.$$

We now see the possibility of having Cartesian relations; of collecting sets of points each of which has its x-co-ordinate and y-co-ordinate connected by a particular rule. It seems probable that figures, defined as geometrical relations as in Section 3, can also be expressed as Cartesian relations, and vice versa. If so, properties of such figures may be studied in either system.

Interest

Interest can be aroused by starting graph work with parabolas since this is new. If preferred now, rather than later, look at the straight line (page 63) from the view-point there since this is probably different from that used in Form II.

Parabolas

Some detail has been included here in case Section 4 is studied before the earlier sections. The approach has also been used with rather more sophistication with a " full mathematics " group, and some extensions for such students are indicated where appropriate.

What is a Parabola ?

The treatment here reverses the somewhat traditional approach for the following reasons. Students who draw the graph of the relation $A = \{(x, y): y = x^2\}$, and are told that this shape is a parabola, usually ask : " What is a parabola ?" The relation $B = \{(x, y) : y = 2x^2\}$ is also a parabola, and yet A is not the same as B, and the confusion deepens as the analytic definition is built up. In any case, the ultimate definition is probably, in effect : " Parabolas are graphs of relations $\{(x, y) : y = kx^2, k \in R\}$ or congruence transformations thereof." But students often wonder if the curve $\{(x, y) : xy = 1\}$ is two parabolas, perhaps under some transformations which include rotations. How can we give students at this level the opportunity to judge for themselves ? After all, the general equation of a parabola is $ax^2 + bxy + cy^2 + dx + ey + f = 0$, with certain restrictions on the co-efficients, so can we give a satisfactory answer at this level if we refer to this equation ?

Let us start by finding out typical shapes from the geometrical definition as a locus. Then we can use analytic geometry to assist recognition and for the study of size and place. This establishes the parabola first as a definite geometrical shape, or rather, as a family of parabola shapes, different distances between the focus and the directrix giving different parabolas. This can be profitably compared with the geometrical definition and shape of the circle, and the family of circles of different radius lengths.

We shall use the geometrical definition suggested at fifth-form level. Introduced now, the study will have a firmer foundation. Students will enjoy plotting curves now, and benefit from doing so, whether they proceed with analysis or not. Detailed guidance is given below.

Definition

Let F be a given point, $\overleftrightarrow{AB}$ a given line not containing F. Let P be a point in the plane ABF such that its distance PF from F is equal to its (perpendicular) distance $(P, \overleftrightarrow{AB})$ from $\overleftrightarrow{AB}$. Then the set S of all such points is a parabola.

$S = \{P : PF = (P, \overleftrightarrow{AB}),\ P \in \text{plane } ABF\}$,

or $S = \{P : d(P, F) = p(P, \overleftrightarrow{AB}),\ P \in \text{plane } ABF\}$.

(See note in Section 3, page 42, regarding notation.)
(Word the circle definition in a similar manner, and compare.)

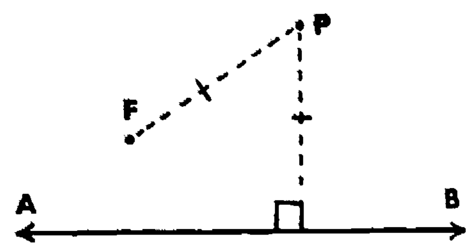


FIGURE 4.7

Construction

Supply students with a duplicated $\frac{1}{4}$ in. grid, or make use of a graph page. Choose F and $\overleftrightarrow{AB}$ (see diagram). Suggest for a start, $\overleftrightarrow{AB}$ about two-thirds of the way down the page, F one inch above $\overleftrightarrow{AB}$. See Figure 4.9 (*a*).
Discuss :

(*a*) The line pair parallel to $\overleftrightarrow{AB}$ and $\frac{3}{4}$ in. from it as the locus or set of all points P such that $(P, \overleftrightarrow{AB}) = \frac{3}{4}$ in. See Figure 4.8 (*a*).

(*b*) The circle with centre F, radius length $\frac{3}{4}$ in., as the set of points P such that $PF = \frac{3}{4}$ in. See Figure 4.8 (*b*).

(*c*) The points of intersection (if any) as points P for which $(P, \overleftrightarrow{AB}) = \frac{3}{4}$ in. $= PF$. See Figure 4.8 (*c*).

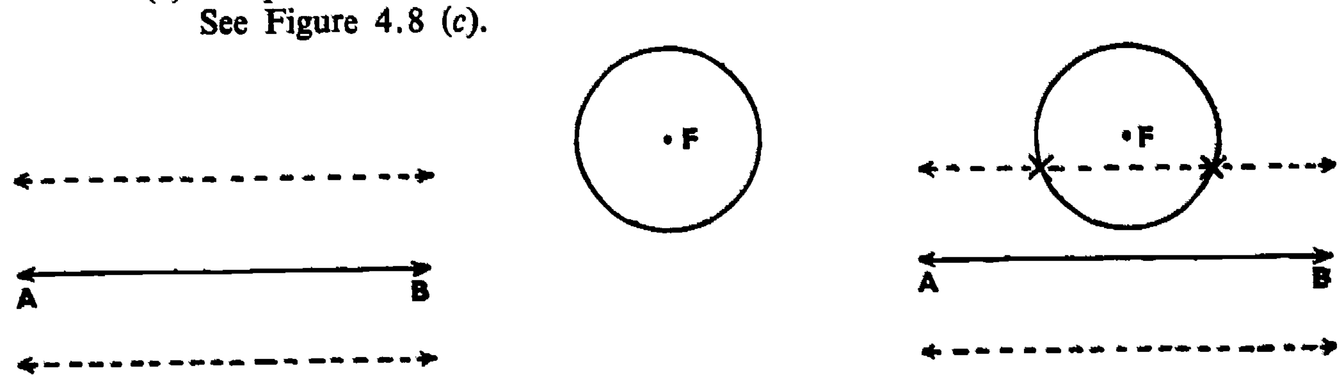


FIGURE 4.8 (a) FIGURE 4.8 (b) FIGURE 4.8 (c)

Now construct a family of circles with F as centre, radius lengths $\frac{1}{4}$ in., $\frac{2}{4}$ in., $\frac{3}{4}$ in.,....

Label line pairs $\frac{1}{4}$ in., $\frac{2}{4}$ in., $\frac{3}{4}$ in.,.... from $\overleftrightarrow{AB}$.

Mark in red the one point which is $\frac{2}{4}$ in. from F and $\frac{2}{4}$ in. from A.B

Mark two points $\frac{3}{4}$ in. from each.

Estimate the positions of two points $\frac{2\frac{1}{2}}{4}$ in. from each.

Mark the two points $\frac{4}{4}$ in. or 1 in. from each. (Notice the small square there. See Figure 4.8 (*d*).

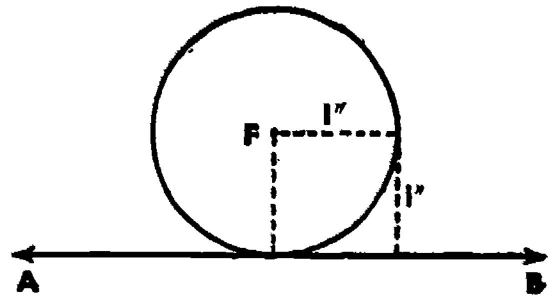


FIGURE 4.8 (d)

Proceed now to points $\frac{5}{4}$ in., $\frac{6}{4}$ in., ... from each, until there are sufficient points marked to guess the set of all such points by a smooth curve K. See Figure 4.9 (*a*).

Test some points on the curve to make sure PF = $(P, \overleftrightarrow{AB})$ for each of them (i.e. an informal treatment of the converse aspect of a locus by making sure no " extra ", non-suitable points have crept in). Test also some points Q, not on the curve, to make sure either QF > $(Q, \overleftrightarrow{AB})$ or QF < $(Q, \overleftrightarrow{AB})$.

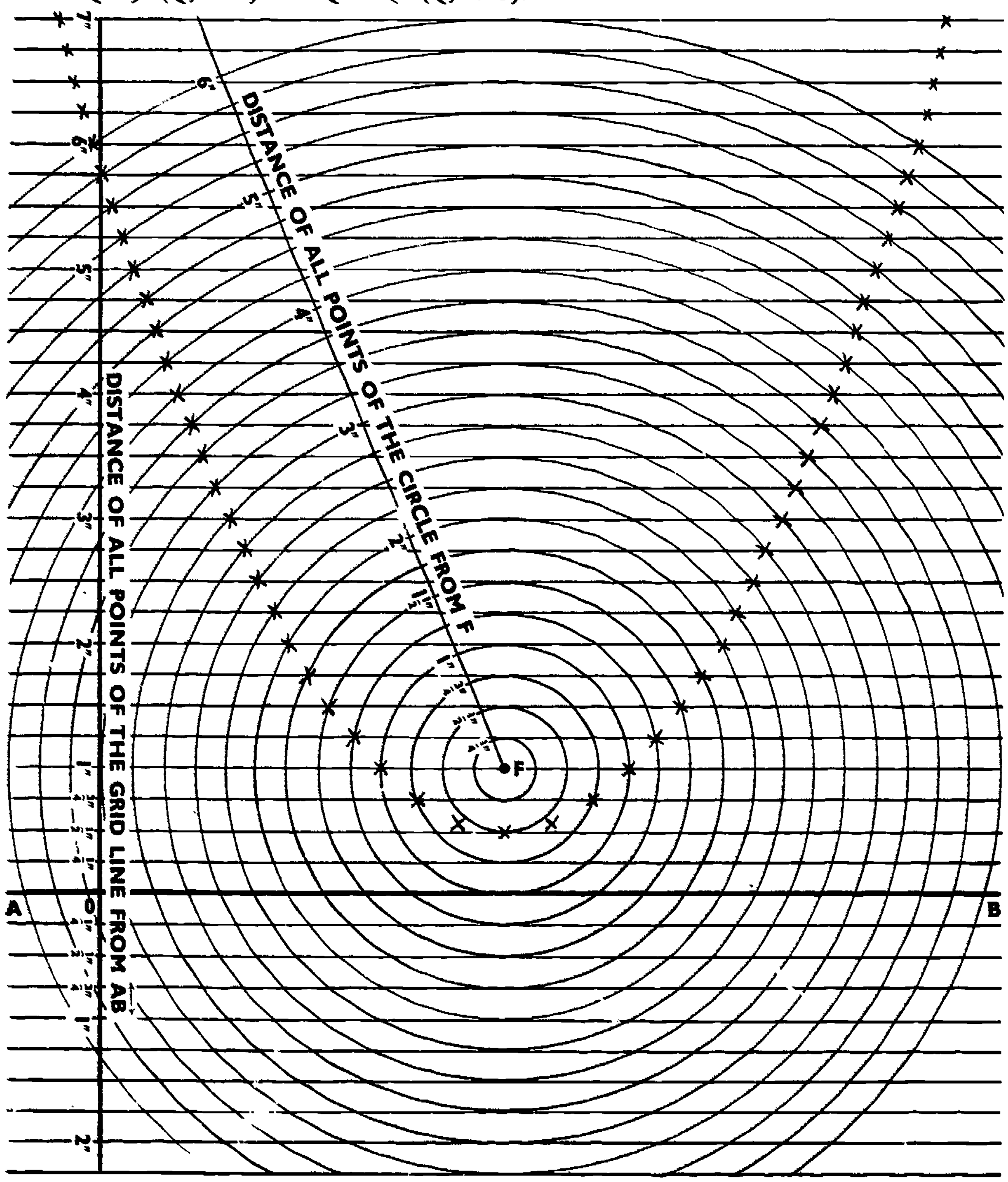


FIGURE 4.9 (a)

Note that the parabola is not complete ; enthusiasts may extend theirs by joining on extra grid sheets.

Cut out and retain the region bounded by the top edge of the page and this section of the parabola. A cardboard trace gives a better template.

Now construct other parabolas, using fresh grid sheets. Follow the same instructions but each time start with $\overleftrightarrow{AB}$ at a different distance from F. A duplicated sheet with circles and lines as above may be used for all of these constructions by placing $\overleftrightarrow{AB}$ suitably on the sheet.

Make templates for—

parabola S with $(F, \overleftrightarrow{AB}) = \frac{1}{2}$ in. ;

parabola T with $(F, \overleftrightarrow{AB}) = \frac{1}{4}$ in. ;

parabola M with $(F, \overleftrightarrow{AB}) = 2$ in. ;

parabola N with $(F, \overleftrightarrow{AB}) = 4$ in.

Looking Ahead (for teachers only)

For a parabola on a Cartesian plane, as in Figure 4.9 (*b*), suppose FO $= a$ units $=$ focal length.

Then $(F, \overleftrightarrow{AB}) = 2\,a$ units,

and $PF = (P, \overleftrightarrow{AB}) \Rightarrow PF^2 = PM^2$,

$\Rightarrow x^2 + (y - a)^2 = (y + a)^2$,

$\Rightarrow x^2 + y^2 - 2ay + a^2 = y^2 + 2ay + a^2$,

$\Rightarrow 4ay = x^2$,

$\Rightarrow y = (\frac{1}{4a})\,x^2$.

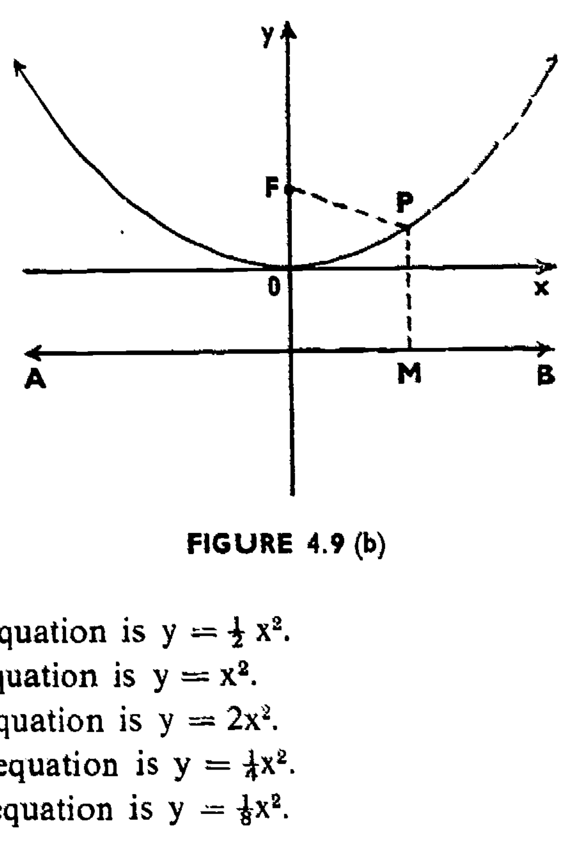


FIGURE 4.9 (b)

Parabola K has $FO = \frac{1}{2}$ of 1 in., i.e. $a = \frac{1}{2}$, and equation is $y = \frac{1}{2}x^2$.

Parabola S has $FO = \frac{1}{2}$ of $\frac{1}{2}$ in., i.e. $a = \frac{1}{4}$, and equation is $y = x^2$.

Parabola T has $FO = \frac{1}{2}$ of $\frac{1}{4}$ in., i.e. $a = \frac{1}{8}$, and equation is $y = 2x^2$.

Parabola M has $FO = \frac{1}{2}$ of 2 in., i.e. $a = 1$, and equation is $y = \frac{1}{4}x^2$.

Parabola N has $FO = \frac{1}{2}$ of 4 in., i.e. $a = 2$, and equation is $y = \frac{1}{8}x^2$.

The Shape of the Parabola

Look now at the members K, S, T, M, and N of the parabola family. Discuss the characteristics of a parabola and contrast them with those of a circle. Points of a parabola are not at the same distance from the given point which we name the focus.

Note :

(*a*) That one point is closest to the focus ; name it the *vertex* O ; call the length FO, the *focal length*, *a* units, and note that p $(F, \overleftrightarrow{AB}) = 2a$.

(*b*) That at each distance greater than *a* units from the focus, there are two points symmetrically placed ; name the ray $\overrightarrow{OF}$ the axis of the parabola, and $\overleftrightarrow{AB}$, whose position determines the direction $\overrightarrow{OF}$, the directrix.

(*c*) That there is no greatest distance ; note the never-ending (call on the imagination !), never-closing-in properties.

(*d*) That the greater the focal length *a* units, the shallower and the wider the curve ; for *a* smaller, the curve is steeper and narrower.

(*e*) That the parabola never " starts again " ; i.e. there is only one section for a parabola as distinct from the two branches of a hyperbola.

Family of Circles

Given centre C, radius length measure r :

(*a*) If the radius length is $\frac{1}{4}$ in. there is just one possible circle. See Figure 4.10 (*a*).

(*b*) There is a family of circles with the same centre but with different radius lengths. See Figure 4.10 (*b*).

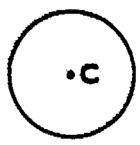


FIGURE 4.10 (a)

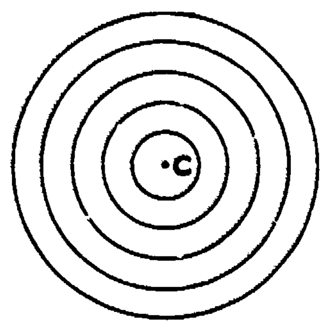


FIGURE 4.10 (b)

Family of Parabolas

Given focus F, focal length measure *a* :

(*a*) If the focal length is $\frac{1}{4}$ in., the directrix is $2 \times \frac{1}{4} = \frac{1}{2}$ in. from F, but may have any direction. There is a family of parabolas each having different " directions ". Note that all are congruent to one geometrical shape. See Figure 4.10 (*b*).

(*b*) There is a family of parabolas with the same focus and *direction* (i.e. with parallel directrices on the same side of F) but with different focal lengths. See Figure 4.10 (*d*).

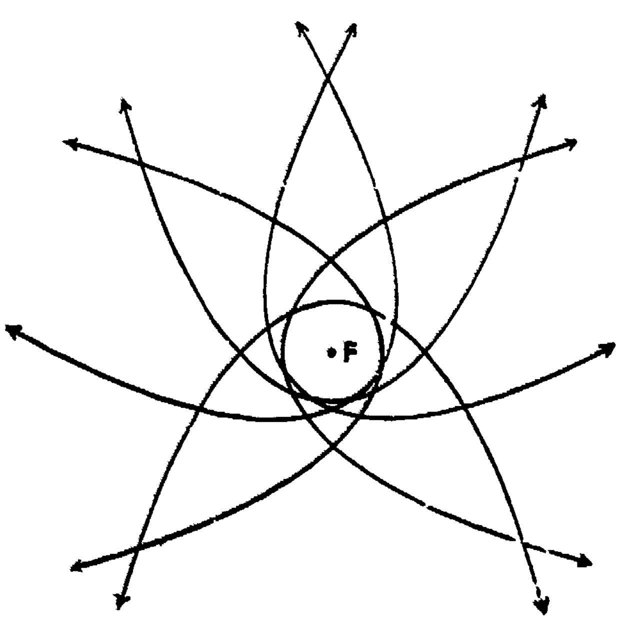


FIGURE 4.10 (c)

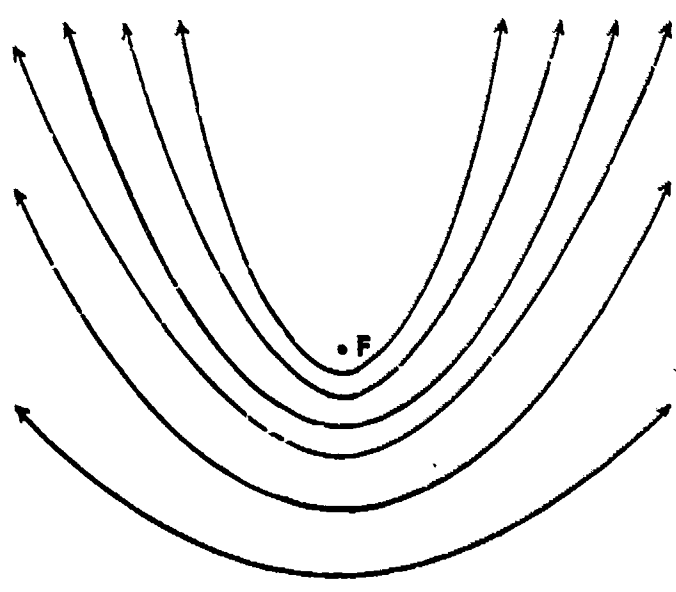


FIGURE 4.10 (d)

The Parabola as a Cartesian Relation

Let us see if, when the parabola is situated on a Cartesian plane, the x and the y co-ordinates of points of it have any pattern or relationship.

1. Take a graph page, choose the x-axis and the y-axis, using *exactly the same unit* as in *the geometrical construction* earlier.

   (N.B. see note regarding scaling later.)

   Here the unit is the inch (the $\frac{1}{4}$ in. labels are merely a helpful device).

   Take the template for S (prepared earlier) and place it with the vertex at O, and the axis of symmetry along $\overrightarrow{Oy}$. (One reason for preferring a separate page to the original one is to avoid the choice of directrix as the x-axis.) Outline S on the graph page and prick F through (the point should be $(0, \frac{1}{4})$).

   Make a list of the co-ordinates of a number of points of the parabola. For example :

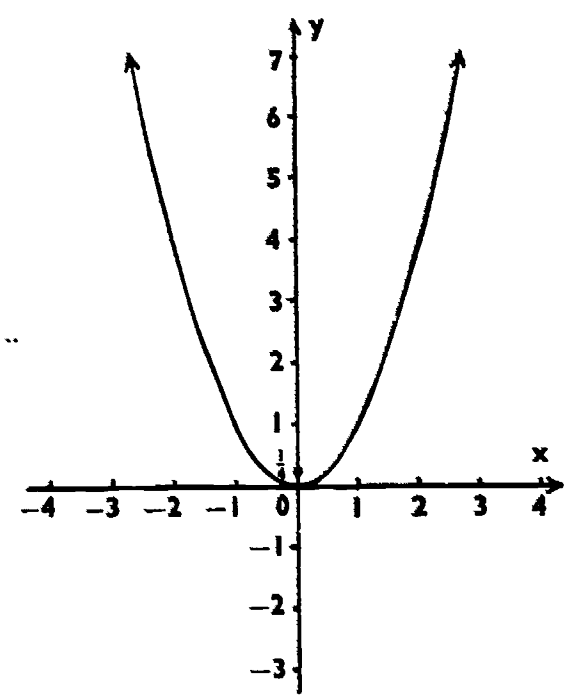


FIGURE 4.11

Points (x, y) on S.	Points (x, y).	
	above S	below S
(1, 1)	$(1, 1\frac{1}{2})$	$(1, \frac{1}{8})$
(2, 4)	—	(2, −1)
(0, 0)	—	—
(−1, 1)	—	—
(−2, 4)	—	—
$(\frac{1}{2}, \frac{1}{4})$	—	—
$(2\frac{1}{2}, 6\frac{1}{4})$	—	—
(1.1, 1.21)	—	—

Notice a pattern ? In each ordered pair the y-co-ordinate, appears to be the same as the square of the x-co-ordinate, i.e. $y = x^2$. Is this true for points above or below the curve ? List and test some. Try some further points both on the curve and off the curve and test that $y = x^2$ and $y \neq x^2$ respectively for these points (or approximately so !). Test the converse aspect, viz. that there are no extra points for which $y = x^2$. Calculate some ordered pairs, plot them, and observe that they do belong to S, e.g. if $x = 0.7$, then $x^2 = 0.49$, so plot (0.7, 0.49) as accurately as possible. Does it appear to belong to S ? Also plot some for which $y > x^2$, and some for which $y < x^2$, e.g. (0.7, 0.6) and (0.7, 0.3) respectively.

Then conclude : " It appears that the parabola S is the set of points for which the y-co-ordinate is equal to the square of the x-co-ordinate ; in symbols—

$$S = \{(x, y) : y = x^2\}."$$

Also :

(i) The region A above S (i.e. in the $\overrightarrow{Oy}$ direction) consists of the set of points for which the y-co-ordinate is greater than the square of the x-co-ordinate, i.e. $A = \{(x, y) : y > x^2\}$. See Figure 4.12 (*a*).

(ii) The region B below S (i.e. in the $-\overrightarrow{Oy}$ direction) consists of the set of points for which the y-co-ordinate is less than the square of the x-co-ordinate, i.e. $B = \{(x, y) : y < x^2\}$. See Figure 4.12 (*b*).

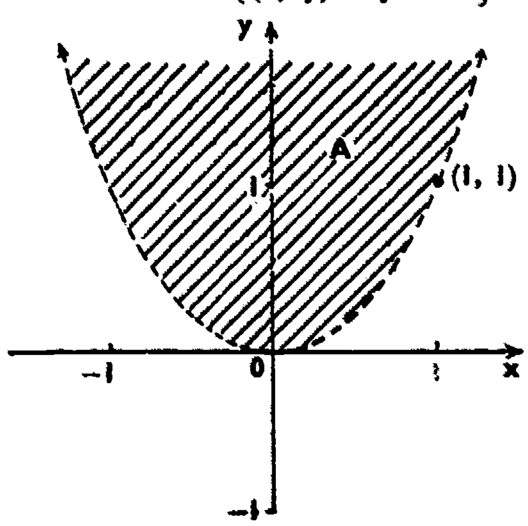


FIGURE 4.12 (a)

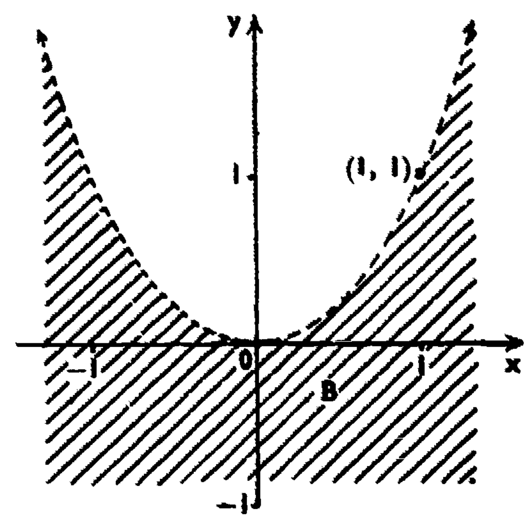


FIGURE 4.12 (b)

2. Treat other parabolas K, T, M, N similarly (on the same graph sheet as S may be helpful).
3. Reverse the procedure by calculating ordered pairs and plotting corresponding points of relations such as $\{(x, y) : y = 3x^2\}$; $\{(x, y) : y \neq 3x^2\}$; $\{(x, y) : y > 3x^2\}$; and others if time permits.
4. Notice the correspondence between—

Focal length	*Equation*	*Shape*
$\frac{1}{4}$ in.	$y = x^2$	Use as the standard shape to compare with other shapes.
$\frac{1}{2}$ in.	$y = \frac{1}{2}x^2$	Shallower and wider.
1 in.	$y = \frac{1}{4}x^2$	Shallower and wider still.
$\frac{1}{8}$ in.	$y = 2x^2$	Steeper and narrower.
?	$y = 3x^2$	
and perhaps		
$\frac{1}{4k}$ in.	$y = kx^2$	where k is a real positive number.
a in.	$y = \frac{1}{4a}x^2$	

5. Make a conclusion : " It appears that any member of the family of parabolas may be identified with one of the family of Cartesian relations of the form $\{(x, y) : y = kx^2\}$, where k is a real positive number."

(We could go further and state $k = \frac{1}{4a}$, *a* units focal length, with brighter students.)

In later years this conclusion is proved to be quite correct.

6. Consolidate with curve *sketching* of given relations such as :

$\{(x, y) :\ y = \frac{1}{3}x^2\}$; $\{(x, y) :\ y \geqslant x^2\}$; and so on.

Show focus and unit on each sketch, and perhaps the directrix.

Scales (Notes for teachers)

It seems important to leave students with a firm mental picture of a standard parabola shape such as S, where the focal length of $\frac{1}{4}$ in. is connected with the relation $\{(x, y) :\ y = x^2\}$ by having the same inch unit of measurement on the Cartesian plane. A great deal of confusion can be caused by scaling too early. *If it is done at all*, compare bounded regions such as those shaded below. It is a consequence of the [illegible]s definition that all (infinite) parabolas are expansions of other parabolas.

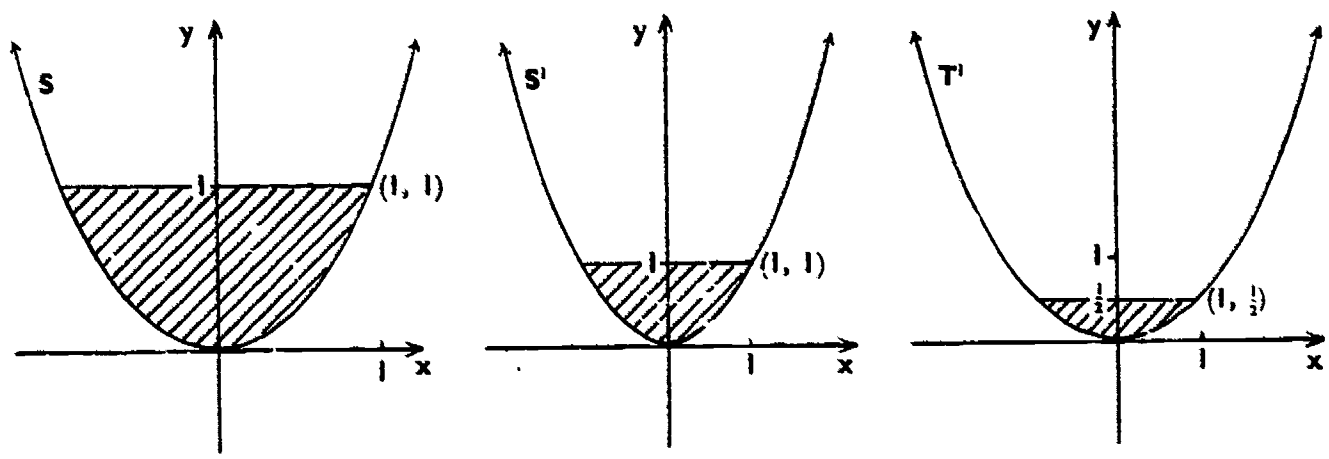


FIGURE 4.13 (a) FIGURE 4.13 (b) FIGURE 4.13 (c)

Consider :

$S = \{(x, y) :\ y = x^2\}$ on a 1 in. to 1 unit scale. Figure 4.13 (*a*).

$S' = \{(x, y) :\ y = x^2\}$ on a 1 in. to 2 units scale. Figure 4.13 (*b*).

$T' = \{(x, y) :\ y = \frac{1}{2}\, x^2\}$ on a 1 in. to 2 units scale. Figure 4.13 (*c*).

S and S′ appear to be different shapes, while S and T′ appear to be congruent.

Drawing in corresponding chords as in Figure 4.13 above shows the closed region of S as an expansion of that of S′, but it is neither congruent nor similar to that of T′.

By calculating the focal lengths of the geometrical shapes which are the graphs after scaling we see why –

for S, $\frac{1}{4}$ in. ; for S′, $\frac{1}{4}$ of $\frac{1}{2}$ in. $= \frac{1}{8}$ in. ; for T′, 2 x $\frac{1}{4}$ of $\frac{1}{2}$ in. $= \frac{1}{4}$ in.

Further, if different scales are used on the x and the y axes, e.g.

$$\left.\begin{array}{l}\text{1 in. to 1 unit on the x-axis}\\ \text{1 in. to 3 units on the y-axis}\end{array}\right\}\ \text{for } S'' = \{(x, y) :\ y = x^2\}$$

then the parabola S is being given a " one-way compression " toward the x-axis (or a *dilation* of $\frac{1}{3}$ from the x-axis). See Figure 4.14

Observe that the new curve appears to be a parabola with focal length $\frac{3}{4}$ in., and appears to be congruent to $W = \{(x, y) :\ y = \frac{1}{3}\, x^2\}$ on the geometrically true scale. It is a lucky property of infinite parabolas that expansions and dilations transform them into other parabolas. Basically, the shape of $\{(x, y) :\ y = x^2\}$ is the geometrical shape called a parabola having focal length $\frac{1}{4}$ of the unit used on the Cartesian plane.

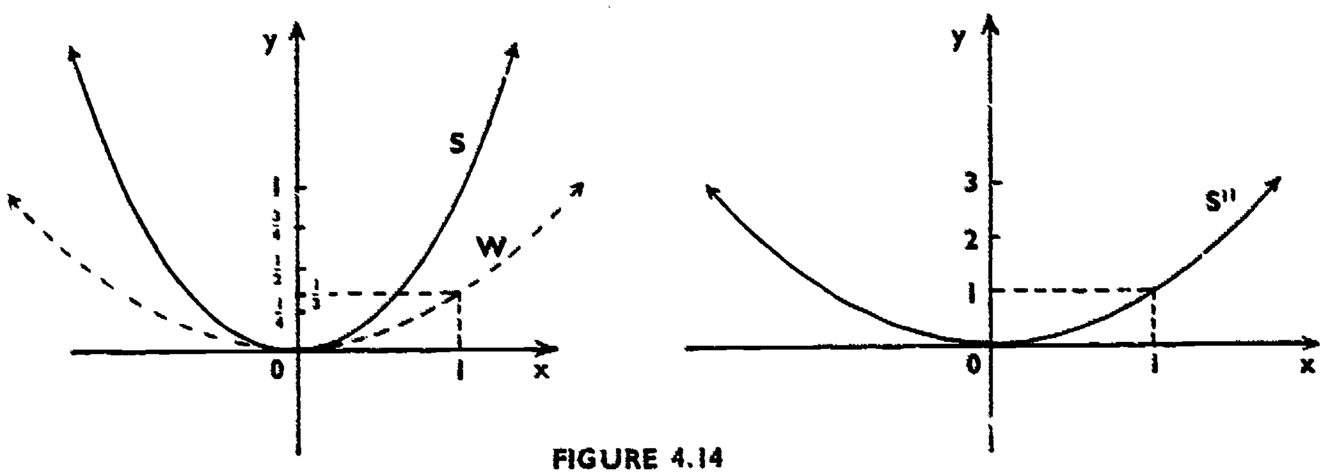


FIGURE 4.14

Transformations

Some of the modified mathematics group of students may be able to extend to the simplest transformations ; the others are included briefly to indicate follow-up for full mathematics groups. The parabola may be placed in any other position on the Cartesian plane. Let us look at the equation, i.e. the relationship between the x-co-ordinate and the y-co-ordinate for points on the parabola.

Take the template S prepared earlier and place it in various positions on a graph page. Repeat with other templates. (N.B. Do not alter the scale in making the templates.) Deduce equations of parabolas in new positions thus :

1. *Reflection in* $\overleftrightarrow{Ox}$ (*or* "*turn over x-axis*") (See Figure 4.15).

Look at the co-ordinates of a point P of S and its image P′ of S′. "Every y-co-ordinate has the same magnitude but opposite sign," or "Every y-co-ordinate i now negative, but still of size x^2."

Arrive at $y = -x^2$, (and $y = -kx^2$). Try plotting $\{(x, y) : y = -2x^2\}$.

2. *Translation in the* $\overrightarrow{Oy}$ *direction* (See Figure 4.16).

"Each y-co-ordinate is c more than the old one." This leads to $y = x^2 + c$ (and to $y = kx^2 + c$). Try plotting $\{(x, y) : y = x^2 + 2\}$. If proceeding to step 3, transform to $y - c = kx^2$.

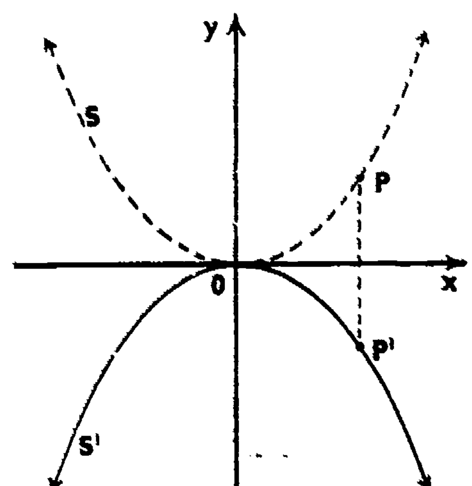


FIGURE 4.15

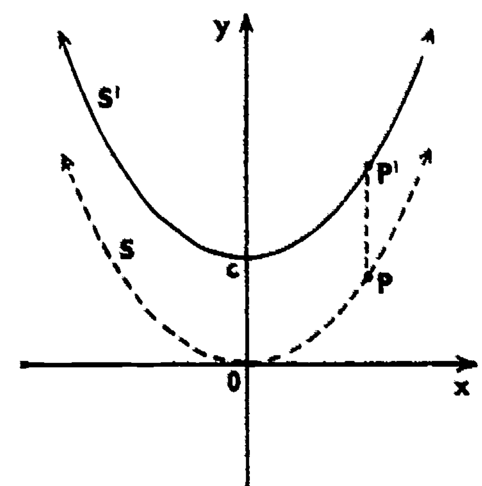


FIGURE 4.16

3. *Translation in the $\overrightarrow{Ox}$ direction* (See Figure 4.17).

"y-co-ordinates stay the same, but the x-co-ordinates are d too great. We can only square $(x - d)$ if we wish to calculate y correctly."

This leads to $y = (x - d)^2$; and $y = k\,(x - d)^2$.

4. *Dilation from $\overleftrightarrow{Ox}$* (See Figure 4.18).

Informal treatment, e.g. a dilation of three from $\overleftrightarrow{Ox}$ as a stretch in the $\overrightarrow{Oy}$ (or $-\overrightarrow{Oy}$) direction keeping the points on $\overleftrightarrow{Ox}$ firmly held in the present position, so that all y-co-ordinates are three times as large. Hence, for the new curve we would need the relationship $y = 3\,x^2$.

Students can now recognise the geometrical shape as that of the parabola of focal length $\frac{1}{3}$ of $\frac{1}{4}$ in. $= \frac{1}{12}$ in.

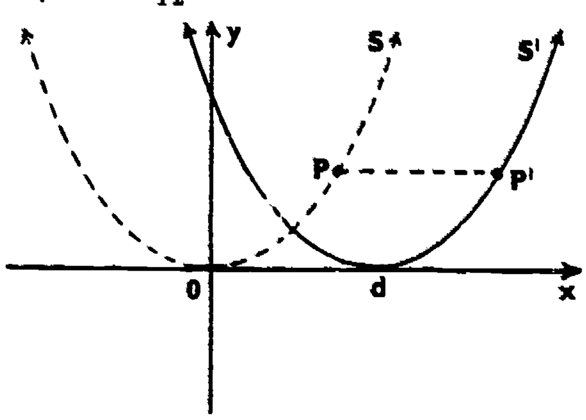


FIGURE 4.17

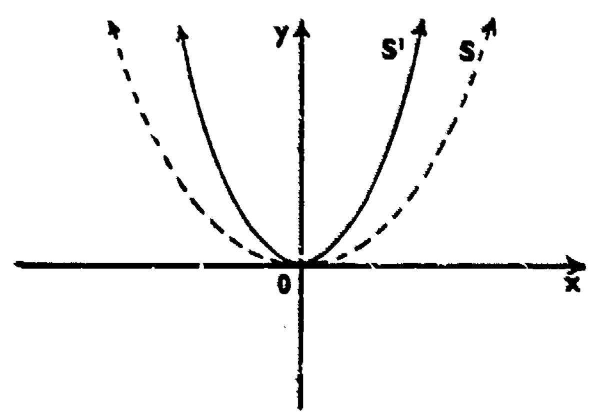


FIGURE 4.18

5. *Rotation about O* (See Figure 4.19).

Sketching, using a template, can be done. Students might discuss the 90° clockwise rotation, giving the equation with x and y interchanged, i.e. $x = y^2$.

6. *Combinations* (Keep to simple ones) (See Figure 4.20).

$y = (x - d)^2 + c = x^2 - 2dx + (d^2 + c)$; and lead gradually to, for example, $y = 2\,(x - 3)^2 - 5 = 2x^2 - 12x + 13$.

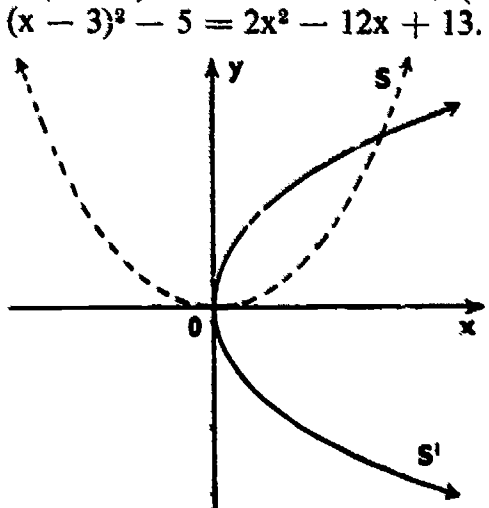


FIGURE 4.19

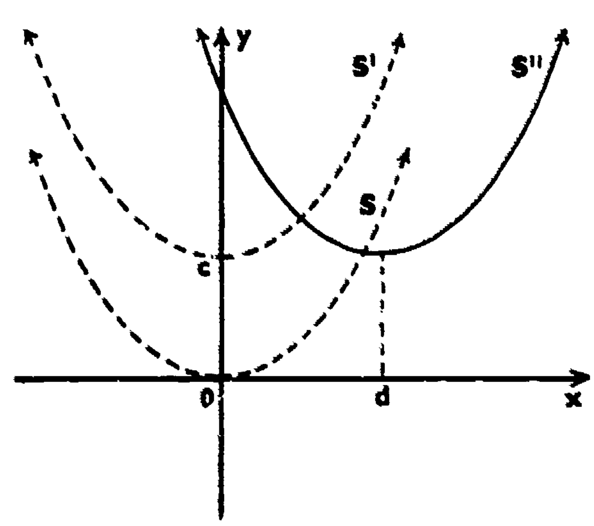


FIGURE 4.20

Sketching the graph of any relation with a quadratic equation now provides strong motivation for " completing the square ". Form III exercises should be kept very simple.

Making Square and Square Root Tables

The parabola S (prepared earlier) may be used for this purpose.

(*a*) Record x in one column, then read off the corresponding y ($= x^2$) and put into the square column, e.g. 1.2, 1.45 approx.

(*b*) Record y, then read off backwards the corresponding values of x. These are the numbers x which when squared are equal to y.

e.g. $y = 2\frac{1}{4}$

$1\frac{1}{2}$ is a corresponding value of x ; $y = x^2 \Rightarrow 2\frac{1}{4} = (1\frac{1}{2})^2$;

$-1\frac{1}{2}$ is another corresponding value of x ; $y = x^2 \Rightarrow 2\frac{1}{4} = (-1\frac{1}{2})^2$;

and these imply that a square root of $2\frac{1}{4}$ is $1\frac{1}{2}$; another is $-1\frac{1}{2}$.

(Note : The sign $\sqrt{}$ reads "the positive square root of".

$\sqrt{2\frac{1}{4}} = 1\frac{1}{2}$; $-\sqrt{2\frac{1}{4}} = -1\frac{1}{2}$)

The Straight Line

We may regard a line as consisting of a set of points, so let us express it also as a Cartesian relation. A line is one of the undefined concepts we accept in our study of geometry.

1. On a graph page, choose an x-axis and a y-axis and use, as before, the inch unit. Start by using a ruler to place line L_0 along the x-axis. Read off the co-ordinates of some of its points, and note that for each point the y-co-ordinate is zero. Read the co-ordinates of some points above L_0 and some points below L_0.

Points on the line. (x, y)	Points not on the line.	
	A_0 Points above.	B_0 Points below.
(1, 0)	(1, 2)	(1, − 3)
(−2, 0)	—	—
($\frac{1}{4}$, 0)	—	—
—	—	—
—	—	—
—	—	—
—	—	—

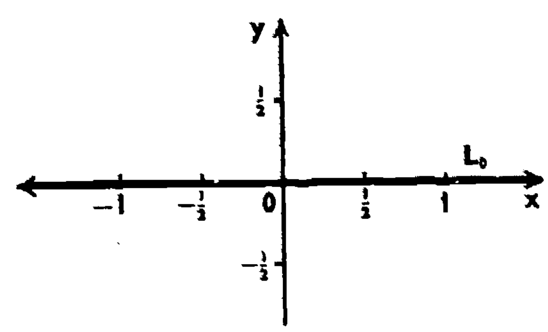


FIGURE 4.21 (a)

Proceed, as for the parabola, establishing :

$L_0 = \{(x, y) : y = 0\}$,
$A_0 = \{(x, y) : y > 0\}$,
$B_0 = \{(x, y) : y < 0\}$.

2. Suppose we now slope the line a little, as $L_{\frac{1}{2}}$. You can think of walking or cycling along L_0, a flat road, *in $\overrightarrow{Ox}$ direction*, and of travelling uphill along $L_{\frac{1}{2}}$, but with *the same direction* $\overrightarrow{Ox}$ in mind. See Figure 4.21 (*b*). Choose a hill with gradient $\frac{1}{2}$, i.e. a rise of $\frac{1}{2}$ in. in a run of 1 in.

Examine points as before, and deduce :

$L_{\frac{1}{2}} = \{(x, y) : y = \frac{1}{2}x\}$;

$A_{\frac{1}{2}} = \{(x, y) : y > \frac{1}{2}x\}$;

$B_{\frac{1}{2}} = \{(x, y) : y < \frac{1}{2}x\}$.

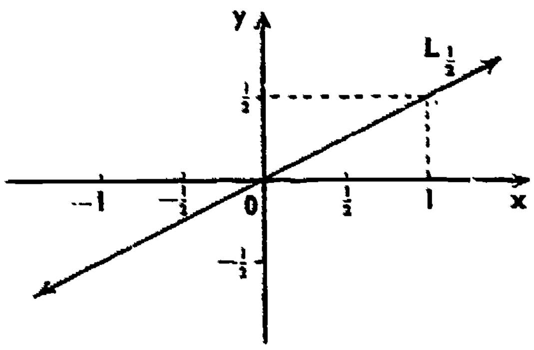


FIGURE 4.21 (b)

Points on line $L_{\frac{1}{2}}$ (x, y)	Points not on line $L_{\frac{1}{2}}$.	
	$A_{\frac{1}{2}}$ Points above.	$B_{\frac{1}{2}}$ Points below.
$(1, \frac{1}{2})$	—	—
$(2, 1)$	—	—
$(3, 1\frac{1}{2})$	—	—
$(-1, -\frac{1}{2})$	—	—
$(0, 0)$		

3. Increase the gradient to 1, i.e. a rise of 1 in. in a run of 1 in. (line L_1).
   Increase the gradient to 2, i.e. a rise of 2 in. in a run of 1 in. (line L_2).
   Express these as relations. See Figure 4.21 (*c*).

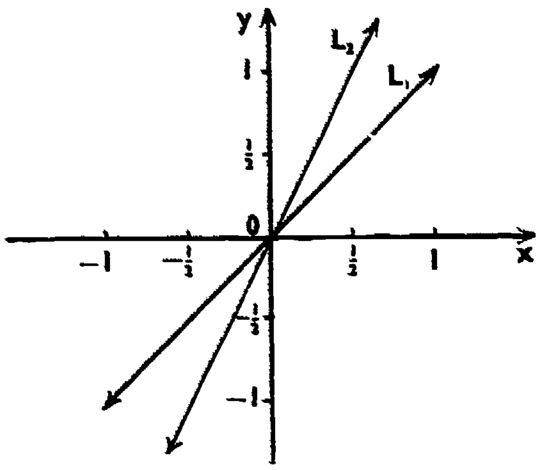


FIGURE 4.21 (c)

4. Now you will need a helicopter for line H. Express it, too, as a relation. Note that A and B are not very appropriate for the inequalities sets. See Figure 4.21 (*d*).

Points on line H. (x, y)	Points not on line H.	
	Points on $\overrightarrow{Ox}$ side of H.	Points on $-\overrightarrow{Ox}$ side of H.

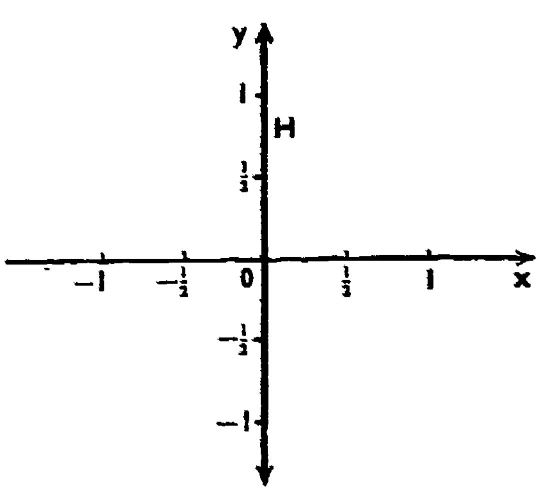


FIGURE 4.21 (d)

5. Now start coming downhill; describe the gradient as down 2 in. in 1 in., or the opposite to a rise of 2 in. in 1 in.; this could well be expressed as -2 in. in 1 in.

Find relations for L_{-2} and L_{-1}.

See Figure 4.21 (*e*).

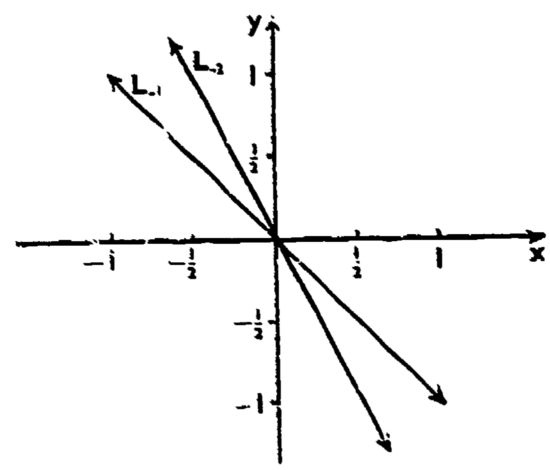


FIGURE 4.21 (e)

6. Reverse the procedure by calculating ordered pairs and plotting corresponding points of relations such as $\{(x, y): y = 3x\}$ and $\{(x, y): y > 3x\}$, etc. Observe the gradient of the lines.

7. Make a list as follows:

Line	*Gradient*	*Equation*
L_0	flat: 0 rise in 1 in.	$y = 0$
$L_{\frac{1}{2}}$	$\frac{1}{2}$ in. rise in 1 in.	$y = \frac{1}{2}x$
⋮	⋮ ⋮ ⋮	⋮ ⋮ ⋮
H	straight up—does not have a gradient.	$x = 0$
L_{-2}	(-2) in. rise in 1 in.	$y = -2x$
⋮	⋮ ⋮ ⋮	⋮ ⋮ ⋮
and perhaps L_k	k in. rise in 1 in.	$y = kx$.

8. Make a conclusion: "It appears that any member of the family of lines through 0 may be identified with one of the family of Cartesian relations either of the form $\{(x, y): y = kx\}$, where k is a real number (positive, negative, or zero) or of the form $\{(x, y): x = 0\}$."

Also, if k is positive, the line slopes uphill in the $\overrightarrow{Ox}$ direction; the greater k, the steeper the slope.

If $k = 0$, the line is flat.

If k is negative, the line slopes downhill in the $\overrightarrow{Ox}$ direction; the greater the magnitude of k the steeper the slope.

(Note that 2 points only need be plotted now, although plotting 3 is safer.)

Further, the relations

$\{(x, y): y > kx\}$ or $\{(x, y): x > 0\}$;

and $\{(x, y): y < kx\}$ or $\{(x, y): x < 0\}$

are regions on opposite sides of the corresponding line.

Plot the line itself (using a broken line), then determine which side by plotting one or two points of the relation. See Figure 4.21 (*f*).

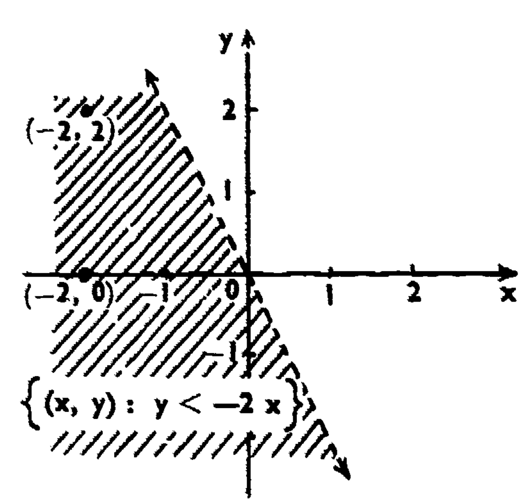


FIGURE 4.21 (f)

Scale : For a true picture of gradient you need the same scale on both axes.

Translation in $\overrightarrow{Oy}$ direction may be used to investigate further positions of lines, and translation along $\overrightarrow{Ox}$ for lines parallel to $\overleftrightarrow{Oy}$. Refer to the parabola section. Sketches are to be encouraged rather than the drawing of accurate graphs.

Sketches : Many quick sketches should now be made of relations such as $\{(x, y): y \geqslant 3x\}$; $\{(x, y) : y + x = 1\}$.

Further Cartesian Relations

If students are keen to investigate other curves, have them suggest equations, e.g. $y = x^3$, to replace the dots in $A = \{(x, y) : \ . \ . \ . \ .\}$.

Calculate ordered pairs and plot corresponding points, and join by a smooth curve as a guess at the complete set A. Test co-ordinates of several points on A as a check. and also co-ordinates of several points not on A to show that for them the equation is *not* satisfied.

SECTION 5

POLAR RELATIONS

In Section 4 we saw that a given starting-point O and a number of units in a particular direction determine exactly the position of a point in a plane. The given starting-point is called the *pole*. A relation which specifies the positions of points by this means is called a *polar relation.*

Example 1 : {points in the same direction as Q from O} all lie in the same ray, with initial point O.

Example 2 : {points the same distance from O} lie on a circle, centre O.

Example 3 : If {P} is specified by distance 3 units in the $\overrightarrow{OQ}$ direction, we have a "one point" set, namely P itself. See Figure 5.1.

We can specify circles about O by their radius lengths. How can we specify direction ?

There are two convenient methods. We must first choose a starting ray $\overrightarrow{Ox}$. Then proceed as follows :

Method 1 : Imagine a very long thin rod L hinged at O. Rotate L to the position OQ ; measure the rotation, $\alpha°$ (anticlockwise rotation is conventional).

Then the number α determines the direction $\overrightarrow{OQ}$. See Figure 5.2 (*a*).

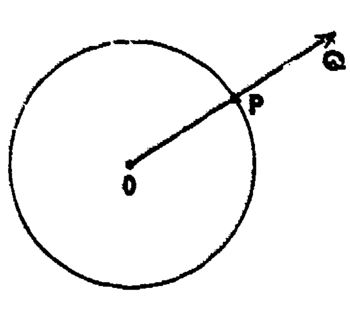


FIGURE 5.1

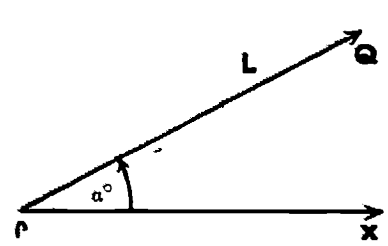


FIGURE 5.2 (a)

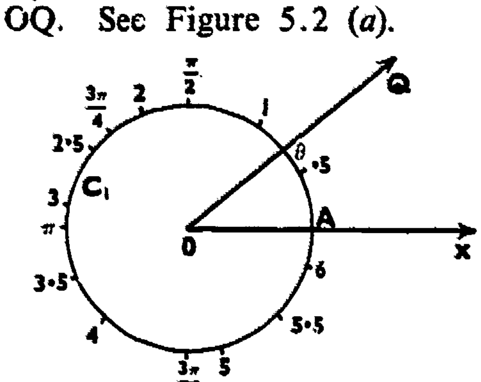


FIGURE 5.2 (b)

Method 2 : Const t a circle C_1, of 1 in. radius length, with O as centre. Let C_1 cut $\overrightarrow{Ox}$ at A. Starting from A, mark arcs of C_1 of length 0.1 in., 0.2 in.,

. . 1 in., $\frac{\pi}{2}$ in., 2 in., $\frac{3\pi}{4}$ in., 2.5 in., 3 in., π in., 3.5 in., 4 in., $\frac{3\pi}{2}$ in., 5 in., 5.5 in., 6 in. (A practical method is described in Section 2, Exercise 2.3.)

The direction $\overrightarrow{OQ}$ is then determined by the number θ on C_1, at which $\overrightarrow{OQ}$ intersects C_1. See Figure 5.2 (*b*).

Method 2 may have advantages in that—

(*a*) it assists the work on circumferences by providing a memory "picture";

(*b*) it gives the "same units" idea as in graphical work where we have used x and y-co-ordinate axes;

(*c*) it can give the θ conventionally used in radian measure;

(*d*) it gives less cumbersome equations.

However Method 1 can give some pleasant results also.

Plotting Points

Allow pupils to explore freely the method of locating a point on a plane by the method of using the distance and the direction from a fixed point. "Graph" paper like that shown below is useful for these exercises. See Figures 5.3 (*a*) and 5.3 (*b*) Figure 5.3 (*a*) is to be used with Method 1 above, and Figure 5.3 (*b*) with Method 2 above.

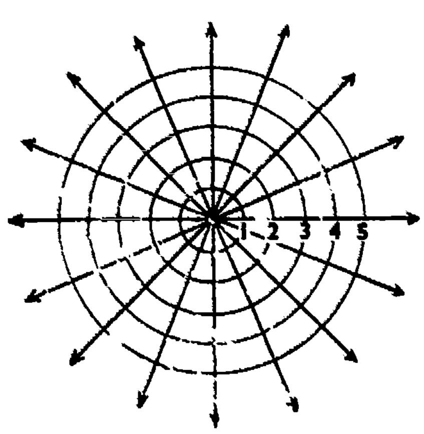


FIGURE 5.3 (a)

OR

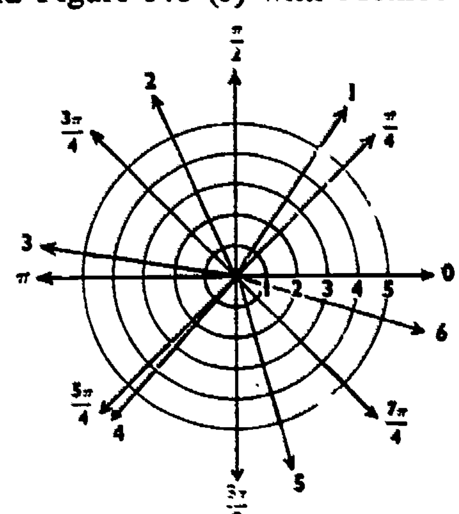


FIGURE 5.3 (b)

The grid consists of a set of concentric circles, together with a set of rays beginning at the centre. Practice plotting several points by "Going" from O.

Exercise (*a*) Go 2 inches in the direction 250°. See Figure 5.4 (*a*).

Exercise (*b*) Go 2 inches in the direction 3 inches along C. See Figure 5.4 (*b*).

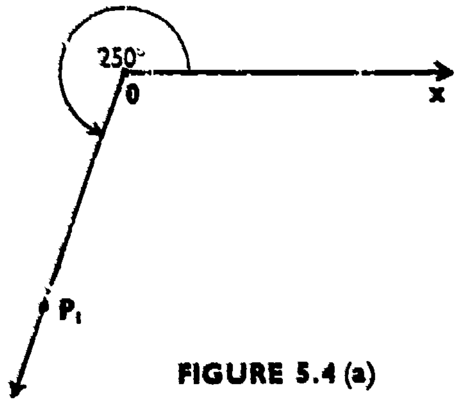


FIGURE 5.4 (a)

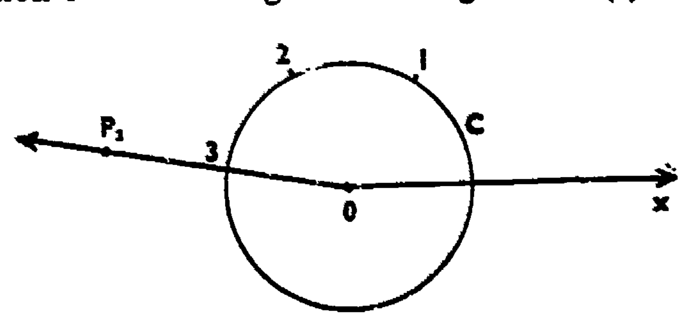


FIGURE 5.4 (b)

The use of square brackets e.g. $[r, \alpha^c]$ or $[r, \theta]$ is suggested to prevent confusion with the use of the usual x-co-ordinates and y-co-ordinates, e.g. (x, y).

Geometrical Figures and Polar Relations

It is interesting to explore some unusual curves by means of polar relations.

Exercise 1 : $\{[r, \alpha^\circ] : r = 3\}$

The direction (α°) may change freely, but "distance from O" (namely, r) remains at a fixed value of 3 units.

Some ordered pairs satisfying the relation are :

$[3, 0^\circ]$, $[3, 10^\circ]$, $[3, 30^\circ]$, $[3, 60^\circ]$, $[3, 90^\circ]$, . . . $[3, 180^\circ]$, . . . $[3, 270^\circ]$ . . .

The corresponding graph must be a circle, with centre at the pole O, and radius 3 units. See Figure 5.5 *(a)*.

Exercise 2 : $\{[r, \theta] : \theta = 3\}$

Here the "direction" remains unchanged but the "distance" can vary from 0 to any number, without limit. See Figure 5.5 *(b)*.

Some ordered pairs satisfying the relation are :

$[0, 3]$; $[1, 3]$; $[2, 3]$; . . $[100, 3]$; . . .

Note : The 3 in this relation is a measure of arc length (of unit circle) determined as in Method 2 on page 66.

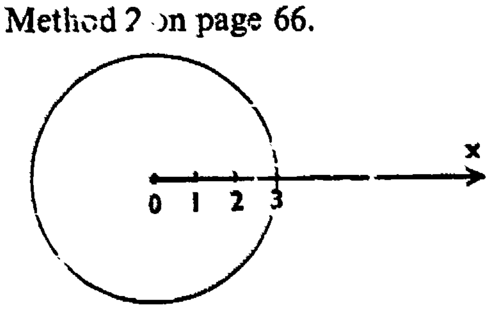


FIGURE 5.5 (a)

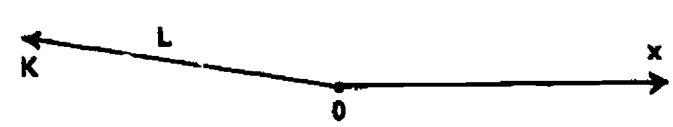


FIGURE 5.5 (b)

Exercise 3 : $\{[r, \theta] : r = \frac{1}{2}\theta\}$

Typical ordered pairs for this relation are :

$[0, 0]$; $[\frac{1}{4}, \frac{1}{2}]$; $[\frac{1}{2}, 1]$; $[\frac{3}{4}, 1\frac{1}{2}]$; $[1, 2]$; . . $[2, 4]$; . . . $[3, 6]$; . . .

(These ordered pairs may be set out in a table of values for r and θ.)

Using polar graph paper, or grid, the resulting graph will be like that shown in Figure 5.6 *(a)*.

With the help of polar graph paper we can explore the graphs of the following relations. In each case we list sufficient ordered pairs to allow the graph to be drawn.

1. $\{[r, \alpha^\circ] : r = \frac{\alpha}{100}\}$.
2. $\{[r, \alpha^\circ] : r + \frac{\alpha}{100} = 4\}$.
3. $\{[r, \alpha^\circ] : r = 2^{\frac{\alpha}{100}}\}$.
4. $\{[r, \alpha^\circ] : \alpha = 600 \log r\}$.
5. $\{[r, \alpha^\circ] : r = \frac{360}{\alpha}\}$.

6. $\{[r, \theta] : \ r + \frac{\theta}{2} = 4\}$. See Figure 5.6 (*b*).

7. $\{[r, \theta] : \ r = 2^{\frac{1}{2}\theta}\}$. See Figure 5.6 (*c*).

8. $\{[r, \theta] : \ \theta = 6 \log r\}$.

9. $\{[r, \theta] : \ r = \frac{3}{\theta}\}$. See Figure 5.6 (*d*).

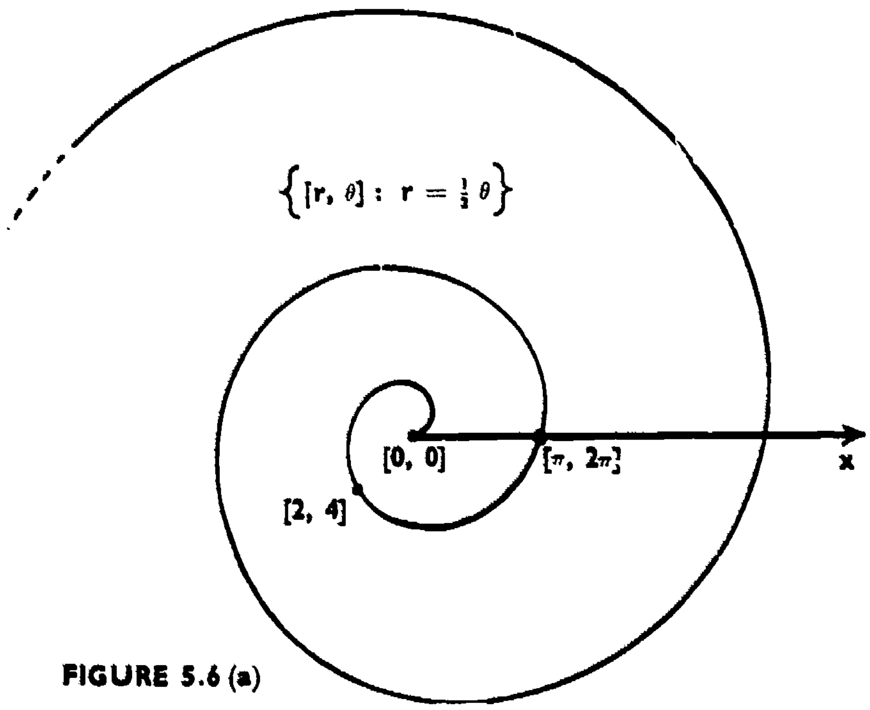


FIGURE 5.6 (a)

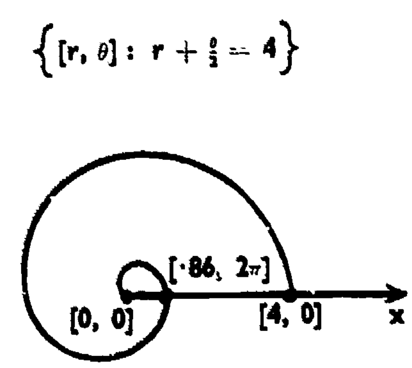


FIGURE 5.6 (b)

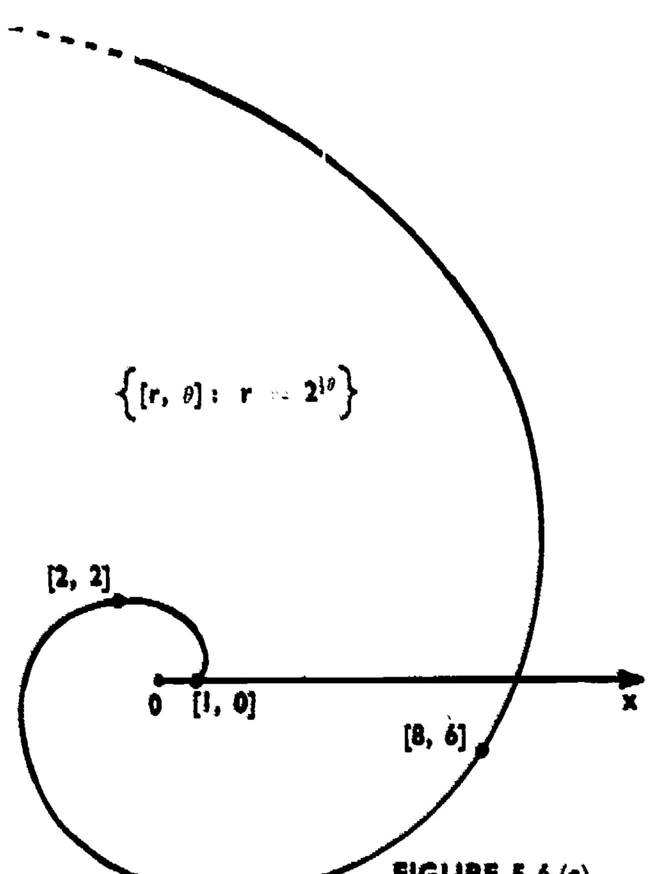


FIGURE 5.6 (c)

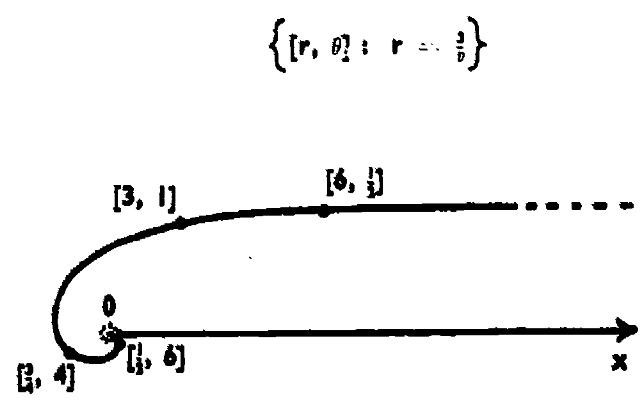


FIGURE 5.6 (d)

If any trigonometry has been learnt, the rose-leaf patterns of the following relations are most intriguing.

(*a*) $\{[r, \theta] : r = | 3 \sin \theta | \}$. See Figure 5.7 (*a*).

(*b*) $\{[r, \theta] : r = | 3 \sin 3\theta | \}$. See Figure 5.7 (*b*).

In these relations the | | signs indicate that the modulus or numerical value of the expression is required.

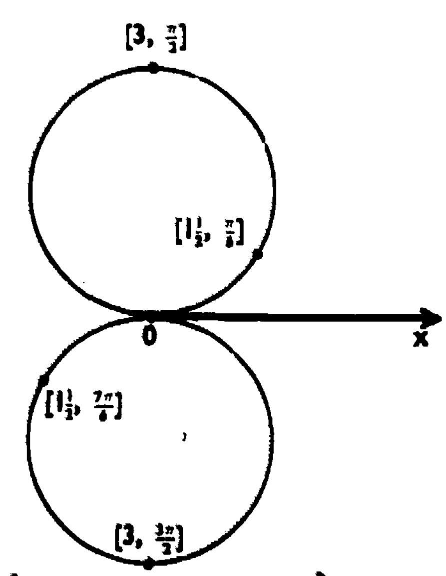
e.g. For $r = | 3 \sin \theta |$ and $\theta = \frac{5\pi}{4}$,

$$\text{then } 3 \sin \theta = 3 \sin \frac{5\pi}{4}$$

$$= 3 \times \left(-\frac{1}{\sqrt{2}}\right)$$

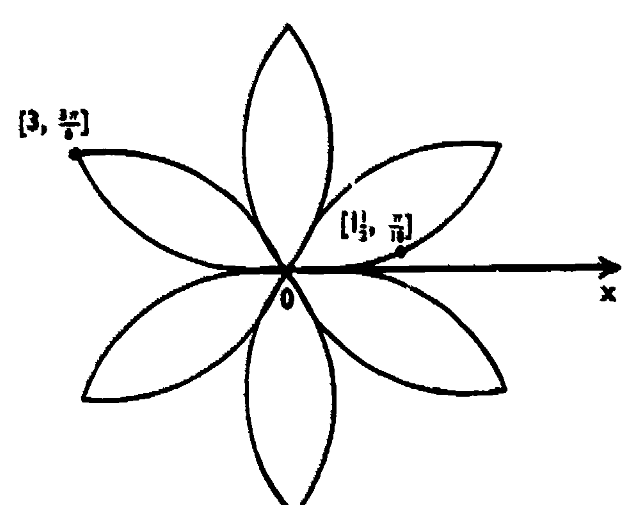
$$= -\frac{3}{\sqrt{2}}$$

$$\text{but } \left| 3 \sin \frac{5\pi}{4} \right| = \left| \frac{-3}{\sqrt{2}} \right| = \frac{3}{\sqrt{2}} \simeq 2{\cdot}12.$$



$\left\{[r, \theta] : r = | 3 \sin \theta | \right\}$

FIGURE 5.7 (a)



$\left\{[r, \theta] : r = | 3 \sin 3\theta | \right\}$

FIGURE 5.7 (b)

If trigonometry has not been learnt, then the following method, using very light construction lines, may be tried.

With centre 0 and radius length 3 in. construct circle C_3.

Construct lightly the ray with direction θ to cut C_3 at K. Draw $\overline{KM}$ perpendicular to $\overrightarrow{Ox}$.

Take radius length equal to KM on your compass. This is the r that corresponds to θ.

Mark P, i.e. $[r, \theta]$, in red on the ray $\overrightarrow{OK}$ so that OP = KM. See Figure 5.8 (*a*). Collect many positions of P.

Is there a geometrical figure corresponding to $\{[r, \theta] : r = d\ (K, M)\}$? See Figure 5.8 (*b*).

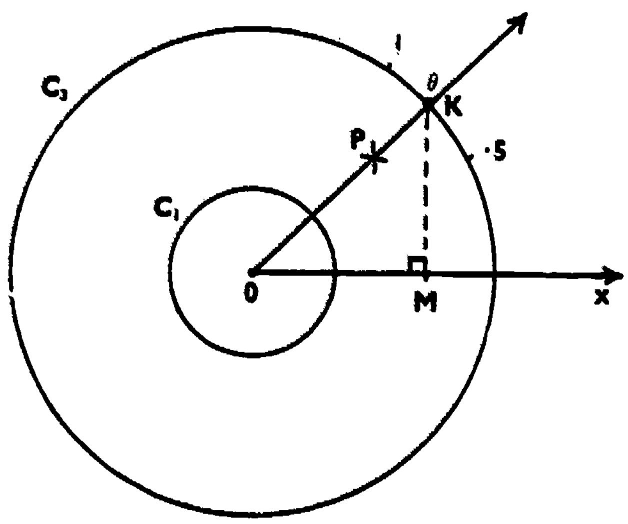
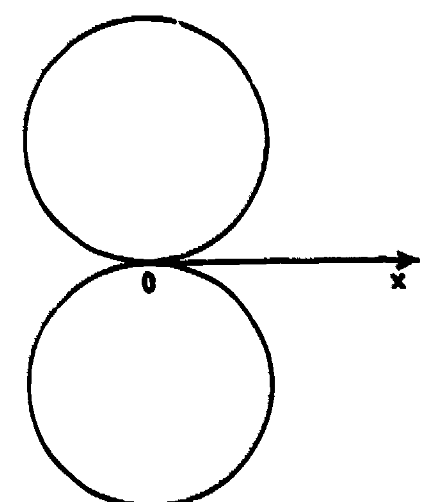


FIGURE 5.8 (a)



$\{[r, \theta] : r = d\ (KM)\}$

FIGURE 5.8 (b)

Extensions

At first it is probably enough to restrict to $0^\circ \leqslant \alpha^\circ < 360^\circ$ or $0 \leqslant \theta < 2\pi$ and to $r \geqslant 0$. However some teachers may wish to extend to :

(i) $\alpha^\circ > 360^\circ$ or $\theta > 2\pi$;

(ii) α or $\theta < 0$;

(iii) $r < 0$ (i.e. $|r|$ units in $\overrightarrow{QO}$ direction).

The construction above might lead to interest in sin θ or sin α° as the y-co-ordinate of the point θ on C_1, and hence to making students' own tables for sine.

If this is extended to graphing on a z—θ Cartesian plane, students find the resulting shape very pleasing. See Figure 5.9.

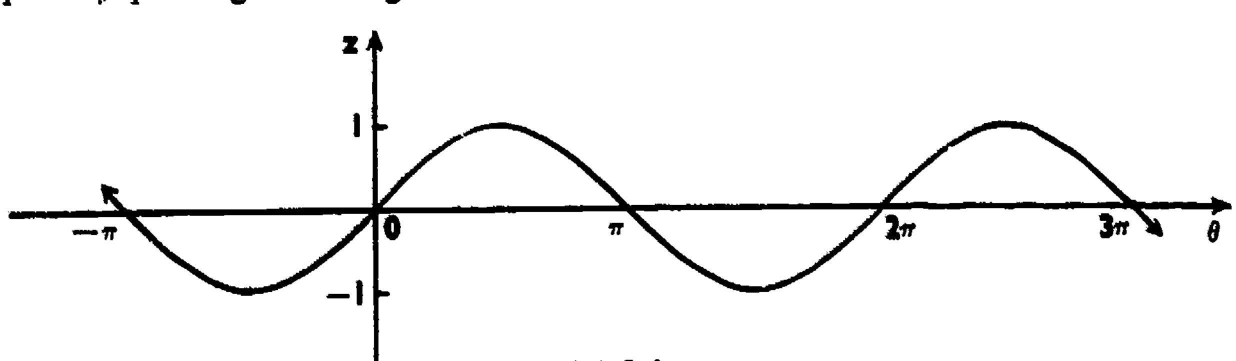


FIGURE 5.9

C. H. Rixon, Government Printer, Melbourne.