DOCUMENT RESUME

ED 054 523		EA 003 638
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TITLE	A Cost/Effectiveness Model for Educational Programs.	
INSTITUTION	Educational Testing Service, Princeton, N.J.	
PUB DATE	Dec 70	
NOTE	66p.	
EDRS PRICE	MF-$0.65 HC-$3.29	
DESCRIPTORS	*Cost Effectiveness; Criteria; *Decision Making; Educational Programs; Management; *Mathematical Applications; *Mathematical Models; Models; *Set Theory	

ABSTRACT

Most decision problems are those in which a choice among multiple-objective alternatives must be made. The central difficulty of such decision problems lies in finding single decision criteria that combine the decisionmaker's objectives and interests in an acceptable way. In this paper, a general procedure for the construction of such single decision criteria is presented. This general procedure is then applied to the construction of a decision criterion for a "two-objective" decision problem such as which pupils, if any, should be enrolled in which educational programs when cost and effectiveness are essential factors. The use of the resulting decision model is illustrated in detail. (Author)

RESEARCH BULLETIN

RB-70-70

A COST/EFFECTIVENESS MODEL FOR EDUCATIONAL PROGRAMS

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Educational Testing Service
Princeton, New Jersey
December 1970

A Cost/Effectiveness Model for Educational Programs[1]

1. Decision Criteria

The decision problem to which this paper is addressed is: Which pupils, if any, should be enrolled in which educational programs when one is interested in the programs' cost and "effectiveness"?

The major difficulty of this problem lies in finding a decision criterion which combines the programs' cost and "effectiveness" in an acceptable way. In this section a general procedure for the construction of single decision criteria, which combine the decision-maker's objectives and interests in an acceptable way, is presented.

A complete development of the problem on which the present paper is based appears in the author's doctoral dissertation (Badran, 1970). Available either from the University of Pennsylvania in microfilm form or from ETS.

The Nature of Decision Criteria

First let us explicate the concepts "attribute," "attribute's domain," "attribute's range," "observable attribute," "one's small world," and "ideal state."

To begin with, an attribute of something is a property of that something. The domain of an attribute is the set, containing at least one

[1]Some of the initial work on this paper was done while the author was participating in Chicago Title I Evaluation under contract with the Chicago Board of Education.

element, to which the attribute meaningfully applies, that is, an element of an attribute's domain is thought of as a carrier of the attribute. For example, the domains of the attributes "self-image," "resource-availability" and "a program's success toward the achievement of a given objective" are, respectively, the individuals, the resources and the programs that are under consideration. For the purpose of the present section, we will use "self-image" and "resource-availability" as paradigm examples of attributes. Let us, for the purpose of avoiding monotonous repetition, denote the attributes "self-image" and "resource-availability" by Q_1 and Q_2, and their respective domains by D_1 and D_2, that is,

Q_1 = Self-image,

Q_2 = Resource-availability,

D_1 = Individuals under consideration

and

D_2 = Resources under consideration.

The range of an attribute is any set, containing at least two distinct elements, which partition the attribute's domain into equivalence classes. The elements of an attribute's range are called the attribute's categories. Depending on the nature of an attribute, its range will be equivalent to an interval of the real line or equivalent to a subset of the set of integers. An attribute whose range is equivalent to an interval of the real line is said to exhibit continuous variation; otherwise it is said to exhibit categorical variation. Examples of continuously varying attributes are the age, the height and the weight of an individual. Although the units of measurements for these attributes are different, they all have the interval $[0,\infty)$ as their common range. Examples of categorically varying attributes, on the other hand, are the sex, the color and the religion of an individual. The range of the sex attribute, for example, could be represented by any three elements set, e.g., $\{F,N,M\}$, where F = Female, N = Neuter and M = Male. Now, let us explicate the defining proposition, namely, the attribute's range partitions its domain into equivalence classes. Suppose the attributes in which we are interested are Q_1 and Q_2. Let

$$R_1 = \{r_{11}, r_{12}, r_{13}\} ,$$

and

$$R_2 = \{r_{21}, r_{22}\}$$

denote the respective ranges of Q_1 and Q_2, where, for example,

r_{11} = The "low" category of self-image,

r_{12} = The "medium" category of self-image,

and

r_{13} = The "high" category of self-image,

r_{21} = The "expended" category of resource-availability

and

r_{22} = The "nonexpended" category of resource-availability.

Upon evaluation or otherwise, e.g., use of psychological tests and accounting data, each of the individuals and the resources that are under consideration will be paired off with one, and only one, of the above categories. Such pairing off for individuals will partition the domain into three equivalence classes, namely, individuals who are "low" with respect to their self-image, individuals who are "medium" with respect to their self-image and individuals who are "high" with respect to their self-image .

Naturally, any meaningful discourse about Q_1 and Q_2 must be based on the facts that (a) two individuals can differ with respect to their self-image, (b) resources, at two points in time, can differ with respect to their availability and (c) there are nontrivial procedures by which such differences can be identified. An observable attribute is, precisely, one with respect to which a meaningful discourse is possible. That is, more specifically, an attribute is observable if there is a **nontrivial** procedure which maps the attribute's domain into its range. The nontriviality requirement about such procedures is introduced so as to rule out procedures with no support from logic and/or fact. For example, the attribute "self-image" is trivially observable under the following procedure, namely: for each individual under consideration, draw a random number and pair off that individual with r_{11} , r_{12} or r_{13} when the number drawn is 0(mod 3), 1(mod 3) or 2(mod 3).[2] The triviality of this procedure stems from the fact that it is non-reliable and is invalid— nonreliable in the sense that the procedure, applied repeatedly over a short period of time, will not pair off the same individual with one and the same category; invalid, on the other hand, in the sense of lack of logical and/or factual support for the premises on which the procedure is based. The nontrivial procedure, the existence of which insures

[2] When X_1 , X_2 , X_3 and X_4 are nonnegative integers and $X_1 = X_2X_4 + X_3$, one says that $X_1 = X_3 \bmod(X_2)$ and/or $X_1 = X_3 \bmod(X_4)$, e.g., $7 = 1 \bmod(2) = 1 \bmod(3)$.

the observability of an attribute, is called the attribute's associated scale. When Q_1 and Q_2 are observable, their respective associated scales are denoted by T_1 and T_2, e.g., T_1 might be a psychological test and T_2 might be an accounting procedure. Knowledge of D_1, T_1 and R_1 amounts to a formal specification of Q_1, that is,

$$Q_1 \equiv (D_1, T_1, R_1) \; .$$

One's small world is, simply stated, any connected set of attributes in which one is interested. For example, a girl's small world, namely, "the choice of a suitor," might be given by the following set of connected attributes, namely,

$$\left\{\begin{array}{l}\text{The Suitor's Health,}\\ \text{The Suitor's Physical Appearance,}\\ \text{The Suitor's Financial Circumstances,}\\ \text{The Suitor's Education,}\\ \text{The Girl's Personal Freedom.}\end{array}\right\}$$

As another example, an individual's small world, namely, "the buying of a car," might be given by the following set of connected attributes, namely,

$$\left\{\begin{array}{l}\text{The Car's Cost,}\\ \text{The Car's Seating Capacity,}\\ \text{The Car's Engine Horsepower,}\\ \text{The Individual's Own Safety.}\end{array}\right\}$$

Similarly, when one is interested in the attributes "self-image" and "resource-availability," one's small world, insofar as these attributes are connected, is given by the set $\{Q_1, Q_2\}$.

At any given time the scales T_1 and T_2 will, respectively, pair off elements of D_1 with elements of R_1 and elements of D_2 with elements of R_2. The result of this pairing off is a description, at the given time, of the state of one's small world $\{Q_1,Q_2\}$. For example, one may find that 60%, 30% and 10% of the individuals in question are in the "low," "medium" and "high" categories of self-image, and that 40% and 60% of the resources in question are in the "expended" and "nonexpended" categories of resource-availability. This fact, that is, "the state of one's small world (as indicated by the pairings induced by T_1 and T_2) is so and so," is represented, conveniently and compactly, by the <u>proposition</u>,

$$[p = \hat{p}] \quad ,$$

where

$$\begin{aligned} \hat{p} &= (\hat{p}_1, \hat{p}_2) \\ &= ((\hat{p}_{11}, \hat{p}_{12}, \hat{p}_{13}), (\hat{p}_{21}, \hat{p}_{22})) \\ &= ((0.60, 0.30, 0.10), (0.40, 0.60)) \\ p &= (p_1, p_2) \\ &= ((p_{11}, p_{12}, p_{13}), (p_{21}, p_{22})) \quad , \end{aligned}$$

p_{ij} = The degree of truth of the proposition[3]: T_i paired off D_i with r_{ij} .

[3] The degree of truth of a proposition is equivalent to the degree by which the proposition is supported by the available objective and/or subjective data.

The meaning of the proposition $[p = \hat{p}]$ is: T_1 paired off 60% of D_1 with r_{11} <u>and</u> T_1 paired off 30% of D_1 with r_{12} <u>and</u> T_1 paired off 10% of D_1 with r_{13} <u>and</u> T_2 paired off 40% of D_2 with r_{21} <u>and</u> T_2 paired off 60% of D_2 with r_{22} ; that is, briefly stated, $\hat{p}$ represents the <u>true</u> state of $\{Q_1,Q_2\}$ (at the given time).

The true state $\hat{p}$ represents the pairings induced by T_1 and T_2 . The set of all conceivable pairings will give rise to the set of all conceivable states. Let us denote this set of all conceivable states by P . Naturally, the true state

$$\hat{p} = ((0.60,0.30,0.10),(0.40,0.60))$$

is an element of P . Examples of other elements of P are

$$\bar{p} = ((0,0,1),(0,1))$$
$$p' = ((0.30,0.30,.40),(0.50,0.50))$$
$$p'' = ((0.70,0,0.30),(0.30,0.70))$$
$$p''' = ((0.10,0.80,0.10),(0.60,0.40))$$
$$\underline{p} = ((1,0,0),(1,0)) \quad .$$

The element $\bar{p}$, for example, represents the <u>possibility</u> where all the individuals and resources in question are, respectively, in the "high" category of "self-image" and the "nonexpended" category of "resource-availability."

The concept of the "ideal state" of one's small world can now be introduced. Simply stated, the ideal state of one's small world is the state whose realization is preferred to the realization of any other state. For example, if $\{Q_1,Q_2\}$ is <u>my own</u> small world, and if I am offered a <u>choice</u> between the realization of

$$\overline{p} = ((0,0,1),(0,1))$$

and the realization of any other state, I will invariably choose the realization of $\overline{p}$. A more compact way of stating this fact is as follows. Let

$p' \gtrsim p''$ = The realization of the state p'' is <u>not preferred</u> to the realization of the state p' .

The ideal state $\overline{p}$, then, is assumed to have the privileged position whereby it is possible to assert that

$$\overline{p} \gtrsim p$$

for every other state p in P .

The thesis on the basis of which decision criteria can be constructed can now be stated: <u>One's small-world-behavior is directed at being as close as possible to the ideal state of his small world.</u> Another way of stating this thesis is as follows: Let

Q = One's small world, (e.g. $\{Q_1,Q_2\}$) .

F = The set of all conceivable states of Q ,

$\overline{f}$ = The ideal state of $Q; \overline{f} \in F$,

f', f'' = Any two states of $Q; f' \in F, f'' \in F$.

One's small-world-behavior, then, is such that realization of the state f' is preferred to the realization of the state f'' if, and only if, f' is "closer" to $\overline{f}$ than f'' to $\overline{f}$. In other words, a distance function $d(\cdot,\cdot)$ for which $d(\overline{f},f') \leq d(\overline{f},f'')$ when, and only when, $f' \gtrsim f''$ will serve as a decision criterion.

Construction of Decision Criteria

Let us note, at the outset, that, whenever appropriate, the word "attribute" will be used as a generic denotation for the words "objective," "criterion," "factor" and/or "dimension." Now, the state of one's small world can be viewed as a point in a multidimensional space, the state space. According to this view each one of the attributes is represented along one of the dimensions of this state space. Naturally, most of these attributes do not have a common scale of measurement. To this end a common scale on which the different attributes are measured, namely, the degree of truth of an event-proposition, is introduced. This common scale is, formally speaking, a probability measure which is defined over events of the form: the attribute's domain is mapped by the attribute's associated scale into a subset of the attribute's range. The density-like function which is generated by this probability-like measure is called the attribute's monitor. To illustrate, suppose the attribute in which we are interested is Q_1, "self-image." Let us recall that, formally speaking, we have

$$Q_1 \equiv (D_1, T_1, R_1)$$

where

D_1 = The domain of Q_1 (e.g., individuals in question).

T_1 = Q_1's associated scale (e.g., psychological test).

R_1 = The range of Q_1 (e.g., $R_1 = \{r_{11}, r_{12}, r_{13}\}$; r_{11} = "low", r_{12} = "medium" and r_{13} = "high").

In this case the common scale, namely, the degree of truth of an event-proposition is defined over the following events:

a. D_1 is mapped by T_1 into ϕ , the empty set,

b. D_1 is mapped by T_1 into r_{11} ,

c. D_1 is mapped by T_1 into r_{12} ,

d. D_1 is mapped by T_1 into r_{13} ,

e. D_1 is mapped by T_1 into $\{r_{11},r_{12}\}$,

f. D_1 is mapped by T_1 into $\{r_{11},r_{13}\}$,

g. D_1 is mapped by T_1 into $\{r_{12},r_{13}\}$,

h. D_1 is mapped by T_1 into R_1 .

Suppose the following data are available: D_1 is a set of individuals; upon evaluation, T_1 paired off 60% of D_1 with r_{11} <u>and</u> T_1 paired off 30% of D_1 with r_{12} <u>and</u> T_1 paired off 10% of D_1 with r_{13} . Accordingly, the degree of truth of the proposition " D_1 is mapped by T_1 into r_{11} " is equal to 0.60, that of " D_1 is mapped by T_1 into r_{12} " is equal to 0.30, and that of " D_1 is mapped by T_1 into r_{13} " is equal to 0.10. Similarly, since r_{11} , r_{12} and r_{13} are mutually exclusive and exhaustive categories of Q_1 , the degree of truth of the proposition " D_1 is mapped by T_1 into $\{r_{11},r_{12}\}$, for example, is equal to 0.90, (0.60 + 0.30). The result of this pairing off is a description of the state of the attribute Q_1 . This description is sufficiently characterized by the 3-tuple

$$\hat{p}_1 = (\hat{p}_{11},\hat{p}_{12},\hat{p}_{13})$$
$$= (0.60,0.30,0.10) \quad .$$

This 3-tuple, $\hat{p}_1$, is called a monitor (of the state of Q_1). In general, a monitor of Q_1, p_1 , is given by a convex linear combination of the unit vectors

$$u_{11} = (1,0,0)$$
$$u_{12} = (0,1,0)$$
$$u_{13} = (0,0,1) \quad ,$$

that is,

$$p_1 = (p_{11}, p_{12}, p_{13}) = \sum_{i=1}^{3} p_{1i} u_{1i} \quad ,$$

$$p_{1i} \geq 0 \quad , \qquad \sum_{i=1}^{3} p_{1i} = 1 \quad .$$

In particular, u_{13} is the monitor that describes the state of affairs in which all the individuals in question are in the "high" category of the attribute "self-image."

The monitor space of the attribute Q_k, F_k is the set of all its conceivable monitors. The Cartesian product of the monitor spaces of the attributes in which one is collectively interested, F , is called the monitor space of one's small world Q . This Cartesian product F is precisely the required canonical representation of the state space of one's small world. To illustrate, the monitor space of Q_1, P_1 is the set of all convex linear combinations of the vectors u_{11} , u_{12} , u_{13} ; the monitor space of Q_2, P_2 is the set of all convex linear combinations of the vectors

$$u_{21} = (1,0)$$

$$u_{22} = (0,1) \quad ;$$

and the monitor space of the small world $\{Q_1,Q_2\}$, P , is the Cartesian product of P_1 and P_2 , that is,

$$P = P_1 \times P_2 \quad ;$$

examples of points in which are,

$$\bar{p} = ((0,0,1),(0,1)) \quad ,$$

$$p' = ((0.30,0.30,0.40),(0.50,0.50)) \quad ,$$

$$p'' = ((0.70,0,0.30),(0.30,0.70)) \quad ,$$

$$p''' = ((0.10,0.80,0.10),(0.60,0.40)) \quad ,$$

$$\underline{p} = ((1,0,0),(1,0)) \quad .$$

In general, the monitor space of the connected set of attributes

$$Q = \{Q_1,Q_2,\ldots,Q_N\} \quad ,$$

is given by

$$F = F_1 \times F_2 \times \ldots \times F_N \quad ,$$

where F_k is the monitor space of the attribute Q_k . In this case the states $\bar{f}$, f' , f'' etc. of Q are given by the N -tuples

$$\bar{f} = (\bar{f}_1,\bar{f}_2,\ldots,\bar{f}_N) \quad ,$$

$$f' = (f'_1,f'_2,\ldots,f'_N) \quad ,$$

$$f'' = (f''_1,f''_2,\ldots,f''_N)$$

etc. The monitor space F is appropriately metricized by the distance function

$$d(f',f'') = \{ \sum_{k=1}^{N} ||f'_k - f''_k||^p \}^{1/p} \quad , \quad 1 \leq p < \infty$$

where

$$||f'_k - f''_k||^p = \int_{R_k} |f'_k(r) - f''_k(r)|^p \, d\mu(r|R)$$

R_k = The range of the attribute Q_k ,

$R = \bigcup_{k=1}^{N} R_k$ = The range of Q ,

$f'_k(r)$ = The value taken by the monitor f'_k when evaluated at the point r of the range R_k ;

$\mu(\cdot|\cdot)$ = σ -finite-normalized measure,

in other words $\mu(\cdot|\cdot)$ is a finite measure which is defined over the σ -algebra of the set R , and which has the multiplicative property, namely,

$$\mu(R'|R''') = \mu(R'|R'')\mu(R''|R''')$$

for every subset R' , R'' and R''' of R , such that,

$$R' \subset R'' \subset R''' \quad .$$

This distance function is reduced into a preference function, i.e.,

$d(\overline{f},f') \leq d(\overline{f},f'')$ when, and only when, $f' \gtrsim f''$,

by identifying the measure $\mu(\cdot|\cdot)$ with what I will call "concern measure," $C(\cdot|\cdot)$. To illustrate, suppose one is collectively interested in the attribut. Q_1 , "self-image," and Q_2 , "resource-availability." Let us recall that, formally speaking, we have

$$Q_1 \equiv (D_1,T_1,R_1)$$
$$Q_2 \equiv (D_2,T_2,R_2) \quad ,$$

where

D_1 = The domain of Q_1 (e.g., individuals in question),

T_1 = Q_1's associated scale (e.g., psychological test),

R_1 = The range of Q_1 (e.g., $R_1 = \{r_{11},r_{12},r_{13}\}$; r_{11} = "low," r_{12} = "medium" and r_{13} = "high"),

D_2 = The domain of Q_2 (e.g., resources in question),

T_2 = Q_2 's associated scale (e.g., accounting procedure)

and

R_2 = The range of Q_2 (e.g., $R_2 = \{r_{21},r_{22}\}$;r_{21} = "expended" and r_{22} = "nonexpended").

Now, insofar as one is concerned about the relation of D_k to R_k , $k = 1,2$, the news that the event-proposition

$$[p = \overline{p}]$$

is true will be regarded as the most valuable news item. Let us recall that the meaning of the news item, namely, " $[p = \overline{p}]$ is true" is: T_1 paired off <u>none</u> of D_1 with r_{11} <u>and</u> T_1 paired off <u>none</u> of D_1 with r_{12} <u>and</u> T_1 paired off <u>all</u> of D_1 with r_{13} <u>and</u> T_2 paired off <u>nothing</u> of D_2 with r_{21} <u>and</u> T_2 paired off <u>all</u> of D_2 with r_{22} . In other words the news item " $[p = \overline{p}]$ is true" is a conjunction of news items of the form " $[p_{ki} = \overline{p}_{ki}]$ is true" where

$$"[p_{ki} = \overline{p}_{ki}] \text{ is true}"$$

is the same thing as

$$"T_k \text{ paired off } \overline{p}_{ki} \text{ of } D_k \text{ with } r_{ki} \text{ ,"}$$

e.g.,

$$"[p_{11} = 0] \text{ is true}" \equiv "T_1 \text{ paired off none of } D_1 \text{ with } r_{11} \text{ ."}$$

Let us, for the purpose of simplifying notations, denote the news item " $[p_{ki} = \overline{p}_{ki}]$ is true" by the symbol Er_{ki} . In this case

$Er_{11} = T_1$ paired off <u>none</u> of D_1 with r_{11} ,

$Er_{12} = T_1$ paired off <u>none</u> of D_1 with r_{12} ,

$Er_{13} = T_1$ paired off <u>all</u> of D_1 with r_{13} ,

$Er_{21} = T_2$ paired off <u>nothing</u> of D_2 with r_{21} ,

$Er_{22} = T_2$ paired off <u>all</u> of D_2 with r_{22} .

In general, the news item, namely, " $[f_k(r) = \overline{f}_k(r)]$ is true," will be denoted by Er_k . Accordingly, the news item " $[f_k(r) = \overline{f}_k(r)\}$ is true for every r in R' , where R' is an element of the σ -algebra of R ," is consistently denoted by ER' and one gets

$$ER' = \bigwedge_{r_k \in R'} Er_k \quad ,$$

where "$\wedge$" is the conjunctive connective "and." The concern measure $C(\cdot|\cdot)$ is defined over news items of the form ER' . The quantity $C(ER'|ER)$ is called one's concern for the event ER' relative to that of the event ER . The quantity $C(ER'|ER)$ is interpreted as the "news-value" of the news item ER' relative to that of ER . By this we mean that the interpretive properties of $C(.|.)$ are:

1. $\mathcal{E}R'$ is a worthless news item when, and only when, $C(\mathcal{E}R'|\mathcal{E}R) = 0$,

2. $\{\mathcal{E}R',\alpha'\} \succsim \{\mathcal{E}R'',\alpha''\}$ when, and only when,

$$\alpha' C(\mathcal{E}R'|\mathcal{E}R) \geq \alpha'' C(\mathcal{E}R''|\mathcal{E}R) \quad ,$$

where

$\{\mathcal{E}R',\alpha'\}$ = The gamble which results in $\mathcal{E}R'$ with probability α' or with a worthless news item with probability $1 - \alpha'$.

3. If $\{\mathcal{E}R',\alpha'\} \succsim \{\mathcal{E}R'',\alpha''\}$, then there is $0 < \beta' \leq \alpha'$ such that $\{\mathcal{E}R',\beta'\} \sim \{\mathcal{E}R'',\alpha''\}$, ("$\sim$" indicates indifference between the two news items).

4. The measures $C(\cdot|\cdot)$ and $\mu(\cdot|\cdot)$ are isomorphic under the one-to-one transformations

$$R' \longleftrightarrow \mathcal{E}R'$$

$$\cup \longleftrightarrow \wedge$$

$$\subset \longleftrightarrow \Leftarrow$$

e.g.,

$$R' \subset R' \cup R'' \longleftrightarrow \mathcal{E}R' \Leftarrow \mathcal{E}R' \wedge \mathcal{E}R'' \quad ,$$

where

$\cup$ = The set theoretical operation of "union,"

$\wedge$ = The "and" connective of propositional calculus,

$\subset$ = The set-theoretical operation of "inclusion,"

$\Leftarrow$ = The "implied by" connective of propositional calculus.

The multiplicative property of the concern measure, namely,

$$C(ER'|ER) = C(ER'|ER'')C(ER''|ER) \quad ,$$

whenever $R' \subset R'' \subset R$, suggests that the quantity on the left side of this equation be evaluated as the product of the terms on the right side. In particular, then,

$$C(Er_k|ER) = C(Er_k|ER_k)C(ER_k|ER) \quad .$$

To illustrate, suppose one is collectively interested in the attributes Q_1 , "self-image," and Q_2 , "resource-availability." The news items over which the concern measure is defined are:

$Er_{11} = T_1$ paired off none of D_1 with r_{11} ,

$Er_{12} = T_1$ paired off none of D_1 with r_{12} ,

$Er_{13} = T_1$ paired off all of D_1 with r_{13} ,

$Er_{21} = T_2$ paired off nothing of D_2 with r_{21} ,

$Er_{22} = T_2$ paired off all of D_2 with r_{22}

and the conjunctions, namely,

$$Er_{11} \wedge Er_{12} \quad , \quad Er_{11} \wedge Er_{13} \quad , \quad Er_{12} \wedge Er_{13} \quad ,$$

$$ER_1 = Er_{11} \wedge Er_{12} \wedge Er_{13} \quad ,$$

$$ER_2 = Er_{21} \wedge Er_{22} \quad ,$$

$$ER = ER_1 \wedge ER_2 \quad .$$

The quantity $C(Er_{21}|ER)$, for example, is determined as follows. First, since

$$C(Er_{21}|ER) = C(Er_{21}|ER_2)C(ER_2|ER) \quad ,$$

one moves, then, to determine the quantities on the right side by systematic application of the properties 1 through 4 above.

We have,

$$C(ER|ER) = 1$$
$$= C(ER_1 \wedge ER_2|ER)$$
$$= C(ER_1|ER) + C(ER_2|ER) \quad .$$

Suppose that

$$\{ER_1,1\} \gtrsim \{ER_2,1\} \quad ;$$

consequently, there is $0 < \beta_1 \leq 1$ such that

$$\{ER_1,\beta_1\} \sim \{ER_2,1\} \quad ;$$

accordingly,

$$\beta_1 C(ER_1|ER) = C(ER_2|ER) \quad ,$$

i.e.,

$$C(ER_2|ER) = \frac{\beta_1}{1 + \beta_1} \quad .$$

Similarly, since

$$ER_2 = Er_{21} \wedge Er_{22} \quad ,$$
$$Er_{21} \Longleftrightarrow Er_{22}$$

one gets

$$C(Er_{21}|ER_2) = C(Er_{22}|ER\ ) \quad ,$$

$$C(ER_2|ER_2) = 1$$
$$= C(Er_{21}|ER_2) + C(Er_{22}|ER_2) \quad ;$$

accordingly,

$$C(Er_{21}|ER_2) = 0.5 \quad ,$$

and, finally, one gets

$$C(Er_{21}|ER) = C(Er_{21}|ER_2)C(ER_2|ER)$$
$$= 0.5\ (\frac{\beta_1}{1+\beta_1}) \quad .$$

Let us introduce the important notion of "transparent preferences over a monitor space P_k ." Suppose P_1 denotes the monitor space of the attribute Q_1 "self-image." An element of P_1 , it was noted above, is given as a convex linear combination of the unit vectors

$$u_{11} = (1,0,0)$$
$$u_{12} = (0,1,0)$$
$$u_{13} = (0,0,1) \quad .$$

Preferences over P_1 are said to be transparent when

1. $u_{13} \gtrsim u_{12} \gtrsim u_{11}$,
2. $\overline{p}_1 = u_{13}$.

In general, when the range of the attribute Q_k , R_k , is given by

$$R_k = \{r_{k1}, r_{k2}, \ldots, r_{kn}\} \quad ,$$

its monitor space P_k is given as the set of all convex linear combinations of the n -unit vectors

$$u_{k1} = (1,0,0,\ldots,0)$$
$$u_{k2} = (0,1,0,\ldots,0)$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$
$$u_{kn} = (0,0,0,\ldots,1) \quad ;$$

and preferences over P_k are said to be transparent when

a. $u_{k,i+1} \gtrsim u_{k,i}$, $i = 1,2,\ldots,n-2$

b. $\overline{p}_k = u_{k,n}$.

The importance of this notion stems from two facts. First, when preferences over P_k are transparent, one gets

a'. $C(Er_{k,i}|ER_k) \geq C(Er_{k,i+1}|ER_k)$; $i = 1,2,\ldots,n-2$

b'. $C(Er_{k,n}|ER_k) = 1/2$.

Second, a particularly simple expression for the distance function $d(\cdot,\cdot)$ results when preferences over $P_k, k = 1,2,\ldots,N$ are transparent; in fact, one gets

$$d(\overline{p},p) = 1/2 + \sum_{k=1}^{N} C(ER_k|ER)p_k \cdot w_k \quad ,$$

where

$$p_k = (p_{k1},p_{k2},\ldots,p_{kn})$$

$$w_k = \begin{bmatrix} C(Er_{k,1}|ER_k) \\ C(Er_{k,2}|ER_k) \\ \vdots \\ C(Er_{k,n-1}|ER_k) \\ -1/2 \end{bmatrix} \quad .$$

2. Decision Model

The decision problem to which we are addressed is: which pupils, if any, should be enrolled in which educational programs when one is interested in the programs' cost and effectiveness? In this case one's small world, Q , is the connected attributes, namely, "the programs' costs" and "the programs' effectiveness." To illustrate, when one is interested in Q_1 , "self-image," and Q_2 , "resource-availability," the programs' costs and effectiveness, in this case, will relate to Q_2 and Q_1 respectively. Let t_1 denote the point in time at which a deliberate action δ is initiated with respect to Q (e.g., the time at which one decides to enroll some pupils in some programs). Let the duration of time over which an action δ is effective be denoted by τ ,

$$\tau = [t_1, t_2] \quad ,$$

e.g.,

$$\tau = [9/15/1969,\ 5/28/1970] \quad .$$

Let $\hat{p}(t)$ denote the true state of Q at time t , $t \in \tau$. However, to simplify notations, we will denote $\hat{p}(t_1)$ and $\hat{p}(t_2)$ by $\hat{p}$ and $\hat{p}(\delta)$ respectively, that is,

$$\hat{p} = \hat{p}(t_1) \quad ,$$

$$\hat{p}(\delta) = \hat{p}(t_2) \quad .$$

For example, when, at time t_1 , " T_1 pair off 70% of D_1 with r_{11} and T_1 pair off 20% of D_1 with r_{12} and T_1 pair off 10% of D_1 with r_{13} and T_2 pair off nothing of D_2 with r_{21} and T_2 pair off all of D_2 with r_{21} ," one gets

$$\hat{p} = (\hat{p}_1, \hat{p}_2) \quad ,$$

$$= ((\hat{p}_{11}, \hat{p}_{12}, \hat{p}_{13}), (\hat{p}_{21}, \hat{p}_{22})) \quad ,$$

$$= ((0.70, 0.20, 0.10), (0, 1)) \quad .$$

Let the subsets of D_1 which, at time t_1, were paired off with the categories r_{11}, r_{12} and r_{13} be denoted by D_{11}, D_{12} and D_{13}. As an example of $\hat{p}(\delta)$, one may have

$$\hat{p}(\text{enroll } D_{11} \text{ and } D_{13} \text{ but not } D_{12}) = ((0.49, 0.28, 0.23), (0.40, 0.60)) \quad .$$

The effect of a course of action δ is such that it transforms $\hat{p}$ into $\hat{p}(\delta)$; such a state of affairs is denoted by

$$\hat{p}(\delta) = (\hat{p}_1(\delta), \hat{p}_2(\delta))$$

$$= (\hat{p}_1, \hat{p}_2) \begin{bmatrix} \Psi_1(\delta) \\ \Psi_2(\delta) \end{bmatrix} \quad ,$$

where $\Psi_k(\delta)$ is a probability-like transition matrix which is construed as a <u>representation</u> of the effects of δ on Q_k, i.e.,

$$\hat{p}_k(\delta) = \hat{p}_k \Psi_k(\delta) \quad .$$

For example, since

$$\hat{p}_1(\text{enroll } D_{11} \text{ and } D_{13} \text{ but not } D_{12}) = (0.49, 0.28, 0.23)$$

$$= (0.70, 0.20, 0.10) \begin{bmatrix} 0.70 & 0.20 & 0.10 \\ 0 & 0.70 & 0.30 \\ 0 & 0 & 1.0 \end{bmatrix} \quad ,$$

and

$$\hat{p}_2(\text{enroll } D_{11} \text{ and } D_{13} \text{ but not } D_{12}) = (0.40, 0.60)$$

$$= (0,1) \begin{bmatrix} 0 & 0 \\ 0.40 & 0.60 \end{bmatrix} ,$$

one gets

$$\Psi_1(\text{enroll } D_{11} \text{ and } D_{13} \text{ but not } D_{12}) = \begin{bmatrix} 0.70 & 0.20 & 0.10 \\ 0 & 0.70 & 0.30 \\ 0 & 0 & 1.0 \end{bmatrix}$$

and

$$\Psi_2(\text{enroll } D_{11} \text{ and } D_{13} \text{ but not } D_{12}) = \begin{bmatrix} 0 & 0 \\ 0.40 & 0.60 \end{bmatrix} .$$

In general, when

$$Q = \{Q_1, Q_2, \ldots, Q_{N-1}, Q_N\} ,$$

one gets

$$\hat{p}(\delta) = (\hat{p}_1(\delta), \ldots, \hat{p}_{N-1}(\delta), \hat{p}_N(\delta)) ,$$

$$\hat{p}_k(\delta) = \hat{p}_k \Psi_k(\delta) .$$

Since, in this case, at least one of the attributes relates to the available resources, let us agree to denote this available resource by D_N and to denote the programs' potential enrollees by $D_1, D_2, \ldots, D_{N-1}$. The subset of the domain D_k which, at time t_1 , was paired off with category r_{kj} , will be denoted by D_{kj} .

<u>The development of the matrix operator $\Psi_k(\delta)$</u> . This development can be stated as follows. Let $\{P_1, P_2, \ldots, P_J\}$ be the set of identifiable

programs. The subdomain $D_{kj}, k = 1,2,\ldots,N-1$ may enroll in any combination (including none) of these programs. A course of action δ is defined in terms of the following zero-one controllable variables, namely,

$$\delta(k,j,m_1,\ldots,m_n) = \begin{cases} 1 \text{ when } \underline{\text{exactly}}\ P_{m_1}\&P_{m_2}\&\ldots\&P_{m_n} \text{ are operative} \\ \text{on } D_{kj};\ \text{"}(m_1,\ldots,m_n)=0\text{" indicates the null program.} \\ 0 \text{ otherwise.} \end{cases}$$

Under the condition, namely,

$$\delta(k,j,0) + \sum_{\substack{m_1<\ldots<m_n \\ n,m_1,\ldots,m_n=1}}^{J} \delta(k,j,m_1,\ldots,m_n) = 1, k = 1,2,\ldots,N-1, \quad j = 1,2,\ldots,n_k$$

where n_k denote the number of categories of Q_k, the matrix operator $\Psi_k(\delta)$ is given by the equation,

$$\Psi_k(\delta) = \Delta(k,0)\Psi_k(0) + \sum_{\substack{m_1<\ldots m_n \\ n,m_1,\ldots,m_n=1}}^{J} \Delta(k,m_1,\ldots,m_n)\Psi_k(m_1,\ldots,m_n) \quad ,$$

where $\Delta(k,m_1,\ldots,m_n)$ is the matrix

$$\begin{bmatrix} \delta(k,1,m_1,\ldots,m_n) & 0 & 0 \\ 0 & \delta(k,2,m_1,\ldots,m_n) & 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & \delta(k,n_k,m_1,\ldots,m_n) \end{bmatrix},$$

$\Psi_k(m_1,\ldots,m_n)$ = Probability-like transition matrix which is construed as a <u>representation</u> of the effects of $P_{m_1}\&P_{m_2}\&\ldots\&P_{m_n}$ on D_k.

From this development one constructs the following simple zero-one "linear" programming model. Let B denote the total dollar value of the available resources (at time t_1). Let $Z(k,j,m_1,\ldots,m_n)$ denote the cost of having $P_{m_1}\&\ldots\&P_{m_n}$ operative on D_{kj}. The incurred cost, C, is then given by the equation

$$C = \sum_{k=1}^{N-1} \sum_{j=1}^{n_k} \sum_{\substack{m_1<\ldots<m_n \\ n,m_1,\ldots,m_n=1}}^{J} Z(k,j,m_1,\ldots,m_n)\delta(k,j,m_1,\ldots,m_n) .$$

The matrix operator for Q_N, $\Psi_N(\delta)$, is given by,

$$\Psi_N(\delta) = \begin{bmatrix} 0 & 0 \\ C/B & 1 - C/B \end{bmatrix} .$$

Under transparent preferences, then, the programming model is:

$$\underset{\delta}{\text{Minimize}}\ d(\overline{p},p(\delta)) = \frac{1}{2} + \sum_{k=1}^{N} C(ER_k|ER)p_k(\delta)\cdot w_k ,$$

where

$$p_k(\delta) = \hat{p}\Psi_k(\delta)$$

subject to,

a. $\delta(k,j,0) + \sum_{\substack{m_1<\ldots<m_n \\ n,m_1,\ldots,m_n=1}}^{J} \delta(k,j,m_1,\ldots,m_n) = 1, k = 1,2,\ldots,N-1, \quad j = 1,2,\ldots,n_k$,

b. $C \leq B$ ..

3. Illustrative Example

In this section the use of the above model is illustrated by applying it to the following simple decision situation. The decision-maker, a superintendent of schools, say, is concerned about the reading achievement of his 30,000, 25,000, and 20,000 pupils in the third, sixth, and ninth grades. He is thinking about establishing a remedial reading program. The funds available for such a program are $3 million; the program's design, among other things, is such that it costs $55, $50, and $60 per pupil enrolled from the third, sixth, and ninth grades. Naturally, he is also concerned about the effectiveness of the program. The decision-maker wants to know which pupils, if any, should be enrolled in the program.

Formulating the Decision Problem

The decision-maker is entertaining, among other things, the following attributes:

$Q(1,t)$ = The reading achievement of third grade pupils,[4]

$Q(2,t)$ = The reading achievement of sixth grade pupils

and

$Q(3,t)$ = The reading achievement of ninth grade pupils.

The range of the attribute $Q(k,t), R(k,t), k = 1,2,3$, is the real line $(-\infty,\infty)$. The interval $(-\infty,\infty)$ can be reasonably decomposed into $n(k) = 8$ categories

$$r(k,t,1) = (-\infty, -36]$$

$$r(k,t,i) = [-6(8-i), -6(7-i)], i = 2,\ldots,7$$

$$r(k,t,8) = (0,\infty] \quad ,$$

where, for example,

$r(3,t,1)$ = The category of ninth grade pupils who are more than 36 months behind grade level with respect to reading achievement,

$r(3,5,i)$ = The category of ninth grade pupils who are $6(8 - i)$ to $6(7 - i)$ months behind grade level with respect to reading achievement,

---

[4] The time dimension will be explicitly introduced. Thus $X(t)$ is used to denote the status of X at time t .

and

$r(3,t,8)$ = The category of ninth grade pupils who are not behind grade level with respect to reading achievement.

At time $t(1)$, upon evaluation or otherwise, the following initial distributions, $p(k,t(1),i), k = 1,2,3; i = 1,2,\ldots,8$, were found:

k \ i	1	2	3	4	5	6	7	8
1	0.11	0.14	0.16	0.20	0.14	0.12	0.08	0.05
2	0.12	0.13	0.25	0.17	0.13	0.10	0.07	0.03
3	0.05	0.13	0.17	0.16	0.16	0.14	0.13	0.07

where, for example,

$p(3,t(1),5)$ = The percentage of ninth grade pupils who, at time $t(1)$, were found to be 18 to 12 months behind grade level with respect to reading achievement.

Dismayed by these observations and the fact that over a period of time τ_f ,

$$\tau_f = (t(1),t(\ell))$$

the effect of the status quo, $\Psi[k,\tau_f,0]$, that is, the effect of instituting no remedial programs, is to transform the initial distribution $p(k,t(1))$ into $p(k,t(\ell),0)$,

$$p(k,t(\ell),0) = p(k,t(1))\Psi[k,\tau_f,0]$$

which will not be significantly different from $p(k,t(1))$, that is,

$$p(k,t(\ell),0) \simeq p(k,t(1)) \quad , \quad k = 1,2,3 \quad ,$$

the decision-maker came to entertain the idea of implementing a remedial reading program.

The period of time over which the program is active is τ_f and the funds available for such a program, B, are \$3 million. The program's design, among other things, is such that it costs \$55, \$50 and \$60 per pupil enrolled from the third, sixth and ninth grades. There are 30,000, 25,000 and 20,000 pupils in the third, sixth and ninth grades. The decision—maker is in a state of doubt as to which pupils, if any, should be enrolled in the program.

Now, resources, valued and scarce as they are, should not be expended (= transformed, exchanged, transferred) unless such expenditures are known or believed to be capable of creating no less value than the value of the would-be-expended resources. The attribute $Q(4,t)$ will be related to resources in the following way: The domain of $Q(4,t), D(4,t)$ is the resources available to the decision-maker at time t ; the range $R(4,t)$ is the two categories $r(4,t,1)$ and $r(4,t,2)$, where

$r(4,t,1)$ = The category of exchanged resources

and

$r(4,t,2)$ = The category of nonexchanged resources.

The initial distribution of resources, $p(4,t(1))$, is $(0,1)$., that is, at the beginning none of the available resources (\$3 million) is expended. The final distribution of the resources, $p(4,t(\ell))$, is $(C/B, 1 - C/B)$, where

B = \$3 million, the dollar value of the total available resources at time $(t(1))$,

C = The claim on B due to enrolling some pupils in the remedial reading program.

The monitor space of $Q(k,t), P(k,t)$ is the set of all conceivable distributions $p(k,t)$. The monitor space of the decision situation, $P(t)$, is the Cartesian product.

$$P(1,t) \times P(2,t) \times P(3,t) \times P(4,t) \quad ,$$

where it is clear that the outcome of any conceivable action is represented as a point in $P(t)$. For example, the outcome of not enrolling any pupil in the program, $p_o(t(\ell))$, is

$$p_o(t(\ell)) = (p_o(1,t(\ell)), p_o(2,t(\ell)), p_o(3,t(\ell)), p_o(4,t(\ell))) \quad ,$$

where

$$p_o(k,t(\ell)) = p(k,t(1))\Psi[k,\tau_f,0]$$

$$\simeq p(k,t(1)) \quad , \quad k = 1,2,3$$

$$p_o(4,t(\ell)) = p(4,t(1)) = (0,1) \quad .$$

The ideal outcome $\overline{p}(t)$, on the other hand, is

$$\overline{p}(t) = (\overline{p}(1,t), \overline{p}(2,t), \overline{p}(3,t), \overline{p}(4,t)) \quad ,$$

where

$$\overline{p}(k,t) = (\overline{p}(k,t,1), \overline{p}(k,t,2), \ldots, \overline{p}(k,t,n(k))) \quad ,$$

$$\overline{p}(k,t,n(k)) = 1$$

$$\overline{p}(k,t,i) = 0 \quad , \quad i \neq n(k) \quad ,$$

that is, insofar as $Q(k,t)$ is concerned, the ideal state of affairs is to have all of $D(k,t)$ in the category $r(k,t,n(k))$. For example, the ideal state of affairs with respect to the reading achievement of third grade pupils is to have all third graders in the category $r(1,t,8)$ = not behind grade level with respect to reading achievement.

The decision-maker's objective, $ER(t)$, is the conjunction of the four objectives $ER(1,t)$, $ER(2,t)$, $ER(3,t)$ and $ER(4,t)$, that is,

$$ER(t) = \bigwedge_{k=1}^{4} ER(k,t)$$

$$= ER(1,t) \wedge ER(2,t) \wedge ER(3,t) \wedge ER(4,t)$$

$$= ER(1,t) \ \underline{\text{and}} \ ER(2,t) \ \underline{\text{and}} \ ER(3,t) \ \underline{\text{and}} \ ER(4,t) \ ,$$

where

$ER(1,t)$ = To have all the third grade pupils not behind grade level with respect to reading achievement,

$ER(2,t)$ = To have all the sixth grade pupils not behind grade level with respect to reading achievement,

$ER(3,t)$ = To have all ninth grade pupils not behind grade level with respect to reading achievement

and

$ER(4,t)$ = To have all the available resources ($3 million) in the nonexpended category.

Each of the component objectives $ER(1,t)$, $ER(2,t)$, $ER(3,t)$ and $ER(4,t)$ is, in turn, a conjunction of other component objectives, that is,

$$ER(1,t) = \bigwedge_{i=1}^{8} Er(1,t,i) \ ,$$

$$ER(2,t) = \bigwedge_{i=1}^{8} Er(2,t,i) \ ,$$

$$ER(3,t) = \bigwedge_{i=1}^{8} Er(3,t,i)$$

and

$$ER(4,t) = \bigwedge_{i=1}^{2} Er(4,t,i) \quad ,$$

where

$Er(1,t,8)$ = To have all third grade pupils in the category $r(1,t,8)$ not behind grade level ...

$Er(1,t,i)$ = To have all third grade pupils <u>not</u> <u>in</u> the category $r(1,t,i), i = 1,2,\ldots,7$,

$Er(2,t,8)$ = To have all sixth grade pupils in the category $r(2,t,8)$, not behind grade level...

$Er(2,t,i)$ = To have all sixth grade pupils <u>not</u> <u>in</u> the category $r(2,t,i), i = 1,2,\ldots,7$,

$Er(3,t,8)$ = To have all ninth grade pupils in the category $r(3,t,8)$, not behind grade level...

$Er(3,t,i)$ = To have all ninth grade pupils <u>not</u> in the category $r(3,t,i), i = 1,2,\ldots,7$,

$Er(4,t,2)$ = To have all the available resources in the category $r(4,t,2)$, the nonexpended category

and

$Er(4,t,1)$ = To have all the available resources <u>not</u> in the category $r(4,t,1)$, the expended category.

The decision-maker's metaobjective is to choose that course of action whose outcome is closest to the ideal outcome $\bar{p}(t)$. Since preferences over $P(1,t)$, $P(2,t)$, $P(3,t)$ and $P(4,t)$ are transparent, the distance between the ideal outcome $\overline{p}(t)$ and any other outcome $p(t(\ell))$[5], is

$$d(p(t(\ell)))=\frac{1}{2}+\sum_{k=1}^{4} C(ER(k,t)|ER(t))p(k,t(\ell))w(k) \quad ,$$

where

$C(ER(k,t)|ER(t))$ = The decision-maker's concern for realizing the objective $ER(k,t)$ relative to his concern for the realization of the spectrum of objectives $ER(t)$.

$p(k,t(\ell)) = p(k,t(1))\Psi[k,\tau_f]$

$\Psi[k,\tau_f]$ = The effect of the processes operative on $D(k,\tau_f)$, that is, operative on $D(k,t)$ during τ_f ,

for example,

$$\Psi[4,\tau_f] = \begin{bmatrix} 1 & 0 \\ C/B & 1 - C/B \end{bmatrix}$$

and for $k = 1,2,3$

$$\Psi[k,\tau_f] = \Psi[k,\tau_f,0] + \sum_{k=1}^{3} \Delta(k,\tau_f,1)\Omega[k,\tau_f,1] \quad ,$$

where

$$\Delta(k,\tau_f,1) = \begin{bmatrix} \delta(k,1,\tau_f,1) & 0 & 0 & & 0 \\ 0 & \delta(k,2,\tau_f,1) & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & \delta(k,n(k),\tau_f,1) \end{bmatrix}$$

[5] $p(t(\ell))$ is the outcome at the end of the period $\tau_f = (t(1),t(\ell))$.

$$\delta(k,i,\tau_f,1) = \begin{cases} 1 \text{ when the remedial program is operative on } D(k,\tau_f,i)^6 \\ 0 \text{ otherwise} \end{cases}$$

$\Psi[k,\tau_f,0]$ = The effect of the status quo, that is, the effect of no remedial program, on the domain $D(k,t)$ during τ_f ,

$\Omega[k,\tau_f,1] = \Psi[k,\tau_f,1] - \Psi[k,\tau_f,0]$

$\Psi[k,\tau_f,1]$ = The effect of the remedial program on $D(k,\tau_f)$,

$$C = \sum_{k=1}^{3} \sum_{i=1}^{n(k)} p(k,t(1),i)Z(k,\tau_f,1)\delta(k,i,\tau_f,1) \leq B \quad ,$$

$B = \$3$ million

$Z(k,\tau_f,1)$ = The claim on B if all of $D(k,t(1))$ is to be enrolled in the program,

that is,

$Z(1,\tau_f,1) = 55 \times 30{,}000 = \$1.65$ million

$Z(2,\tau_f,1) \quad 50 \times 25{,}000 = \$1.25$ million

$Z(3,\tau_f,1) = 60 \times 20{,}000 = \$1.20$ million.

and

$$w(k) = \begin{bmatrix} w(k,1) \\ w(k,2) \\ \vdots \\ w(k,n(k)) \end{bmatrix}$$

$w(k,i) = C(Er(k,t,i)|ER(k,t)) \qquad i \neq n(k)$

$w(k,n(k)) = -1/2$,

[6] $D(k,t(1),i)$ is the subdomain of $D(k,t(1))$ that, at time $t(1)$, were in the category $r(k,t,i)$.

that is, $w(k,i), i \neq n(k)$ is the decision-maker's concern for realizing the objective $\mathcal{E}r(k,t,i)$ relative to his concern for the realization of the spectrum of objectives $\mathcal{E}R(k,t)$.

In terms of the bivalent variables $\delta(k,i,\tau_f,1)$ as controllable ones, the solution of the decision-maker's problem, then, is the solution of the following, knapsack type, programming problem

$$\underset{\delta}{\text{Minimize}}\ d(\delta) = (1-\alpha)/2 + (1-\alpha) \sum_{k=1}^{3} \sum_{i=I}^{n(k)} \phi'(k,i,0)$$

$$+ \sum_{k=1}^{3} \sum_{i=1}^{n(k)} \{(1-\alpha)[\phi'(k,i,1) - \phi'(k,i,0)]$$

$$+ \alpha Z(k,i,1)/B\}\delta(k,i,\tau_f,1) \quad .$$

Subject to:

$$\sum_{k=1}^{3} \sum_{i=1}^{n(k)} Z(k,i,1)\delta(k,i,\tau_f,1) \leq B \quad ,$$

where

$\alpha = C(\mathcal{E}R(4,t)|\mathcal{E}R(t))$

= The decision-maker's concern for conserving his available resources relative to his concern for achieving the whole spectrum of objectives $\mathcal{E}R(t)$,

$$Z(k,i,1) = p(k,t(1),i)Z(k,\tau_f,1)$$

$$\phi'(k,i,1) = c'(k)p(k,t(1),i)u(k,i)\Psi[k,\tau_f,1]w(k)$$

$$\phi'(k,i,0) = c'(k)p(k,t(1),i)u(k,i)\Psi[k,\tau_f,0]w(k)$$

$$c'(k) = C(\mathit{E}R(k,t)|\mathit{E}(R(t) - R(4,t))),k=1,2,3$$

= The decision-maker's concern for realizing the objective $\mathit{E}R(k,t)$ relative to his concern for the realization of the spectrum of objectives $\mathit{E}(R(t) - R(4,t))$,

and $u(k,i)$ is the i -th $\mathbf{n}(k)$ -unit vector, that is, an $n(k)$ -vector whose i -th component is 1, all other components being 0.

Measuring Concern

Let

$\mathit{E}R' \wedge x\mathit{E}(R(t) - R')$ = Having an outcome about which the only thing definitely known is that it is ideal insofar as R' is concerned.

For example, when $R' = R(1,t) \cup R(2,t) \cup R(3,t)$, then

$\mathit{E}(R(1,t) \cup R(2,t) \cup R(3,t)) \wedge x\mathit{E}R(4,t)$ = Having an outcome where all third, sixth and ninth grade pupils are not behind grade level with respect to reading achievement and the availability of resources is undetermined, where undetermined availability of resources means that it could be anywhere in the monitor space $P(4,t)$. From now on, to simplify notations, the outcome $E\mathrm{R}' \wedge xE(\mathrm{R}(t) - \mathrm{R}')$, whenever it is under consideration, will be written as $E\mathrm{R}'$.

Measuring $C(E\mathrm{R}(4,t)|E\mathrm{R}(t))$. Have the decision maker rank the outcomes $E\mathrm{R}(4,t)$ and $E(\mathrm{R}(t) - \mathrm{R}(4,t))$ in order of preference, where the latter outcome is the one where the pupils'-related objectives $E\mathrm{R}(1,t)$, $E\mathrm{R}(2,t)$ and $E\mathrm{R}(3,t)$ are fulfilled. Let such ranking be

$$E(\mathrm{R}(t) - \mathrm{R}(4,t)) \gtrsim E\mathrm{R}(4,t) \quad .$$

Consequently,

$$\{E(\mathrm{R}(t) - \mathrm{R}(4,t)),1\} \gtrsim \{E\mathrm{R}(4,t),1\} \quad ,$$

where

$\{E\mathrm{R}',\alpha'\}$ = The gamble which results in the outcome $E\mathrm{R}'$ with probability α' or in a worthless (= C -null) outcome with probability $1 - \alpha'$.

Let $0 < \beta' \leq 1$ be such that

$$\{E(\mathrm{R}(t) - \mathrm{R}(4,t)),\beta'\} \sim \{E\mathrm{R}(4,t),1\} \quad ;$$

that is, the decision-maker is indifferent between having the resource-related objective, $E\mathrm{R}(4,t)$, realized with certainty and having the pupil-related objective $E(R(t) - R(4,t))$ realized with probability β' or the realization of a worthless outcome with probability $1 - \beta'$. Accordingly, one gets

$$\alpha = C(ER(4,t)|ER(t)) = \frac{\beta'}{\beta} ,$$

$$C(E(R(t) - R(4,t))|ER(t)) = \frac{1}{\beta} ,$$

where

$$\beta = 1 + \beta' .$$

<u>Measuring $C(ER(k,t)|E(R(t) - R(4,t))), k = 1,2,3$</u>. Have the decision-maker rank the outcomes $ER(1,t)$, $ER(2,t)$ and $ER(3,t)$ in order of preference. Let such ranking be

$$ER'(1) \gtrsim ER'(2) \gtrsim ER'(3) .$$

Consequently,

$$\{ER'(2),1\} \gtrsim \{ER'(3),1\} .$$

Let $0 < \beta'(2) \leq 1$ be such that

$$\{ER'(2),\beta'(2)\} \sim \{ER'(3),1\} ,$$

that is, the decision-maker is indifferent between having the outcome $\mathcal{E}R'(3)$ with certainty and having the outcome $\mathcal{E}R'(2)$ with probability $\beta'(2)$ or a worthless outcome with probability $1 - \beta'(2)$. Accordingly,

$$C(\mathcal{E}R'(2)|\mathcal{E}R(2)) = 1/\beta(2)$$
$$C(\mathcal{E}R''(2)|\mathcal{E}R(2)) = \beta'(2)/\beta(2) \quad ,$$

where

$$\mathcal{E}R''(2) = \mathcal{E}R'(3)$$
$$\mathcal{E}R(2) = \mathcal{E}R'(2) \wedge \mathcal{E}R''(2)$$
$$\beta(2) = \beta'(2) + 1 \quad .$$

Next have the decision-maker rank the outcomes $\mathcal{E}R'(1)$ and $\mathcal{E}R''(1)$ in order of preference, where

$$\mathcal{E}R''(1) = \mathcal{E}R'(2) \wedge \mathcal{E}R'(3) \quad .$$

Let $0 < \beta'(1) \leq 1$ and $0 < \beta''(1) \leq 1$ be such that

$$\{\mathcal{E}R'(1),\beta'(1)\} \sim \{\mathcal{E}R''(1),\beta''(1)\} \quad ,$$

where, depending on the direction of preference or indifference, at least one of the quantities $\beta'(1)$ and $\beta''(1)$ is equal to one. Accordingly,

$$C(\mathcal{E}R'(1)|\mathcal{E}R(1)) = \beta''(1)/\beta(1) \quad ,$$
$$C(\mathcal{E}R''(1)|\mathcal{E}R(1)) = \beta'(1)/\beta(1) \quad ,$$

where

$$\mathcal{E}R(1) = \mathcal{E}R'(1) \wedge \mathcal{E}R''(1) = \mathcal{E}(R(t) - R(4,t))$$
$$\beta(1) = \beta'(1) + \beta''(1) \quad .$$

The thought of weights, then, are:

$$C(ER'(1)|E(R(t) - R(4,t))) = \beta''(1)/\beta(1) \quad ,$$

$$C(ER'(2)|E(R(t) - R(4,t))) = \beta''(1)/\beta(1)\beta(2) \quad ,$$

$$C(ER'(3)|E(R(t) - R(4,t))) = \beta'(1)\beta'(2)/\beta(1)\beta(2) \quad ,$$

where, to be consistent with the initial preference ranking

$$ER'(1) \gtrsim ER'(2) \gtrsim ER'(3) \quad ,$$

one must have

$$\beta''(1)/\beta'(1) \geq 1/\beta(2) \quad .$$

The above data can be represented in Table 1.

Table 1

	$ER'(1)$	$ER'(2)$	$ER'(3)$
$ER(2)$		$1/\beta(2)$	$\beta'(2)/\beta(2)$
$ER(1)$	$\beta''(1)/\beta(1)$	$\beta'(1)/\beta(1)$	$\beta'(1)/\beta(1)$
$C(\cdot\|\cdot)$	$\beta''(1)/\beta(1)$	$\beta'(1)/\beta(1)\beta(2)$	$\beta'(1)\beta'(2)/\beta(1)\beta(2)$

where row $ER(2)$ indicates evaluation with respect to $ER'(2) \wedge ER'(3)$; row $ER(1)$ indicates evaluation with respect to $ER'(1) \wedge ER(2)$ and the last row contains the required preferential weights.

Measuring $C(Er(k,t,i)|ER(k,t)), k = 1,2,3$. The outcomes $Er(k,t,i) k = 1,2,3, i = 1,2,\ldots,8$ were defined above; for example, $Er(1,t,8)$ is the outcome where all the third grade pupils are not behind grade level with respect to reading achievement and $Er(1,t,1)$ is the outcome where none of the third grade pupils is 36 or more months behind grade level with respect to reading achievement.

The procedure for measuring $C(Er(k,t,i)|ER(k,t))$ is similar to the above procedure used for measuring $C(ER(k,t)|ER(t))$. In the present case, however, the initial preference ordering is transparent, that is, it is the case that

$$Er(k,t,8) \gtrsim Er(k,t,1) \gtrsim \dots \gtrsim Er(k,t,7) \quad .$$

To simplify notations, let

$$Er'(k,1) = Er(k,t,8)$$

$$Er'(k,i) = Er(k,t,i-1) \;,\; i \neq 1$$

$$Er''(k,s) = \bigwedge_{i=s+1}^{8} Er'(k,i) \quad , \quad s = 1,2,\dots,7$$

$$Er(k,s) = Er'(k,s) \wedge Er''(k,s) \quad .$$

Accordingly, $Er(k,s)$ is the outcome of having none of the pupils in the categories $r(k,t,s), r(k,t,s+1),\dots,r(k,t,7)$ and that $Er(k,1)$ is the outcome of having all the pupils in the category $r(k,t,8)$. In terms of these notations the initial preference ranking can be written as

$$Er'(k,1) \gtrsim Er'(k,2) \gtrsim \dots \gtrsim Er'(k,8) \quad .$$

Now, since

$$Er'(k,7) \gtrsim Er'(k,8)$$

let $0 < \beta'(k,7) \leq 1$ be such that

$$\{Er'(k,7),\beta'(k,7)\} \sim \{Er'(k,8),1\} \quad ,$$

that is, the decision-maker is indifferent between having the outcome $Er'(k,8)$, none of the pupils is in the category $r(k,t,7)$, with certainty and having the outcome $Er'(k,7)$ with probability $\beta'(k,7)$ or a worthless outcome with probability $1 - \beta'(k,7)$. Accordingly,

$$C(Er'(k,7)|Er(k,7)) = 1/\beta(k,7) \quad ,$$

$$C(Er''(k,7)|Er(k,7)) = \beta'(k,7)/\beta(k,7) \quad ,$$

where

$$Er(k,7) = Er'(k,7) \wedge Er''(k,7)$$

$$\beta(k,7) = \beta'(k,7) + 1$$

$$Er''(k,7) = Er'(k,8) \quad .$$

Next, have the decision-maker rank the outcomes $Er'(k,s)$ and $Er''(k,s)$, $s = 6,5,4,3,2$. Let $0 < \beta'(k,s) \leq 1$ and $0 < \beta''(k,s) \leq 1$ be such that

$$\{Er'(k,s),\beta'(k,s)\} \sim \{Er''(k,s),\beta''(k,s)\}^{7} \quad ,$$

accordingly,

$$C(Er'(k,s)|Er(k,s)) = \beta''(k,s)/\beta(k,s)$$

$$C(Er''(k,s)|Er(k,s)) = \beta'(k,s)/\beta(k,s) \quad ,$$

where

$$Er(k,s) = Er'(k,s) \wedge Er''(k,s)$$

$$\beta(k,s) = \beta'(k,s) + \beta''(k,s) \quad .$$

Finally, it is clear that

$$Er'(k,1) \sim Er''(k,1) \quad ,$$

that is, the decision-maker is indifferent between the outcome of having all the pupils not behind grade level, $Er'(k,1)$, and the outcome of having none of the pupils behind grade level, $Er''(k,1)$; accordingly,

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[7] Depending on the direction of preference or indifference, at least one of the quantities $\beta'(k,s)$ and $\beta''(k,s)$ is equal to one.

$$C(Er'(k,1)|Er(k,1)) = 1/2$$
$$C(Er''(k,1)|Er(k,1)) = 1/2 \quad .$$

Furthermore, it should be noted that the quantities $\beta'(k,s)$ and $\beta''(k,s)$, so determined, are not independent of each other in the sense that one must have

$$\beta''(k,s)/\beta'(k,s) \geq \beta''(k,s+1)/\beta(k,s+1) \quad , \quad s = 8,\ldots,2 \quad ,$$

if the initial preference ranking

$$Er(k,t,8) \gtrsim Er(k,t,1) \gtrsim \ldots \gtrsim Er(k,t,7)$$

is not to be violated.

The above data are presented in Table 2,

where

$$C(Er(k,t,8)|ER(k,t)) = 1/2$$

$$C(Er(k,t,s-1)|ER(k,t)) = \frac{\beta''(k,s)}{2\beta(k,s)} \prod_{j=s-1}^{2} \frac{\beta'(k,j)}{\beta(k,j)} \quad , \quad s = 6,5,4,3,2 \quad ,$$

$$C(Er(k,t,6)|ER(k,t)) = \frac{1}{2\beta(k,7)} \prod_{j=6}^{2} \frac{\beta'(k,j)}{\beta(k,j)} \quad ,$$

$$C(Er(k,t,7)|ER(k,t)) = \frac{\beta'(k,7)}{2\beta(k,7)} \prod_{j=6}^{2} \frac{\beta'(k,j)}{\beta(k,j)} \quad ;$$

that is, the preferential weight $C(Er(k,t,i)|ER(k,t))$ is obtained as the product of the numerical entries in the column headed by $Er(k,t,i)$.

Estimating $\Psi[k,\tau_f,0]$, the Status Quo's Effect

The decision-maker came to entertain the idea of instituting a remedial reading program when it was "observed" that the reading achievement of certain groups of pupils was unsatisfactory and that no signifi-

Table 2

	$Er(k,t,8)$	$Er(k,t,1)$	$Er(k,t,2)$	$Er(k,t,3)$	$Er(k,t,4)$	$Er(k,t,5)$	$Er(k,t,6)$	$Er(k,t,7)$
	$Er'(k,1)$	$Er'(k,2)$	$Er'(k,3)$	$Er'(k,4)$	$Er'(k,5)$	$Er'(k,6)$	$Er'(k,7)$	$Er'(k,8)$
$Er(k,7)$							$1/\beta(k,7)$	$\beta'(k,7)/\beta(k,7)$
$Er(k,6)$						$\beta''(k,6)/\beta(k,6)$	$\beta'(k,6)/\beta(k,6)$	$\beta'(k,6)/\beta(k,6)$
$Er(k,5)$					$\beta''(k,5)/\beta(k,5)$	$\beta'(k,5)/\beta(k,5)$	$\beta'(k,5)/\beta(k,5)$	$\beta'(k,5)/\beta(k,5)$
$Er(k,4)$				$\beta''(k,4)/\beta(k,4)$	$\beta'(k,4)/\beta(k,4)$	$\beta'(k,4)/\beta(k,4)$	$\beta'(k,4)/\beta(k,4)$	$\beta'(k,4)/\beta(k,4)$
$Er(k,3)$			$\beta''(k,3)/\beta(k,3)$	$\beta'(k,3)/\beta(k,3)$	$\beta'(k,3)/\beta(k,3)$	$\beta'(k,3)/\beta(k,3)$	$\beta'(k,3)/\beta(k,3)$	$\beta'(k,3)/\beta(k,3)$
$Er(k,2)$		$\beta''(k,2)/\beta(k,2)$	$\beta'(k,2)/\beta(k,2)$	$\beta'(k,2)/\beta(k,2)$	$\beta'(k,2)/\beta(k,2)$	$\beta'(k,2)/\beta(k,2)$	$\beta'(k,2)/\beta(k,2)$	$\beta'(k,2)/\beta(k,2)$
$Er(k,1)$	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2
$C(\cdot\|\cdot)$	XX	XX	XX	XX	XX	XX	XX	XX

cant improvement was expected in the absence of a remedial program. The embodiment of the decision-maker's pattern $\Psi[k,\tau_f,0]$ which will, over a period of time τ_f, transform the initial distribution $p(k,t(1))$ into the distribution $p(k,t(\ell),0)$, i.e.,

$$p(k,t(\ell),0) = p(k,t(1))\Psi[k,\tau_f,0] \quad .$$

In the absence of a control group, the transition pattern $\Psi[k,\tau_f,0]$ can be estimated by asking the decision-maker questions such as: If, at time $t(1)$, a hundred sixth grade pupils who are in the category $r(2,t,i)$ with respect to reading achievement were <u>not</u> enrolled in the remedial reading program, how many, out of these hundred, might you be expected to find in the categories $r(2,t,j), j = 1,2,\ldots,8$?

Estimating $\Psi[k,\tau_f,1]$, the Program's Effect

The transition pattern expected of the program, $\Psi[k,\tau_f,1]$, will have to be estimated by asking the decision-maker, insofar as he conceived and designed the program, questions such as: If, at time $t(1)$, a hundred sixth grade pupils who are in the category $r(2,t,i)$ with respect to reading achievement were enrolled in the program, how many, out of these hundred, might you be expected to find in the categories $r(2,t,j)$, $j = 1,2,\ldots,8$?

4. Numerical Illustration

Suppose that

$$ER'(1) = ER(1,t), ER'(2) = ER(2,t), ER'(3) = ER(3,t) \quad ;$$

that is,

$$ER(1,t) \succsim ER(2,t) \succsim ER(3,t) \quad ;$$

that is, insofar as the outcomes $ER(1,t)$, $ER(2,t)$ and $ER(3,t)$ are concerned, the decision-maker prefers having all third grade pupils not behind grade level with respect to reading achievement, $ER(1,t)$, to having all sixth grade pupils not behind grade level with respect to reading achievement, $ER(2,t)$, and that the latter outcome is preferred to the one where all ninth grade pupils are not behind grade level with respect to reading achievement, $ER(3,t)$. Consequently, let $\beta'(2) = .95$ be such that

$$\{ER(2,t), 0.95\} \sim \{ER(3,t), 1\} \quad ;$$

that is, the decision-maker is willing to take 0.05 chance of getting a worthless outcome in favor of getting $ER(2,t)$, as opposed to getting $ER(3,t)$ with certainty. Accordingly,

$$C(ER(2,t)|ER(2,t) \wedge ER(3,t)) = 1/1.95 = 0.513$$

$$C(ER(3,t)|ER(2,t) \wedge ER(3,t)) = 0.95/1.95 = .487 \quad .$$

Now, having evaluated the outcomes $ER(2,t)$ and $ER(3,t)$ with respect to each other, one moves to evaluate the outcomes $ER(1,t)$ and $(ER(2,t) \wedge ER(3,t))$, where the latter outcome designates the case where all sixth and ninth grade pupils are not behind grade level with respect to reading achievement. Suppose that

$$ER(1,t) \precsim ER(2,t) \wedge ER(3,t) \quad ,$$

that is, the decision-maker prefers having all sixth and ninth grade pupils not behind grade level to having all third grade pupils not behind grade level. Let $\beta''(1) \geq 0.513$ be such that

$$\{ER(1,t),1\} \sim \{ER(2,t) \wedge ER(3,t),\beta''(1)\} \quad ;$$

for example,

$$\{ER(1,t),1\} \sim \{ER(2,t) \wedge ER(3,t),0.7\}$$

indicates that the decision-maker is willing to take a 0.90 chance of getting $ER(2,t) \wedge ER(3,t)$, as opposed to getting $ER(1,t)$ with certainty. Accordingly,

$$C(ER(1,t)|E(R(t) - R(4,t))) = 0.7/1.7 = 0.412$$

$$C(ER(2,t) \wedge ER(3,t)|E(R(t) - R(4,t))) = 1/1.7 = .588 \quad .$$

The corresponding tabular form (Table 3) is presented below:

Table 3

	$ER(1,t)$	$ER(2,t)$	$ER(3,t)$
$ER(2)$		0.513	0.487
$ER(1)$	0.412	0.588	0.588
$C(\cdot\|\cdot)$	0.412	0.302	0.286

Now, the "internal" evaluation of each of the outcomes $ER(1,t)$, $ER(2,t)$ and $ER(3,t)$ is to be carried out. In this case, however, the initial ordering is transparent; that is, it is the case that

$$ER(k,t,8) \gtrsim Er(k,t,1) \gtrsim Er(k,t,2) \gtrsim \cdots \gtrsim Er(k,t,7) \quad .$$

Consequently, let $\beta'(k,7) = 0.95$ such that

$$\{Er(k,t,6), 0.95\} \sim \{Er(k,t,7), 1\} \quad ;$$

that is, the decision-maker is willing to take a 0.05 chance of getting nothing in favor of getting $Er(k,t,6)$, no pupils in the category $Er(k,t,6)$, as opposed to getting $Er(k,t,7)$ with certainty. Accordingly,

$$C(Er(k,t,6) \mid Er(k,t,6) \wedge Er(k,t,7)) = 1/1.95 = 0.513$$

$$C(Er(k,t,7) \mid Er(k,t,6) \wedge Er(k,t,7)) = 0.95/1.95 = 0.487 \quad .$$

Having evaluated the outcomes $Er(k,t,6)$ and $Er(k,t,7)$ with respect to each other, one moves to evaluate the outcomes $Er(k,t,s)$ and $\bigwedge_{i=s+1}^{7} Er(k,t,i)$, $s = 5,\ldots,1$, where the latter outcome designates the state of affairs where no pupils are in the categories $r(k,t,s+1)$, $r(k,t,s+2),\ldots,r(k,t,7)$. Suppose that

$$Er(k,t,s) \lesssim \bigwedge_{i=s+1}^{7} Er(k,t,i) \quad , \quad s = 5,\ldots,1 \quad .$$

Let $\beta''(k,5) = 0.54$, $\beta''(k,4) = 0.38$, $\beta''(k,3) = .30$, $\beta''(k,2) = 0.25$ and $\beta''(k,1) = 0.22$ be such that

$$\{Er(k,t,s), 1\} \sim \{\bigwedge_{s+1}^{7} Er(k,t,i), \beta''(k,s)\} \quad , \quad s = 5,4,3,2,1$$

The corresponding tabular form, then, is presented as Table 4.

Table 4

	Er(k,t,8)	*Er*(k,t,1)	*Er*(k,t,2)	*Er*(k,t,3)	*Er*(k,t,4)	*Er*(k,t,5)	*Er*(k,t,6)	*Er*(k,t,7)
Er(k,7)							0.513	0.487
Er(k,6)						0.350	0.650	0.650
Er(k,5)					0.276	0.724	0.724	0.724
Er(k,4)				0.230	0.770	0.770	0.770	0.770
Er(k,3)			0.200	0.800	0.800	0.800	0.800	0.800
Er(k,2)		0.179	0.821	0.821	0.821	0.821	0.821	0.821
Er(k,1)	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
C(·|·)	0.500	0.089	0.082	0.076	0.070	0.064	0.061	0.058

Now, as to the transformations $\Psi[k,\tau_f,0]$ and $\Psi[k,\tau_f 1], k = 1,2,3$ let us assume, for simplicity, that

$$\Psi[1,\tau_f,0] = \Psi[2,\tau_f,0] = \Psi[3,\tau_f,0] \quad ,$$

$$\Psi[1,\tau_f,1] = \Psi[2,\tau_f,1] = \Psi[3,\tau_f,1] \quad .$$

Furthermore, let us assume that

$$\Psi[k,\tau_f,0] = \begin{bmatrix} 0.95 & 0.03 & 0.02 & 0 & 0 & 0 & 0 & 0 \\ 0.01 & 0.95 & 0.02 & 0.02 & 0 & 0 & 0 & 0 \\ 0 & 0.02 & 0.96 & 0.02 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.05 & 0.90 & 0.03 & 0.02 & 0 & 0 \\ 0 & 0 & 0 & 0.02 & 0.92 & 0.06 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.04 & 0.95 & 0.01 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.02 & 0.97 & 0.01 \\ 0 & 0 & 0 & 0 & 0 & 0.01 & 0.02 & 0.97 \end{bmatrix}$$

$$\Psi[k,\tau_f,1] = \begin{bmatrix} 0.50 & 0.30 & 0.15 & 0.05 & 0 & 0 & 0 & 0 \\ 0 & 0.55 & 0.25 & 0.10 & 0.07 & 0.03 & 0 & 0 \\ 0 & 0 & 0.57 & 0.30 & 0.08 & 0.03 & 0.02 & 0 \\ 0 & 0 & 0 & 0.65 & 0.20 & 0.10 & 0.03 & 0.02 \\ 0 & 0 & 0 & 0 & 0.70 & 0.25 & 0.03 & 0.02 \\ 0 & 0 & 0 & 0 & 0 & 0.70 & 0.20 & 0.10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.70 & 0.30 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0 \end{bmatrix}$$

Using the above numerical values for $C(ER(k,t)|E(R(t) - R(4,t)))$, $C(Er(k,t,i)|ER(k,t))$, $Z(k,i,1)$, $\Psi[k,\tau_f,0]$ and $\Psi[k,\tau_f,1]$, $k = 1,2,3, i = 1,2,\ldots,8$, one gets the following numerical values for the programming problem's coefficients:

$\phi'(k,i,1)$

k \ i	1	2	3	4	5	6	7	8
1	0.00381	0.00447	0.00477	0.00463	0.00299	0.00021	-0.00361	-0.01030
2	0.00304	0.00304	0.00547	0.00288	0.00203	0.00013	-0.00231	-0.00453
3	0.00120	0.00288	0.00352	0.00257	0.00222	0.00017	-0.00407	-0.01001

$\phi'(k,i,0)$

k \ i	1	2	3	4	5	6	7	8
1	0.00401	0.00471	0.00501	0.00576	0.00369	0.00302	0.00173	-0.0 995
2	0.00321	0.00321	0.00574	0.00359	0.00251	0.00184	0.00111	-0.00438
3	0.00127	0.00304	0.00370	0.00320	0.00274	0.00245	0.00195	-0.00967

$Z(k,i,1)/B$

k \ i	1	2	3	4	5	6	7	8
1	0.06050	0.07700	0.08800	0.11000	0.07700	0.06600	0.04400	0.02750
2	0.05000	0.05417	0.10417	0.07083	0.05417	0.04170	0.02917	0.01250
3	0.02000	0.05200	0.06800	0.06400	0.06000	0.05600	0.05200	0.02800

The programming problem, using $\alpha = C(ER(4,t)|ER(t))$ as a parameter, is:

$$\underset{\delta}{\text{Minimize}}\; G(\delta|\alpha) = \sum_{k=1}^{3}\sum_{i=1}^{8} \{\phi'(k,i,1) - \phi'(k,i,0) + \alpha[Z(k,i,1)/B - \phi'(k,i,1) + \phi'(k,i,0)]\}\delta(k,i,\tau_f,1) .$$

Subject to,

$$\sum_{k=1}^{3} \sum_{i=1}^{8} [Z(k,i,1)/B]\delta(k,i,\tau_f,1) \leq 1 \quad ,$$

$$\delta(k,i,\tau_f,1) = 0 \text{ or } 1 \quad ,$$

where $\phi'(k,i,1) - \phi'(k,i,0)$ and $Z(k,i,1)/B - (\phi'(k,i,1) - \phi'(k,i,0))$ are given in the following tables:

$\phi'(k,i,1) - \phi'(k,i,0)$

k \ i	1	2	3	4	5	6	7	8
1	-0.00020	-0.00025	-0.00024	-0.00113	-0.00070	-0.00281	-0.00534	-0.00035
2	-0.00017	-0.00017	-0.00019	-0.00071	-0.00048	-0.00171	-0.00342	-0.00015
3	-0.00007	-0.00016	-0.00018	-0.00063	-0.00052	-0.00228	-0.00592	-0.00034

$Z(k,i,1)/B - (\phi'(k,i,1) - \phi'(k,i,0))$

k \ i	1	2	3	4	5	6	7	8
1	0.06070	0.07725	0.08824	0.11113	0.07770	0.06881	0.04934	0.02785
2	0.05017	0.05434	0.10436	0.07154	0.05465	0.04341	0.03259	0.01265
3	0.02007	0.05216	0.06818	0.06463	0.06052	0.05828	0.05792	0.02834

Let

$$a(k,i,1|\alpha) = \phi'(k,i,1) - \phi'(k,i,0) + \alpha\{Z(k,i,1)/B - [\phi'(k,i,1) - \phi'(k,i,0)]\} \quad ;$$

that is,

$$G(\delta|\alpha) = \sum_{k=1}^{3} \sum_{i=1}^{8} a(k,i,1|\alpha)\delta(k,i,\tau_f,1) \quad .$$

Since $G(\delta|\alpha)$, the objective function, is linear in the bivalent variables $\delta(k,i,\tau_f,1)$ and since the coefficients in the budget constraints,

$Z(k,i,1)/B$, are nonnegative, the variables that contribute to the minimization of $G(\delta|\alpha)$, then, are those with negative $a(k,i,1|\alpha)$. The smallest α's that render $a(k,i,1|\alpha)$ positive are

$\underline{\alpha}(k,i,1)$

k \ i	1	2	3	4	5	6	7	8
1	0.00330	0.00324	0.00272	0.01017	0.00901	0.04084	0.10823	0.01257
2	0.00339	0.00313	0.00183	0.00993	0.00879	0.03940	0.10495	0.01186
3	0.00349	0.00307	0.00265	0.00975	0.08560	0.03913	0.10221	0.01200

Accordingly, if the decision-maker's concern for conserving his available resources, α , is greater than or equal to $\underline{\alpha}(1,7,1) = 0.10823$, no pupils should be enrolled in the program.

Using the above numerical data, the programming problem will be solved for α = 0.002, 0.004, 0.008, 0.012, 0.018, 0.102, and 0.108 . The objective function's relevant coefficients are

$-a(k,i,1|0.002)$

k \ i	1	2	3	4	5	6	7	8
1	0.00008	0.00010	0.00006	0.00091	0.00054	0.00267	0.00524	0.00029
2	0.00007	0.00006	---	0.00057	0.00037	0.00162	0.00335	0.00012
3	0.00003	0.00006	0.00004	0.00050	0.00040	0.00216	0.00580	0.00028

$-a(k,i,1|0.004)$

k \ i	1	2	3	4	5	6	7	8
1	---	---	---	0.00068	0.00039	0.00253	0.00514	0.00024
2	---	---	---	0.00042	0.00026	0.00154	0.00329	0.00010
3	---	---	---	0.00037	0.00028	0.00205	0.00569	0.00023

$-a(k,i,1|0.008)$

k \ i	1	2	3	4	5	6	7	8
1	---	---	---	0.00024	0.00008	0.00226	0.00494	0.00013
2	---	---	---	0.00014	0.00004	0.00136	0.00316	0.00005
3	---	---	---	0.00011	0.00004	0.00181	0.00546	0.00011

$-a(k,i,1|0.012)$

k \ i	1	2	3	4	5	6	7	8
1	---	---	---	---	---	0.00198	0.00475	0.00002
2	---	---	---	---	---	0.00119	0.00303	---
3	---	---	---	---	---	0.00158	0.00522	---

$-a(k,i,1|0.018)$

k \ i	1	2	3	4	5	6	7	8
1	---	---	---	---	---	0.00157	0.00445	---
2	---	---	---	---	---	0.00093	0.00283	---
3	---	---	---	---	---	0.00123	0.00488	---

$-a(k,i,1|0.102)$

k \ i	1	2	3	4	5	6	7	8
1	---	---	---	---	---	---	0.00031	---
2	---	---	---	---	---	---	0.00010	---
3	---	---	---	---	---	---	0.00001	---

$-a(k,i,1|0.108)$

k \ i	1	2	3	4	5	6	7	8
1	---	---	---	---	---	---	0.00001	---
2	---	---	---	---	---	---	---	---
3	---	---	---	---	---	---	---	---

The solutions $\delta^*(k,i,1|\alpha')$ for $\alpha = \alpha'$ when $\alpha' = 0.002$, 0.004, 0.012, 0.018, 0.102 and 0.108 are:

$\delta^*(k,i,1|0.002)$

k \ i	1	2	3	4	5	6	7	8
1	1	1	0	1	1	1	1	1
2	1	0	0	1	1	1	1	1
3	0	0	0	1	1	1	1	1

$\delta^*(k,i,1|0.004)$

k \ i	1	2	3	4	5	6	7	8
1	0	0	0	1	1	1	1	1
2	0	0	0	1	1	1	1	1
3	0	0	0	1	1	1	1	1

$\delta^*(k,i,1|0.008) = \delta^*(k,i,1|0.004)$,

$\delta^*(k,i,1|0.012)$

k \ i	1	2	3	4	5	6	7	8
1	0	0	0	0	0	1	1	1
2	0	0	0	0	0	1	1	0
3	0	0	0	0	0	1	1	0

$\delta^*(k,i,1|0.018)$

k \ i	1	2	3	4	5	6	7	8
1	0	0	0	0	0	1	1	0
2	0	0	0	0	0	1	1	0
3	0	0	0	0	0	1	1	0

$\delta^*(k,i,1|0.102)$

k \ i	1	2	3	4	5	6	7	8
1	0	0	0	0	0	0	1	0
2	0	0	0	0	0	0	1	0
3	0	0	0	0	0	0	1	0

$\delta^*(k,i,1|0.108)$

k \ i	1	2	3	4	5	6	7	8
1	0	0	0	0	0	0	1	0
2	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0

For example, when the decision-maker's concern for conserving his available resources, α, is equal to 0.108,[8] the above specific data regarding $C(ER(k,t)|E(R(t) - R(4,t)))$, $C(Er(k,t,i)|ER(k,t))$, $Z(k,i,1)$, $\Psi[k,\tau_f,0]$ and $\Psi[k,\tau_f,1]$, $k = 1,2,3$, $i = 1,2,\ldots,8$ support the course of action where the pupils to be enrolled in the remedial program are only those third grade pupils who, at time $t(1)$, are six months or less behind grade level with respect to their reading achievement.

Naturally, the resulting solution depends on the specific numerical values used for the preferential weights, $C(ER(k,t)|ER(t))$ and $C(Er(k,t,i)|ER(k,t))$, the costs involved, $Z(k,i,1)$, as well as the effects of the program and the status quo, $\Psi[k,\tau_f,1]$ and $\Psi[k,\tau_f,0]$. For example, when the program's effect is given, instead, by

$$\Psi[k,\tau_f,1] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} ;$$

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[8] $\alpha \leq 1/2$ is equivalent to asserting that

$$ER(1,t) \wedge ER(2,t) \wedge ER(3,t) \underset{\sim}{>} ER(4,t) ,$$

that is, the realization of the pupil-related objective is preferred to the realization of the resource-related one.

that is, when the program is ideal insofar as its effects are concerned, one finds the following numerical values for the programming problem's coefficients:

$\phi'(k,i,1)$

k \ i	1	2	3	4	5	6	7	8
1	-0.02266	-0.02884	-0.03296	-0.04120	-0.02884	-0.02472	-0.01648	-0.01030
2	-0.01812	-0.01963	-0.03775	-0.02567	-0.01963	-0.01510	-0.01057	-0.00453
3	-0.00715	-0.01859	-0.02431	-0.02288	-0.02145	-0.02002	-0.01859	-0.01001

$\phi'(k,i,1) - \phi'(k,i,0)$

k \ i	1	2	3	4	5	6	7	8
1	-0.02266	-0.03355	-0.03797	-0.04696	-0.03253	-0.02774	-0.01820	-0.00035
2	-0.02133	-0.02284	-0.04349	-0.02926	-0.02214	-0.01694	-0.01168	-0.00015
3	-0.00842	-0.02163	-0.02801	-0.02608	-0.02419	-0.02247	-0.02054	-0.00034

$Z(k,i,1)/B - [\phi'(k,i,1) - \phi'(k,i,0)]$

k \ i	1	2	3	4	5	6	7	8
1	0.08717	0.11055	0.12597	0.15696	0.10953	0.09374	0.06220	0.02785
2	0.07133	0.07701	0.14766	0.10009	0.07631	0.05864	0.04085	0.01265
3	0.02842	0.07363	0.09601	0.09008	0.08419	0.07847	0.07254	0.02834

$\underline{\alpha}(k,i,1)$

k \ i	1	2	3	4	5	6	7	8
1	0.30595	0.30348	0.30142	0.29918	0.29699	0.29592	0.29251	0.01256
2	0.29903	0.29658	0.29452	0.29233	0.29013	0.28888	0.28592	0.01185
3	0.29627	0.29376	0.29174	0.28952	0.28732	0.28635	0.28315	0.01199

$-a(k,i,1|0.002)$

k \ i	1	2	3	4	5	6	7	8
1	0.02650	0.03333	0.03772	0.04665	0.03231	0.02756	0.01807	0.00029
2	0.02119	0.02269	0.04319	0.02906	0.02199	0.01682	0.01160	0.00012
3	0.00836	0.02148	0.02782	0.02590	0.02402	0.02231	0.02040	0.00028

$-a(k,i,1|0.004)$

k \ i	1	2	3	4	5	6	7	8
1	0.02632	0.03311	0.03747	0.04633	0.03209	0.02737	0.01795	0.00024
2	0.02104	0.02253	0.04280	0.02886	0.02183	0.01671	0.01152	0.00010
3	0.00830	0.02133	0.02763	0.02572	0.02385	0.02216	0.02025	0.00023

$-a(k,i,1|0.008)$

k \ i	1	2	3	4	5	6	7	8
1	0.02597	0.03267	0.03696	0.045 0	0.03165	0.02699	0.01767	0.00013
2	0.02076	0.02222	0.04231	0.02846	0.02153	0.01647	0.01135	0.00005
3	0.00819	0.02104	0.02724	0.02536	0.02352	0.02184	0.01996	0.00011

$-a(k,i,1|0.012)$

k \ i	1	2	3	4	5	6	7	8
1	0.02563	0.03222	0.03646	0.04508	0.0312	0.02662	0.01745	0.00002
2	0.02047	0.02192	0.04172	0.02806	0.02122	0.01624	0.01119	---
3	0.00808	0.02075	0.02686	0.02500	0.02318	0.02153	0.01967	---

$-a(k,i,1|0.018)$

k \ i	1	2	3	4	5	6	7	8
1	0.02510	0.03156	0.03570	0.04413	0.03056	0.02605	0.01708	---
2	0.02005	0.02145	0.04083	0.02746	0.02077	0.01588	0.01094	---
3	0.00791	0.02030	0.02628	0.02446	0.02267	0.02106	0.01923	---

$-a(k,i,1|0.102)$

k \ i	1	2	3	4	5	6	7	8
1	0.01778	0.02227	0.02512	0.03095	0.02136	0.01818	0.01186	---
2	0.01405	0.01498	0.02843	0.01905	0.01436	0.01096	0.00751	---
3	0.00552	0.01412	0.01822	0.01689	0.01560	0.01447	0.01314	---

$-a(k,i,1|0.108)$

k \ i	1	2	3	4	5	6	7	8
1	0.01726	0.02161	0.02437	0.03001	0.02070	0.01762	0.01148	---
2	0.01363	0.01452	0.02754	0.01845	0.01390	0.01061	0.00727	---
3	0.00535	0.01368	0.01764	0.01635	0.01510	0.01400	0.01270	---

Under these circumstances, one finds the solutions $\delta^*(k,i,1|\alpha)$, $\alpha = 0.002, 0.004, 0.008, 0.012, 0.018, 0.102, 0.108$ to be identical and equal to:

$\delta^*(k,i,1|\alpha)$, $0.002 \leq \alpha \leq 0.108$

k \ i	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	0
2	1	1	1	1	1	1	1	0
3	1	1	0	0	0	0	0	0

that is, when the decision-maker's concern for conserving his available resources, α , is less than or equal to 0.108, the above specific data support the course of action where third and sixth grade pupils, who are behind grade level and ninth grade pupils who are 30 months or more behind grade level, are enrolled in the program. Similarly, when, besides the initial transparent ordering,

$$Er(k,t,8) \gtrsim Er(k,t,1) \gtrsim \cdots \gtrsim Er(k,t,7) \quad , \quad k = 1,2,3 \quad ,$$

it is indicated that

$$\{Er(k,t,s),0.9\} \sim \{ \bigwedge_{i=s+1}^{7} Er(k,t,i),1\} \quad , \quad s = 6,5,\ldots,1 \quad ;$$

that is, when $C(Er(k,t,i)|ER(k,t))$ are given, instead, by

k \ i	1	2	3	4	5	6	7	8
1	0.263	0.125	0.059	0.028	0.013	0.006	0.006	0.500
2	0.263	0.125	0.059	0.028	0.013	0.006	0.006	0.500
3	0.263	0.125	0.059	0.028	0.013	0.006	0.006	0.500

one finds the following numerical values for the coefficients of the programming problem:

$\phi'(k,i,0)$

k \ i	1	2	3	4	5	6	7	8
1	0.01155	0.00708	0.00394	0.00236	0.00075	0.00033	0.00002	-0.00999
2	0.00924	0.00482	0.00451	0.00147	0.00051	0.00020	0.00001	-0.00439
3	0.00365	0.00457	0.00290	0.00131	0.00056	0.00026	0.00002	-0.00971

$\phi'(k,i,1)$

k \ i	1	2	3	4	5	6	7	8
1	0.00812	0.00503	0.00287	0.00096	0.00006	-0.00220	-0.00481	-0.01030
2	0.00650	0.00342	0.00328	0.00060	0.00064	-0.00104	-0.00309	-0.00453
3	0.00256	0.00324	0.00211	0.00053	0.00004	-0.00178	-0.00543	-0.01001

$\phi'(k,i,1) - \phi'(k,i,0)$

k \ i	1	2	3	4	5	6	7	8
1	-0.06619	-0.00205	-0.00107	-0.00140	-0.00069	-0.00253	-0.00483	-0.00031
2	-0.00274	-0.00140	-0.00123	-0.00087	-0.00047	-0.00154	-0.00305	-0.00014
3	-0.00109	-0.00127	-0.00079	-0.00078	-0.00052	-0.00204	-0.00545	-0.00030

$Z(k,i,1)/B - [\phi'(k,i,1) - \phi'(k,i,0)]$

k \ i	1	2	3	4	5	6	7	8
1	0.12669	0.07905	0.08907	0.11140	0.07769	0.06853	0.04883	0.02781
2	0.05274	0.05557	0.10540	0.07170	0.05464	0.04324	0.03222	0.01264
3	0.02109	0.05327	0.06879	0.06478	0.06052	0.05804	0.05745	0.02830

$\underline{\alpha}(k,i,1)$

k \ i	1	2	3	4	5	6	7	8
1	0.52246	0.02593	0.01201	0.01257	0.00888	0.03692	0.09891	0.01115
2	0.05195	0.02519	0.01167	0.01213	0.00860	0.03562	0.09466	0.01108
3	0.05168	0.02384	0.01148	0.01204	0.00859	0.03515	0.09487	0.01060

$-a(k,i,1|0.002)$

k \ i	1	2	3	4	5	6	7	8
1	0.06594	0.00189	0.00089	0.00118	0.00053	0.00239	0.00473	0.00025
2	0.00263	0.00129	0.00102	0.00073	0.00036	0.00144	0.00299	0.00011
3	0.00105	0.00116	0.00065	0.00065	0.00040	0.00192	0.00533	0.00024

$-a(k,i,1|0.004)$

k \ i	1	2	3	4	5	6	7	8
1	0.06568	0.00173	0.00071	0.00095	0.00038	0.00226	0.00463	0.00020
2	0.00253	0.00118	0.00081	0.00058	0.00025	0.00137	0.00292	0.00009
3	0.00101	0.00106	0.00051	0.00052	0.00028	0.00181	0.00523	0.00019

$-a(k,i,1|0.008)$

k \ i	1	2	3	4	5	6	7	8
1	0.06518	0.00142	0.00036	0.00051	0.00007	0.00198	0.00444	0.00009
2	0.00232	0.00096	0.00039	0.00030	0.00003	0.00119	0.00279	0.00004
3	0.00092	0.00084	0.00024	0.00026	0.00004	0.00158	0.00500	0.00007

$-a(k,i,1|0.012)$

k \ i	1	2	3	4	5	6	7	8
1	0.06467	0.00110	0.00000	0.00006	---	0.00171	0.00424	---
2	0.00211	0.00073	---	0.00001	---	0.00102	0.00266	---
3	0.00084	0.00063	---	0.00000	---	0.00134	0.00476	---

$-a(k,i,1|0.018)$

k \ i	1	2	3	4	5	6	7	8
1	0.06391	0.00063	---	---	---	0.00130	0.00395	---
2	0.00179	0.00040	---	---	---	0.00076	0.00247	---
3	0.00071	0.00031	---	---	---	0.00100	0.00442	---

$-a(k,i,1|0.102)$

k \ i	1	2	3	4	5	6	7	8
1	0.05327	---	---	---	---	---	---	---
2	---	---	---	---	---	---	---	---
3	---	---	---	---	---	---	---	---

$-a(k,i,1|0.108)$

k \ i	1	2	3	4	5	6	7	8
1	0.05251	---	---	---	---	---	---	---
2	---	---	---	---	---	---	---	---
3	---	---	---	---	---	---	---	---

Under these circumstances, the solutions $\delta^*(k,i,1|\alpha)$,

α = 0.002, 0.004, 0.008, 0.012, 0.018, 0.102, 0.108 are:

$\delta^*(k,i,1|0.002)$

k \ i	1	2	3	4	5	6	7	8
1	1	1	1	1	0	1	1	1
2	1	1	0	1	0	1	1	1
3	1	1	0	1	0	1	1	0

$$\delta^*(k,i,1|0.004) = \delta^*(k,i,1|0.002) \quad ,$$

$$\delta^*(k,i,1|0.008) = \delta^*(k,i,1|0.004) \quad ,$$

$\delta^*(k,i,1|0.012)$

k \ i	1	2	3	4	5	6	7	8
1	1	1	0	1	0	1	1	0
2	1	1	0	1	0	1	1	0
3	1	1	0	0	0	1	1	0

$\delta^*(k,i,1|0.018)$

k \ i	1	2	3	4	5	6	7	8
1	1	1	0	0	0	1	1	0
2	1	1	0	0	0	1	1	0
3	1	1	0	0	0	1	1	0

$\delta^*(k,i,1|0.102)$

k \ i	1	2	3	4	5	6	7	8
1	1	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0

and

$$\delta^*(k,i,1|0.108) = \delta^*(k,i,1|0.102) \quad .$$

Reference

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