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ABSTRACT

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FACTOR TRANSFORMATION

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The purpose of this paper is twofold. First, a new--but hardly revolutionary--general approach to the problem of oblique factor transformation is identified and presented as an alternative to the common "blind" transformation techniques currently available. Secondly, techniques for implementing such an approach are developed. The first section of the paper contains a brief review of the procrustes problem. The next section contains the rationale for a somewhat related--"quasi-procrustean"--approach to oblique factor transformation. The third section contains the development of new techniques for implementing this approach. Following this derivation, solutions for three well-known sets of data are presented. Finally, some aspects of such an approach in need of further research are discussed.

1. The Procrustes Problem

The objective in general procrustean transformation is to find that transformation which, when applied to some preliminary matrix--for example, of unrotated components, image factors, or common factors--yields a solution matrix

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that is, according to some criterion (traditionally that of least-squares, although it need not be), the closest possible fit to a "target" matrix. The target matrix in such an application is usually generated by a prior hypothesis or factor study, and the procedure can be characterized as hypothesis confirmation, as opposed to the more prevalent factor analytic objective of hypothesis generation (see Cattell, 1966). If the obtained solution matrix is "close" to the target matrix, one concludes that the hypothesis manifested in the target matrix has been confirmed. "Closeness" appears, at present, not to have been operationalized.

Several methods of obtaining an oblique procrustean solution have been proposed. The best known [Hurley & Cattell, 1962; Mosier, 1939], yields a least-squares transformation to a reference-vector structure solution only prior to column-normalization. Browne [1967] presented an exact unit-length least-squares transformation to a reference structure matrix, and Mulaik [1969] extended this work to a least-squares fit to a primary-factor pattern matrix. It is noted in passing that least-squares solutions to the orthogonal procrustes problem have been proposed for various specialized cases by Gibson [1960, 1962], Green [1952], and Johnson [1964], and for the most general case by Schönemann [1966]. It appears that the problem of obtaining procrustean solutions--both orthogonal and oblique--satisfying the least-squares criterion has, for all purposes, been solved. Alternative criteria remain a possibility for further research on the procrustes problem.

2. Rationale for a Quasi-Procrustean Approach

The usual objective in factor analytic studies is to discover, rather than confirm the existence of, factors, and it is the former application that is the major concern of the rest of this paper. The proposal developed in this section,

then, differs in aim from the techniques noted in the preceding section, in that no prior expectations determine the shape of the final solution.

An oblique quasi-procrustean procedure will be defined as one in which a final oblique solution is the result of first obtaining, from the data at hand, rather than from prior hypothesis or experiment, a preliminary estimate of how the final structure should appear, and then obtaining the "cleanest" (in some sense) manifestation of that structure, or, alternatively, the most compelling simple structure. Two procedures, which are free to vary, are implied by such an approach. The first is the method in which the preliminary estimate is generated. The second is the method of obtaining the optimal manifestation of this preliminary estimate for the data. Probably the best known quasi-procrustean technique is <u>promax</u> [Hendrickson & White, 1964], in which the preliminary estimate is a "blind" orthogonal solution--typically varimax--and the cleanest manifestation of this structure is obtained by raising each element of the orthogonal solution matrix to a power--typically the second or fourth--retaining the signs, and obtaining the least-squares transformation to this target using the fitting technique of Mosier [1939] and Hurley and Cattell [1962]. A quasi-procrustean solution may involve obtaining some "ideal" target matrix (perhaps a matrix of 0's and 1's) from the preliminary estimate of structure and using one of the previously noted matrix fitting procedures to simulate, as closely as possible, that target. An alternative to this procedure is presented in the following section.

The most compelling rationale for a quasi-procrustean approach seems, to the author, to rest in the fact that, among the current "blind" automatic techniques, (1) none appears capable of consistently yielding cleanest or most interpretable solutions for all kinds of data (see Hakstian, in press), and (2) not only may the <u>clarity of resolution</u> between salient and hyperplane elements differ between two

analytic solutions for the same data, but the actual factors themselves may be somewhat different, a factor being marked by a certain subset of variables in the one solution and the corresponding factor (as closely as can be determined) being marked by only some of these variables in the other. The final interpretation of the factors obtained may well be a function of the particular transformation procedure employed. Such "blind" procedures are used, then, to obtain simultaneously, the underlying structure of the data--that is, the variables marking each factor--and the most clear-cut and interpretable manifestation of this structure. What appears needed is a final solution whose structure is somewhat generalizable over several transformation techniques--a viewpoint that appears in line with Harris [1967], whose concern was generalizability over factor models and extraction procedures--and which represents the most clearly resolved representation of that structure.

It is proposed that the quasi-procrustean approach in general, then, should ideally involve first, identifying for each factor the variables that appear to define the factor with some constancy over several transformations of the unrotated matrix, and second, obtaining that solution most clearly displaying this structure.

3. Development of a Quasi-Procrustean Computing Procedure

The overall strategy dealt with in this paper requires that for each factor, a decision be made as to which variables are likely to be consistently (over transformations) "salient," or, alternatively, to have large (in absolute value) projections on the factor axis. Although such a procedure appears subjective, attempts to put this phase on an operational footing are discussed later in this section.

A Technique for Clear Resolution of the Structure

Once the salient variables for each factor have been determined, one simple structure representation of it may be obtained by constructing a target vector--

perhaps containing only 0's, 1's and -1's--and obtaining the least-squares fit, using the techniques of Section 1. An alternative resolution of salient and hyperplane elements, however, may be obtained by maximizing their separation. It is this approach that will be developed in this section. Still a third hyperplane-fitting procedure--but using only the non-salient variables--was noted by Harris and Kaiser [1964, p. 360].

The technique to be considered may be seen as a "maximal mean difference" criterion. Let $\underline{A}$, of order $\underline{n} \times \underline{m}$, be a given matrix of unrotated factor coefficients, where $\underline{n}$ is the number of variables, and $\underline{m}$, of factors. We wish to find a transformation matrix $\underline{B}$, of order $\underline{m} \times \underline{m}$, yielding a matrix $\underline{V}$, of order $\underline{n} \times \underline{m}$, of transformed reference structure coefficients

(1) $$V = AB$$

that has a set of $\underline{m}$ extremum properties as follows: Let $\underline{V} = [\underline{v}_1, \ldots, \underline{v}_m]$. In the $\underline{j}$th column of $\underline{V}$, $\underline{v}_j$, label some $\underline{p}_j$ elements as salient, and the $\underline{q}_j$ remaining elements as non-salient (so that, of course, $\underline{p}_j + \underline{q}_j = \underline{n}$). Then choose the $\underline{j}$th column of $\underline{B}$, $\underline{b}_j$, subject to the constraint

(2) $$\underline{b}_j'\underline{b}_j = 1, \quad \underline{j} = 1, \ldots, \underline{m},$$

so that the difference between the mean square (thus treating positive and negative salients as equivalently marking the factor) of the salient elements, and the mean square of the non-salient elements, of $\underline{v}_j$, is an absolute maximum, i.e., such that

(3) $$\underline{\Phi}_j = \frac{1}{\underline{p}_j} \sum_{(s)} \underline{v}_{ij}^2 - \frac{1}{\underline{q}_j} \sum_{(ns)} \underline{v}_{ij}^2$$

attains its supremum value, where the first sum on the right is over the $\underline{p}_j$ salient elements, and the second is over the $\underline{q}_j$ non-salient elements.

Now let

$$\underline{v}_j = \underline{v}_{j(s)} + \underline{v}_{j(ns)}, \tag{4}$$

where $\underline{v}_{j(s)}$ is obtained from $\underline{v}_j$ by replacing all non-salient elements by zeros, and $\underline{v}_{j(ns)}$ is obtained from $\underline{v}_j$ by replacing all salient elements by zeros, $\underline{j} = 1, \ldots, \underline{m}$. Correspondingly, let

$$\underline{A} = \underline{A}_{j(s)} + \underline{A}_{j(ns)}, \tag{5}$$

where $\underline{A}_{j(s)}$ is obtained from $\underline{A}$ by replacing each row whose row-index is the same as that of a non-salient element of $\underline{v}_j$, by a null row, and conversely for $\underline{A}_{j(ns)}$, $\underline{j} = 1, \ldots, \underline{m}$. We may then rewrite (3) in the form

$$\underline{\Phi}_j = \frac{1}{\underline{p}_j}\underline{v}'_{j(s)}\underline{v}_{j(s)} - \frac{1}{\underline{q}_j}\underline{v}'_{j(ns)}\underline{v}_{j(ns)}, \tag{6}$$

and clearly we also have

$$\begin{aligned} \underline{v}_{j(s)} &= \underline{A}_{j(s)}\underline{b} \\ \underline{v}_{j(ns)} &= \underline{A}_{j(ns)}\underline{b}, \end{aligned} \tag{7}$$

so that (6) becomes

$$\begin{aligned} \underline{\Phi}_j &= \frac{1}{\underline{p}_j}\underline{b}'\underline{A}'_{j(s)}\underline{A}_{j(s)}\underline{b} - \frac{1}{\underline{q}_j}\underline{b}'\underline{A}'_{j(ns)}\underline{A}_{j(ns)}\underline{b} \\ &= \underline{b}'\underline{W}_j\underline{b}, \end{aligned} \tag{8}$$

where

$$\underline{W}_j = \frac{1}{\underline{p}_j}\underline{A}'_{j(s)}\underline{A}_{j(s)} - \frac{1}{\underline{q}_j}\underline{A}'_{j(ns)}\underline{A}_{j(ns)}, \tag{9}$$

$$\underline{j} = 1, \ldots, \underline{m}.$$

From well-known theory, the absolute maximum of $\underline{\Phi}_j$ in (8), subject to the constraint (2), is given by the largest latent root of $\underline{W}_j$, at the point $\underline{b} = \underline{b}_j$, where $\underline{b}_j$ is the corresponding (normalized) latent vector, $\underline{j} = 1, \ldots, \underline{m}$.

To form the entire transformation matrix $\underline{B}$, then, we form successively the $\underline{m}$ matrices $\underline{W}_1, \ldots, \underline{W}_m$, correspondingly obtain the latent vectors $\underline{b}_1, \ldots, \underline{b}_m$, that yield the largest latent root of each matrix so formed, and set $\underline{B} = [\underline{b}_1, \ldots, \underline{b}_m]$.

It was found in practice that the mean difference criterion as developed above was generally maximized by transforming to very large values for the salient variables and values only moderately close to zero for the potential hyperplane variables. Consequently, a weighting factor was sought to apply to the squared non-salient entries in order to increase their importance in the maximization procedure, under the assumption that a solution with salient values dispersed about .50 and hyperplane values, about .05 is greatly preferable to one with the salients dispersed about .60 and the non-salients, about .15. Thus, analogously to (3), we seek the absolute maximum of

$$\text{(10)} \qquad \underline{\Phi}^*_j = \frac{1}{\underline{p}_j} \sum_{(s)} \underline{v}^2_{ij} - \underline{k} \cdot \frac{1}{\underline{q}_j} \sum_{(ns)} \underline{v}^2_{ij},$$

where $\underline{k}$ is the weighting factor. Equation (9) is replaced by

$$\text{(11)} \qquad \underline{W}^*_j = \frac{1}{\underline{p}_j} \underline{A}'_{j(s)} \underline{A}_{j(s)} - \frac{\underline{k}}{\underline{q}_j} \underline{A}'_{j(ns)} \underline{A}_{j(ns)},$$

$$\underline{j} = 1, \ldots, \underline{m}.$$

The absolute maximum of $\underline{\Phi}^*_j$ in (10), subject to the constraint (2), then, is obtained by choosing for $\underline{b}_j$, the latent vector corresponding to the largest latent root of $\underline{W}^*_j$. Several values of $\underline{k}$ have been tested: 3, 10, 20, 50, 100, and 500; the optimal value appears to be around 50. It is interesting to note that a value of 50 for $\underline{k}$ will generally make the two terms on the right side of (10) approximately equal, thus assigning roughly equal importance to increasing the salient coefficients in absolute value and lowering the non-salient.

Although the preceding technique was developed in the context of oblique quasi-procrustean transformed factor solutions, other applications may be seen for it. The criterion is, first of all, somewhat akin to the discriminant function, which maximizes a different index of division between groups, namely $\underline{MS}_b/\underline{MS}_w$, and it should be clear that if the variables in a factor solution were replaced with persons, and the factors with variables, the maximal mean difference criterion could be used analogously to a discriminant function in categorizing persons. Further, this technique may be used in the "pure" procrustean application, as an alternative to the matrix fitting procedures mentioned in Section 1, merely by identifying the salient variables--both positive and negative--for a factor from prior hypothesis or experiment. In the following development, however, the procedure is developed in the quasi-procrustean context.

The preceding technique has the advantage over matrix fitting to a target vector of 0's, 1's, and -1's of restricting neither the salient nor non-salient variables to as close an approximation as possible to equality among themselves (which happens, for example, by fitting all salient entries to the <u>same</u> value, 1). Justification for this rather severe restriction will seldom, if ever, exist, nor will reliable evidence exist for establishing specific <u>different</u> target values within each group.

<u>Determination of Salient and Non-Salient Variables for Each Factor</u>

Two approaches to this phase of the overall procedure follow.

(a) <u>Defining salient as consistently moderate or large over several different orthogonal rotations</u>

Since orthogonal factor solutions generally permit a more tentative interpretation of the factors than do oblique solutions, obtaining several different orthogonally transformed representations of the data and identifying a common

structure over solutions is one possible approach at this stage. A fairly wide range of solutions can be obtained by varying the $\underline{w}$ parameter in the general "orthomax" criterion [Harman, 1960; Harris & Kaiser, 1964]:

$$\underline{n} \sum_{j=1}^{n} \sum_{p=1}^{m} \underline{b}_{jp}^{4} \quad - \quad \underline{w} \sum_{p=1}^{m} \left(\sum_{j=1}^{n} \underline{b}_{jp}^{2} \right)^{2} = \text{maximum},$$

where $\underline{b}_{jp}$ is the orthogonal factor loading of variable $\underline{j}$ on factor $\underline{p}$, $\underline{n}$ is the number of variables, and $\underline{m}$, of factors. A value for $\underline{w}$ of 0 gives the quartimax criterion, 1 gives varimax, and $\underline{m}/2$, equamax.

Orthogonal solutions obtained using the orthomax criterion with $\underline{w}$ set to, say, 0, 1, $\underline{m}/2$, and $\underline{m}$, can be expected to differ somewhat and, in general, the variance among the factors will be increasingly more evenly dispersed as $\underline{w}$ increases. For each factor, the variables that consistently have large (in absolute value) loadings over the several solutions obtained may be identified as the salient variables for that factor, the remaining non-salient variables being relegated to that factor's hyperplane. As yet, no thoroughly tested rule as to what constitutes "consistently large loadings" has been developed, but one decision rule has been used with fair success and is elaborated in connection with some empirical solutions presented in a later section.

(b) Defining salient as consistently large over several maximal mean difference solutions varying in the choice of other salient variables

Starting from any good orthogonal solution, e.g., a varimax solution, this approach involves rank ordering the orthogonal loadings for a factor and considering, in turn, the largest $\underline{r}$ as salient, the largest $\underline{r}-1$, $\underline{r}-2$, and so on until only two or three salient variables remain. The value of $\underline{r}$ may be as large as $\underline{n}/2$. Those variables that consistently have large reference structure

coefficients (in absolute value) after the application of the maximal mean difference criterion, regardless of what and how many other variables are considered salient in the application of the procedure, and even when they are included among the non-salients in a particular application of the technique, may be considered salient for that factor. Greater elaboration is provided in a later section, where an empirical solution using this approach is presented.

Summary of the Procedure

The steps in the proposed quasi-procrustean procedure can be summarized as follows:

(i) Determine the salient and non-salient variables for each factor by one of the methods described earlier.

(ii) Using the results of (i), for each factor, j, construct the matrix $W_j^* = (1/p_j)A'_{j(s)}A_{j(s)} - (k/q_j)A'_{j(ns)}A_{j(ns)}$, $j = 1, \ldots, m$. The optimal value of k is likely to be around 50.

(iii) For each factor, j, determine the vector of direction cosines, b_j, of order $m \times 1$, where b_j is the normalized latent vector corresponding to the largest latent root of W_j^*, $j = 1, \ldots, m$.
Note that if method (b) of the immediately preceding section is used to determine salient and non-salient variables for a factor, steps (ii) and (iii) must be repeated several times for each factor, j, with the variables considered salient varied each time.

(iv) Construct the transformation matrix, B, by $B = [b_1, \ldots, b_m]$.
A reference structure solution, V, is then obtained by $V = AB$. A primary pattern solution and the other matrices of interest are subsequently obtained by well-known relationships.

The quasi-procrustean procedure developed in this section contains several rather clear-cut steps and consequently may appear somewhat time consuming. Actually, however, given an operationalized decision rule for the determination of salient variables for each factor (possibly an input parameter), a complete solution can be obtained in one computer run, with no need for intervention by the user.

4. Empirical Examples

The procedures developed in Section 3 were programmed and applied to three sets of data, each with a well-known structure ascertained from several analytic solutions and a graphic oblique solution: (1) the (20 x 3) Thurstone Twenty-Variable Box Problem, for which the centroid and graphic oblique solutions were found in Thurstone [1947], (2) the (20 x 5) Sixteen P. F. variables, for which centroid and graphic oblique solutions were obtained from Horn [1963], and (3) the Holzinger-Harman (24 x 4) Twenty-Four Psychological Tests, with centroid and graphic solutions in Harman [1960].

For data sets (1) and (2), procedure (a) of the preceding section was employed to determine the salient variables for each factor. Three orthomax solutions were obtained--with w set to 1 (varimax), $m/2$ (equamax), and m. For each factor, a variable was considered salient if it loaded either .25 or larger (in absolute value) in all three solutions or .30 or larger (in absolute value) in at least one solution and .20 or larger in the others. For data set (3), procedure (b) of the preceding section was employed. For all three data sets, k, in the maximal mean difference criterion, was set to 50--other values for k were not investigated with these data sets.

For each of the three data sets, the obtained quasi-procrustean solution was compared with a solution obtained using the Harris-Kaiser [1964] method--found in an earlier study [Hakstian, in press] to be the best among several currently available analytic oblique techniques which were compared. The obtained solutions were compared and evaluated on three operational criteria of simple structure: (1) hyperplane counts (number of coefficients on the hyperplanes--0±.10), as rough indices of clarity of resolution between salient and non-salient elements, (2) obliquity of the solution--measured in terms of within-method primary factor intercorrelations and corresponding angular separations (θ_{pq} = arccos r_{pq}), using the graphic solution as the standard of correctness, and (3) closeness of each analytic primary factor to the corresponding graphic primary--measured in terms of between-method correlations and corresponding angular separations--and an overall measure of closeness of the obtained with the graphic solution, the mean angular separation of primary axes. The rationale for procedure (3) is given elsewhere [Hakstian, in press].

Thurstone Twenty-Variable Box Problem

In Table 1, the graphic, Harris-Kaiser ($P'P$ proportional to Φ version with varimax rotations), and quasi-procrustean primary-factor pattern solutions for the Box Problem are presented, along with the within-method correlations and angular separations, and the correlations and angular separations between the factors of the graphic solution and those, in turn, of the Harris-Kaiser and quasi-procrustean solutions. It can be seen from Table 1 that both the Harris-Kaiser and quasi-procrustean solutions are extremely close to the graphic, the quasi-procrustean having values identical, to two decimal places, to the graphic in 43 of the 60 primary pattern coefficients. Some discrimination is possible in terms of within-method obliquity, with the quasi-procrustean solution closer to the graphic than was the Harris-Kaiser. If we define, for a pair of factors

TABLE 1

Primary Pattern Solutions, Hyperplane-Counts, and Within- and Between-Method Correlations and Angular Separations for the Graphic, Harris-Kaiser*, and Quasi-Procrustean Solutions of the Thurstone Twenty-Variable Box Problem (Decimal Points Omitted)

Primary Pattern Solutions

	Graphic				Harris-Kaiser*				Quasi-Procrustean			
	I	II	III		I	II	III		I	II	III	
1	100	00	00		100	00	-04		99	01	01	
2	03	99	01		01	98	03		02	99	01	
3	01	02	99		-03	-01	100		01	02	98	
4	49	76	00		48	76	00		49	77	01	
5	40	01	87		37	-03	87		40	00	87	
6	01	41	82		-03	38	84		01	41	82	
7	74	54	-03		74	53	-04		74	54	-01	
8	87	02	40		87	00	37		87	02	41	
9	-01	80	46		-04	78	48		-01	80	46	
10	63	66	-01		63	65	-01		63	66	00	
11	69	01	66		67	-02	64		69	01	67	
12	-01	64	65		-04	62	67		-01	64	65	
13	98	04	-01		99	03	-05		98	04	00	
14	-03	97	07		-04	96	09		-03	97	07	
15	03	-02	96		-01	-06	98		03	-02	96	
16	29	38	72		26	35	73		29	38	73	
17	66	48	28		65	47	27		66	49	29	
18	97	-03	01		98	-04	-02		97	-03	02	
19	07	95	-04		06	95	-02		07	95	-04	
20	00	03	98		-04	00	99		00	03	98	
H-Count	9	9	9	27	9	9	9	27	9	9	9	27

Within-Method Primary Factor Correlations and Angular Separations

	Graphic			Harris-Kaiser*			Quasi-Procrustean		
I	100			100			100		
II	23	100		26	100		23	100	
III	11	22	100	19	25	100	09	22	100
I	0°			0°			0°		
II	76°46'	0°		75°11'	0°		76°58'	0°	
III	83°58'	77°3'	0°	79°19'	75°44'	0°	84°54'	77°6'	0°

	Between-Method Correlations			Between-Method Angular Separations			
	Graphic Primary Factor			Graphic Primary Factor			
Solution	I	II	III	I	II	III	Mean
Harris-Kaiser*	9991	9987	9979	2°26'	2°55'	3°43'	3° 1'
Quasi-Procrustean	9995	9991	9996	1°49'	2°26'	1°37'	1°57'

*The Harris-Kaiser solution is the $\underline{P}'\underline{P}$ proportional to $\underline{\Phi}$ version with varimax rotations.

$\underline{p}$, $\underline{q}$, the "departure from orthogonality" as $|\text{angular separation}_{pq} - 90°|$, we have a mean departure from orthogonality (over the three factor pairs) of 10° 44' for the graphic solution, 13° 15' for the Harris-Kaiser, and 10° 21' for the quasi-procrustean. The mean, over the three factors pairs, on an analogous measure--departure from obliquity of the graphic solution ($|\text{angular separation}_{pq}$ - graphic angular $\text{separation}_{pq}|$)--was for the Harris-Kaiser solution, 2° 31', and for the quasi-procrustean, 0° 24'. Clearly, the obliquity of the quasi-procrustean solution was closer to that of the graphic than was that of the Harris-Kaiser solution. Finally, the Harris-Kaiser factors were separated from the graphic, on the average, by 3° 1', whereas the quasi-procrustean - graphic mean angular separation was 1° 57'.

Sixteen P. F. Data

In Table 2, the graphic, Harris-Kaiser ($\underline{P}'\underline{P}$ proportional to $\underline{\Phi}$ with equamax rotations), and quasi-procrustean primary pattern solutions for the Sixteen P. F. data, with hyperplane-counts, are presented. From this table, it can be seen that on Factors I, II, V, and possibly IV, the quasi-procrustean coefficients are, in general, closer than the Harris-Kaiser, to the graphic. On Factor III, the first decision rule for identifying salient variables described in a preceding section--using the three orthomax solutions--indicated that variables 2, 5, 7, 9, and 20 should be considered salient for this factor, missing variables 12 and 19 that, had they been included, would have permitted a close approximation to the graphic factor and a considerably closer overall solution. The decision rule used worked effectively for the remaining four factors. The total solution hyperplane-counts only slightly favored the quasi-procrustean solution (42) over the Harris-Kaiser (41). Perhaps a more effective quick index of a solution's "cleanness"

TABLE 2

Primary Pattern Solutions and Hyperplane-Counts for the Graphic, Harris-Kaiser*, and Quasi-Procrustean Solutions of the Sixteen P. F. Data
(Decimal Points Omitted)

	Graphic						Harris-Kaiser*						Quasi-Procrustean					
	I	II	III	IV	V		I	II	III	IV	V		I	II	III	IV	V	
1	-26	39	-06	31	-11		-21	36	-12	33	-11		-22	39	-13	31	-15	
2	-05	67	51	08	-13		22	66	41	07	-03		04	68	37	13	-04	
3	09	35	21	-20	13		25	37	13	-21	18		12	36	07	-17	17	
4	-36	50	03	02	03		-23	51	-08	03	06		-33	51	-15	02	00	
5	06	-54	35	07	-11		-01	-53	45	02	-08		05	-56	54	06	-02	
6	-10	-10	03	-26	03		-08	-06	01	-26	05		-12	-10	-02	-28	03	
7	-07	03	56	11	-06		04	04	55	07	02		-03	03	59	15	06	
8	09	07	10	70	10		00	-03	13	66	03		13	02	23	75	10	
9	33	-09	29	34	06		30	-15	34	28	04		36	-12	42	38	13	
10	56	09	08	-09	-03		58	06	12	-10	-06		55	09	12	-07	-01	
11	70	05	04	09	-05		71	-01	10	-12	-11		69	04	10	-09	-05	
12	75	-08	-32	10	06		61	-19	-23	09	-07		72	-12	-19	09	00	
13	-36	-04	21	-23	24		-28	05	12	-25	31		-36	-02	09	-22	29	
14	-57	-01	10	-03	10		-53	07	03	-03	17		-57	01	01	-03	13	
15	-09	01	-07	09	55		-14	03	-15	06	52		-10	01	-13	13	54	
16	-59	-08	02	-11	04		-56	00	-03	-10	10		-59	-06	-06	-13	05	
17	00	-01	03	-10	59		-02	03	-06	-14	56		-02	-01	-07	-06	59	
18	09	-03	08	-51	05		17	05	05	-50	09		06	00	-01	-52	07	
19	-52	03	35	-01	24		-43	11	26	-04	33		-49	04	26	02	31	
20	-28	-09	47	-17	-33		-14	-03	47	-16	-19		-24	-07	47	-17	-21	
H-Count	8	14	9	8	9	48	5	12	6	8	10	41	5	12	7	8	10	42

*The Harris-Kaiser solution is the P'P proportional to Φ version with equamax rotations.

would be the number of coefficients (a small number being desirable) between .20 and .30, the most tentative region, since, for a given factor, it is not clear whether such coefficients represent salient or hyperplane variables, a slight shift in the axis in one direction changing a .25 to .35, whereas in the other direction, to .15. On this criterion, the graphic and quasi-procrustean solutions had totals of eight, the Harris-Kaiser, 12.

In Table 3, the within- and between-method correlations and angular separations for the three solutions to the Sixteen P. F. data are presented. In terms of within-method obliquity, the mean departure from orthogonality (as described earlier), over the ten factor paris, was 10° 9' for the graphic solution, 4° 34' for the Harris-Kaiser, and 12° 30' for the quasi-procrustean. Thus, the quasi-procrustean solution tended to be somewhat more oblique than the graphic, the Harris-Kaiser, somewhat more orthogonal, a tendency of the $\underline{P}'\underline{P}$ proportional to $\underline{\Phi}$ version noted elsewhere [Hakstian, in press]. The mean departure from the obliquity of the graphic solution (as described earlier), over the ten factor pairs, was 5° 59' for the Harris-Kaiser solution and 4° 35' for the quasi-procrustean. Finally, over the five factors, the Harris-Kaiser solution had a mean between-method angular separation with the graphic of 12° 32', the quasi-procrustean, of 11° 13'.

Twenty-Four Psychological Tests

As noted earlier, method (b) of the preceding section was used to identify the salient variables for each factor in turn. Specifically, for each factor, the maximal mean difference criterion was applied starting with the largest 12 varimax loadings being considered salient and subtracting one each time until three loadings were left. For Factor I, such a procedure yielded five reference structure

TABLE 3

Within- and Between-Method Correlations and Angular Separations for the Graphic, Harris-Kaiser*, and Quasi-Procrustean Solutions of the Sixteen P. F. Data (Decimal Points Omitted)

Graphic Primary Factors

	Within-Method Correlations					Within-Method Angular Separations				
I	100					0°				
II	02	100				88°51'	0°			
III	27	-03	100			74°20'	91°43'	0°		
IV	-09	-17	-09	100		95°10'	99°47'	95°10'	0°	
V	-48	17	-09	-33	100	118°42'	80°13'	95°10'	109°16'	0°

Harris-Kaiser* Primary Factors

	Within-Method Correlations					Within-Method Angular Separations				
I	100					0°				
II	-09	100				95°18'	0°			
III	06	02	100			86°38'	88°41'	0°		
IV	-01	-02	00	100		90°17'	90°54'	90° 4'	0°	
V	-33	18	-06	-20	100	109°20'	79°55'	93°19'	101°38'	0°

Quasi-Procrustean Primary Factors

	Within-Method Correlations					Within-Method Angular Separations				
I	100					0°				
II	03	100				88° 1'	0°			
III	21	19	100			78° 7'	78°53'	0°		
IV	-21	-20	-26	100		102° 7'	101°25'	105°19'	0°	
V	-39	21	-11	-33	100	113° 0'	77°41'	96°36'	109°17'	0°

	Between-Method Correlations					Between-Method Angular Separations					
	Graphic Primary Factor					Graphic Primary Factor					
Solution	I	II	III	IV	V	I	II	III	IV	V	Mean
Harris-Kaiser*	9857	9600	9692	9756	9856	9°42'	16°16'	14°15'	12°41'	9°44'	12°32'
Quasi-Procrustean	9928	9623	9825	9754	9849	6°53'	15°47'	10°44'	12°44'	9°58'	11°13'

*The Harris-Kaiser Solution is the P'P proportional to Φ version with equamax rotations.

coefficients larger than .30 in absolute value when the number of salient variables was considered either seven, six, five, four, or three, indicating that the corresponding five variables are likely to be the salient ones for this factor. For Factor II, this procedure yielded five reference structure coefficients larger than .30 when the number of salient variables was considered to be any number between twelve and three, strongly indicating these five variables as marking this factor. For Factors III and IV, this procedure identified five and six salient variables, respectively, with little equivocation.

In Table 4, the graphic, Harris-Kaiser (independent cluster version), and quasi-procrustean primary pattern solutions for this data set are presented. It is seen from this table that all three solutions are very "clean" and similar to one another. Although hyperplane-counts were identical for the Harris-Kaiser and quasi-procrustean solutions (37 as opposed to 42 for the graphic solution), the Harris-Kaiser solution did have fewer entries in the .20 - .30 region (9) than did the quasi-procrustean (12).

In Table 5, within- and between-method correlations and angular separations for the three solutions to the Twenty-Four Psychological Tests are given. In terms of within-method obliquity, both the Harris-Kaiser and quasi-procrustean solutions were very close to the graphic, with the former slightly closer. The quasi-procrustean solution was perhaps slightly more oblique than the Harris-Kaiser. The between-method angular separations were very small for both the Harris-Kaiser and quasi-procrustean solutions. Over the four factors, the quasi-procrustean solution was in slightly closer alignment with the graphic (<u>mean</u> angular separation of 3° 47') than was the Harris-Kaiser (<u>mean</u> angular separation of 4° 3').

Over the three sets of data, then, the quasi-procrustean solutions compared very favorably with the Harris-Kaiser, and tended to be extremely close to graphic

TABLE 4

Primary Pattern Solutions and Hyperplane-Counts for the Graphic, Harris-Kaiser*, and Quasi-Procrustean Solutions of the Twenty-Four Psychological Tests
(Decimal Points Omitted)

	Graphic					Harris-Kaiser*					Quasi-Procrustean				
	I	II	III	IV		I	II	III	IV		I	II	III	IV	
1	70	-07	09	04		76	-12	08	00		73	-06	04	03	
2	46	-02	-01	03		50	-05	-02	01		48	-01	-04	02	
3	60	02	-08	-01		65	-02	-10	-03		63	03	-13	-03	
4	58	07	00	-05		63	03	-02	-08		61	08	-04	-07	
5	-04	83	09	-06		-01	81	12	-08		-07	86	12	-10	
6	-05	83	-08	09		00	80	-05	07		-07	84	-04	05	
7	-05	93	04	-13		-02	91	06	-14		-07	97	07	-18	
8	22	52	16	-08		25	49	18	-10		21	55	16	-11	
9	-08	90	-20	16		-02	87	-18	15		-10	91	-17	12	
10	-32	08	78	08		-36	08	84	04		-40	10	81	12	
11	-18	07	61	25		-19	04	66	21		-25	07	63	29	
12	14	-15	79	-12		11	-17	82	-16		09	-12	78	-09	
13	35	01	65	-20		35	-01	66	-25		33	06	62	-21	
14	-24	18	01	58		-21	14	06	56		-29	14	05	61	
15	-07	03	-10	61		-03	-01	-07	59		-10	-01	-08	65	
16	31	-08	-07	47		32	-14	-06	45		30	-12	-09	50	
17	-24	05	01	77		-20	01	06	75		-30	-01	04	83	
18	15	-19	13	59		19	-25	16	56		11	-24	12	65	
19	08	02	03	42		12	-02	05	40		05	-01	03	45	
20	37	25	-05	17		43	20	-05	15		37	25	-07	16	
21	32	-01	32	13		35	-06	34	10		30	-01	30	15	
22	26	28	-15	34		32	23	-14	32		25	26	-16	34	
23	49	21	08	08		54	16	08	04		50	22	05	06	
24	-01	27	39	18		-01	23	41	18		-08	27	39	24	
H-Count	7	12	14	9	42	6	10	12	9	37	7	10	12	8	37

*The Harris-Kaiser Solution is the independent cluster version.

TABLE 5

Within- and Between-Method Correlations and Angular Separations for the Graphic, Harris-Kaiser*, and Quasi-Procrustean Solutions of the Twenty-Four Psychological Tests (Decimal Points Omitted)

Graphic Primary Factors

Within-Method Correlations					Within-Method Angular Separations			
I	100				0°			
II	59	100			54°12'	0°		
III	46	46	100		62°44'	52°25'	0°	
IV	58	51	60	100	54°41'	59° 4'	53° 8'	0°

Harris-Kaiser* Primary Factors

Within-Method Correlations					Within-Method Angular Separations			
I	100				0°			
II	61	100			52°19'	0°		
III	54	48	100		57° 1'	61°29'	0°	
IV	60	53	61	100	53°29'	57°46'	52°17'	0°

Quasi-Procrustean Primary Factors

Within-Method Correlations					Within-Method Angular Separations			
I	100				0°			
II	61	100			52°10'	0°		
III	55	44	100		56°40'	63°44'	0°	
IV	57	61	60	100	48°19'	52°44'	53°20'	0°

	Between-Method Correlations				Between-Method Angular Separations				
	Graphic Primary Factor				Graphic Primary Factor				
Solution	I	II	III	IV	I	II	III	IV	Mean
Harris-Kaiser*	9964	9988	9976	9968	4°52'	2°48'	3°58'	4°35'	4° 3'
Quasi-Procrustean	9975	9988	9984	9960	4° 4'	2°46'	3°13'	5° 6'	3°47'

*The Harris-Kaiser Solution is the independent cluster version.

oblique solutions for the same data. Although no reliable assessment of the efficacy of the quasi-procrustean approach in general can be made from only three examples, the preliminary results, as reported, appear to indicate that the approach may have promise.

5. Conclusions

It is hoped that the preliminary work reported in this paper on the general quasi-procrustean approach as well as the specific procedures discussed will lead to further refinements and insights into both, in the search for an oblique transformation technique yielding a solution with at once a somewhat generalizable structure and an optimally clear-cut representation of it. On the maximal mean difference criterion, further research is needed on (1) determining the optimal weighting factor, k, (2) extending this procedure to the general procrustes application, and (3) extending the procedure to the orthogonal case. More important, however, research is needed on the determination of the salient and non-salient variables for each factor. Finally, in terms of the total quasi-procrustean approach, the effects of size of the problem, and reliability and inherent factorial complexity of the variables need attention.

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