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ABSTRACT

Research reported deals with identifying stages in attaining a concept of function by students, eleven through fourteen years of age, of above average ability, taking the experimental mathematics program of the Secondary School Mathematics Curriculum Improvement Study. In order to obtain a hierarchy of the learning stages, both a written test and an individual interview were used. All 201 subjects received the written test and 20 subjects were selected for individual testing on a set of sixteen function tasks. From the written test, five stages were identified: (1) ability to find images of a function, (2) ability to consistently identify instances of mappings in several different representations, (3) operational ability in finding images and domain is exhibited, (4) ability to consistently identify non-instances of mappings in several different representations, (5) ability to find images under the composition of two functions and solve related problems using functions in several different representations. The individual interview test results altered the previous findings to yield the final analysis of stages. These were: (1) concept identification (ability to discriminate instances and non-instances of function), (2) process (ability to work with various representations and names of function in finding images, preimages, domain, range, and sets of images), (3) operations (ability to carry out operations on functions with an indication that the result of the operation is understood again to be a function. (JG)

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Abstract

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The problem investigated was the identification of stages in attaining a concept of function. Written tests were given to 201 subjects ages eleven through fourteen and were followed by individual interview-tests with twenty of these subjects. All subjects were above average in ability and had received specific classroom instruction on functions.

From the written test results stages were identified as a learning hierarchy with composition of functions at the highest level. From individual testing, three components in attaining a concept of function were identified and four inclusive stages in terms of these components. Initial understanding of functions as conceptual entities was indicated at the highest stage.

The concept of function is of major importance in mathematics, and, in recent curricular trends, a basic concept in school mathematics. Certain aspects of the concept of function are accessible to subjects at the concrete operational stage but a complete explication and elaboration of the concept is thought to require formal reasoning in at least two ways: (1) in the use of infinite sets as domain so that quantified statements must be manipulated, and (2) in the composition of functions where operations on functions must be understood. Also, as noted by Taback (1969), functions are involved in the study of limits. In addition, proportionality and continuity are part of any serious study of functions, both of which require formal reasoning.

The study here reported was concerned with the learning of a concept of function by seventh and eighth grade subjects in a modern program of instruction which specifically taught and applied the concept. (Secondary School Mathematics Curriculum Improvement Study, 1966). In the text materials of this program, a function is described in terms of the assignments of objects in one collection, or set, of objects, to the objects in another collection, or set, of objects. For example, a function is determined by the assignment of a father to each person in a given collection of students. In this assignment, each student is assigned exactly one father. It is the uniqueness of the object assigned that makes the

assignment determine a function and not simply a relation. Note that if <u>each</u> student is assigned <u>each</u> parent the assignment would not determine a function.

The basic diagrammatic technique utilized for representing a function was the "arrow diagram." An arrow diagram for a function of the type described above is shown in Figure 1, which is taken from the text materials employed. The fact that Mr. Brown is assigned as Mary's father is indicated by an arrow from "Mary" to "Mr. Brown."

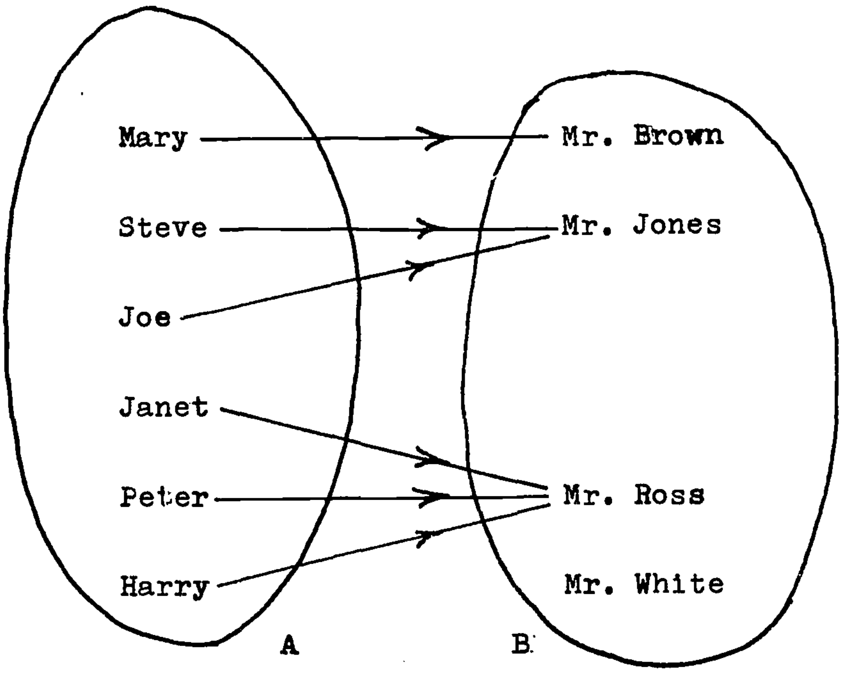

FIG. 1. Initial mapping diagram in the SSMCIS text.

Formally, the concept of function considered can be defined as an ordered triple of sets (F, A, B) such that F is a subset of the set of ordered pairs with first

elements in A and second elements in B and such that each element of A appears as the first element of exactly one ordered pair in F. In this definition the set of ordered pairs, F, may be considered analogous to the "assignment" referred to in the informal description.

The fundamental hypothesis on which the study was based was the hypothesis that the learning of an abstract concept in the classroom context proceeds through a sequence of identifiable stages. Furthermore, for such formal operational concepts as function, the sequence of stages for subjects at the level of abstract cognitive functioning proceeds from the concrete-intuitive level to an abstract cognitive level. Evidence for the final stage is that the concept has become an entity that can be manipulated in thought, an abstraction upon which new abstract concepts may be built.

The major problem to which the study was directed was the identification of stages in the attainment of a concept of function. The function concept is considered as including processes associated with functions and the elaborations of the concept such as composition of functions, in addition to the categorization of the class of relations called functions.

Method

Subjects

The subjects utilized in the study were 201 students from the nine schools participating in teaching the experimental mathematics program of the Secondary School Mathematics Curriculum Improvement Study (SSMCIS). Two classes (and schools) were not included in the study since one class was used for pilot testing and the other was highly dissimilar to the others in composition. There were 177 seventh grade and 24 eighth grade subjects in the study population ranging in age from 11 years, 6 months to 14 years, 5 months. There were 128 girls and 73 boys (one large class was from a school for girls). These subjects were, as a group, well above average in ability.

Procedures

The steps carried out in the investigation were, briefly, as follows.

1. Specification of hypothesized stages in attaining a formal concept of function.
2. Review of the hypothesized stages by a panel of mathematicians and mathematics educators.
3. Construction and pilot testing of a written test on the stages specified in the hypothesized analysis.
4. Construction and trial of individual test tasks.
5. Administration of the written test to the study population and scoring in terms of stage performance.
6. Selection and individual testing of twenty subjects on a set of sixteen function tasks.

In the hypothesized analysis of stages, five stages were proposed:

I. Unique assignments, described in terms of the ability to determine whether or not an assignment of elements of a set B to the elements of a set A determines a mapping (function) of A into B. Assignments restricted to those involving physical objects, whole numbers, and the elementary arithmetic operations, with little or no special symbolism.

II. Vocabulary and symbolism, described in terms of the ability to operate efficiently with the basic vocabulary and symbolism of functions. This included using arrows to indicate images and the computation of images from rules such as "$n \longrightarrow 3n + 5$" and the identification of domain and range of a function.

III. Diagrams, graphs, and ordered pairs, described in terms of the ability to identify assignments in these representational modes as functions or not functions and the ability to find images, preimages, domain, and range for a given function. It was expected that at this stage subjects would recognize a function as essentially determined by its set of ordered pairs.

IV. Operations on functions, described in terms of the ability to carry out such operations on functions as addition, multiplication, and composition. Furthermore, as part of the recognition of a function as an entity, it was expected that at this stage subjects be able to recognize the properties of such operations.

V. Internal properties of functions, described in terms of the ability to identify internal properties of functions such as linearity or preservation of operations (since subjects had received little or no instruction on these concepts this stage was not actually further involved in the study).

Following scoring of the written test each subject was categorized on the basis of a predetermined criterion as having achieved or not achieved a given stage. A score of 1 or 0 was assigned to indicate this. Since there were four stages tested for, and corresponding subtests 1

through 4, a response pattern with four entries was obtained for each subject. The distribution of response patterns is shown in Table 1.

TABLE 1

Distribution of Subtest Response Patterns:

Hypothesized Analysis

Pattern	Subtest				f
	1	2	3	4	
X1	0	0	0	0	8
X2	1	0	0	0	12
X3	1	1	0	0	35
X4	1	1	1	0	36
X5	1	1	1	1	35
X6	1	0	1	0	2
X7	1	1	0	1	25
X8	1	0	1	1	1
X9	1	0	0	1	1
X10	0	1	1	1	13
X11	0	1	1	0	6
X12	0	1	0	1	3
X13	0	1	0	0	26
X14	0	0	1	1	0
X15	0	0	1	0	1
X16	0	0	0	1	1

Clearly, the assumption that the attainment of a concept of function is stage related would imply that patterns X1 through X5 should be the only ones produced. A majority, 61 per cent, do fall into one of these patterns. The fact that a large number of responses do not indicates departure from the theoretically expected result. Particularly troublesome is subtest 1, where many subjects failed to perform successfully. In fact, 45 non-scale responses, roughly 23 per cent of the response patterns, can be attributed to this subtest.

The last observation, and similar problems with subtest 3, led the investigator to attempt a revision of the stages and the corresponding subtests before selection of subjects for individual testing. This revised analysis is briefly described below.

I. At this stage, subjects can find images in a mapping of the whole numbers to the whole numbers given by a simple arithmetic or linear algebraic rule. The word "image" is used correctly and simple interpretations of arrow diagrams can be made.

II. This stage is characterized by the ability to consistently identify instances of mappings in several different representations.

III. Here, operational ability in finding images, preimages, range (or set of images), and domain is exhibited, the mappings being given by some display of the set of ordered pairs (finite domains only).

IV. This stage is characterized by the ability to conconsistently identify non-instances of mappings in several different representations.

V. At the highest stage subjects are able to find images under the composition of two functions and solve related problems using functions in several different representations. Subjects can translate from one representation of a function to another.

Following the revised analysis, the test items were regrouped into the appropriate subtests 1 - 5 and the distribution of response patterns again obtained. This distribution is shown in Table 2. Although 35 per cent of the response patterns still departed from the theoretically expected result, the revised analysis of stages was accepted as a basis for the selection of subjects for individual testing.

To select subjects for individual testing the subjects were first categorized by sex, age (upper and lower), and response pattern. The response patterns were (1) 11000, (2) 11100, (3) 11110, (4) 11111, and (5) mixed, having a non-scale response pattern with respect to subtest 3 or subtest 4. Combination of the categories produced a 2 x 2 x 5 partition of the remaining 164 subjects into twenty subsets from each of which one individual was selected randomly for individual testing.

The individual test consisted of sixteen function tasks, five on concept identification related to Stages II and IV of the revised analysis of stages and eleven tasks related to Stages I, III, and V of the revised analysis. Additional tasks on composition of functions and addition of functions were prepared especially for the individual test.

From study of the transcribed individual test-interviews, response categories were established for each task. The attributes looked for in establishing response

TABLE 2

Distribution of Subtest Response Patterns: Revised Analysis

Pattern	Subtest					f	Error[a]
	1	2	3	4	5		
X1	0	0	0	0	0	4	0
X2	1	0	0	0	0	4	0
X3	1	1	0	0	0	32	0
X4	1	1	1	0	0	35	0
X5	1	1	1	1	0	28	0
X6	1	1	1	1	1	27	0
X7	1	1	0	1	1	6	6
X8	1	1	0	0	1	5	5
X9	1	1	0	1	0	13	13
X10	1	1	1	0	1	18	18
X12	1	0	1	0	1	3	6
X14	1	0	1	0	0	5	5
X23	0	1	0	1	0	4	8
X25	0	1	0	0	0	7	7
Others[b]						11	11

[a]The error column refers to the scalogram analysis.

[b]Those with frequency no more than two.

categories included the ready use of symbolism, a tendency to reason from the general to the specific, and indications that function was thought of as an entity. The description of the response levels for each task served to identify the categories of response. Four response levels were stated for each task with 1 considered the lowest level and 4 the highest.

The distribution of response levels is shown in Table 3, together with the subtests of the individual test corresponding to the stages of the revised analysis, e. g., 1I corresponds to Stage I. Since the concept identification tasks of the individual test were not separated by instances and non-instances, subtest CI, on concept identification, relates jointly to stages II and IV.

Results

The appropriateness of the revised analysis of stages as determined from study of the paper and pencil test results was checked by scalogram analysis. Each stage-subtest was treated as an item in a scale with dichotomous categories of response. The response patterns of Table 2, with the total error for each, are those required for the scalogram analysis.

Guttman's Coefficient of Reproducibility (Rep) was computed from these data by the procedure outlined by Torgerson (1953) and found to be 0.921. Despite some limitations in the data employed, this value of Rep was

TABLE 3

Distribution of Response Levels

Task	Response Level				Subtest
	1	2	3	4	
1	3	5	5	7	1I
2	3	5	4	8	
3	2	2	5	11	
4	1	5	12	2	
5	4	7	6	3	3I
6	3	5	9	3	
7	3	3	4	10	
8	4	2	7	7	
9	2	6	6	6	5I
10	3	6	6	5	
11	1	8	6	5	
12	4	6	3	7	
13	2	4	2	12	
14	2	2	2	14	CI
15	1	1	11	7	
16	2	2	3	13	

considered supportive of the hypothesis of stages in the attainment of a concept of function. This result did not, however, serve to account for certain empirical and logical difficulties with stages II, III, and IV. These difficulties led, after careful study, to a different characterization of stages in a final analysis of stages, based primarily on the responses from the individual testing.

The characterization of stages in the final analysis of stages was based on three components found both in the paper and pencil test results and in the individual test protocols. These were: (1) concept identification, (2) process, and (3) operations.

Concept identification refers to the ability to discrimination instances and non-instances of function. Process refers to the ability to work with various representations and names of functions in finding images, preimages, domain, range, and sets of images. Operations refers to the ability to carry out operations on functions with an indication that the result of the operation is understood again to be a function.

Using these three components as basic elements, a final analysis of stages was prepared taking into account the information obtained from individual testing. From the three components four inclusive stages were obtained. These final stages are briefly described below.

I. Subjects are able to carry out processes associated with the function concept when they are basically arithmetic in nature or when assignments are given quite specifically as in an arrow diagram or a table. Their interpretation of rules such as "$n \longrightarrow 3n + 5$" is as a sequence of arithmetic operations to be performed. Extension to new or less familiar representations of functions, such as an ordered pair point graph, is limited.

II. In addition to the processes described in Stage I, subjects at Stage II are able to find images, preimages, domain, and range in all representations employed with adequate explanations of the processes used. Arithmetic or algebraic rules are stated verbally in quantified form, e. g., "You take any number and you add 23 to it." The differences between Stage I and Stage II are especially marked with respect to the ordered pair point graph representation of a function.

IIIA. Subjects are able to identify relations in the several representational modes employed as functions or not functions and can give an adequate criterion for each such discrimination, whereas subjects at Stages I and II cannot. In addition, their mastery of the process component is at least equivalent to that of Stage I.

IIIB. Subjects show a grasp of the relational aspects of the function concept over all representations employed, in addition to mastery of the concept identification component in these representations. These subjects fall short of the highest level on the operations component.

IIIC. Subjects perform at a high level on the operations component, with intimations that they have begun to treat functions as conceptual entities. Their grasp of the process component is good except for the ordered pair concepts in the geometrical context of the ordered pair point graph. Concept identification has been mastered as in Stage IIIA.

IVA. Performance is at a high level on all components and with a fluency and generality of response that indicates an apparent integration of the subconcepts that remained separated for the subjects at the substages of Stage III.

The last stage is noted as Stage IVA because subjects are only at the beginning level of the stage of

operations on functions, based on the limited tasks used. Since the analysis of stages originally hypothesized was successively modified in light of evidence obtained from paper and pencil testing and again on the basis of information from individual testing, the final analysis of stages must be considered a new, hypothesized sequence of stages in attaining a concept of function.

Discussion

A recent study by Orton (1970) on the function concept was reported at the Conference on Piaget-Related Research in Mathematical Education in October, 1970. The subjects in Orton's study were also students in a modern curricular program that taught and made strong use of a modern concept of function. But Orton was able to work with subjects from ages 12 through 17 who had been in the program from 2 to 5 years. Because of this inverse functions, and, in particular, composition of functions could be studied extensively.

Using similar procedures based on individual testing only, Orton found that, excluding composition of functions, the stages identified closely paralleled stages I, II, IIIA, and IIIB. In addition, stages identified for the composition of functions contained elements of the same components utilized in the final analysis of stages.

An important question that arises in considering results such as these is the meaning of the word "stage."

Indeed, it is doubtful that the word applies in the sense used by Piaget to identify stages in cognitive development with age. But, whatever the word used, the evidence does seem to indicate that in mastering the function concept there are certain reasonably clear and identifiable levels of understanding that appear. These levels, or stages, can be thought of as culmination points in an individual's progress toward attainment of a concept of function at a formal operational level. At these culmination points there is apparent integration of subconcepts, or a move forward through mastery of an important subconcept, or a new elaboration of the concept developed, as in composition of functions.

One of the more striking outcomes observed by this investigator in both individual and paper and pencil testing was the difficulty that subjects had in using the ordered pair point graph of a function. Sufficient success was shown, nevertheless, to indicate that the transition from arrow diagrams to ordered pair points can be accomplished in a meaningful way.

Both the mapping and the ordered pair approaches are an important part of the concept of function. An interesting problem would be the determination of the relative effectiveness of each of the two as intial approaches in teaching the concept of function.

A recent study by Nelson(1969) provides some information relative to this problem. Nelson, using

eighth graders as subjects, studied the possible relations between verbal, visual-spatial, and numerical intellective factors and the effectiveness of instructional sequences keyed to those intellective factors. He expected, for example, that a verbally oriented instructional sequence would be most effective for subjects with a high pretest score on the verbal factor and a low pretest score on the other two factors.

His findings were, in part, that the visual-spatial instructional sequence, based on a mapping approach using arrow diagrams, was the most effective for all subjects. But in this study the ordered pair approach was strictly non-geometric, in accordance with the basic design. Thus, the effectiveness of ordered pairs in the context of cartesian graphs was not studied.

The complexity of the concept of function and the experience of Orton suggests that one direction for further research on the learning of the concept of function is the relationship of the mastery of subconcepts, such as one-to-one functions, and elaborations of the concept, such as composition of functions, to the mastery of function at a formal operational level. The relationships between the learning of subordinate concepts such as ordered pairs and sets to mastery of the function concept also needs to be investigated. On a higher level, what is the relationship of logic, variables, and quantification to to the learning of a concept of function?

Finally, the developmental precursors of the formal concept of function need to be investigated so that formal instruction can adjusted to the cognitive structure in which the concept of function may be most firmly anchored. Beilin (1970) has noted that " . . . little effort has been expended in testing the relation between the conceptual systems of mathematics and the cognitional system of the child except in the most limited of circumstances [55]." The conceptual system based on the function concept provides a rich and important opportunity for the study and testing of such relations.

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