DOCUMENT RESUME

ED 052 256 TM 000 662

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TITLE	Some General Results on Fit in Factor Rotation.
INSTITUTION	Educational Testing Service, Princeton, N.J.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	Oct 70
NOTE	16p.
EDRS PRICE	EDRS Price MF-$0.65 HC-$3.29
DESCRIPTORS	Correlation, Factor Analysis, *Factor Structure, *Goodness of Fit, Mathematics, *Oblique Rotation, Orthogonal Rotation, *Transformations (Mathematics)

ABSTRACT

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RESEARCH BULLETIN

RB-70-60

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Educational Testing Service
Princeton, New Jersey
October 1970

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Abstract

In this paper a rather general theory of oblique factor rotation is outlined. The main results are formulated as four theorems. Necessary and sufficient conditions are derived for two factor matrices to admit identical factor structures and/or factor patterns with factors having unit variances. These conditions are expressed in terms of eigenvectors and eigenvalues of certain matrices obtainable from the data. It is also shown that two matrices admitting identical factor structures will admit identical factor patterns and vice versa. After introducing the notion of a pair of transformations to identical structures and/or identical patterns, rules are given as to finding such pairs if they exist. Finally, some immediate consequences of the theorems are noted. They concern, for example, the suitable choice of a target structure and/or pattern and a hierarchical order of jointly necessary and sufficient conditions for fitting a specified target perfectly.

SOME GENERAL RESULTS ON FIT IN FACTOR ROTATION[1]

1. Introduction

Transformation of factor matrices is often an indispensable step in the course of a factor analytic study. Especially when the study is of the confirmatory type, we may wish to rotate an initial factor matrix to a specified target. Closeness of fit will be a measure of the success of the transformation.

At times, fit may turn out to be quite unsatisfactory. In such cases we may wish to know the conditions responsible, and we may also wish to know if fit can be improved by a possibly minor change of the target which the investigator might be willing to concede. Necessary and sufficient conditions for perfect fit will be derived, and imperfect fit of any degree must then be attributable to violations of such conditions.

Techniques for obtaining optimal fit have been worked out for three different problems when rotation of a factor matrix (uncorrelated factors of unit variances) to a specified target is required. The problems differ with respect to the class of transformations admitted. Least squares solutions of the problem of finding an optimal orthogonal transformation matrix were given by several writers (e.g., Cliff, 1966; Fischer & Roppert, 1964; Green, 1952; Kristof, 1964; Schönemann, 1966). When correlated factors are involved, however, then a distinction must be made between factor structures and factor patterns. Mosier (1939) was the first to derive an approximate least squares method for transforming a given factor matrix toward a given target factor structure or, more specifically, the reference factor structure usually referred to as the matrix $\underline{V}$. Browne

[1]Research reported in this paper has been supported by grant GB-18230 from National Science Foundation.

(1967) developed a least squares procedure which is, at least in theory, exact. He illustrated it by way of a worked example. Least squares solutions for the case in which the specified target is a factor pattern are of a still more recent date. Browne and Kristof (1969) worked out a solution and applied the method to a set of data. Gruvaeus (in press) dealt with the same basic problem and arrived at a different procedure by minimizing a different criterion.

It may be worth noting that the orthogonal "Procrustes" problem is basically different from the more general oblique case. The usual least squares goodness of fit criterion as employed first by Green (1952) and then by others in the orthogonal situation is invariant under orthogonal transformations of the target. Analogous invariance properties of the corresponding least squares criterion do not hold, however, when the target is either a factor structure or a factor pattern, and replacement of the original target by another one obtained from the same correlation matrix is allowed. On account of this fact, the theory developed in the following sections of this paper will be concerned with problems in fitting factor structures and factor patterns to target structures and patterns respectively.

Target matrices will not necessarily be regarded as unalterable. In fact, a good deal of the following theory will be applicable to the problem of selecting a target. We will give necessary and sufficient conditions for two correlation or covariance matrices to admit identical factor structures or factor patterns and we will show how to recover such structures or patterns if they exist.

In defining least squares goodness of fit criteria, Browne (1967), Browne and Kristof (1969) and Gruvaeus (in press) used a rigidly specified

target without assumed knowledge of the correlation or covariance matrix from which the target was obtained. We do assume such knowledge, however, as will be the case when the target is determined by some earlier study. A factor structure or a factor pattern alone does not fully represent the information contained in a correlation or covariance matrix.

For the sake of clarity, the basic matrix notions used in this paper will be explained in some detail. Let $\underline{\Sigma}$ be a (reduced) correlation or covariance matrix of order $n \times n$ and rank p, $p \leq n$. $\underline{\Sigma}$ is called "reduced" when the diagonal elements have been adjusted such as to be determined only by common factors in the sense of the traditional linear factor analysis model. Consider a decomposition $\underline{\Sigma} = \underline{A\Phi A}'$ with $\underline{A}'$ and $\underline{\Phi}$ having orders $n \times p$ and $p \times p$, respectively, and rank p. Matrix $\underline{\Phi}$ is symmetric and positive definite because $\underline{\Sigma}$ is symmetric and should be left positive semidefinite, notwithstanding any adjustment of the diagonals. It is further postulated that diag $\underline{\Phi} = \underline{I}$. Under these provisions, $\underline{A}$ is called a <u>factor pattern.</u> Its elements are regression coefficients of tests on factors. Matrix $\underline{B} = \underline{A\Phi}$ is called a <u>factor structure</u>. Its elements are covariances between tests and factors. Evidently, $\underline{\Sigma} = \underline{B\Phi}^{-1}\underline{B}'$. Matrix $\underline{\Phi}$ gives the <u>correlations between factors</u>. If $\underline{\Phi} = \underline{I}$, i.e., when the factors are taken to be uncorrelated, then factor pattern and factor structure coincide. One speaks simply of a <u>factor matrix</u>. It will be observed that $\underline{\Sigma} = \underline{BA}'$ for $\underline{\Phi}$ given.

The entirety of factor structures determined by $\underline{\Sigma}$ can be written as follows. Let $\underline{F}$ be a factor matrix, $\underline{\Sigma} = \underline{FF}'$. Then $\underline{B} = \underline{FU}$ with diag $\underline{U}'\underline{U} = \underline{I}$, $\underline{U}$ full rank, comprises all possible factor structures. This formulation is generally valid because any two factor matrices $\underline{F}_1$

and $\underline{F}_2$ with $\underline{F}_1\underline{F}_1' = \underline{F}_2\underline{F}_2'$ are related by means of $\underline{F}_2 = \underline{F}_1\underline{S}$, $\underline{S}$ orthogonal. One observes that $\underline{U}^{*\prime}\underline{U}^* = \underline{U}'\underline{U}$ when $\underline{U}^* = \underline{SU}$.

The entirety of factor patterns is given by $\underline{A} = \underline{FV}$ with $\text{diag}(\underline{V}'\underline{V})^{-1} = \underline{I}$, $\underline{V}$ full rank of course. Again, this formulation is generally valid. In each case, $\underline{F}$ may be taken as $\underline{P\Gamma}^{1/2}$ when $\underline{\Sigma} = \underline{P\Gamma P}'$ is a canonical decomposition, $\underline{P}$ having orthonormal column vectors and $\underline{\Gamma}$ being a positive definite diagonal matrix.

In writing $\underline{\Sigma}$ for a correlation or covariance matrix we do not wish to imply that $\underline{\Sigma}$ must be a population matrix. In a typical situation, $\underline{\Sigma}$ will be obtained from a sample.

The present paper is primarily theoretical in nature. Applications of the theory will be contained in another report.

2. A Lemma

A proof of the following lemma will shorten the development of the intended theory of factor rotation.

Lemma: Consider the two simultaneous equations

$$(1)\qquad \begin{aligned} \mu_1\xi_1^2 + \ldots\ldots + \mu_p\xi_p^2 &= 0 \\ \xi_1^2 + \ldots\ldots + \xi_p^2 &= 1 \;, \end{aligned}$$

$p \geq 2$, the coefficients μ_i being real and not all of them zero. These equations admit p linearly independent real solution vectors $\underline{\xi}' = (\xi_1, \ldots, \xi_p)$ if and only if there is a coefficient $\mu_j > 0$ as well as a coefficient $\mu_k < 0$.

Proof: Necessity of the condition is clear. Sufficiency is inferred from considering the following exhaustive distinction of possibilities.

(i) Precisely two coefficients are different from zero, $\mu_1 > 0$ and $\mu_2 < 0$, say. Two linearly independent solution vectors of the form $(\xi_1, \xi_2, 0, \ldots, 0)$ can be found easily. Additional $p - 2$ linearly independent solution vectors can be chosen to be of the form $(0, 0, \xi_3, \ldots, \xi_p)$. Thus a set of p linearly independent solution vectors exists. This result holds for any $p \geq 2$.

(ii) Suppose there are exactly K coefficients different from zero, $2 < K \leq p$. They may be taken as $\mu_1, \ldots, \mu_K$ with μ_1 and μ_2 having the same sign. We proceed by induction with respect to K, the first induction step being contained in (i). In the sequence $\mu_2, \mu_3, \ldots, \mu_K$ both signs must occur. By induction assumption, there are $p - 1$ linearly independent solution vectors of the form $(0, \xi_2, \xi_3, \ldots, \xi_p)$. Analogously, the sequence $\mu_1, \mu_3, \ldots, \mu_p$ will contain coefficients of each sign, hence there are also $p - 1$ linearly independent solution vectors of the form $(\xi_1, 0, \xi_3, \ldots, \xi_p)$. Among these, there must be at least one solution vector with $\xi_1 \neq 0$. Adding such a solution vector to the first set gives a total of p linearly independent solution vectors. This completes the proof.

3. Theorems on Fit

In this section, some theorems on problems of fit as regards factor structures and factor patterns obtained from matrices $\underline{\Sigma}_1$ and $\underline{\Sigma}_2$ will be given. The theorems may serve to identify such characteristics of $\underline{\Sigma}_1$ and $\underline{\Sigma}_2$ which preclude satisfactory fit. The question of goodness of fit in relation to the choice of a target will also be considered. The number of factors in two matrices under comparison will be assumed equal.

Theorem 1: Let $\underline{\Sigma}_1$ and $\underline{\Sigma}_2$, $\underline{\Sigma}_1 \neq \underline{\Sigma}_2$, have canonical decompositions $\underline{\Sigma}_1 = \underline{P}_1\underline{\Gamma}_1\underline{P}_1'$ and $\underline{\Sigma}_2 = \underline{P}_2\underline{\Gamma}_2\underline{P}_2'$. The following conditions for $\underline{\Sigma}_1$ and $\underline{\Sigma}_2$ to admit identical factor structures, $\underline{B}_1 = \underline{B}_2$, are jointly necessary and sufficient:

1. There is an orthogonal matrix $\underline{T}$ with $\underline{P}_2 = \underline{P}_1\underline{T}$.
2. $\underline{W} = \underline{\Gamma}_2^{1/2}\underline{T}'\underline{\Gamma}_1^{-1}\underline{T}\underline{\Gamma}_2^{1/2}$ has an eigenvalue greater than unity as well as an eigenvalue smaller than unity.

Proof: (i) Necessity will be proved first. We write

$$(2) \qquad \begin{aligned} \underline{B}_1 &= \underline{P}_1\underline{\Gamma}_1^{1/2}\underline{U}_1 , &\qquad \operatorname{diag} \underline{U}_1'\underline{U}_1 &= \underline{I} \\ \underline{B}_2 &= \underline{P}_2\underline{\Gamma}_2^{1/2}\underline{U}_2 , &\qquad \operatorname{diag} \underline{U}_2'\underline{U}_2 &= \underline{I} \quad . \end{aligned}$$

$\underline{B}_1 = \underline{B}_2$ implies that the column vectors of both $\underline{P}_1$ and $\underline{P}_2$ are orthonormal bases of the same p -dimensional space. Hence there is an orthogonal matrix $\underline{T}$ relating the two bases, $\underline{P}_2 = \underline{P}_1\underline{T}$.

Now, starting with (2), it follows from $\underline{B}_1 = \underline{B}_2$ and $\underline{P}_1'\underline{P}_1 = \underline{I}$ that

$$(3) \qquad \underline{U}_1 = \underline{\Gamma}_1^{-1/2}\underline{T}\underline{\Gamma}_2^{1/2}\underline{U}_2 \quad .$$

Then, using (3) and introducing

$$(4) \qquad \underline{W} = \underline{\Gamma}_2^{1/2}\underline{T}'\underline{\Gamma}_1^{-1}\underline{T}\underline{\Gamma}_2^{1/2} \quad ,$$

the conditions on the diagonals as stated in (2) become

$$(5) \qquad \operatorname{diag} \underline{U}_2'\underline{W}\underline{U}_2 = \underline{I} , \qquad \operatorname{diag} \underline{U}_2'\underline{U}_2 = \underline{I} \quad .$$

An equivalent formulation is

$$(6) \qquad \operatorname{diag} \underline{U}_2'(\underline{W} - \underline{I})\underline{U}_2 = \underline{0} , \qquad \operatorname{diag} \underline{U}_2'\underline{U}_2 = \underline{I} \quad .$$

Let $\underline{W}$ have a canonical decomposition $\underline{W} = \underline{C}'\underline{\Lambda}\underline{C}$ with elements λ_i in the diagonal of $\underline{\Lambda}$. Further, put $\underline{U}_2^* = \underline{C}\underline{U}_2$ and define a diagonal matrix $\underline{M} = \underline{\Lambda} - \underline{I}$ with elements μ_i in the diagonal. We will have $\underline{M} \neq \underline{0}$ because $\underline{\Lambda} = \underline{I}$ implies $\underline{W} = \underline{I}$ and therefore $\underline{\Sigma}_1 = \underline{\Sigma}_2$ as it follows rather easily from (4). Conditions (6) become

(7) $\quad \operatorname{diag} \underline{U}_2^{*\prime}\underline{M}\underline{U}_2^* = \underline{0} \ , \qquad \operatorname{diag} \underline{U}_2^{*\prime}\underline{U}_2^* = \underline{I} \ .$

Let $(\xi_1,\ldots,\xi_p)$ be a row vector of $\underline{U}_2^{*\prime}$. Conditions (7) can be expressed in form (1) where the two simultaneous equations are to admit p linearly independent solution vectors. According to the previous lemma this will be the case precisely when there is a coefficient $\mu_j > 0$ as well as a coefficient $\mu_k < 0$ or, equivalently, when there is an eigenvalue $\lambda_j > 1$ as well as an eigenvalue $\lambda_k < 1$. This completes the necessity part of the proof.

(ii) As to sufficiency, suppose that the conditions given in the theorem are satisfied. Thus $\underline{M} = \underline{\Lambda} - \underline{I}$, $\underline{M}$ and $\underline{\Lambda}$ as defined before, has positive as well as negative elements in the diagonal. It follows from the previous lemma that there is a nonsingular matrix $\underline{U}_2^*$ obeying (7). Reversing the corresponding steps in (i) we arrive at a nonsingular matrix $\underline{U}_2$ satisfying (5). Now define a matrix $\underline{U}_1$ by means of (3). Evidently, $\operatorname{diag} \underline{U}_1'\underline{U}_1 = \operatorname{diag} \underline{U}_2'\underline{W}\underline{U}_2 = \underline{I}$. Choosing the possible factor structures $\underline{B}_1 = \underline{P}_1\underline{\Gamma}_1^{1/2}\underline{U}_1$ and $\underline{B}_2 = \underline{P}_1\underline{T}\underline{\Gamma}_2^{1/2}\underline{U}_2$ one has indeed $\underline{B}_1 = \underline{B}_2$. This concludes the proof.

The problem of existence of identical factor structures, $\underline{B}_1$ and $\underline{B}_2$, is certainly a symmetrical one. Yet in the formulation and proof of Theorem 1 the matrices $\underline{\Sigma}_1$ and $\underline{\Sigma}_2$ have been treated unsymmetrically. It may therefore be desirable to demonstrate that interchanging the roles of $\underline{\Sigma}_1$ and $\underline{\Sigma}_2$ does not affect the theorem materially.

Such interchange would evidently not alter condition 1. But $\underline{W}$ would have to be replaced by $\underline{W}^* = \underline{\Gamma}_1^{1/2}\underline{T}\underline{\Gamma}_2^{-1}\underline{T}'\underline{\Gamma}_1^{1/2}$. However, placing $G = \underline{\Gamma}_2^{1/2}\underline{T}'\underline{\Gamma}_1^{-1/2}$, one has $\underline{W} = \underline{GG}'$ and $\underline{W}^* = (\underline{G}'\underline{G})^{-1}$. Hence the eigenvalues of $\underline{W}$ are reciprocals of those of $\underline{W}^*$ which implies that condition 2 remains materially the same.

It will be noted that condition 1 in Theorem 1 is equivalent to $\underline{P}_1\underline{P}_1' = \underline{P}_2\underline{P}_2'$.

An analogous theorem involving factor patterns could be established by essentially following the scheme of the proof of Theorem 1. However, a somewhat more general statement will be shown to be true.

<u>Theorem 2</u>: $\underline{\Sigma}_1$ and $\underline{\Sigma}_2$, $\underline{\Sigma}_1 \neq \underline{\Sigma}_2$, admit identical factor patterns if and only if they admit identical factor structures.

In other words, conditions 1 and 2 in Theorem 1 are jointly necessary and sufficient for the existence of both identical factor structures and identical factor patterns.

Proof: Let $\underline{\Sigma}_1$ and $\underline{\Sigma}_2$ have canonical decompositions $\underline{\Sigma}_1 = \underline{P}_1\underline{\Gamma}_1\underline{P}_1'$ and $\underline{\Sigma}_2 = \underline{P}_2\underline{\Gamma}_2\underline{P}_2'$. In analogy to the first paragraph of part (i) in the proof of Theorem 1 it is at once inferred that the existence of an orthogonal matrix $\underline{T}$ with $\underline{P}_2 = \underline{P}_1\underline{T}$ is also necessary for $\underline{\Sigma}_1$ and $\underline{\Sigma}_2$ to admit identical factor patterns. Now suppose that $\underline{\Sigma}_1$ and $\underline{\Sigma}_2$ admit identical factor structures, thus conditions 1 and 2 in Theorem 1 are satisfied. Introduce matrices

$$(8) \qquad \underline{\tilde{\Sigma}}_1 = \underline{P}_1\underline{\Gamma}_1^{-1}\underline{P}_1' \quad , \qquad \underline{\tilde{\Sigma}}_2 = \underline{P}_2\underline{\Gamma}_2^{-1}\underline{P}_2'$$

and form

$$(9) \qquad \underline{\tilde{W}} = \underline{\Gamma}_2^{-1/2}\underline{T}'\underline{\Gamma}_1\underline{T}\underline{\Gamma}_2^{-1/2} \quad .$$

It is seen that $\tilde{\underline{W}} = \underline{W}^{-1}$, hence the eigenvalues of $\tilde{\underline{W}}$ are reciprocals of those of $\underline{W}$. Therefore, according to Theorem 1, $\tilde{\underline{\Sigma}}_1$ and $\tilde{\underline{\Sigma}}_2$ will also admit identical factor structures. In other words, there are nonsingular matrices $\tilde{\underline{U}}_1$ and $\tilde{\underline{U}}_2$ with

$$\underline{P}_1\underline{\Gamma}_1^{-1/2}\tilde{\underline{U}}_1 = \underline{P}_1\underline{T}_1\underline{\Gamma}_2^{-1/2}\tilde{\underline{U}}_2 \tag{10}$$

$$\operatorname{diag} \tilde{\underline{U}}_1'\tilde{\underline{U}}_1 = \operatorname{diag} \tilde{\underline{U}}_2'\tilde{\underline{U}}_2 = \underline{I} \quad .$$

But this is equivalent to

$$\underline{P}_1\underline{\Gamma}_1^{1/2}\tilde{\underline{U}}_1'^{-1} = \underline{P}_1\underline{T}\underline{\Gamma}_2^{1/2}\tilde{\underline{U}}_2'^{-1}$$

$$\operatorname{diag} (\tilde{\underline{U}}_1^{-1}\tilde{\underline{U}}_1'^{-1})^{-1} = \operatorname{diag} (\tilde{\underline{U}}_2^{-1}\tilde{\underline{U}}_2'^{-1})^{-1} = \underline{I} \quad .$$

Upon setting

$$\underline{V}_1 = \tilde{\underline{U}}_1'^{-1} \quad , \qquad \underline{V}_2 = \tilde{\underline{U}}_2'^{-1} \tag{11}$$

we obtain instead

$$\underline{P}_1\underline{\Gamma}_1^{1/2}\underline{V}_1 = \underline{P}_1\underline{T}\underline{\Gamma}_2^{1/2}\underline{V}_2 \tag{12}$$

$$\operatorname{diag} (\underline{V}_1'\underline{V}_1)^{-1} = \operatorname{diag} (\underline{V}_2'\underline{V}_2)^{-1} = \underline{I} \quad .$$

Therefore $\underline{\Sigma}_1$ and $\underline{\Sigma}_2$ admit identical factor patterns also, as given in (12). The proof is completed upon noting that each step in this derivation is reversible.

Theorems 1 and 2 have been formulated as existence theorems. The problem of actually determining identical factor structures and/or factor patterns, if they exist in a given instance, will be our next concern.

Let $\underline{\Sigma}_1 = \underline{P}_1\underline{\Gamma}_1\underline{P}_1'$ and $\underline{\Sigma}_2 = \underline{P}_2\underline{\Gamma}_2\underline{P}_2'$ be canonical decompositions. We wish to characterize all pairs of transformation matrices $\underline{U}_1, \underline{U}_2$ with diag $\underline{U}_1'\underline{U}_1 =$ diag $\underline{U}_2'\underline{U}_2 = \underline{I}$ which yield $\underline{B}_1 = \underline{P}_1\underline{\Gamma}_1^{1/2}\underline{U}_1 = \underline{P}_2\underline{\Gamma}_2^{1/2}\underline{U}_2 = \underline{B}_2$. A pair of transformation matrices satisfying these conditions will be called <u>a pair of transformations to identical structures</u>. Similarly, we wish to characterize all pairs of transformation matrices $\underline{V}_1, \underline{V}_2$ with $\text{diag}(\underline{V}_1'\underline{V}_1)^{-1} = \text{diag}(\underline{V}_2'\underline{V}_2)^{-1} = \underline{I}$ which yield $\underline{A}_1 = \underline{P}_1\underline{\Gamma}_1^{1/2}\underline{V}_1 = \underline{P}_2\underline{\Gamma}_2^{1/2}\underline{V}_2 = \underline{A}_2$. A pair of transformation matrices satisfying the latter conditions will be named <u>a pair of transformations to identical patterns</u>. The following theorem gives necessary and sufficient conditions for $\underline{U}_1, \underline{U}_2$ and $\underline{V}_1, \underline{V}_2$ to be a pair of transformations to identical structures and a pair of transformations to identical patterns, respectively.

<u>Theorem 3</u>: Let $\underline{\Sigma}_1$ and $\underline{\Sigma}_2$, $\underline{\Sigma}_1 \neq \underline{\Sigma}_2$, have canonical decompositions $\underline{\Sigma}_1 = \underline{P}_1\underline{\Gamma}_1\underline{P}_1'$ and $\underline{\Sigma}_2 = \underline{P}_2\underline{\Gamma}_2\underline{P}_2'$. Full rank matrices $\underline{U}_1, \underline{U}_2$ or $\underline{V}_1, \underline{V}_2$ are a pair of transformations to identical structures or a pair of transformations to identical patterns, respectively, if and only if the following conditions are met:

1. There is an orthogonal matrix $\underline{T}$ with $\underline{P}_2 = \underline{P}_1\underline{T}$.
2. Let $\underline{W} = \underline{\Gamma}_2^{1/2}\underline{T}'\underline{\Gamma}_1^{-1}\underline{T}\underline{\Gamma}_2^{1/2}$ have a canonical decomposition $\underline{W} = \underline{C}'\underline{\Lambda}\underline{C}$. Then $\underline{U}_2$ is to be chosen such that $\hat{\underline{U}}_2 = \underline{C}\underline{U}_2$ satisfies simultaneously diag $\hat{\underline{U}}_2'(\underline{\Lambda} - \underline{I})\hat{\underline{U}}_2 = 0$ and diag $\hat{\underline{U}}_2'\hat{\underline{U}}_2 = \underline{I}$. Matrix $\underline{U}_1$ is to be taken as $\underline{U}_1 = \underline{\Gamma}_1^{-1/2}\underline{T}\underline{\Gamma}_2^{1/2}\underline{U}_2$.
3. As regards factor patterns, $\underline{V}_2$ is to be chosen such that $\hat{\underline{V}}_2 = \underline{C}\underline{V}_2$ satisfies simultaneously diag $\hat{\underline{V}}_2^{-1}(\underline{\Lambda}^{-1} - \underline{I})\hat{\underline{V}}_2'^{-1} = 0$ and $\text{diag}(\hat{\underline{V}}_2'\hat{\underline{V}}_2)^{-1} = \underline{I}$. Matrix $\underline{V}_1$ is to be taken as $\underline{V}_1 = \underline{\Gamma}_1^{-1/2}\underline{T}\underline{\Gamma}_2^{1/2}\underline{V}_2$.

Proof: The theorem follows from suitably adapting and interpreting the combined proofs of Theorems 1 and 2 quite immediately.

It should be noted that the determination of matrices $\hat{U}_2$ or $\hat{V}_2^{-1}$ satisfying the above conditions 2 or 3, respectively, is a rather simple matter. It amounts to finding linearly independent solutions of a system of equations of the form (1). Then the next step would consist in obtaining $U_2 = C'\hat{U}_2$ or $V_2 = C'\hat{V}_2$. The applicability of this procedure depends exclusively upon the existence of T and, secondly, the eigenvalues of W .

In the preceding developments repeated use has been made of matrix W as originally defined in Theorem 1. In fact, knowledge of the eigenvalues of W is required if Theorem 3 is to be applied.

However, being existence theorems, Theorems 1 and 2 permit a combined reformulation which does not explicitly involve matrix W and which may be more appealing formally. The following result is obtained.

Theorem 4: Let Σ_1 and Σ_2 , $\Sigma_1 \neq \Sigma_2$, have matrices of eigenvectors P_1 and P_2 , respectively. The following conditions for Σ_1 and Σ_2 to admit identical factor structures and/or factor patterns are jointly necessary and sufficient:

1. P_1 and P_2 are orthogonal transforms of each other.
2. $\Sigma_1 - \Sigma_2$ is not a semidefinite matrix.

Proof: Condition 1 is obvious in view of Theorems 1 and 2. As to condition 2, Theorem 1 stipulates that $W = \Gamma_2^{1/2}T'\Gamma_1^{-1}T\Gamma_2^{1/2}$ have an eigenvalue greater than unity as well as an eigenvalue smaller than unity. This is equivalent to $W^{-1} - I$ having a positive as well as a negative eigenvalue. Or, by a generalization of Sylvester's law of inertia,

$$\underline{P}_2\underline{\Gamma}_2^{1/2}(\underline{W}^{-1} - \underline{I})\underline{\Gamma}_2^{1/2}\underline{P}_2' = \underline{P}_1\underline{\Gamma}_1\underline{P}_1' - \underline{P}_2\underline{\Gamma}_2\underline{P}_2'$$
$$= \underline{\Sigma}_1 - \underline{\Sigma}_2$$

is not a semidefinite matrix. This concludes the proof.

4. Some Immediate Consequences of the Theorems

I. Theorem 3 gave rules as to the possible choice of pairs of transformations to identical structures and pairs of transformations to identical patterns. The following consequence may deserve particular interest: Not every full rank matrix $\underline{U}_2$ or $\underline{V}_2$ satisfying diag $\underline{U}_2'\underline{U}_2 = \underline{I}$ or diag$(\underline{V}_2'\underline{V}_2)^{-1} = \underline{I}$ can be taken as a possible member of a pair of transformations to identical structures or identical patterns, respectively. Therefore, if unsatisfactory fit is encountered in rotating a factor matrix to a target structure or target pattern, improvement of fit may be attempted through a possibly but minor change of the target. The original target may have been obtained from rotating a factor matrix $\underline{P}_2\underline{\Gamma}_2^{1/2}$ by means of a transformation $\underline{U}_2$ or $\underline{V}_2$ which, when rotation of a factor matrix $\underline{P}_1\underline{\Gamma}_1^{1/2}$ toward the target is sought, is not close enough to being a possible member of a pair of transformations to identical structures or patterns. However, there may be another transformation, $\underline{U}_2^*$ or $\underline{V}_2^*$ say, which answers this requirement more accurately and which will yield a target that is still psychologically meaningful or otherwise useful to the researcher. Of course, such a substitution is feasible only if the factor matrix is known from which the original target was derived.

II. According to the previous theory, the following <u>three</u> conditions are jointly necessary and sufficient for actually attaining perfect fit when a target is specified:

(i) $\underline{P}_1$ and $\underline{P}_2$ span identical spaces, i.e., there is an orthogonal matrix $\underline{T}$ satisfying $\underline{P}_2 = \underline{P}_1\underline{T}$.

(ii) $\underline{W} = \underline{\Gamma}_2^{1/2}\underline{T}'\underline{\Gamma}_1^{-1}\underline{T}\underline{\Gamma}_2^{1/2}$ has an eigenvalue greater than unity as well as an eigenvalue smaller than unity ($\underline{\Sigma}_1 - \underline{\Sigma}_2$ is not semidefinite).

(iii) Transformation matrix $\underline{U}_2$ or $\underline{V}_2$ is a member of a pair of transformations to identical structures or identical patterns.

These three conditions form a hierarchical order. Condition (ii) becomes meaningful only if condition (i) is already met. Condition (iii) can be checked upon only if both conditions (i) and (ii) are satisfied. If conditions (i) and/or (ii) are not fulfilled then perfect fit can under no circumstances be achieved, even if any choice of a target were permitted.

III. Suppose that two covariance (correlation) matrices, $\underline{\Sigma}_1$ and $\underline{\Sigma}_2$, admit identical factor structures or factor patterns. Then, typically, the corresponding correlation (covariance) matrices $\underline{D}_1\underline{\Sigma}_1\underline{D}_1$ and $\underline{D}_2\underline{\Sigma}_2\underline{D}_2$, $\underline{D}_1$ and $\underline{D}_2$ diagonal nonscalar matrices, will not allow for identical factor structures or factor patterns. For, consider the canonical decompositions $\underline{\Sigma}_1 = \underline{P}_1\underline{\Gamma}_1\underline{P}_1'$, $\underline{\Sigma}_2 = \underline{P}_2\underline{\Gamma}_2\underline{P}_2'$, $\underline{D}_1\underline{\Sigma}_1\underline{D}_1 = \underline{Q}_1\underline{\Delta}_1\underline{Q}_1'$ and $\underline{D}_2\underline{\Sigma}_2\underline{D}_2 = \underline{Q}_2\underline{\Delta}_2\underline{Q}_2'$. Now $\underline{P}_2 = \underline{P}_1\underline{T}$ with $\underline{T}$ orthogonal does in general not imply $\underline{Q}_2 = \underline{Q}_1\underline{S}$ with $\underline{S}$ orthogonal.

References

Browne, W. M. On oblique Procrustes rotation. Psychometrika, 1967, 32, 125-132.

Browne, W. M., & Kristof, W. On the oblique rotation of a factor matrix to a specified pattern. Psychometrika, 1969, 34, 237-248.

Cliff, N. Orthogonal rotation to congruence. Psychometrika, 1966, 31, 33-42.

Fischer, G., & Roppert, J. Bemerkungen zu einem Verfahren der Transformationsanalyse. Archiv für die gesamte Psychologie, 1964, 116, 98-100.

Green, B. F. The orthogonal approximation of an oblique structure in factor analysis. Psychometrika, 1952, 17, 429-440.

Gruvaeus, G. T. A general approach to Procrustes pattern rotation. Psychometrika, in press.

Kristof, W. Die beste orthogonale Transformation zur gegenseitigen Überführung zweier Faktorenmatrizen. Diagnostica, 1964, 10, 87-90.

Mosier, C. I. Determining a simple structure when loadings for certain tests are known. Psychometrika, 1939, 4, 149-162.

Schönemann, P. H. A generalized solution of the orthogonal Procrustes problem. Psychometrika, 1966, 31, 1-10.