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Michael V. Levine
University of Pennsylvania

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Educational Testing Service
Princeton, New Jersey
June 1971

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Abstract

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TRANSFORMING CURVES INTO CURVES WITH THE SAME SHAPE[1,2]

Michael V. Levine
University of Pennsylvania[3]

INTRODUCTION

Two curves are commonly called parallel when it is possible to shift one along the x-axis until it coincides with the other. In other words a pair of curves (or subsets of the plane) are parallel when there is a constant k such that the translation $\langle x, y \rangle \longrightarrow \langle x + k, y \rangle$ carries one onto the other. For an example, let one curve be the graph of the distribution of a normal random variable and the other be that of a normal random variable with the same variance as the first but a different mean.

Curves will be called similar when a increasing similarity transformation along the x-axis carries one onto the other. In other words they have the same shape in the sense that for some constants $a > 0$ and k the mapping $\langle x, y \rangle \longrightarrow \langle ax + k, y \rangle$ carries one onto the other. For an example, consider the distributions of two normal random variables with different means and variances. More examples are indicated below.

A real valued function u renders a set of curves parallel when the transformation $\langle x, y \rangle \longrightarrow \langle u(x), y \rangle$ simultaneously carries all the curves into new curves such that each pair of new curves is parallel. The phase "u renders a set of curves similar" is analogously defined.

There is (Levine, 1970) a general theory for curves that can be rendered parallel. This paper extends that theory to curves that can be rendered similar.

For definiteness the reader may wish to consider transforming curves obtained as the graphs of special (increasing, one-to-one, onto) real functions. However, there are two important reasons for temporarily setting aside this interpretation in favor of a more abstract interpretation of the paper's subject.

In the first place in most of the applications presently foreseen the psychologist will not be considering mappings from the space of real numbers onto the space of real numbers. Instead he will be considering mappings between spaces assumed to be topologically equivalent to the real number. For examples, there are mappings between the objective and subjective probabilities of uncertain events,between valued objects and utilities and between dial settings on a rheostat controlling the physical intensity of a stimulus and a continuum of sensations. Secondly it is extremely important to recognize that even when spaces of numbers are considered in this theory, the algebraic,arithmetic,and metric properties of the numbers are irrelevant. Only the ordinal (i.e., topological) attributes of the numbers have consequences within this theory.

For these reasons we briefly consider the following problem about two abstract topological spaces A and B and a family of homeomorphisms (i.e., continuous functions with continuous inverses).

Problem One: Let $\underset{\sim}{F}$ be a given set of homeomorphisms mapping a topological space A onto a space B. Find all the pairs of homeomorphisms (u,v) such that

1. u _maps_ A _onto the reals and_ v _maps the reals onto_ B _and_
2. _for each_ F _in_ $\underset{\sim}{F}$ _there are real constants_ a,b _such that_
   i. a _is positive_
   ii. $F(x) = v(au(x) + b)$ _for all_ x _in_ A.

For example, suppose $\underset{\sim}{F}$ is a set of strictly monotonic functions mapping the reals onto the reals. Then a transformation u renders the graphs of the function in $\underset{\sim}{F}$ similar if and only if for some v, (u,v) is a solution to Problem One.

In the earlier paper five examples were given to demonstrate that this sort of problem occurs so frequently in psychology that it is desirable to have a general theory to deal with it. That paper dealt with the special case obtained by setting $A = B =$ real numbers, $vu(x) = x$ for all x and $a = 1$ in condition 2. However, the general situation is obvious in many of the examples.

As an additional example suppose one is studying a family of distribution functions such as, for concreteness, $\{F_\lambda\}$ where $F_\lambda(x)$ is the proportion of children having weight less than x pounds that one expects to observe in the population of children of age λ. As an alternative to regarding the normal family of distributions as a convenient set of distributions containing good approximations to the distributions F_λ one may use this theory to study all the (bicontinuous) transformations u of weight such that the distributions of the transformed measures are exactly normal.

To do this one considers Problem One with A = positive numbers, B = numbers between zero and one and $\underset{\sim}{F} = \{F_\lambda\}$. Then this theory will deliver all the functions u such that distributions of the transformed weights, u(weight), will be normal. It will tell when such transformations can be defined. It will show that the transformed weights are generally unique up to a linear transformation. And even when the distributions cannot be made normal it will sometimes deliver transformations u such that distributions of the transformed random variables have the same (not normal) shape. In this event the transformation u will generally be essentially unique. Furthermore, from the numbers a,b in part 2 one can obtain measures of dispersion and central tendency which are precisely analoguous to the standard deviation and mean of the normal family.

SUMMARY AND RELATION TO MATHEMATICAL LITERATURE

For definitions of technical terms used in the summary part of this section, please see section III.

Section I contains a reformulation of Problem One and the problem of finding the transformations rendering curves similar as a more standard mathematical problem. Section II contains some definitions and some published mathematical facts needed for solving the reformulated problem. Section III and V contain solutions to the problem. Section IV contains an informal discussion of concepts used in section III.

To be more detailed, the set of functions $\underset{\sim}{F}$ in Problem One is replaced by a group of real homeomorphisms $\underset{\sim}{G}$. In section I it is shown that this results in no loss of generality. Composition of functions is the group operation. Problem One is then rephrased as an equivalent question about the group $\underset{\sim}{G}$. The question is: find all the increasing homeomorphisms u such that $u\underset{\sim}{G}u^{-1}$ is a subgroup of the group of affine homeomorphisms $\{x \longrightarrow ax + b$: a is positive and b is real$\}$.

This question has three different answers according to the nature of the group $\underset{\sim}{G}$.

1. If $\underset{\sim}{G}$ is not abelian, then functions u with the desired properties exist if and only if the derived group $\underset{\sim}{G}'$ of $\underset{\sim}{G}$ is conjugate to a group of translations. In this case $u\underset{\sim}{G}u^{-1}$ is affine if and only if $u\underset{\sim}{G}'u^{-1}$ is a group of translations.

2. If $\underset{\sim}{G}$ is abelian but not conjugate to a group of translations, then either there are no solutions u or a pair of groups $\underset{\sim}{G}^{+}$, $\underset{\sim}{G}^{-}$ can be constructed such that all u's can be obtained from pairs v,w, where both $v\underset{\sim}{G}^{+}v^{-1}$ and $w\underset{\sim}{G}^{-}w^{-1}$ are groups of translations.

3. If $\underset{\sim}{G}$ is abelian and conjugate to a group of translations, then all functions u with the desired properties can be obtained by applying the results in an earlier paper (Levine, 1970). That paper also characterizes these groups.

From the mathematician's point of view, the major result is a characterization of the conjugates of the subgroups of the group $\{x \longrightarrow ax + b$: a is positive$\}$ within the group of increasing real homeomorphisms with product given by composition of functions. This result has obvious implications for the transformation equation (Aczél, 1966) $F[F(x,a),b] = F(x,a \circ b)$ where x ranges over the reals and a,b range over a subsemigroup of the affine transformations of the real line. For a discussion of the connections between the work generalized in this paper and work on functional equations, continuous iteration, iteration groups and webs see Levine, 1970, section I.4.

I. SIMPLIFYING PROBLEM ONE

In this section Problem One is reformulated until a simpler, more standard problem is obtained. In the process an attempt is made to demonstrate that this is an ordinal theory and to show how this theory may prove useful in measuring mental events.

As a first step both of the topological spaces A and B are replaced by the space of real numbers. This will be done by considering homeomorphisms α: Reals $\longrightarrow A$ and β: $B \longrightarrow$ Reals. With such a pair of functions, the set of functions $\underset{\sim}{F}$ may be used to define a set of real functions $\beta\underset{\sim}{F}\alpha = \{\beta F\alpha : F \in \underset{\sim}{F}\}$.

There are two types of applications in which one might consider spaces other than the reals. In the first type A and B are objective spaces with numbers already assigned to points by physical measurement. In this case the points in A and B will be identified by numbers ranging over intervals and a choice of α,β will be easy. Thus in the example on distributions of weights, each weight (that is, class of equally heavy individuals) in A is measured by a positive number x giving weight in pounds and each probability in B by a number y between zero and one. In this case we may choose $\alpha(x) = e^x$ and $\beta(y) = \log y - \log(1-y)$. The particular choice of α and β will be inconsequential in a sense soon to be made precise.

For any choice of (α,β) and any F in $\underset{\sim}{F}$ we have $v[au(x)+b] = F(x)$ for all x in A if and only if $\beta v[au\alpha(x)+b] = \beta F\alpha(x)$ for all real x. Consequently a pair of homeomorphisms (u,v) satisfy the conditions of Problem One posed in terms of $\underset{\sim}{F}$, A and B if and only if $(u\alpha,\beta v)$ satisfy the conditions of Problem One with $\beta\underset{\sim}{F}\alpha$, Reals, Reals replacing $\underset{\sim}{F}$, A, B. Thus if in Problem One A and B are objective spaces we may choose any homeomorphisms α,β and solve Problem One with $\beta\underset{\sim}{F}\alpha$, Reals, Reals replacing $\underset{\sim}{F}$, A, B for solution pairs (u',v'). The solutions to the original problem will be exactly the pairs $(u'\alpha^{-1},\beta^{-1}v')$.

Notice that the same functions u and v are obtained as solutions to Problem One for each choice of α and β. It is in this sense that the choice of α are β is inconsequential. Thus, for example, if A is a set of weights we obtain the same functions u assigning numbers to individuals whether we initially measure weights in pounds or the logarithm of pounds. In this sense the algebraic, arithmetic and metric properties of the numbers initially assigned to objects are irrelevant.

In the second type of application, the space B is a subjective space. Thus in studying audition we might assume that there is some basic psychophysical function ρ connecting the physical intensity of a continuum of sounds A with a continuum of loudnesses B. Further it might be assumed that there are manipulations λ which change the relationship between physical intensity and loudness so that in condition λ the correspondence F_λ between physical intensity and subjective intensity is $x \longrightarrow a_\lambda \rho(x) + b_\lambda$ where a_λ and b_λ are constants depending only on λ. Statements about recruitment and thresholds could then be rephrased in terms of hypotheses about a_λ and b_λ. However if one is primarily interested in the function ρ, then one may consider Problem One relative to the hypothetical function $\{F_\lambda\}$, the physical A and the subjective continuum B.

In this type of application experimental finesse and a deep understanding of phenomena are required to recast Problem One as a question about real homeomorphisms. One general method is to use cross context or cross modality matching to first recast Problem One as a question about experimentally defined correspondences between two objective spaces. The rationale and procedure is roughly this: Under some standard conditions of observation there is a homeomorphism H mapping an objective continuum A_0 onto the subjective continuum B. Under each experimental condition λ there is a correspondence F_λ between the objective continuum A and the subjective continuum B. In order to obtain an empirical mapping between two objective spaces, an observer selects pairs x in A and y in A_0 such that $F_\lambda(x)$

matches $H(y)$. Under appropriate conditions this empirically defines homeomorphism $H^{-1}F_\lambda$ sending x in the objective space A to y in the objective space A_0. By varying λ we obtain a family $\{H^{-1}F_\lambda\}$ of homeomorphisms mapping A onto A_0.

Thus at the cost of introducing a new unknown H we convert the second type of application into the first. By solving Problem One with $\{H^{-1}F_\lambda\}$, A, A_0 we can still obtain a great deal of quantitative information about the psychophysical function ρ. For (ρ, H^{-1}) will be a solution for Problem One. Now suppose (u,v) and (u',v') are also solutions to Problem One. Then under very general conditions u and u' will be related by a linear transformation. Consequently if (u,v) is any solution to Problem One, we have $\rho(\cdot) = au(\cdot) + b$ for some constants a and b.

From this discussion it should be clear that there is no generality lost by considering only real homeomorphisms. It should be also clear that the theory might prove useful in measuring subjective spaces and that prior physical measurements of the objective spaces appearing in applications have no consequences.

In order to use standard group theoretical methods and results, each set of real homeomorphisms will be associated with a definite group of homeomorphisms. The group operation here will be composition of functions. The <u>associated group</u> of $\underset{\sim}{F}$ is defined as the subgroup of the group of all real homeomorphisms generated by the set $\{F^{-1}G\colon F$ and G are in $\underset{\sim}{F}\}$. These groups were introduced and discussed in detail in section IV.1. of the earlier paper. In that paper it was shown that these groups could be used to reduce a problem defined by many curves to an equivalent problem defined by just two or three curves.

To further motivate the use of associated groups it is shown that with these groups it becomes possible to search for the functions u and v of Problem One successively rather than simultaneously. This claim is made precise in two propositions below.

Proposition 1: Let the set of real homeomorphisms $\underset{\sim}{F}$ have associated group $\underset{\sim}{G}$. Then for any u there exists a v such that (u,v) satisfies Problem One for $\underset{\sim}{F}$, Reals, Reals if and only if for each g in $\underset{\sim}{G}$ there is some positive a and real b such that $g(\cdot) = u^{-1}(au(\cdot) + b)$.

Proof: Suppose (u,v) satisfies One for homeomorphisms $\underset{\sim}{F}$. Then for each F and G in $\underset{\sim}{F}$ we have positive a,c and real b,d such that

$$F(\cdot) = v[au(\cdot) + b]\ ;\ F^{-1} = u^{-1}[\tfrac{1}{a}v^{-1}(\cdot) - b/a]$$

$$G(\cdot) = v[cu(\cdot) + d]\ ;\ F^{-1}G = u^{-1}[\tfrac{c}{a}u(\cdot) + (d-b)/a].$$

Consequently the generators of $\underset{\sim}{G}$ are of form $u^{-1}(au(\cdot) + b)$. Since inverses and products (i.e., composites) of functions of this form are also of this form, every g in $\underset{\sim}{G}$ is of the form $g(\cdot) = u^{-1}(au(\cdot) + b)$ for some positive a and real b.

Conversely, let u be a homeomorphism such that any g in $\underset{\sim}{G}$ is of form $g(\cdot) = u^{-1}(au(\cdot) + b)$. For any F_0 in $\underset{\sim}{F}$ put $v_0 = F_0u^{-1}$. Then v_0 is a real homeomorphism and $F_0^{-1}F$ is in $\underset{\sim}{G}$. Consequently, $F(\cdot) = v_0(au(\cdot) + b)$ where $F_0^{-1}F(\cdot) = u^{-1}(au(\cdot) + b)$. Thus, (u,v_0) satisfies the conditions of Problem One. This completes the proof.

Proposition 1 characterizes the first components of the pairs (u,v). Most of the remainder of the paper deals with the problem of finding the functions u. Proposition 2 below trivializes the problem of finding v after a suitable u has been discovered.

Proposition 2: *If $\underset{\sim}{F}$ is a set of real homeomorphisms and u is such that for some v, (u,v) satisfies the conditions of Problem One then*

i. *for any F_0 in $\underset{\sim}{F}$ (u,F_0u^{-1}) satisfies Problem One.*

ii. *if (u,v) satisfies One, then (u,w) also satisfies Problem One if and only if for some positive a and real b, $w(x) = v(ax + b)$ for all real x.*

Proof: The proof of i is contained in the proof of Proposition 1. The "if" half of ii is proven by deriving $v(x) = w(x/a - b/a)$ from $w(x) = v(ax + b)$ and substituting in $F(\cdot) = v(a_1u(\cdot) + b_1)$. The remainder of ii is obtained by manipulating the equations

$$F(\cdot) = v\left[a_1u(\cdot) + b_1\right]; \; F(\cdot) = w\left[au(\cdot) + b\right]$$

to conclude $w(x) = v\left[\frac{a}{a_1}x - (ab_1/a_1) + b\right]$.

In view of Proposition 1 and 2, Problem One can be considered solved when Problem Two below is solved.

<u>Problem Two: Given a group of real homeomorphisms $\underset{\sim}{G}$ find all the real homeomorphisms u such that for each g in $\underset{\sim}{G}$ there is positive a and real b such that</u> $g(\cdot) = u^{-1}[au(\cdot) + b]$.

Actually, Two is no more general than One because every group $\underset{\sim}{G}$ of real homeomorphisms can be regarded as the associated group of a set of real homeomorphism $\underset{\sim}{F}$. This is so because the associated group of a <u>group</u> of homeomorphisms is the group itself.

Problem One can be further simplified (again, without loss of generality) by considering only the strictly increasing real homeomorphisms rather than the increasing and decreasing homeomorphisms. First let m denote the homeomorphism $x \longrightarrow -x$. Since $u^{-1}[au(\cdot) + b]$ equals $(mu)^{-1}(amu(\cdot) - b)$, u satisfies Problem One if and only if mu does. Consequently we can seek only the increasing u; the remaining functions can be obtained by multiplication by minus one.

In the final simplification (Proposition 3) the groups with decreasing homeomorphisms are eliminated. It means that either there are no u or all functions used to define the associated group are decreasing or all increasing.

<u>Proposition 3: If a group $\underset{\sim}{G}$ contains a homeomorphism which fails to be strictly increasing then there are no functions u satisfying Problem One.</u>

<u>Proof</u>: Every real homeomorphism is either strictly increasing or strictly decreasing. If u is strictly decreasing then so is u^{-1}. For each positive a and real b, $x \longrightarrow ax + b$ is strictly increasing. The composition of an even number of decreasing functions is increasing.

Consequently each function $u^{-1}[au(\cdot) + b]$ is strictly increasing whenever u is a homeomorphism and a is positive. Thus if there is any u satisfying Two for $\underset{\sim}{G}$, then each g in $\underset{\sim}{G}$ is strictly increasing. This proves the contrapositive.

For brevity, a strictly increasing homeomorphism of the reals onto the reals will be called a scale. Thus Problem One when stripped to essentials becomes

<u>Problem Three: Given a group of scales, find all the scales u such that for each g in the group there is some positive a and real b such that $g(\cdot) = u^{-1}[au(\cdot) + b]$.</u>

II. GLOSSARY

This section contains a list of definitions, notations and results from the earlier paper and from elementary group theory. With few exceptions most introductory general algebra texts contain more group theory than is needed here.

A <u>scale</u> is defined to be a strictly increasing homeomorphism of the reals onto the reals. It is easy to verify that the set of all scales is a group when the product fg of f and g is defined by $fg(x) = f[g(x)]$. The identity in this group is the identity on the reals and will be denoted by e.

The relation $\leqq$, defined by $f \leqq g$ iff $f(x) \leqq g(x)$ for all real x, is a partial order relation in the set of all scales because the usual order relation is a partial relation in the set of all numbers.

Furthermore, since scales are one-to-one and onto functions, $f \leqq g$ implies $hf \leqq hg$ and $fh \leqq gh$.

By direct calculation the set of all scales of form $f(x) = ax + b$ is easily shown to be a subgroup of the group of all scales and will henceforth be called the affine group. Note that the affine group as used here has only increasing functions in it. Two subgroups of the affine group which are referred to frequently enough to require names are the group of translations $\{x \longrightarrow x + b: b \text{ is real}\}$ and the linear group $\{x \longrightarrow ax: a \text{ is positive}\}$. A scale or group of scales will be called affine if it is an element or subgroup of the affine group.

Two subgroups $\underset{\sim}{G}$ and $\underset{\sim}{H}$ of the group of all scales are said to be conjugate if there is a scale u such that $u\underset{\sim}{G}u^{-1} = \underset{\sim}{H}$. In other words, $\underset{\sim}{G}$ and $\underset{\sim}{H}$ are conjugate if for some scale u, the isomorphism $g \longrightarrow ugu^{-1}$ maps $\underset{\sim}{G}$ onto $\underset{\sim}{H}$. Thus there are scales u satisfying Problem Three for a group $\underset{\sim}{G}$ if and only if $\underset{\sim}{G}$ is conjugate to a subgroup of the affine group.

An affine system is a group of scales conjugate to a subgroup of the affine group. A uniform system is an affine system conjugate to a subgroup of the group of translations. From the preceeding section it can be seen that a theory of affine (uniform) systems can be regarded as a theory about triples $\underset{\sim}{F}$, A, B where A and B are topological spaces and $\underset{\sim}{F}$ is a set of homeomorphisms of A onto B such that there exist homeomorphisms u: Reals $\longrightarrow$ A and v: B $\longrightarrow$ Reals such that for each F in $\underset{\sim}{F}$, vFu is in the affine group (respectively, group of translations).

A detailed theory of uniform systems is available (Levine, 1970). It contains necessary and sufficient conditions for a group of scales to be a uniform system and a discussion of the set of all scales u such that $u\underset{\sim}{G}u^{-1}$ is a group of translations. In this paper, Problem Three is solved by reducing it to answered questions about uniform systems.

If $\underset{\sim}{G}$ is a group of scales and uGu^{-1} is a subgroup of the translations then any two scales F, G of $\underset{\sim}{G}$ are rendered parallel by u in the sense used in the introduction: the transformation of the plane $\langle x,y \rangle \longrightarrow \langle u(x),y \rangle$ carries the graphs of F and G into parallel curves. Consequently we will say u <u>renders</u> $\underset{\sim}{G}$ <u>parallel</u>. Analogously, if uGu^{-1} is a subgroup of the affine group u will be said to <u>render</u> $\underset{\sim}{G}$ <u>similar</u>. Since in both cases u can be thought of as a solution to a large number of simultaneous functional equations, u will be called a <u>solution</u> or a <u>solution for</u> $\underset{\sim}{G}$.

A group is <u>abelian</u> if xy equals yx for any two elements in the group. Equivalently, a group is abelian if each product $xyx^{-1}y^{-1}$ is equal to the identity in the group.

In any group $\underset{\sim}{G}$ a <u>commutator</u> is simply any element which can be written in the form $xyx^{-1}y^{-1}$ for some group elements x and y. Henceforth the commutator $xyx^{-1}y^{-1}$ will be abbreviated as $[x,y]$. Since $[x,x]$ is the identity and $[y,x]$ is the inverse of $[x,y]$, the set of products $[x_1,y_1][x_2,y_2] \ldots [x_n,y_n]$ is a subgroup. It is denoted by $\underset{\sim}{G}'$ and called <u>the derived group of</u> $\underset{\sim}{G}$. An informal discussion of some useful aspects of derived groups is in section III.

There are only two facts about derived groups which will be used in the mathematical sections below. They are

1. $\underset{\sim}{G}'$ is invariant, i.e., if f is in $\underset{\sim}{G}$ and g is in $\underset{\sim}{G}'$, then $f^{-1}gf$ is in $\underset{\sim}{G}'$.

2. If $\underset{\sim}{G}$ is a subgroup of the affine group, $\underset{\sim}{G}'$ is a subgroup of the group translations.

Fact 1 follows from the equations

$$\begin{cases} f^{-1}ghf = (f^{-1}gf)(f^{-1}hf) \\ f^{-1}[g,h]f = [f^{-1}gf, f^{-1}hf] . \end{cases}$$

To verify fact 2 one notes that if $f(x) = ax + b$ and $g(x) = cx + d$ then for suitable k, $fg(x) = acx + k$. From this it follows that each commutator is a translation.

A set of elements $\{g_i\}$ in a group are called <u>generators</u> if every element in the group can be written as a product of the g_i and their inverses. A group is finitely generated if it has a finite set of generators. For example the associated group of a finite set of scales is finitely generated. A group is cyclic or <u>monogenic</u> if it is generated by one of its elements; i.e., if there is some g in $\underset{\sim}{G}$ such that $\underset{\sim}{G} = \{g^n$: n is an integer$\}$. For examples there are the group of real integers and its subgroups.

A <u>nonmonogenic</u> group is a group which is not monogenic. For example, consider the additive reals. If this group $\underset{\sim}{G}$ were monogenic there would be some real x such that $\underset{\sim}{G} = \{nx$: n is an integer$\}$ and $\underset{\sim}{G}$ would have a smallest positive element, $|x|$. This line of reasoning leads to the following assertion which is needed in the sequel: <u>A subgroup of the additive reals which has a sequence of non-zero elements converging to zero is nonmonogenic</u>.

This distinction between monogenic and nonmonogenic groups is important because the following very strong uniqueness theorem for nonmonogenic groups has been proven (Levine, 1970, section V.4).

Uniqueness Theorem: *If $\underset{\sim}{G}$ is a nonmonogenic group of scales and $u\underset{\sim}{G}u^{-1}$ is a subgroup of the translations, then $v\underset{\sim}{G}v^{-1}$ is a subgroup of the translations if and only if for some positive a and real b, $v(\cdot) = au(\cdot) + b$.*

Finally a well known result attributed to Hion will also be needed in the sequel.

Hion's Theorem (Fuchs, 1963, page 46): *If $x \longrightarrow x'$ is a strictly increasing isomorphism of two subgroups of the additive group of reals, then for some positive a and each x, x' equals ax.*

III. SOLUTION FOR NONABELIAN GROUPS

The abelian and nonabelian groups are treated separately. Curiously, the theory for nonabelian groups is much simpler than the theory for abelian groups. An informal discussion of the significance of these results is in the next section.

Theorem 1: *A nonabelian group of scales is an affine system if and only if its derived group is a nonmonogenic uniform system.*

Proof: Let $\underset{\sim}{G}$ be a nonabelian group of scales. To prove the condition is necessary, suppose for some scale u, $u\underset{\sim}{G}u^{-1}$ is a subgroup of the affine group. Then $(u\underset{\sim}{G}u^{-1})'$ is a subgroup of the group of translations. Since $(u\underset{\sim}{G}u^{-1})'$ equals $u\underset{\sim}{G}'u^{-1}$, $\underset{\sim}{G}'$ is a uniform system. Since $\underset{\sim}{G}$ is nonabelian, we may choose f and g in $\underset{\sim}{G}$ such that $fg \neq gf$. Without loss of generality, we may take $f(\cdot) = u^{-1}(au(\cdot) + b)$ for some $a \neq 1$, for otherwise $\underset{\sim}{G}$ would be a uniform system and isomorphic to some subgroup of the additive reals. Since $fgf^{-1}g^{-1}$ is in $\underset{\sim}{G}'$ and $fg \neq gf$, $fgf^{-1}g^{-1}(\cdot) = u^{-1}[u(\cdot) + c]$ for some c other than zero. Since $\underset{\sim}{G}'$ is invariant, for each integer n, $h_n = f^n(fgf^{-1}g^{-1})f^{-n}$ is in $\underset{\sim}{G}'$. But since $h_n(\cdot)$ equals $u^{-1}(u(\cdot) + a^n c)$ and a is neither zero nor one, the isomorphism $u^{-1}(u(\cdot) + k) \longrightarrow k$ of $\underset{\sim}{G}'$ into the additive reals places $\underset{\sim}{G}'$ in correspondence with a subgroup of the additive reals which cannot be monogenic. Thus $\underset{\sim}{G}'$ is a nonmonogenic uniform system.

Conversely, suppose $\underset{\sim}{G}'$ is a nonmonogenic uniform system. Let u be a scale such that $u\underset{\sim}{G}'u^{-1}$ is a subgroup of the group of translations. Then for any g in $\underset{\sim}{G}'$, $g(\cdot)$ equals $u^{-1}(u(\cdot) + k)$ for some k. Let f be an arbitrary element of $\underset{\sim}{G}$. Since $\underset{\sim}{G}'$ is normal, for each g in $\underset{\sim}{G}'$, fgf^{-1} is also in $\underset{\sim}{G}'$. Thus for some d, $fgf^{-1}(\cdot) = u^{-1}[u(\cdot) + d]$; i.e., $g(\cdot) = (uf)^{-1}[uf(\cdot) + d]$. Consequently, uf is also a solution for $\underset{\sim}{G}'$. By the uniqueness theorem, $uf(\cdot) = au(\cdot) + b$ for some positive a; i.e., $f(\cdot) = u^{-1}(au(\cdot) + b)$. Thus $u\underset{\sim}{G}u^{-1}$ is contained in the affine group. This completes the proof.

As a corollary, we obtain a complete solution for Problem Three when $\underset{\sim}{G}$ is nonabelian. Its proof is contained in the proof on Theorem One.

<u>Corollary</u>: <u>If $\underset{\sim}{G}$ is a nonabelian affine system, then u satisfies the conditions of Problem Three if and only if u renders $\underset{\sim}{G}'$ parallel.</u>

In fact a somewhat sharper result has been proven. It is Theorem One without the word "monogenic". The only proof now available is long and technical. It will be presented elsewhere. The idea of the proof is to show that if $\underset{\sim}{G}'$ is a uniform system, then the subset $\underset{\sim}{F}$ of scales with fixed points is an invariant subgroup of $\underset{\sim}{G}$ and $\underset{\sim}{G}/\underset{\sim}{F}$ is abelian. From this it follows that $\underset{\sim}{G}' \subseteq \underset{\sim}{F}$ and $\underset{\sim}{G}$ is abelian.

IV. DISCUSSION OF RESULTS

This section contains some informal comments on the results of the previous section. No proofs will be given. No assertions made here will be used in the sequel. A knowledge of some results in the earlier paper and the definitions of rings and polynomials is assumed.

Some readers will find the results of the last section pleasing and other readers disappointing. The problem of finding solutions for the nonabelian affine systems has been reduced to finding solutions for a particular abelian system. By using the results in the earlier paper one can calculate solutions, specify necessary geometric properties on empirical curves and design experiments to test hypotheses which imply that a set of curves is an affine system.

However the reader concerned with numerical computations and gaining insight by manipulating symbols may be disappointed by the reduction to the derived group. The introduction of groups in the study of uniform systems was justified by two considerations which no longer seem valid. Firstly, the number of scales that needed to be considered to obtain solutions was drasticly reduced. A procedure was specified for finding a subgroup with two generators within the associated group of a finite set of scales such that the subgroup had exactly the same solutions as the original set of scales. Consequently each finite set of curves could be reduced to an equivalent set of no more than three curves to be rendered parallel. Secondly, with associated groups our intuitions about a highly familiar object could be used to understand uniform systems. The associated group of a uniform system is isomorphic to a subgroup of the additive reals. Thus, the scales in a uniform system can be thought of as numbers and the composition of scales as addition.

The reader may now be disappointed since the derived group of an affine system may be infinitely generated even when the associated group is finitely generated. Furthermore, all of the derived group's finitely generated subgroups may be monogenic. Consequently, it will often be impossible to find a small subgroup of the derived group equivalent (in the sense of having the same solutions) to the larger group. Furthermore, derived groups can be very complicated objects with a complicated, unfamiliar calculus.

These objections are answered in the remainder of this section by showing that there is a ring structure inherent in affine systems. Using the derived group and the results of the last section we may apply our intuitions concerning a very simple and familiar object, namely the polynomials in one variable with integer coefficients. Furthermore, as rings, affine systems will always have equivalent finitely generated subsystems. A simple algorithm can be given for locating a pair of scales such that each of the scales needed to compute a solution can be defined by finitely many operations (of a simple, familiar nature) applied to these two scales.

Consider for a moment $\underset{\sim}{Z}[t]$, the polynomials in t with integer coefficients; i.e., the set expressions of form $p(t) = n_0 + n_1t + n_2t^2 + \ldots + n_rt^r$ where $r \geqq 0$ each of the n_i are integers, with the usual laws of multiplication and addition. One way to think about $\underset{\sim}{Z}[t]$ is as an abelian group with an additive operation φ mapping $\underset{\sim}{Z}[t]$ into itself such that $\varphi p(t)$ is simply the polynomial $p(t)$ times the polynomial t. Then each polynomial can be written in the form $n_0 + n_1\varphi(1) + n_2\varphi(\varphi(1)) + \ldots + n_r\varphi(\ldots\varphi(1))$ and the definition of multiplication can be reduced to statements about the iterates of the operation φ.

If the ring structure is ignored and $\underset{\sim}{Z}[t]$ is regarded as an additive group then $\underset{\sim}{Z}[t]$ is not finitely generated. For any finite subset $\underset{\sim}{Z}[t]$ there will always be a polynomial (namely, t^r for some large r) which cannot be obtained by finitely many of the operations of addition and subtraction applied to the generators. On the other hand as a ring it is generated by 1 and t. And as a group with one operat-- φ, it is generated by the element 1.

To see the relevance of rings to affine systems consider an affine system with associated group $\underset{\sim}{G}$ generated by a pair of scales, say $g(x) = u^{-1}[tu(x) + b]$ for $(t \neq 1)$ and $f(x) = u^{-1}(u(x) + 1)$. For simplicity suppose f is in the derived group $\underset{\sim}{G}'$. The general element of $\underset{\sim}{G}$ can be written in the form

$$(1) \qquad h = g^{n_1} f^{m_1} g^{n_2} \ldots g^{n_r} f^{m_r}$$

where n_1 or m_r may be zero. Recall that h is in the derived group iff it can be written as a product of commutators $[x,y] = xyx^{-1}y^{-1}$ where x and y are each products of powers of f and g. If we add the exponents of g in a commutator clearly we must get zero. Consequently, each element in the derived group of $\underset{\sim}{G}$ is of the form (1) with $n_1 + n_2 + \ldots n_r = 0$. Consequently we can write an element in $h \in \underset{\sim}{G}'$ as

$$h = (g^{n_1} f^{m_1} g^{-n_1})(g^{n_1+n_2} f^{m_2} g^{-(n_1+n_2)}) \ldots (g^0 f^{m_r} g^{-0})$$

$$= (g^{n_1} f g^{-n_1})^{m_1} (g^{n_1+n_2} f g^{-(n_1+n_2)})^{m_2} \ldots (f)^{m_r} .$$

By direct calculation

$$gfg^{-1} = u^{-1}[u(\cdot) + t]$$

$$g^n f g^{-n} = g(g^{n-1} f g^{-(n-1)})g^{-1} = u^{-1}[u(\cdot) + t^n]$$

$$(g^n f g^{-n})^m = u^{-1}[u(\cdot) + mt^n]$$

and consequently h equals $u^{-1}[u(\cdot) + p(t)]$ where $p(t)$ is $m_1 t^{n_1} + m_2 t^{n_2} + \ldots m_r t^{n_r}$.

The analogy with polynomials is almost perfect. When $\underset{\sim}{G}$ is an affine system, then $\underset{\sim}{G}'$ is a (generally not finitely generated) abelian group. However, if we may also regard $\underset{\sim}{G}'$ as a ring by thinking of it as an additive group (the group operation is composition of functions) with a multiplication defined by a pair of operations $\varphi(h) = ghg^{-1}$ and its inverse $\varphi^{-1}(h) = g^{-1}hg$. (The inverse operation must be included because some of the n_i may be negative).

Now we return to the general situation and show that affine systems are even "more finite" than uniform systems. Let $\underset{\sim}{G}$ be a non-abelian affine system. By selecting a pair of elements which don't commute, say f_1 and f_2, we obtain an element in the derived group $f = f_1 f_2 f_1^{-1} f_2^{-1} = u^{-1}[u(\cdot) + k]$ for some solution u and some $k \neq 0$. Since $\underset{\sim}{G}$ cannot be an (abelian) uniform system there must be some g say $g = u^{-1}[tu(\cdot) + b]$ not in $\underset{\sim}{G}'$. Since $g^{-1} = u^{-1}[t^{-1}u(\cdot) - b/t]$ is also not in $\underset{\sim}{G}'$ we may choose g such that t is between one and zero.

Such a g can always be selected prior to finding a solution u. It is easy to show by using the fact that u is a scale that these functions have the following characteristic property: For very large x, x exceeds $g(x)$ and for very small x, $g(x)$ exceeds x.

We consider the associated group $\underset{\sim}{H}$ of the subset of $\underset{\sim}{G}'$, $\{g^n f g^{-n} : n \geqq 0\}$. $\underset{\sim}{H}$ will be closed under the operation $\varphi(h) = ghg^{-1}$. As a group $\underset{\sim}{H}$ will have generators $\{\varphi^n(f)\}$. Since $\underset{\sim}{H}$ is an abelian group (in fact a uniform system), we may indicate the composition of functions in $\underset{\sim}{H}$ by addition so that the typical element of $\underset{\sim}{H}$ will be $h = n_0 f + n_1\varphi(f) + \ldots n_r\varphi^r(f)$ where nh denotes $h + h \ldots + h$ n times and $\varphi^r(f)$ denotes $\varphi(\ldots \varphi(f))$ r times. As with polynomials, the operation φ can be used to define a ring structure on $\underset{\sim}{H}$. Since $\varphi(h_1 + h_2) = \varphi(h_1) + \varphi(h_2)$ and since

$$h_1 = u^{-1}[u(\cdot) + k_1] \text{ and } h_2 = u^{-1}[u(\cdot) + k_2]$$

imply $h_1 + h_2 = u^{-1}[u(\cdot) + k_1 + k_2]$ and $\varphi(h_1) = u^{-1}[u(\cdot) + tk_1]$ there will be a ring homomorphism from $\underset{\sim}{Z}[t]$ onto H. Consequently $\underset{\sim}{H}$ can be regarded as a monogenic ring. Every element in $\underset{\sim}{H}$ can be expressed as a finite combination of a fairly simple kind of the two scales f and g. Furthermore, since the sequence $\varphi^n(f) = u^{-1}(u(\cdot) + t^n k)$ converges to the identity scale, the scales rendering $\underset{\sim}{H}$ parallel are exactly the scales rendering $\underset{\sim}{G}$ similar.

Notice that such an f and g can be found in nonabelian $\underset{\sim}{G}$ whether $\underset{\sim}{G}$ is finitely generated or not. Thus there is a sense in which (non-abelian) affine systems are simpler objects than uniform systems. They can always be regarded as having finitely generated equivalent subsystems.

V. SOLUTION FOR ABELIAN GROUPS

Here we consider the affine systems which are abelian but not uniform. A simple necessary condition will be given which will make their presence obvious in even rough data. Then necessary and sufficient conditions will be given. Finally the problem of finding the transformations which render them similar will be solved by dissecting them into two uniform systems. All the solutions and only solutions are obtained by "pasting together" solutions to the uniform systems.

Necessary condition: *If $\underset{\sim}{G}$ is an abelian affine system, but not a uniform system, then there is exactly one number x_0 such that if f* *and* g *are two different scales of* $\underset{\sim}{G}$ *then* $f(x)$ *equals* $g(x)$ *if and only if* x *equal* x_0. *This point* x_0 *is a fixed point* (i.e., *satisfies the equation* $f(x) = x$) *of all the functions* f *of* $\underset{\sim}{G}$.

In other words, there is exactly one point common to all graphs of the functions of $\underset{\sim}{G}$.

Proof: First let $\underset{\sim}{G}$ be a subgroup of the affine group of scales $\{x \longrightarrow ax + b\}$. Then by a direct calculation it is seen that two functions of $\underset{\sim}{G}$ commute iff they have the same fixed points. Since the graphs of affine functions are straight lines, for any two different affine functions f and g the equation $f(x) = g(x)$ has at most one solution. Note that $\underset{\sim}{G}$ is a uniform system iff it is a subgroup of the translations, i.e., e and the affine functions without fixed points. Consequently, if the subgroup of the affine group $\underset{\sim}{G}$ is an abelian affine system but not a uniform system, then the condition is valid. Since for any scale u, f, and g we have

$$f(x) = g(x) \quad \text{iff} \quad u^{-1}fu[u^{-1}(x)] = u^{-1}gu[u^{-1}(x)]\ ,$$

the condition is generally valid.

Only those groups satisfying the necessary condition are considered in the sequel. Some additional notation is needed to express the remaining results clearly.

Let $\underset{\sim}{G}$ be any group satisfying the necessary condition with fixed point x_0. For any scale g of $\underset{\sim}{G}$ let $g_+ : (x_0,\infty) \longrightarrow (x_0,\infty)$ and $g_- : (-\infty,x_0) \longrightarrow (-\infty,x_0)$ be the homeomorphisms obtained by restricting g

to the indicated intervals. Let ℓ and m denote the real-valued homeomorphisms of (x_0,∞) and $(-\infty,x_0)$, respectively, given by

$$\begin{cases} \ell(x) = \text{natural logarithm of } (x - x_0) \\ m(x) = \ell(2x_0 - x) \; . \end{cases}$$

Two homomorphisms are now defined which make two groups of scales of the group $\underset{\sim}{G}$.

For each g in $\underset{\sim}{G}$ let g^+ and g^- be defined by

$$g^+ = \ell g_+ \ell^{-1} \; ; \; g^- = m g_- m^{-1} \; .$$

Since $(gh)_+ = g_+h_+$ and $(gh)_- = g_-h_-$ we have $(gh)^+ = g^+h^+$ and $(gh)^- = g^-h^-$. Since g^+ is the composition of increasing homeomorphisms mapping the reals eventually onto the reals, it is a scale. Since g^- is the composition of an even number of decreasing homeomorphisms and an increasing homeomorphism, it too is a scale. Consequently $^+$ and $^-$ are homomorphisms into the group of all scales.

Recall that $\underset{\sim}{G}^+$ and $\underset{\sim}{G}^-$ as sets of scales are partially ordered by $f \leqq g$ iff $f(x) \leqq g(x)$ for all real x. When the mappings $^+$ and $^-$ are isomorphisms, $\underset{\sim}{G}^+$ and $\underset{\sim}{G}^-$ are obviously isomorphic groups. Still $\underset{\sim}{G}^+$ and $\underset{\sim}{G}^-$ need not be isomorphic as <u>ordered</u> groups. It is easy to show that they are isomorphic as ordered groups whenever condition 3 of the next theorem is satisfied.

The main result of this section can now be formulated. It contains a complete characterization of the abelian affine systems.

Theorem 2: Let $\underset{\sim}{G}$ be a group of scales satisfying the necessary condition. Then $\underset{\sim}{G}$ is an abelian affine system but not a uniform system if and only if

1. $\underset{\sim}{G}^+$ is a uniform system

2. $\underset{\sim}{G}^-$ is a uniform system

3. for all f, g in $\underset{\sim}{G}$, $f^+ \leqq g^+$ iff $f^- \leqq g^-$.

Proof of Necessity: When $\underset{\sim}{G}$ is conjugate to a subgroup of the linear group each g in $\underset{\sim}{G}$ is of the form $g(\cdot) = u^{-1}[au(\cdot)]$ for some fixed scale u. With these equations, verification of the three conditions is a simple calculation. To show each abelian affine system which is not a u.s. is conjugate to a subgroup of the linear group, let u be a solution. Since the mapping $x \longrightarrow u(x) - u(x_0)$ is also a solution we may choose u such that $u(x_0)$ is zero. Then the equation $ugu^{-1}(x) = ax + b$ implies b is zero and $\underset{\sim}{G}$ is conjugate to a subgroup of the linear group.

Proof of Sufficiency: Let $\underset{\sim}{G}^+$ and $\underset{\sim}{G}^-$ have solutions v and w respectively. Then the equations $g^+ = v^{-1}[v(\cdot) + c]$ and $g^- = w^{-1}[w(\cdot) + k]$ define homeomorphisms of $\underset{\sim}{G}^+$ and $\underset{\sim}{G}^-$ into the ordered additive reals given by $g^+ \longrightarrow c$ and $g^- \longrightarrow k$. By condition 3, the mappings $g^+ \longrightarrow c$ and $g \longrightarrow g^+$ must be 1-1. Consequently, $c \longrightarrow g^+ \longrightarrow g \longrightarrow g^- \longrightarrow k$ defines a strictly monotonic homomorphism of two subgroups of the reals. By Hion's theorem the ratio of c to k remains constant as $g \neq e$ ranges over $\underset{\sim}{G}$. Since for any positive a, av is a solution and since

$g^+ = v^{-1}[v(\cdot) + c]$ implies $g^+ = (av)^{-1}[av(\cdot) + ac]$, the ratio can be considered to be plus or minus one. But condition 3 implies that c and k have the same sign. Consequently, without loss of generality we assume c equals k.

To obtain a solution we define a function u by

$$u(x) = \begin{cases} \ell^{-1}v\ell(x) & , x > x_0 \\ x_0 & , x = x_0 \\ m^{-1}wm & , x < x_0 \end{cases} .$$

This function is obviously a scale. Using the functional equation for the logarithm it is routine to verify that u is a solution and to complete the proof.

The three conditions can be shown to be independent. However, if it is known that $\underset{\sim}{G}^+$ and $\underset{\sim}{G}^-$ are uniform systems, then the third condition can be simplified by choosing any x_1 smaller then x_0 and x_2 larger than x_0. Then condition 3 can be replaced by 3':

3'. $f(x_1) \leqq g(x_1)$ iff $g(x_2) \leqq f(x_2)$.

The final result reduces uniqueness and computation questions to questions about uniform systems. We use the notation already introduced. Since the proof employs no new arguments, it is omitted.

<u>Corollary:</u> <u>(Uniqueness and Computation for Abelian Groups)</u> <u>Let</u> $\underset{\sim}{G}$ <u>be an abelian affine system but not a uniform system.</u> <u>Let</u> $g \neq e$ <u>be one fixed element</u> $\underset{\sim}{G}$. <u>Then a scale</u> u <u>is a solution for</u> $\underset{\sim}{G}$ <u>if and only if</u>

1. $v = \ell u_+ \ell^{-1}$ <u>is a solution for</u> $\underset{\sim}{G}^+$

2. $w = m u_- m^{-1}$ <u>is a solution for</u> $\underset{\sim}{G}^-$

3. $vg^+v^{-1}(0)$ <u>equals</u> $wg^-w^{-1}(0)$.

Thus to find a solution for $\underset{\sim}{G}$ as an affine system, one chooses $g \neq e$ in G and solves uniform systems $\underset{\sim}{G}^+$ and $\underset{\sim}{G}^-$ for v and w. To satisfy condition 3, w is multiplied by the positive constant $vg^+v^{-1}(0)/wg^-w^{-1}(0)$. A solution is given then by the formula in the preceeding proof.

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FOOTNOTES

[1] I am very grateful for the interest Eric Holman, David H. Krantz, Frederic M. Lord, M. Frank Norman and Burton S. Rosner have shown in this work.

[2] This paper has been referred to in Levine (1970) as Transformations that render curves similar.

[3] Most of this research was completed at the Educational Testing Service, Princeton, New Jersey where the author was Visiting Psychologist.