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ABSTRACT

Investigated was the feasibility of pres`nting proof materials to college-capable sixth-grade students. A unit on mathematical proof was developed using an iterative procedure. Formative evaluation procedures were used to improve various components of the unit. Included in the unit are terminal student behaviors, cartoon stories, and several classroom activities which use a desk computer. Results indicated that the iterative developmental procedures employed were highly successful. (Author/FL)

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A FORMATIVE DEVELOPMENT OF A UNIT ON PROOF FOR USE IN THE ELEMENTARY SCHOOL
PART I

Report from the Project on Analysis of Mathematics Instruction

U.S. Office of Education Center No. C-03 Contract OE 5-10-154

Technical Report No. 111

A FORMATIVE DEVELOPMENT OF A UNIT ON PROOF FOR USE IN THE ELEMENTARY SCHOOL PART I

Report from the Project on Analysis of Mathematics Instruction

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Center No. C-03 / Contract OE 5-10-154

STATEMENT OF FOCUS

The Wisconsin Research and Development Center for Cognitive Learning focuses on contributing to a better understanding of cognitive learning by children and youth and to the improvement of related educational practices. The strategy for research and development is comprehensive. It includes basic research to generate new knowledge about the conditions and processes of learning and about the processes of instruction, and the subsequent development of research-based instructional materials, many of which are designed for use by teachers and others for use by students. These materials are tested and refined in school settings. Throughout these operations behavioral scientists, curriculum experts, academic scholars, and school people interact, insuring that the results of Center activities are based soundly on knowledge of subject matter and cognitive learning and that they are applied to the improvement of educational practice.

This Technical Report is from Phase 2 of the Project on Prototypic Instructional Systems in Elementary Mathematics in Program 2. General objectives of the Program are to establish rationale and strategy for developing instructional systems, to identify sequences of concepts and cognitive skills, to develop assessment procedures for those concepts and skills, to identify or develop instructional materials associated with the concepts and cognitive skills, and to generate new knowledge about instructional procedures. Contributing to the Program objectives, the Mathematics Project, Phase 1, is developing and testing a televised course in arithmetic for Grades 1-6 which provides not only a complete program of instruction for the pupils but also inservice training for teachers. Phase 2 has a long-term goal of providing an individually guided instructional program in elementary mathematics. Preliminary activities include identifying instructional objectives, student activities, teacher activities materials, and assessment procedures for integration into a total mathematics curriculum. The third phase focuses on the development of a computer system for managing individually guided instruction in mathematics and on a later extension of the system's applicability.

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ABSTRACT

To test the feasibility of presenting proof materials to college-capable Sixth-Grade students, a unit on mathematical proof was developed using an iterative procedure. Formative evaluation procedures were used to improve various components of the unit. A sequence of six theorems was selected, terminating with "Given any set of prime numbers, there is always another prime number."

Action words were used to write the objectives of the unit in terms of terminal student behaviors, and these terminal behaviors were task analyzed into subordinate behaviors. A new three-dimensional model was developed to task analyze the higher-order cognitive tasks involved in proving mathematical theorems. An instructional analysis performed after the mathematical analysis was based on Bruner's hypothesis that "any subject can be taught effectively in some intellectually honest form to any child at any stage of development." The basic approach was to recast the mathematical content into a form which was consistent with the cognitive structures of Sixth-Grade children. Hence, pedagogical and psychological considerations outweighted mathematical ones.

A formative pilot study was conducted with six Sixth-Grade students. The mathematical and instructional components were then reexamined and a second formative pilot study was conducted with seven students. Proofs of several theorems were simplified and the instructional procedures were modified. Cartoon stories were used to introduce new concepts into the unit, and several classroom activities were centered around a desk computer.

From ten Sixth-Grade students, a Control Group was selected by matching procedures and a Nonequivalent Control Group design was used to test the unit. A certified elementary school teacher with a strong background in mathematics taught the unit to the Experimental Group. Bloom's concept of mastery learning was used to bring each student up to pre-established criterion levels of performance.

On the posttest, the Experimental Group showed mastery of all prerequisite skills and all proofs and showed significant differences from the Control Group ($p < 0.0001$).

Thus, the iterative developmental procedures employed in developing the unit were highly successful; Sixth-Grade students are able to understand and prove the mathematical proofs as presented in the unit; and Bloom's concept of mastery learning is a successful operational procedure for the mathematics classroom. Because the teacher of the Experimental Class encountered difficulties in teaching the unit, it is probable that the task of teaching proof is well beyond the capabilities of the typical elementary school teacher.

Chapter I

INTRODUCTION AND STATEMENT OF THE PROBLEM

BACKGROUND OF THE PROBLEM

Proof and deductive reasoning are at the very heart of modern mathematics. The purpose of this study was to develop a unit of instruction on mathematical proof and to use this unit to investigate the feasibility of presenting proof materials to sixth-grade students.

The modern conception of mathematics is that it is a creation of man's intellect: undefined terms, definitions, fundamental postulates, and a set of logical principles give a mathematical system its distinctive structure. Hence, one might create a new system from an old system by altering a few postulates. Bell (1940 p. 305) describes this conception of mathematics as follows:

> In precisely the same way that a novelist invents characters, dialogues, and situations of which he is both author and master, the mathematician devises at will the postulates upon which he bases his mathematical systems. Both the novelist and the mathematician may be conditioned by their environments in the choice and treatment of their material; but neither is compelled by any extra-human, eternal necessity to create certain characters or to invent certain systems.

The modern point of view in mathematics has its origin in the works of Gauss, Bolyai, Lobachevski, and Riemann. During the last century, these mathematicians challenged over two thousand years of tradition by creating consistent geometric systems which contradicted a fundamental postulate of Euclidean geometry. Further refinements of these systems by Pasch, Peano, and Hilbert established the purely hypothetico-deductive nature of geometry.

Euclid had recognized postulates as being "self-evident truths" about the real world. The new method views postulates merely as "assumptions." The mathematician is not necessarily concerned with the truth of falsity of the postulates; he is concerned primarily with their consistency. Postulates are creations of the mathematician's mind and are useful in deriving other mathematical statements.

A clear distinction is also made between that which is defined and that which must remain undefined. Euclid had attempted to define all of his terms. Modern mathematicians realize that some terms must remain undefined to avoid circularity of definition. From the modern point of view, Euclid's geometry is inadequate.

In spite of the fact that the modern conception of mathematics was developed in the last century, the school mathematics curriculum was slow in reflecting this viewpoint. As early as 1901, Russell (1901, p. 100) commented upon the fact that Euclid's geometry was still being taught in England: "... it is nothing less than a scandal that he should still be taught to boys in England." In 1954, MacLane (1954, p. 66) wrote:

> ...the lively modern development of mathematics has had no impact on the content or on the presentation of secondary-school mathematics. Algebra and geometry, as covered in schools, consist exclusively of ideas already well known two hundred years ago---many of them two thousand years ago. No matter how much better these particular ideas are taught to more and more pupils, their presentation leaves mathematics in a state far more antiquarian than that of any other part of the curriculum. The pupils can conclude only that there is no such thing as a new mathematical idea.

At the very time that MacLane was writing these remarks, efforts were already underway to correct some of the deficiencies of which he wrote. The first major shift away from the traditional program had been initiated by the University of Illinois Committee of School Mathematics (UICSM) under the direction of Max Beberman. In 1952, this group of mathematicians and teachers set out to develop materials of instruction for secondary school students and to train secondary school teachers in their use, so as to produce enthusiastic students who understand mathematics. Two basic principles guided the program: discovery and precision of language. It was Beberman's belief (1964) that a student will come to understand mathematics when both his textbook and his teacher use unambiguous language and when he is permitted to discover generalizations by himself.

Eleven units were written with the structure of mathematics uppermost in mind. These units comprise the UICSM program entitled High School Mathematics. Deductive proof is introduced in the first unit in a very informal way with the derivation of specific numerical statements. For instance, the field properties are used to illustrate easy ways to perform computations (1960, p. 57):

$$\begin{aligned} 987 \times 593 + 593 \times 13 &= 593 \times (987 + 13) \\ &= 593 \times 1000 \\ &= 593{,}000 \end{aligned}$$

Unit 2 discusses generalizations, counter-examples, and proof. The following is a typical proof (1960, p. 35):

For each K, 3K + (9K - 2) + 12K - 2.

$$\begin{aligned} 3K + (9K - 2) &= 3K + (9K + -2) && [\text{ps}] \\ &= 3K + 9K + -2 && [\text{apa}] \\ &= (3 + 9)\ K + -2) && [\text{dpma}] \\ &= 12K + -2 && [3 + 9 = 12] \\ &= 12K - 2 && [\text{ps}] \end{aligned}$$

(ps means "the principle of subtraction," apa means "the associative principle for addition," dpma means "the distributive property for multiplication over addition"). As an example of the exercises, the students are asked to prove (1960, p. 61):

$$\forall x \, \forall a \forall b \, \forall c, \; ax + bx + cx = (a + b + c)\, x.$$

Unit 3 contains proofs involving inequalities (1960, p. 101):

$$\forall x \, \forall y \, \forall z, \; (x + z > y + z \text{ if and only if } x > y).$$

Unit 4 contains proofs on the irrationality of various square roots as well as proofs on the sum and product of odd and even integers. Unit 5 contains set algebra proofs, Unit 6 devotes a section to logic, and Unit 7 is on mathematical induction.

The UICSM material introduces the concept of mathematical proof at the ninth-grade level and continues to develop it throughout the high school years. The logical development of the real number system is the vehicle by which the concept of proof is developed. In summary, the UICSM materials present a very thorough treatment of deduction and proof.

The UICSM program has demonstrated the feasibility of presenting a rigorous deductive approach to mathematics to high school students. The units have served as a Platonic blueprint from which has emanated an entire revolution in school mathematics. Many programs followed the lead of the UICSM. In commenting upon the first few programs, Butler and Wren (1960, p.81) remarked:

> Whatever the final results may be, it is safe to say that some of the immediate effects will be a greatly increased emphasis on structure, a réconsideration and refinement of definitions, a clearer attention to the deductive process....

In 1959, the Commission On Mathematics of the College Entrance Board (CEEB) published a booklet entitled <u>Program for College Preparatory Mathematics</u>. Because it can and does include in its testing program any topics it considers important, this group has been very influential. Whereas the UICSM demonstrated the feasibility of presenting proof to high school students, the CEEB was the first highly authoritative group to stress the desirability of increasing the amount of deductive rigor in the school curriculum. As concerns reasoning and proof, this group has recommended that deductive reasoning may be employed in courses other than geometry (1959a, p. 22):

> One way to foster an emphasis upon understanding and meaning in the teaching of algebra is through the introduction of instruction in deductive reasoning. The Commission is firmly of the opinion that deductive reasoning should be taught in all courses in school mathematics and not in geometry courses alone.

The CEEB report makes specific recommendations for the high school curriculum. For ninth-grade algebra, the Commission suggested the following uses of deduction:

1. Simple theorems on odd and even integers and the property of integers.
2. Theorem: $\sqrt{2}$ is irrational.

The theorems in (1) are rather simple and involve the direct applications of the field axioms. By comparison, the proof that $\sqrt{2}$ is irrational is more difficult. A rigorous proof would require a knowledge of the Law of the Excluded Middle, the Law of Contradiction, as well as the establishment of some prerequisite theorem, e.g., the prime factors of a square occur in pairs. The Commission, however, probably had a more intuitive proof in mind, for it uses the phrase "informal deduction" to describe the type of proofs to be included at this level.

Although the Commission felt that algebra should be taught so as to give sound training in deductive methods, it was of the opinion that geometry would continue to be the subject in which the student would receive the most satisfactory introduction to thinking in terms of postulates, proofs, undefined terms and definitions. Appendix 11, "A Note on Deductive Reasoning," includes the following features of deductive reasoning which were to be emphasized (1959b, p. 112):

1. The nature of definition.
2. The impossibility of defining all technical terms.
3. The nature of proof.
4. The impossibility of proving all statements.

A suggested theme for the Grade 10 course was "Geometry and Deductive Reasoning". It includes the development of the postulational nature of geometry as well as the suggestion that a miniature deductive system proving certain properties about abstract objects be taught. Deduction and proof are not objects of study in Grades 11 and 12, but they are used extensively.

As a whole, the Commission envisioned a secondary school curriculum which placed a greater emphasis upon proof than did the traditional program, yet which was somewhat less rigorous than the UICSM program.

In discussing the reasons for the increased emphasis upon deduction, the Commission disagrees with a reason which has often been advanced for teaching deduction (1959a, p. 23):

> Not all reasoning is syllogistic or deductive. Training in mathematics based on deductive logic does not necessarily lead to an increased ability to argue logically in situations where insufficient data exist, and where strong emotions are present. It is a disservice to the student and to mathematics for geometry to be presented as though its study would enable a student to solve a substantial number of his life problems by syllogistic and deductive reasoning.
>
> Deductive methods are taught primarily to enable the pupil to learn mathematics. Mathematics, and consequently deductive methods, can be applied to life only in those life situations that are capable of accurate transformation into mathematical

> models. These situations, though of tremendous importance, are far from frequent in the everyday life of high school students.

Deduction is thus defended on mathematical rather than on practical grounds. Reasoning and proof are necessary and integral components of a healthy mathematics program, and this is sufficient reason for their inclusion in the curriculum.

The growing importance which many mathematicians attach to axiomatics in the secondary schools is vividly illustrated by the pamphlet entitled The Role of Axiomatics and Problem Solving in Mathematics. Allen, Blank, Buck, Dodes, Gleason, Henkin, Kline, Shanks, Suppes, Vaughn and Young address themselves to the task of discussing the role of axiomatics in the curriculum. Although Blank and Kline are not so inclined, the others in this group favor the use of axiomatics. For example, Allen (1966, p.3) comments:

> We believe that the axiomatic method of exposition will help pupils acquire a deeper understanding of elementary mathematics and a better appreciation of the nature of mathematics.

Suppes (1966, p.73) states:

> From years of grading mathematical proofs given on examinations at the university level, I am firmly convinced ...that the ability to write a coherent mathematical proof does not develop naturally even at the most elementary levels and must be a subject of explicit training.

And Shanks (1966, p. 65) adds:

> My major claim for good axiomatics in the classroom is that students actually thrive on it.

These developments have had a profound effect upon high school text books, as is described by Allen (1966, p.2):

> Most of the algebra texts published in the United States since 1960 delineate the structure of algebra

> by presenting the properties of an ordered field as a basis for justifying statements which, in former years, would have merely been announced as rules for computation. Some of these books have sections on logic which give the student an idea of the nature of proof. A few texts present logic as an integral part of the course and use it throughout in the proof of theorems. In some cases, the proofs are short, informal essays designed to convince the student that a certain proposition is true. Sometimes proofs are structured in ledger form, as they are in traditional geometry texts, with the reasons numbered to correspond to the steps.

Until the late 1950's, most of the concern with the mathematics curriculum had been focused upon the secondary school program. At that time the process of revising the curriculum reached the elementary schools. With recommendations for a marked increase in deductive procedures at the secondary school level, a greater emphasis upon rigor and structure was called for at the elementary school level. Several experimental programs were started in response to this challenge. Among these were the Syracuse University-Webster College Madison Project (The Madison Project), The School Mathematics Study Group (SMSG), and the Greater Cleveland Mathematics Program (GCMP). The role that proof plays varies with each program, but as a rule, very little attention is given to deductive procedures.

The Madison Project leans toward inductive procedures and stresses trial and error discovery, but very few mathematical generalizations are actually drawn. Mathematical statements are proven intuitively with the use of repeated illustrations, but deduction is not stressed at the elementary school level. The concepts of implication and contradiction are used, but the use of axioms and theorems are not encountered until algebra.

In describing the curricular objectives of the Madison Project, Davis (1965, p. 2) states:

> The Project seeks to broaden this curriculum by introducing, in addition to arithmetic, some of the fundamental concepts of algebra (such as variable, function, the arithmetic of signed numbers, open sentences, axiom, theorem, and deviations), some fundamental concepts of coordinate geometry (such as graph of a function), some ideas of logic (such as implication), and some work on the relations of mathematics to physical science.

The SMSG material does very little with proof at the elementary school level. The second volume of the junior high school textbook includes some simple proofs, such as $\sqrt{2}$ is irrational, and the high school program increases the amount of proof material at each successive grade level. But at the elementary school level, the only important proof material is the use of the commutative, associative, and distributive axioms of operations on integers to show the correctness of the algorithms of addition, multiplication, etc. For example (1962, p. 96), the following is presented in Book 5:

60 x 70	= (6 x 10) x (7 x 10)	(Rename 60 and 70).
	= (6 x 7) x (10 x 10)	(Use the associative and commutative propeites).
	= 42 x 100	(The product of 6 and 7 is 42; the product of 10 and 10 is 100).
	= 4200	(The product of 42 and 100 is 4200).

The Greater Cleveland Program encourages students to reason out the processes they learn by building the development of basic properties, but no effort is made to introduce or use formal proof. The objectives for the upper elementary school is stated as follows (1964, Preface):

> There are two parallel objectives of mathematics education at this level, both of which should be achieved. The first objective is a clear understanding of the structural interrelationships of numbers. This achievement enables pupils

to develop proficiency in applying numbers to problems. The second objective is skillful and rapid computation.

These experimental programs are illustrative of the general trend at the elementary school level: precision of language is emphasized and an attempt is made to explain why certain rules and procedures are followed. But proof is, for the most part, omitted.

Experimental programs such as the ones discussed above served as models for commercial textbooks. As a result, most of the commercially published textbooks go no further with logic and proof than the ones discussed above. There is one series, however, which devotes a considerable amount of attention to logic, and that is <u>Sets and Numbers</u>, authored by Suppes. In each of the Books from Grade 3 to Grade 6, one chapter is devoted to logic. An intuitive introduction to logical reasoning begins in Book 3, but there is no explicit use of logic terminology. "If - then" (1966b, p. 301) and "or" (1966b, p. 305) statements are used as follows:

if $n > 4$ then $n = 5$

$n > 4$

What is n?

$n = 5$ or $n = 7$

$n \neq 5$

What is n?

Book 4 introduces some simple logical terminology, and rules of inference are introduced in the context of the use of ordinary English. In some cases the use of ordinary English is supplemented by the use of mathematical symbols introduced in other parts of the book, as in the examples above, but special logical symbols, such as $\wedge$ for conjunction and V for disjunction, are not used in this series. Book 4 also discusses conjunctions and disjunctions, atomic and molecular sentences, conditional sentences, and connectives. The student is given tollendo tollens exercises, and is asked to identify various types of sentences.

In Book 5, the terms "antecedent", "consequent", "denial of an atomic sentence", and "premises and conclusions" are presented. Specific names are given to several reasoning patterns: the "if-then" rule (ponendo ponens), the "if-then-not" rule (tollendo tollens), and the "or" rule (tollendo ponens).

Longer reasoning patterns involving three premises and one or two conclusions are introduced in Book 6, and the student is asked to supply reasons for each conclusion. For example (1966b, p. 288):

(1)	if $f = g$, then $f = h$	P
(2)	if $f \neq g$, then $f - g = 3$	P
(3)	$f \neq h$	P
(4)	$f \neq g$	?
(5)	$f - g = 3$	?

This series has several interesting features. First, content spiralling is much in evidence. Each of the books reviews the concepts taught in the preceding book, extends the concepts, and then adds new ones. But although successive chapters on logic fit nicely together, they are isolated from the remainder of the text. Logic is developed, but it is not used elsewhere in the book. It is treated as a separate topic, and the relationship of logic to other aspects of mathematics is ignored. Not a single mathematical result, for example, is proved deductively with the logical machinery developed in the chapters on logic. Even the teacher's manual fails to adequately clarify the role which logic plays in mathematics. The Teacher's Edition of Book 4 states that the main purpose of the study of logic is to introduce the student to a way of thinking that encourages precision (1966b, p. 344):

> For a long time the logical structure of any argument was obscured by the words used in the argument... . In the last century, symbolic logic was developed to solve this problem. In symbolic logic the sentences are replaced by mathematical symbols, so that the whole structure of the argument stands out clearly.

The series gives the impression that logic is a separate discipline. That this is not Suppes' intention is clear from the following quote (1966a, p. 72):

> As a part of training in the axiomatic method in school mathematics, I would not advocate an excessive emphasis on logic as a self-contained discipline...What I do feel is important is that students be taught in an explicit fashion classical rules of logical inference, learn how to use these rules in deriving theorems from given axioms, and come to feel as much at home with simple principles of inference like rodus ponendo ponens as they do with elementary algorithms of arithmetic. I hasten to add that these classical and ubiquitous rules of inference need not be taught in symbolic form, nor do students need to be trained to write formal proofs in the sense of mathematical logic.

Because no attempt is made to prove mathematical results, the series has little direct relevance to the present study. Yet the logical content of this series is based upon experimental evidence gathered at Standford University, and it is encouraging to know that sixth graders can be expected to use fairly sophisticated reasoning patterns.

In the preceding paragraphs, an attempt has been made to outline the major developments and trends which have taken place in the past few years. In spite of the fact that the emerging programs were significantly more rigorous than the traditional programs, the men who attended the Cambridge Conference on School Mathematics (CCSM) envisioned an even more rigorous and accelerated mathematics curriculum. In their 1963 reports, Goals for School Mathematics (The Cambridge Report), the conferees urged that logic and inference be taught for both practical and mathematical reasons. From the practical point of view, logic is held to be essential to a good liberal education. They affirm that a knowledge of logical principles will help the student correct much of the slip-shod reasoning he normally employs in everyday life (1963, p. 9):

> Many poor patterns of thought common in ordinary life may be modified by the study of mathematics...Just a little experience with logic and inference can do away with some of the unfortunate reasoning we meet all too often.

This is almost the identical argument which has been traditionally advanced in support of including geometry in the high school curriculum, and it is an argument refuted by the CEEB report mentioned above.

From a mathematical point of view, the following reasons are cited:

(1) to economize on the number of axioms required.
(2) to illustrate the power of concepts in proving more elaborate statements.
(3) to unify the K - 6 mathematics materials.
(4) to prepare the students for the large dosage of formal proof which they will encounter in geometry and algebra in grades 7 and 8.

The Report recommends that proof be taught in the elementary school (1963, p. 39):

> As the child grows, he learns more and more fully what constitutes a mathematical proof...experience in making honest proofs can and probably should begin in the elementary grades, especially in algebraic situations.

The following topics were recommended for inclusion in the elementary school curriculum:

(1) The vocabulary of elementary logic: true, false, implication, double implication, contradiction.
(2) Truth tables for simplest connectives.
(3) Modus ponendo ponens and modus tollendo tollens.
(4) Simple uses of mathematical induction.

5. Preliminary recognition of the roles of axioms and theorems in relation to the real number system.
6. Simple uses of logical implication or "derivations" in studying algorithms, or complicated identities, etc.
7. Elements of flow charting.
8. Simple uses of indirect proof, in studying inequalities, proving $\sqrt{2}$ irrational, and so on.

In all, the Report suggests proving 20 theorems in Grades 4-6 including indirect proofs and proof by mathematical induction. It is of interest to note that the Cambridge Report recommendations represent a three-year acceleration of the CEEB Report mentioned earlier. Both suggest proving that $\sqrt{2}$ is irrational, but whereas the CEEB has the ninth grade in mind, the Cambridge group has the sixth grade in mind.

Because of this rapid acceleration, these recommendations might appear, at first reading, to be extremely over-ambitious. There are, however, several considerations which make this proposal a more credible one. The first is the fact that these are goals for the future. The curriculum reform which had preceded the Conference had been affected by many factors, the most serious of which were the scarcity of qualified elementary school teachers and the rigidity of the existing school systems. The men attending the Cambridge Conference consciously divorced themselves from such considerations and concentrated upon certain goals for which they believed our schools should be striving.

Another consideration is the fact that the educational ideas of Jerome Bruner permeate the report. One of Bruner's central themes is the belief that "any subject can be taught effectively in some intellectually honest form to any child at any stage of development" (1960, p. 33). This shall be referred to as "Bruner's Hypothesis". If this is a valid educational hypothesis, then proving simple theorems might well be a legitimate goal for sixth-grade students. According to

Bruner (1966), the maturing human organism proceeds through three stages of cognitive development; the enactive, the ikonic and the symbolic. Knowledge can also be represented in these three basic forms. The concept of any academic discipline can be successfully taught to children at any stage of mental development if we can simplify and reformulate these concepts into terms which match the cognitive structures of the children. It should, therefore, be possible to present the proofs of simple theorems to sixth graders in some manner. Varying degrees of rigor exist and the CCSM concludes that "at the elementary school level the amount of logical or inductive reasoning that will be appreciated is uncertain" (1963, p. 15). The CCSM recommends using a spiral method which proposes that the same concept or theorem be dealt with upon several occasions in the curriculum, separated by varying intervals of time. Thus, a highly intuitive proof can be made more rigorous at a later date.

In light of these considerations, the CCSM recommendations may represent a workable alternative for the future. The Cambridge Conference argues that proof should be included in the elementary school curriculum. Are there reasons to believe that it can be done? There is some indication in the theory of Jean Piaget that the answer to this question is "yes".

Logic and psychology are independent disciplines: Logic is concerned with the formalization and refinement of internally consistent systems by means of pure symbolism; psychology, on the other hand, deals with the mental structures that are actually found in human beings. These mental structures develop independently of formal training in logic. Yet human beings do employ logical principles, and there is some relationship between mental functioning and logic. One of Piaget's major concerns has been to determine this relationship. Two basic steps are involved in this attempt: (1) to construct a psychological theory of operations in terms of their genesis and structure; and (2) to examine logical operations and apply these to psychological operations. To construct his theory Piaget uses the mathematical concept of a lattice (1953). What concerns us here, however, is that in the process of developing this theory Piaget analyzes the growth of logical thinking in children. His description of the adolescent mind can be summarized as follows:

(1) The ability to cope with both the real and the possible.

(2) The ability to consider the possible as a set of hypotheses.

(3) The ability to form propositions about propositions and to make various kinds of logical connections between them, such as implication, conjunction, disjunction, etc.

(4) The ability to isolate all the variables of a problem and to subject them to a combinatorial analysis.

The investigator believes that possession of these cognitive abilities constitute an adequate readiness for a formal study of mathematical proof. Hence, if Piaget is correct in his analysis, sixth-grade students possess the logical operations necessary to begin the study of mathematical proof, for the sixth-grade student is just entering the formal operational stage of cognitive development. This is not to say that his description of the formal operational mind is a readiness barometer; it clearly is not. The dangers inherent in such a suggestion have been pointed out by Sullivan (1967). It does suggest to the curriculum developer, nonetheless, that it might be profitable to experiment with proofs in the elementary school. Since sixth-graders are in the formal operational stage of cognitive development, experimentation might profitably be started at the sixth-grade level.

RATIONALE FOR THE STUDY

The rationale for the study is contained in the following facts:

(1) the authoritative recommendations of the Cambridge Report that proofs of mathematical theorems should be presented in the elementary school;

(2) the theory of Piaget which suggests that it can be done;

(3) to the investigator's knowledge a careful study of proof in the elementary school has not been conducted.

Together they suggest the question, Is it possible to develop a unit of proof and test the feasibility of presenting proofs to sixth-grade students? The question is comprised of two parts. First, how does one develop a unit on proof for use with sixth-grade students? And second, what is meant by "feasibility"?

HOW THE UNIT WAS DEVELOPED

Plans were made to develop the unit on proof in accordance with the curriculum development model constructed by Romberg and DeVault (1967).

The model is comprised of four sequential phases: Analysis, Pilot Examination, Validation, and Development. This study carried the development of the unit on proof through the first two phases of the model.

The first step in the Analysis phase is the mathematical analysis. The goals and content of the unit must be identified. These goals are then stated behaviorally in terms of well-defined action words. These behavioral objectives are then subjected to a task analysis. This procedure was developed by Gagné (1965b) for training persons to perform complex skills. The basic idea of task analysis is to break a complex task down into component subtasks and establish a relationship among them. This relationship should, when possible, indicate the order in which each subtask is to be learned and the dependencies among the subtasks. Learning the subtasks are essential to learning the total task. If properly constructed, the task analysis should reveal all the prerequisite skills which are necessary for learning the terminal behavior.

Once a task analysis has been completed, an instructional analysis is undertaken. Here one tries to determine an effective way to teach each of the objectives contained in the unit. As Gagné (1965a) has pointed out, the pedagogical tactics should depend upon the type of learning implied by the content. Learner variables, teacher variables, stimulus variables and reinforcement variables must all be considered in an attempt to create a well-planned and effective instructional unit.

The second phase of development is the pilot examination. Each component of the unit is tried out with an appropriate group of students. Formative evaluation procedures are used to assess the effectiveness of each component of the unit.

The results of the pilot are then analyzed. If they are favorable, one proceeds to the validation phase of the model; if unfavorable, one must recycle and begin again. The arrows in Figure 1 indicate the iterative nature of this model. One continues to cycle and recycle through the analysis and pilot phases of the model until a workable unit is developed.

In the present study, three pilot studies were conducted before a satisfactory unit was developed. In conjunction with the third study, an experiment was conducted. An intact group of students served as the Experimental Group, and a Control Group was selected by matching procedures.

FEASIBILITY

To determine the feasibility of presenting proofs to sixth-grade students, three basic questions were asked:

(1) Can the students display an understanding of the meaning of the theorems?

(2) Can the students reproduce the proofs of the theorems?

(3) Can the students display an understanding of the proofs?

Chapter III discusses means which were used to gather data to answer these questions. For the purpose of this study, the degree of feasibility will be determined by the extent to which the students are able to perform these tasks.

THE PURPOSE OF THE STUDY

The study had two main purposes: (1) to demonstrate that the curriculum development model advocated by Romberg and Devault can be successfully used to develop a unit on proof for use with sixth-grade students; and (2) to use this unit to test the feasibility of presenting selected proof materials to sixth-grade students.

If the unit on proof meets the criteria of feasibility, the effectiveness of the curriculum development will be demonstrated.

SIGNIFICANCE OF THE STUDY

Deduction and proof are central to a mature study of mathematics, yet teachers of mathematics have long bemoaned the fact that high school and college mathematics students are unable to write coherent mathematical proofs. If proof materials can be developed which are appropriate for use with sixth-grade students, then the possibility of producing students with better proof writing abilities exists. For, if sixth graders can learn to prove simple mathematical theorems, a spiral

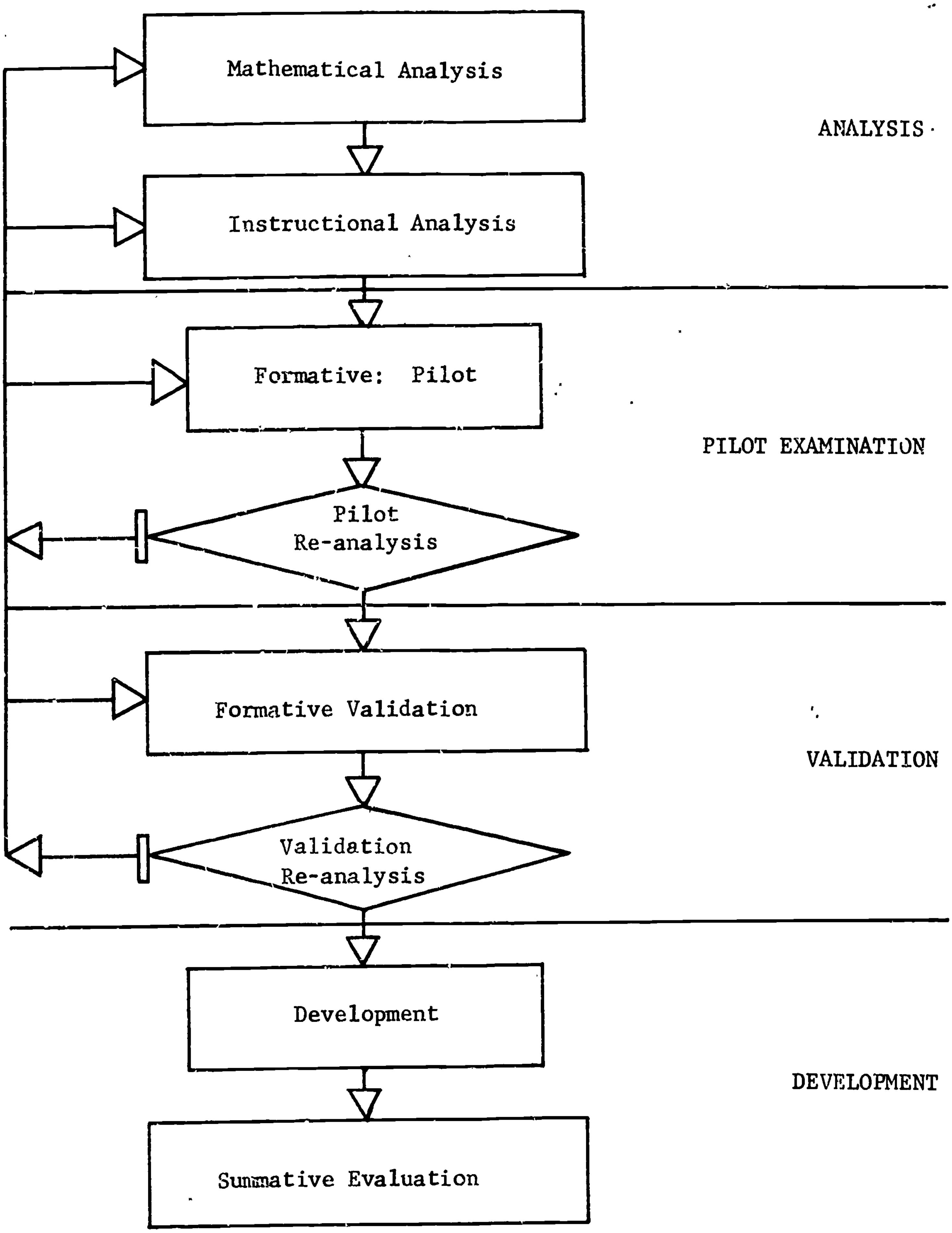


Figure 1. Steps in Developing an Instructional System

approach to the mathematics curriculum could be used to reinforce and expand these skills and understandings. A viable program on proof could conceivably be developed for the upper elementary and junior high schools which would produce high school students capable of writing valid proofs.

The present study was undertaken in an attempt to gather some preliminary information of these possibilities.

Chapter III gives a detailed account of how the unit was developed; nine sequential steps illustrate the formative nature of the development. The results of the experiment are reported in Chapter IV, including all of the data which was used to judge the feasibility of the unit. And finally, Chapter V contains the conclusions of the study and makes recommendations for further study.

Chapter II

RELATED RESEARCH

INTRODUCTION

The research directly related to the teaching of mathematical proofs to elementary school children is scarce. So scarce, in fact, that a review of the literature produced only one study in which an effort was made to teach elementary school children how to prove mathematical theorems, and this occurs in a course on logic (see Suppes' experiment below). However, there are two types of research studies which may be related to the problem of this study. First, research studies are presented which have made an effort to teach logic to elementary school children. Second, studies which make an attempt to determine the status of the logical abilities of children are discussed.

It should be stated at the outset that the author is not certain as to the extent to which the cognitive skills reported in such studies are related to the ability to learn mathematical proofs. Nonetheless, since those who have conducted such studies apparently believe that such a relationship exists, the results are reported in this chapter.

TRAINING IN LOGIC

There are several studies which report attempts at teaching logic to elementary school children. Hill (1967, p. 25) reviews a number of early studies and concludes that they "tend to support the conclusion that specific training contributes to improvement in logical skills." Two recent studies have made a more systematic attack upon the problem than did earlier ones.

The first was conducted by Suppes and Hill from 1960 to 1963. During the 1960-61 school year, Dr. Hill instructed a class of talented fifth graders in mathematical logic, a course developed by herself and Professor Suppes. The materials for the course were extended and revised into a textbook entitled Mathematical Logic for the Schools. After developing such topics as symbolizing sentences, logical inference, sentential derivation, truth tables, predicates, and universal quantifiers, the book applies this logic to a specific mathematical system and proves a variety of simple (and obvious) facts about addition, such as the statement that $4 = (1 + (1 + 1)) + 1$ (1962, p. 250). A chapter on universal generalizations includes the proofs of such statement as $(\forall x)(\forall y)(-x = y \leftrightarrow x = -y)$ (1962, p. 273).

During the following two years, this book was used with experimental classes. In 1961-62, twelve classes of fifth-grade students studied logic, and in 1962-63, eleven classes of sixth graders continued with a second year of logic, and eleven new fifth-grade classes were added to the study. The teachers were given special in-service training in logic, but no details of the mathematical content or of the pedagogical procedures used with the children were reported. For purposes of comparison, two Stanford University logic classes were selected as control groups.

The main conclusion of this three-year study was that the upper quartile of elementary school students could achieve a significant conceptual and technical mastery of elementary mathematical logic.

However, no objective data is reported and no comments are made concerning the ability of these sixth graders to prove mathematical theorems.

This conclusion (and others as well) was challenged by Smith (1966) who pointed out that the evidence provided in the report was incomplete and overgeneralized. A purposive sample was used instead of a random sample, hence generalizing to "the upper quartile of elementary school students" is inappropriate. In addition, Smith points out that even if his sample was random, Suppes should be generalizing to only the highest 7 percent of the elementary school population, for although the upper quartile of the students in the experiment did meet with considerable success in studying logic, these children had an I. Q. range from 142 to 184. Therefore, the study indicates that extremely bright children can learn mathematical logic.

A more recent study was conducted by Ennis and Paulus (1965) in which an attempt was made to teach class logic and conditional logic. They defined class logic as follows (1965, p. II-5):

> The basic units in class logic are parts of sentences, subjects and predicates. The sentences do not reappear essentially unchanged; instead the subjects and predicates are separated from each other and rearranged. Here is an example to which the criteria of class logic are to be applied. It is from "The Cornell Class-Reasoning Test":
>
> Suppose you know that
>
> All the people who live on Main Street were born in Milltown. None of the students in Room 352 live on Main Street.
>
> Then would this be true?
>
> None of the students in Room 352 were born in Milltown.
>
> For purposes of simplification and ease of teaching, the subject and predicate are revised in order to form classes. Following this procedure the classes involved in the above

> example are (1) the people who live on Main Street, (2) the people who were born in Milltown, and (3) the students in Room 352. The two given statements present relationships between the first and second classes and the third and second classes respectively. The statement about which one must decide suggests a relationship between the third and first classes. Thus the subjects and predicates as represented by the classes are the basic units in this kind of reasoning.

Conditional logic is described as follows (1965, p. II-4):

> Sentence logic is concerned with arguments in which the basic units are sentences. That is, distinct sentences, often connected or modified by such logical connectives as 'if', 'then', 'and', 'or', 'not', and 'both', appear essentially unchanged throughout the course of the argument.
>
> ...Sometimes sentence logic itself is broken up into parts, depending on the logical connective which is used. When the connective is 'if', 'only if', or 'if and only if', or any synonyms of these, we have what is sometimes called, and what we shall call, a 'conditional statement'. Arguments which contain only conditional statements and simple sentences or negations thereof shall be called conditional arguments. Reasoning associated with such arguments shall be called conditional reasoning.

Twelve basic principles of conditional logic and eight basic principles of class logic were identified. Class logic was taught to one class each at grades 4, 6, 8, 10 and 12, and conditional logic was taught to one class each at grades 5, 7, 9 and 11. In grades 4 - 9, existing classroom units were used, and at the high school level, volunteers were selected from study halls. Efforts at randomization were not reported and Control groups were selected. A Nonequivalent Control Group design was used (see Campbell and Stanley, 1968, p. 47):

0	X	0	(Treatment)
0		0	(Control)

The treatment groups were pretested, then taught 15 days of logic, and six weeks later were posttested. The control groups were also pretested and posttested.

They summarize the results as follows (1965, p. VIII-20):

> Class reasoning is apparently teachable to some extent from age 11-12 onwards. Students younger than that did not benefit from the 15 days of instruction that we were able to give them. Perhaps under different conditions - or with more time - they also would have benefitted. From 11 - 12 onward, there is apparently modest, fairly even improvement as a result of whatever natural-cultural sources are operating, and deliberate teaching of the sort we did can contribute modestly to this improvement. By age 17 - 18, there was a result of existing natural-cultural influences on our LDT's considerable mastery of the basic principles of class logic. Our teaching made a modest improvement. Overall, talking in terms of readiness, we can say that from age 11 - 12 onward, our students were ready for modest improvements in mastery of the principles of class reasoning, and that by age 17 - 18, the group as a whole was ready to make the modest improvement that, when made, justifies our saying that for practical purposes they have mastered the basic principles of class logic.
>
> Conditional logic makes a different story. Apparently, when given the sort of instruction we provided, our LDT's were not ready to make much improvement until upper secondary; but by age 16 - 17 were ready to make great strides. These improvements in mastery were particularly evident among the fallacy principles (where there is much room for improvement); but they also occurred among the contraposition principles, the affirming-the-antecedent principle and to a slight extent the transitivity principles. No improvement was registered among the 'only-if' principles, though this might be because insufficient time was devoted to them and at the outset the 16 - 17 year olds were fairly good at them.

The failure to produce better results was attributed in part to inadequate time allowances. The study could also be criticized for its lack of planning for instruction. It is impossible to determine from the report just what "sort of instruction" was provided. The content was not analyzed, prerequisite behaviors were not identified, instructional procedures were not outlined, instructional materials were not prepared, each teacher constructed his own set of exercises, etc. The content was identified and each teacher was told to do the best he could with it (1965, VI-11):

> Each staff teacher was instructed to teach the logical principles roughly in order, using whatever style of teaching seemed to him to be most appropriate and going as far down the list as he could in the available time.

As pointed out by Romberg (1968, p. 1) an outline of the content is merely one of many variables which comprise an instructional unit. The units may have failed because inadequate attention was given to the other variables.

STATUS STUDIES

There are a number of status studies which attempt to measure the existence or non-existence of specific logical principles in children of various ages.

Hill (1967) conducted a status study prior to her collaboration with Suppes on the experiment described above. The study was designed to determine how successfully 6, 7 and 8 year old children could recognize valid inference schemes. A 100 item test was individually administered to 270 better-than-average students. The items did not require the student to actually draw conclusions from given premises, but he was required to determine if certain statements followed from others. Three categories in elementary logic were tested: sentential logic, classical syllogism, and logic of quantification. The students were instructed to respond either "yes" or "no" to the following kinds of items:

Sentential logic:

If this is Room 9, then it is fourth grade.
This is Room 9.
Is this fourth grade?

Classical Syllogism:

All of Ted's pets have four legs.
No birds have four legs.
Does Ted have a bird for a pet?

Logic of Quantification:

None of the pictures was painted by anyone I know.
I know Hank's sister.
Did she paint one of the pictures?

The following table summarizes the results of the study and indicates which patterns of inference were tested (1967, p. 57):

Table 11 - Percentage of Correct Responses for Different Principles of inference by age level

Principles of Inference	Percentage of Correst Responses		
	Age 6	Age 7	Age 8
Modus ponendo ponens	78	89	92
Modus tollendo ponens	82	84	90
Modus tollendo tollens	74	79	84
Law of Hypothetical Syllogism	78	86	88
Hypothetical Syllogism and tollendo tollens	76	79	85
Tollendo tollens and tollendo ponens	65	77	81
Ponendo ponens and tollendo tollens	65	67	76
Classical Syllogism	66	75	86
Quantificational Logic-Universal quantifiers	69	81	84
Quantificational Logic-Existential quantifiers	64	79	88

The main conclusion is that children of ages 6 - 8 are able to recognize valid conclusions determined by formal logical principles. Such children make use of certain principles of inference, not in the sense that they can verbalize them, or know principles as principles, but in the sense that they behave in drawing conclusions in accordance with the criteria of validity of formal logic.

Hill concludes that her results are counter to two of Piaget's major contentions: (1) that children at this age level are not capable of hypothetico-deductive reasoning; (2) that logical abilities develop in stages.

O'Brien and Shapiro (1968, p. 533) point out that the behavioral manifestation chosen by Hill was but one of several possible behaviors:

> The problem was that in each of the test items, a necessary conclusion followed from the premises and the task of the student in each case was to simply discern whether the third statement given was the conclusion or the negation of the conclusion. In no case was the student called upon to test the logical necessity of the conclusion, a behavior which the present investigators felt was vital to an adequate consideration of hypothetical-deductive reasoning. Therefore, the present study was undertaken to investigate the second behavior and to determine the differences, if any, between a logically necessary conclusion and its negation and their ability to test the logical necessity of a conclusion.

O'Brien and Shapiro conducted a study in which two tests were used: (1) Test A, the test used in the Hill study; (2) Test B, the same test with 33 of the original 100 items "opened up" so that no necessary conclusion followed from the premises. All the items on the second test had a third option of "Not enough clues", as well as "yes" and "no" options. The following items are the "opened up" versions of the items given as examples in the above discussion of the Hill study:

Sentential logic (1968, p. 533):

If this is Room 9, then it is fourth grade.
This is not Room 9.
Is it fourth grade?
a. Yes b. No c. Not enough clues

Classical Syllogism:

Some of Ted's pets have four legs.
No birds have four legs.
Does Ted have a bird for a pet?
a. Yes b. No c. Not enough clues

Logic of Quantification:

Some of the pictures were painted by people I know.
I know Hank's sister.
Did she paint one of the pictures?
a. Yes b. No c. Not enough clues

The results of Test A essentially confirm the findings of the Hill study. The percentage of correct responses are not significantly different from what would be expected from random guessing. The results of Test B, however, are quite different. The percentage of correct responses on the 33 open items is significantly lower than what would be expected from random guessing. Although it is not mentioned in their report, these results are exactly what Piagetian theory predicts: Children at this level of cognitive development are unable to cope with the world of the possible (Inhelder, 1958).

The experimenters thus concluded that Hill's conclusions must be interpreted with caution (1968, p. 537):

> That these two behavioral manifestations of hypothetical-deductive thinking occur at such different levels among children of the same age seems to bring into question the challenge that the original research gave to Piaget's theory regarding the growth of this kind of logical thinking in children.

Prior to the O'Brien and Shapiro study, Ennis and Paulus made a comprehensive review of the literature, including Hill's study. They concluded that research indicates the following (1965, p. V-42):

1. That there is a development of logical ability as children grow older.

2. That no stages in this development have definitely been identified.

3. That children can do at least some conditional reasoning before 11 - 12 (in conflict with Piaget's claim), but that there is no work on the extent of the mastery of conditional logic in adolescence.

4. That some class logic is mastered by age 11 - 12 and that Piaget apparently thinks that it is mastered by age 11 - 12.

5. That the ability to consider questions of validity regardless of belief in truth of the parts of an argument is not attained by age 11 - 12.

6. That there is practically no study on the different developmental patterns of different principles and components of logic.

On the basis of this information, Ennis and Paulus conducted a status study in an attempt to gather more information. This occurred prior to their attempts to teach logic (discussed above). The same students were used for both studies. Hill's test was reviewed and judged unsuitable for their purposes (1965, p. IV - 2):

> Although arguments in which the conclusion contradicts the premises are a sub-class of invalid arguments, a more important sub-class is the group of arguments in which the conclusion does not follow, but also in which the conclusion does not contradict the premises. People are rarely trapped into thinking that an argument is valid in which the conclusion actually contradicts the premises. The important distinction is between a valid argument and between one which someone might be inclined to call valid, but which is really not valid. The mastery of this distinction is not tested for in this test. Thus, it too was unsatisfactory for our purposes.

A new test was, therefore, constructed. It included items in which the proposed conclusion did not follow necessarily from the premises. For example:

52. Suppose you know that

None of Tom's books are on the shelf.
No science books are on the shelf.
Then would this be true?
At least some of Tom's books are science books.

a. Yes b. No c. Maybe

The main findings were as follows (1965, p. V-42):

a. In this age range there is a development of ability to do logic as students grow older.

b. If there are stages in this range, they are not noticeable at the level of refinement of our measuring techniques.

c. The basic principles of conditional reasoning are not generally mastered by age 11 - 12, nor by age 17.

d. The basic principles of class reasoning are not generally mastered by age 11 - 12, nor are they fully mastered by age 17.

e. The truth validity characteristic (the ability to consider questions of validity regardless of belief in truth of the parts of an argument) is not attained by age 11 - 12, nor by the age of 17.

f. The patterns of development and mastery of principles of logic vary, but there is considerable similarity between the two types of logic studied. The principles expressing the basic logical fallacies are the most difficult at ages 10 - 12, but are also the ones in which there is generally the most improvement over the range studied. The most extreme example is the principle that a statement does not imply its converse. The principle of contraposition is one which in this range starts at medium difficulty and does not become much easier for older students. The transitivity principle starts in this range at medium difficulty, but is considerable easier fo. older students.

DISCUSSION

Both Hill and Ennis have questioned the findings of Piaget. In particular their findings indicate that children attain some logical skills earlier and some later than at the ages cited by Piaget. However, Piaget is not alone in his analysis of children. There is a considerable amount of literature which tends to support his findings. Peel (1960) states that the elementary school child is limited to concrete situations in his immediate experience, and the adolescent begins to carry out logical operations on abstract and symbolic material. Gesell (1956) describes the early adolescent years in much the same way. Broudy, Smith and Burnett (1964) point out that the young adolescent is capable of thinking in purely abstract terms, systematically using the kinds of formal operations characteristic of scientists and mathematicians. Wallach (1963) sees the adolescent as being able to formulate and verify propositions and hypotheses. Yudin and Kates (1963) support these views and list 12 as the age at which formal operations first appear.

There are, therefore, those who agree with Piaget, and those who disagree with him. The research studies reviewed here make no effort to explain these differences. There is a possibility that these differences do not really exist. The researchers who agree with Piaget are engaged in the task of describing a generalized thinking activity, and the other group seem to be concerned with the child's ability to handle specific logical principles. The former begin with the child in a problem situation and observe his behavior, whereas the latter begin with a set of logical principles and are interested in the child's response to them.

Piaget begins with a problem situation, usually one involving physical apparatus. A problem is posed to the child, and the observer records the child's responses. On the basis of the responses, Piaget hypothesizes that the child has certain cognitive structures. One such experiment involves a pendulum (Inhelder, 1958, p.67):

> The technique consists simply in presenting a pendulum in the form of an object suspended from a string; the subject is given the means to vary the length of the string, the weight of the suspended object, the amplitude, etc. The problem is to find the factor that determines the frequency of the oscillations.

In this case, the child must determine that only one factor plays the casual role. The other variables must be isolated and excluded. Piaget identifies different behaviors with different stages of cognitive development. For example: (Inhelder, 1958, p. 69):

> The preoperational stage I is interesting because the subject's physical actions still entirely dominate their mental operations and because the subjects more or less fail to distinguish between these actions and the motion observed in the apparatus itself. In fact, nearly all of the explanations in one way or another imply that the impetus imported by the subject is the real cause of the variations in the frequency of the oscillations.

The research of Ennis, Hill, Miller, and others is of quite a different nature: the subject is not given physical materials with a related problem to solve. Instead, he is asked to judge the validity of a given derivation or is asked to derive conclusions from a given set of premises. Each item involves a particular logical principle or fallacy. The researcher tabulates the responses to each item, and then attempts to conclude that children of different ages know or do not know various logical principles.

There are two important differences involved here:

(1) The stimuli are different. In one case, the student is given physical materials which he can manipulate and experiment with,

and in the othe- he is given a list of verbal or written statements which cannot be physically manipulated.

(2) The tasks are different. In the first case, he is asked to discover features which govern physical objects, and in the other he must determine the validity of reasoning patterns.

It seems reasonable to maintain that different skills are being measured. Perhaps there is no such ability as <u>the</u> logical ability of an individual in the sense that he has certain logical principles at his disposal which he calls into play with equal facility and effectiveness in all situations. Perhaps he has logical abilities which vary from situation to situation. Hill cites a number of sources which hold this view and then adds (1967, p.23):

> The critics might legitimately argue that the influence of content forces us to talk only about specific logical abilities, i.e., specific to particular content. Rather, Piaget assumes that children do dissociate formal relationships from concrete data and can recognize them in many guises. Certainly, this study, in inference, makes a similar assumption . . . .

Hill only considers logical abilities specific to content, yet there are other dimensions to the problem, e.g., levels of abstractness. If one student is given a problem involving a real pendulum, the content is the same in both cases. Yet the two students may be engaged in quite different tasks.

Degree of personal involvement is another factor which can drastically affect one's reasoning ability. For example, consider a young man in two different situations. In the first he is given a hypothetical story about a man with marital difficulties and is asked to supply reasonable and logical behaviors which will help the fellow out of his difficulties.

Then consider the same young man who actually has these same problems with his own wife. Again the content is the same in both cases, yet the real world provides us with enough examples to suggest that the young man may not employ the same logic in both situations.

Such considerations indicate that different situations will evoke different kinds of logical responses. In short, no two tests measure exactly the same thing.

Those who subscribe to the belief that an indivudual possesses a logical ability, quite frequently advocate the teaching of logic or critical thinking skills so that the student will become a better thinker in all realms of human activity. The belief that a geometry course teaches people "how to think" is a prime example, and Miller (1955) argues that the survival of our democratic institutions may very well depend on our determination and ability to teach logic in our schools.

The issue at point is whether training in logic actually prepares one to be a more logical reasoner in other or all areas of human concern. The author has known of too many mathematics students who have succumbed to the logic of door-to-door salesmen to believe that this is the case. Rather than assuming that the mind has one logical ability which extends equally well to all domains of cognition, for the purposes of this paper it is assumed that the mind is so structured as to have different logical abilities for different areas of cognition. Hence, if it is our goal to teach students how to prove mathematical theorems, then they should be given explicit practice in proving theorems. It will not suffice to teach them logic or general reasoning skills and hope for transfer. In Niven's words (1961, p. 7):

> If the nature of proof cannot be described or formulated in detail, how can anyone learn it? It is learned, to use an oversimplified analogy, in the same manner as a child learns to identify colors, namely, by observing someone else identify green things, blue things, etc., and then by imitating what he has observed.

Polya expresses a similar view when he states (1953, p. V):

> Solving problems is a practical art, like swimming, or skiing, or playing the piano: you learn it only by imitation and practice.

SUMMARY

It is difficult to determine how relevant the studies reported in this chapter are to the problem of teaching proof to sixth-grade children, for it is not known exactly what relationship exists between one's ability to learn logic and one's ability to write and understand mathematical proofs. Similarly, the status studies mentioned above measure one's ability to identify correct and incorrect reasoning patterns, and it is not known whether or not this is essential for writing proofs.

At any rate, a review of the research literature as reported in this chapter indicates the following:

(1) Attempts to teach logic to elementary school children have met with varying degrees of success although extremely bright children can profit from such instruction.

(2) There are conflicting findings concerning the logical abilities of children; these differences may be the result of different testing procedures.

(3) The feasibility of teaching proof to college capable sixth-grade students has not been demonstrated.

And so in spite of the fact that Hill, Suppes, Ennis and others have been concerned with improving the reasoning abilities of children, the major question posed by the present study remains unanswered: Is it possible to develop materials on mathematical proof so that the proofs are accessible to average and above-average sixth-grade students?

Chapter III

DEVELOPMENT OF THE UNIT

This chapter gives a detailed, step by step description of how the unit was developed in accordance with the curriculum development model of Romberg and DeVault. Figure 2 shows the steps which were taken in developing the unit.

MATHEMATICAL ANALYSIS (Step 1)

The overall objective of the unit was to investigate the feasibility of presenting proof materials to sixth-grade students. The analysis consists of selecting the theorems to be included in the unit, constructing a content outline for the unit, defining a set of action words, stating specific behavioral objectives in terms of these action words, and then task analyzing these behaviors into prerequisite skills.

Selection of the theorems. Selecting the theorems was the first major decision which had to be made. What type of subject matter should serve as a vehicle for introducing proof into the curriculum? Geometry? Modern Algebra? Number Theory? To aid in selecting the theorems, the following set of criteria was established:

(1) The theorems should be directly related to the fundamental concepts and subject matter with which sixth-grade students are familiar. The investigator believes that reasoning for the sake of

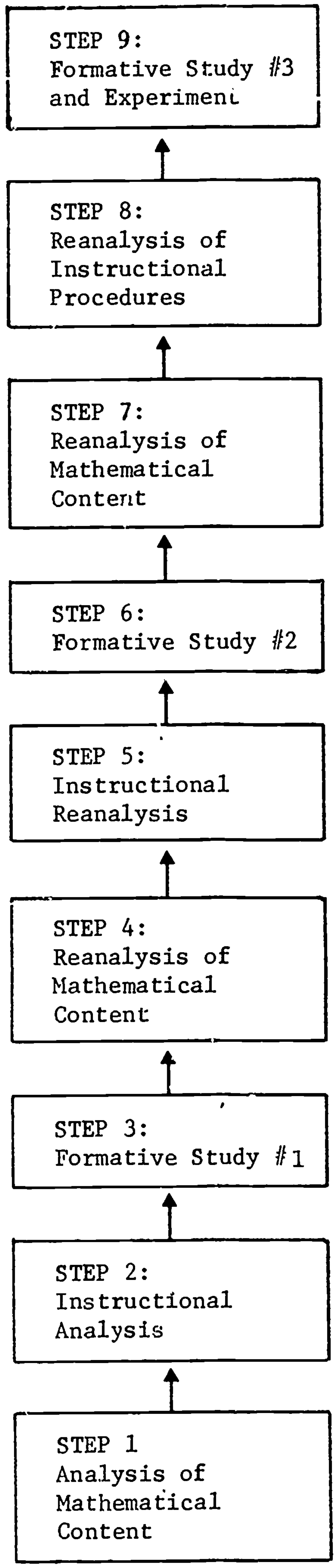


Figure 2. Development and Testing of the Unit on Proof

reasoning has very little meaning to sixth-grade students. A study of logic would be excluded by this criterion;

(2) The mathematical concepts should be learned in an appropriate context, rather than in isolation. The theorems should relate to concepts with which the students are familiar;

(3) The prerequisite knowledge should be minimal. The efforts of the teacher should be concerned with the proof itself, and not with an assortment of knowledge which is necessary for the proof;

(4) The theorems should provide an active role for the students (rather than having the students passively listening to lectures); and

(5) The theorems should provide some opportunities for discovery by the children. It is hoped that the students will be able to formula..e the theorems themselves.

These criteria were applied to a wide variety of theorems. The following sequence c´ theorems appeared as recommendations in the Cambridge Report and appeared to meet the criteria better than any other set of theorems: For any whole numbers A, B, and N,

If $N|A$ and $N|B$, then $N|(A + B)$;

If $N\nmid A$ and $N|B$, then $N\nmid(A + B)$;

There is no largest prime number.

Since sixth graders are usually not familiar with negative numbers, this set of theorems can be interpreted as applying to the set of whole numbers. Hence, the theorems are statements about mathematical entities with which typical sixth-grade students are familiar. The students should have no difficulty in understanding what the theorems mean.

The prerequisites needed to prove these results were fewer in number and less difficult to learn than the prerequisites for the other theorems which were examined. Furthermore, the theorems can be illustrated with numerical examples, thus providing opportunities for student involvement. Therefore, plans were made to build a unit of instruction around these three theorems.

Outline of the unit. An outline of a tentative unit was then written around these three theorems. Several considerations guided the writing. First, the investigator wished to examine how well students understand the strategies of the proofs. In particular, can the students prove new theorems which require the application of the same strategies? In an attempt to answer this question, three theorems were added to the unit: for any whole numbers A, B, and N,

If $N|A$ and $N|B$, then $N|(A - B)$;

If $N|A$ and $N|B$ and $N|C$, then $N|(A + B + C)$; and

If $N \nmid A$ and $N|B$, then $N \nmid (A - B)$.

The proof of the first theorem requires the student to use the fact that multiplication distributes over the difference of two numbers and involves the same strategy as the first theorem in the unit. The second theorem involves the same strategy, but requires the student to distribute multiplication over the sum of three numbers. The proof of the third theorem above involves the same strategy as the second theorem in the unit and requires the student to solve an equation by addition rather than by subtraction.

Throughout the remainder of this paper the theorems will be referred to by the following numbers, it always being understood that

A, B, and N refer to whole numbers:

Theorem 1: If $N|A$ and $N|B$, then $N|(A + B)$;

Theorem 2: If $N|A$ and $N|B$, then $N|(A - B)$:

Theorem 3: If $N|A$, $N|B$, and $N|C$, then $N|(A + B + C)$;

Theorem 4: If $N\nmid A$ and $N|B$, then $N\nmid(A + B)$;

Theorem 5: If $N\nmid A$ and $N|B$, then $N\nmid(A - B)$; and

Theorem 6: Given any set of prime numbers $\{P_1, P_2, \ldots, P_n\}$, there is always another prime number.

It should be pointed out that since the students have not studied negative numbers, the proof for the sum of two numbers does not establish the result for the difference of two numbers. Hence, the theorems are different from the students' point of view.

A second consideration which guided the creation of the unit was the ambiguity of the word "proof" in everyday usage of the word. It was decided to include a general discussion of the meanings of proof, from "that which convinces" to a logical argument. With different connotations of the word "proof" in usage, a major problem lies in motivating the need for a rigorous, logical proof of the theorems in the unit. It was decided that two activities would be tried out in the first formative study, and both of them employ the same strategy: lead the student into inductively hypothesizing a generalization, and then torpedoing that generalization. The upshot of such activities is to stress that repeated instances of a general statement does not constitute proof.

The first activity is partitioning a circle into regions by connecting a given number of points on its circumference. Two points will produce two regions, three points four regions, four points eight regions, and five points will produce sixteen regions (See Figure 3). The idea is to get the students to hypothesize that the next partitioning will yield thirty-two regions, and then have them partition the circle with six points. This time there are thirty regions (See Figure 4).

The second example uses the expression $N^2 - N + 17$, which will yield prime numbers when evaluated for all integers from 0 to 16, but which will yield a composite for $N = 17$. The students were to be enticed into making the induction that $N^2 - N + 17$ will always yield prime numbers, and then the hypothesis was to be torpedoed.

An outline containing nine lessons was drafted.

1. Meaning of proof
   (a) authority
   (b) empirical evidence
   (c) reasoning
2. Prerequisite concepts
   (a) prime numbers
   (b) ACD Laws
   (c) divisibility
   (d) solving equations
   (e) substitution
3. Motivation for proof and counter example
4. Theorem 1: If $N|A$ and $N|B$, then $N|(A + B)$.
5. Theorem 2: If $N|A$ and $N|B$, then $N|(A - B)$.

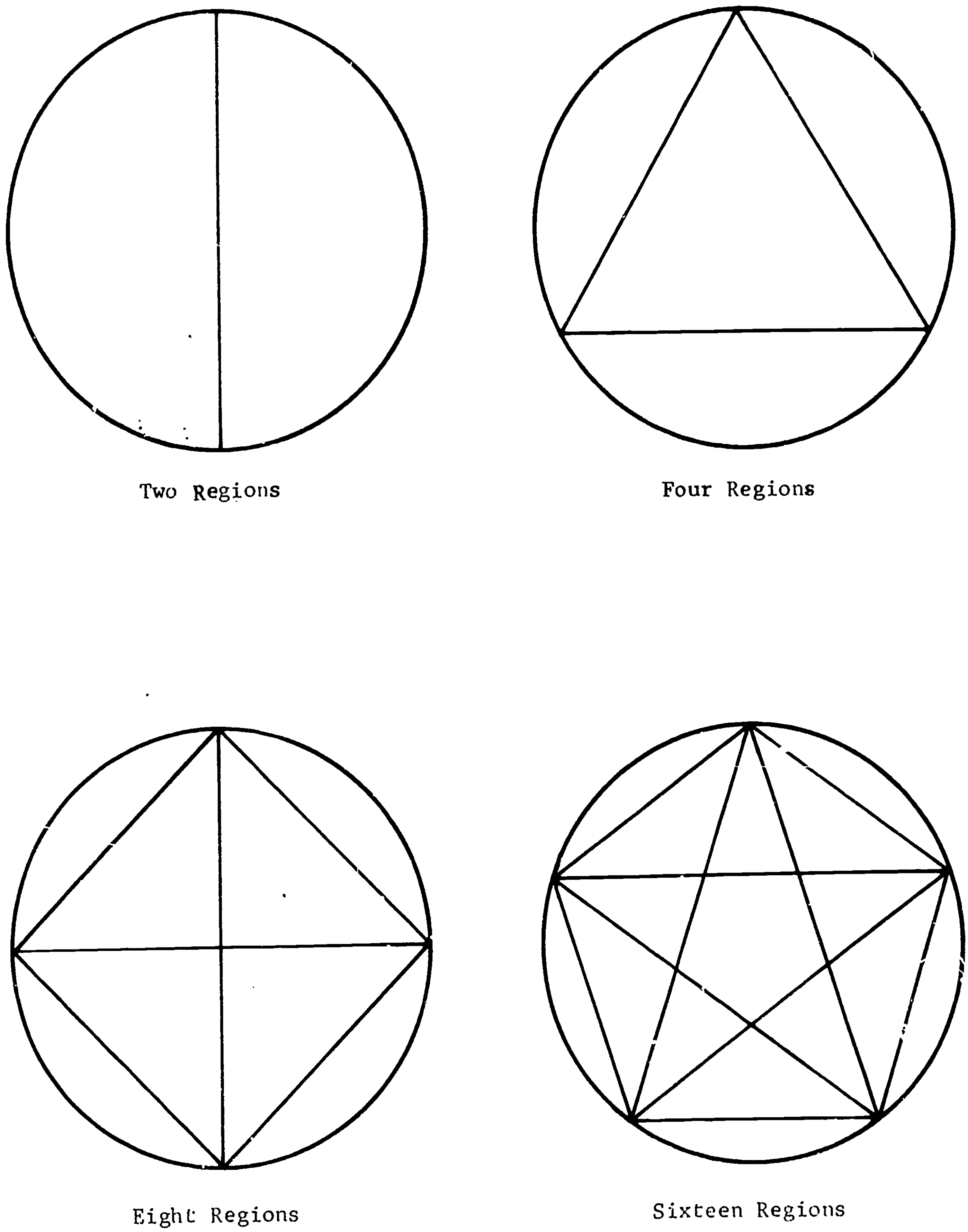
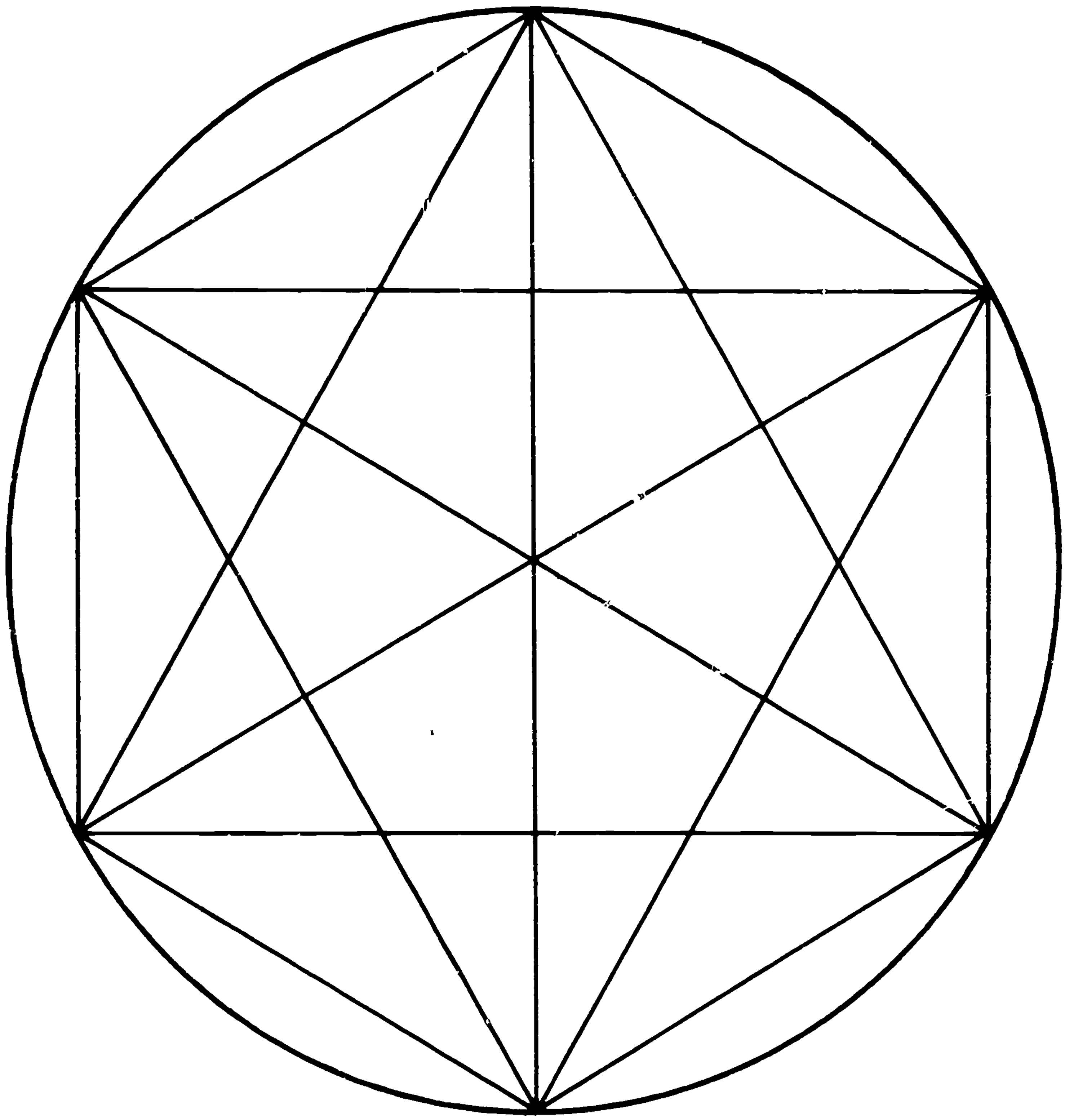


Figure 3. Partitioning the Circle into Regions



Thirty Regions

Figure 4. Partitioning the Circle into 30 Regions

6. Theorem 3: If $N|A$ and $N|B$ and $N|C$, then $N|(A + B + C)$.

7. Theorem 4: If $N\nmid A$ and $N|B$, then $N\nmid(A + B)$.

8. Theorem 5: If $N\nmid A$ and $N|B$, then $N\nmid(A - B)$.

9. Theorem 6: There is always another prime number.

Once the theorems were selected, an operational definition of proof was needed to guide the analysis of the proofs of these theorems. The definition chosen was the definition given by Smith and Henderson in the Twenty-Fourth Yearbook of the National Council of Teachers of Mathematics (1959, pp. 111-112):

> Proof in mathematics is a sequence of related statements directed toward establishing the validity of a conclusion. Each conclusion is (or can be) justified by reference to recognized and accepted assumptions (including the assumptions of logic), definitions, and undefined terms, previously proved propositions (including those of logic), or a combination of these reasons.

In order to stress that each step in a proof must be justified in some appropriate way, a STATEMENT - REASON column format was chosen for the proofs.

Eves and Newsom (1964, p. 156) state that any logical discourse must conform to the following pattern:

> (A) The discourse contains a set of technical terms (elements, relations among the elements, operations to be performed on the elements) which are deliberately chosen as undefined terms. These are the primitive terms of the discourse.
> (B) All other technical terms of the discourse are defined by means of the primitive terms.
> (C) The discourse contains a set of statements about the primitive terms which are deliberately chosen as unproved statements. These are called the postulates, or primary statements, P, of the discourse.
> (D) All other statements (about the primitive and the defined terms) of the discourse are logically deduced from the postulates. These derived statements are called the theorems, T, of the discourse.

(E) For each theorem T_i of the discourse there exists a corresponding statement (which may or may not be formally expressed) asserting that theorem T_i is logically implied by the postulates P. (Often the corresponding statement appears at the end of the proof of the theorem in some such words as, "Hence the theorem," or "This completes the proof of the theorem," etc.)

A full understanding of this pattern of postulational thinking can only come with experience. The purpose of this unit is to use a sequence of six theorems to introduce the students to this pattern. The basic approach was <u>not</u> to start from first principles and develop a system, but rather to start with concepts with which the students were familiar and prove some results about these concepts.

<u>Action words</u>. Action words play a vital role in determining the success or failure of instruction. By stating the objectives of the unit in terms of behaviors described by action words, measurement procedures can be used to determine how successful the instruction has been.

The goal of the unit was to teach sixth-grade students how to prove six theorems. Hence, the main action word is "proving." In addition, there are a number of other action words needed to describe the behaviors used in developing the prerequisites for the proofs. Six action words were defined.

(1) <u>Identifying.</u> The individual selects (by pointing or marking with a pencil) statements which might be confused with one another.

(2) <u>Naming.</u> Supplying (oral or written) the proper name for an instance of a given principle.

(3) Applying a rule or principle. Using a rule or principle to derive a statement from a set of given facts.

(4) Writing a proof (proving). Writing a sequence of statements and reasons in logical order to establish the validity of a given theorem.

(5) Constructing a number. Applying operations to a given set of numbers to form a new number not in the given set.

(6) Giving an example. To state or write an example which illustrates a given principle.

Task analysis. A task analytic approach was employed in the development of the unit. First, the mathematical goals of the unit are expressed in terms of well-defined action words. The idea is to express the objectives of instruction in terms of observable performance tasks. If instruction is successful, the students will demonstrate the ability to perform the specified behavioral objectives. Hence, the success of the instruction is measured in terms of student performance on predetermined performance objectives. Once the curriculum developer has specified these objectives, a task analysis is performed. The task analytic procedure was developed by Gagné (1965b) to train human beings to perform complex tasks. The basic idea of this approach is to break down each behavioral objective into prerequisite subtasks; these subtasks may in turn be analyzed into finer subtasks. The procedure continues until one reaches a set of elemental tasks which cannot or need not be further analyzed. If properly done, the task analysis should yield a hierarchy of tasks which indicate the steps a student must take in order to learn the terminal behavioral objective. The hierarchy indicates how instruction should proceed: one starts with the simplest tasks and learns each subtask until the terminal objective has been mastered.

Gagné (1961, 1962, 1963) has applied this procedure to several mathematical tasks. However, all of the tasks which appear in the literature are rather simple ones, such as determining the intersection of two sets, adding integers, and solving equations. The behaviors which were specified for the unit on proof were considerably more complex. The main behavioral objectives were "proving mathematical theorems." These behaviors consist of writing a sequence of statements and corresponding reasons which together establishes the theorem.

Repeated attempts to task analyze a variety of proofs reveal that there are two basic components of any proof: (1) a knowledge of and ability to manipulate subject-matter content; and (2) a method, a plan, or a strategy which permits the student to weave the subject-matter content into a valid argument. The former is a product, the latter a process, and both are necessary for any proof.

To separate the subject matter from the strategy is sometimes a difficult task. It is also somewhat artificial in the sense that they always occur together as a whole. In stressing that the act of knowing is a process and not a product, Bruner comments upon this fact (1966, p. 72):

> Finally, a theory of instruction seeks to take account of the fact that a curriculum reflects not only the nature of the knowledge itself--the specific capabilities--but also the nature of the knower and of the knowledge-getting process. It is the enterprise par excellence where the line between the subject-matter and the method grows necessarily indistinct.

For instructional purposes it is nonetheless desirable to analytically separate the two. Gagné, the major proponent of the task analysis method, is aware of the distinction between product and process and has made the following comments about their importance in education (1965a, p. 168):

> Among the other things learned by a person who engages in problem solving is 'how to instruct oneself in solving problems.' Such a capability is basically composed of higher-order principles, which are usually called strategies. The manner of learning this particular variety of higher-order principles is not different in any important respect from the learning of other principles. But whereas the higher-order principles previously discussed deal with the content knowledge of the topic being learned, strategies do not. The first type may therefore be called content principles, whereas strategies are heuristic principles. Strategies do not appear as a part of the goals of learning, but they are nevertheless learned.

If strategies can be learned in much the same manner as other principles, why aren't they included "as part of the goals of learning?" The answer is that they are included in any good mathematics program. The Twenty-Fourth Yearbook of the National Council of Teachers of Mathematics, for example, devotes an entire chapter to proof (1959, p. 166):

> When a teacher or student wants to prove a proposition, he will find it useful to know various plans or strategies of proof.

The Yearbook goes on to list eight strategies of proof: (1) counter-example; (2) modus ponens; (3) developing a chain of propositions; (4) proving a conditional; (5) reductio ad absurdum; (6) indirect proof; (7) proving a statement of equivalence; and (8) mathematical induction. These methods are taught in all good mathematics programs.

Gagné's statement that strategies "are nevertheless learned" is challenged in the following comments by Suppes (1966a, p. 73):

> From years of grading mathematical proofs given on examinations at the university level, I am firmly convinced, as I have already indicated, that the ability to write a coherent mathematical proof does not develop naturally even at the most elementary levels and must be a subject of explicit training.

Gagné continues with a line of reasoning which for practical purposes rules out strategies as a goal of learning (1965a, p. 170):

> Obviously, strategies are important for problem solving, regardless of the content of the problem. The suggestion from some writings is that they are of overriding importance as a goal of education. After all, should not formal instruction in the school have the aim of teaching the student 'how to think'? If strategies were deliberately taught, would not this produce people who could then bring to bear superior problem-solving capabilities to any new situation? Although no one would disagree with the aims expressed, it is exceedingly doubtful that they can be brought about by teaching students 'strategies' or 'styles' of thinking. Even if these could be taught (and it is possible that they could), they would not provide the individual with the basic firmament of thought, which is subject-matter knowledge. Knowing a set of strategies is not all that is required for thinking; it is not even a substantial part of what is needed. To be an effective problem solver, the individual must somehow have acquired masses of structurally organized knowledge. Such knowledge is made up of content principles, not heuristic ones.

One must agree with Gagné that subject-matter knowledge should be an important goal of instruction in any mathematics classroom, but one need not, and many do not, agree with his conclusions about the teaching of strategy. Polya's two volume work Mathematical Discovery is devoted to problem solving heuristics, and he thinks a greater emphasis should be placed on strategies (1953, p. viii):

> What is know-how in mathematics? The ability to solve problems--not merely routine problems but problems requiring some degree of independence, judgment, originality, creativity. Therefore, the first and foremost duty of the high school in teaching mathematics is to emphasize methodical work in problem solving.

Fine also takes issue with Gagné's position (1966, p. 100):

> Another possible drawback is that the time spent by a student on problem solving might be better used in the systematic study of some important discipline. It is debatable whether this is a valid objection. There is ample evidence that R. L. Moore's methods have produced many outstanding creative mathematicians. Rademacher's problem seminar, for many years a required course for graduate students at the University of Pennsylvania, has trained many generations in research and exposition. It is my own feeling that the habits of mind engendered by such methods far outweigh all other considerations.

Mathematicians are not the only ones who take issue with Gagné on this matter. In Toward a Theory of Instruction Bruner makes the following statement (1966, p. 72):

> A body of knowledge, enshrined in a university faculty, and embodied in a series of authoritative volumes is the result of much prior intellectual activity. To instruct someone in these disciplines is not a matter of getting them to commit the results to mind; rather, it is to teach him to participate in the process that makes possible the establishment of knowledge. We teach a subject, not to produce little living libraries from that subject, but rather to get a student to think mathematically for himself, to consider matters as a historian does, to take part in the process of knowledge-getting. Knowing is a process, not a product.

Gagné's position is thus at odds with the opinions of many of today's leading educators. He simply places all of his emphasis upon subject matter. In his words, "strategies do not appear as a part of the goals of learning."

It is interesting (but not surprising), therefore, to discover, after many attempts to task analyze the behaviors involved in proving theorems, that the basic task analysis model is inadequate when applied to higher-order cognitive tasks. In the first place the content of a proof does not necessarily fit into a hierarchical pattern. Indeed, many proofs pull together pieces of content which are totally independent in the sense that knowledge of one piece is not a prerequisite for knowledge of another. For example, a simple proof will illustrate the point.

Theorem: If A, B, and N are positive integers, and if $N|A$ and $N|B$, then $N|(A + B)$.

Proof:

(1) $A = NP$ (Definition of divides)
$B = NQ$

(2) $A + B = NP + NQ$ (Substitution)

(3) $A + B = N(P + Q)$ (Distributive law)

(4) Hence $N|(A + B)$ (Definition of divides)

The behaviors involved in proving this theorem are independent of each other. The first step is to form the correct algebraic expression. Each succeeding step in the proof requires a knowledge of a particular principle and the ability to apply this principle to the expression. The principles are not dependent upon one another--knowledge of one is not a prerequisite for knowledge of another. Considered in terms of observed behaviors, the proof would yield a behavioral hierarchy similar to the one illustrated in Figure 5. The terminal behavior, "proving the theorem," is dependent upon each of the other behaviors, hence each behavior is linked to the terminal behavior. But aside from this, there are no dependency relationships in the "hierarchy." Furthermore, the relative cognitive complexity of each task is difficult to determine, hence all the behaviors have been placed on the same level in the hierarchy.

A further difficulty arises if one or more of the prerequisite behaviors are to be taught as part of the unit. Each such behavior would then have to be task analyzed. The result would be a hierarchy of hierarchies (See Figure 6). The diagram represents the extreme case where the learner must be taught each of the prerequisite behaviors. For the present study it is anticipated that the students will know some, but not all, of the prerequisites.

Despite the above modifications in the Gagnéan task analysis model, the model still does not fit the behavior referred to as "proving," for it does not depict the role which strategy plays in the proof. Gagné, of course, denies that strategies are a goal of learning. If this were true, the model in Figure 6 would be sufficient. But it is not true--at least it is not true for the mathematics classroom in general, and this unit on proof in particular. Strategy is an important part of the instructional unit.

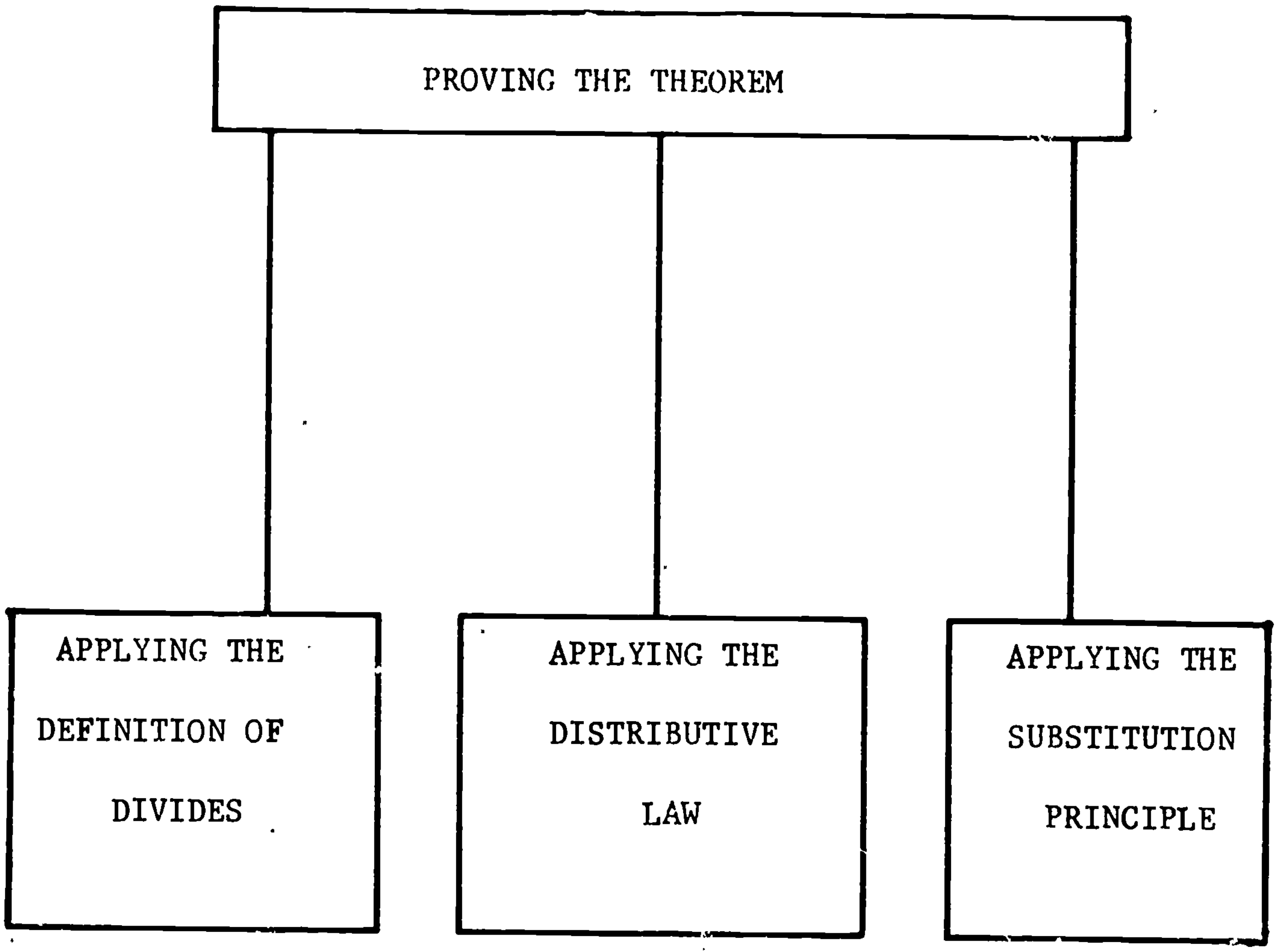


Figure 5. A First Approximation of a Task Analysis for "Proving a Theorem"

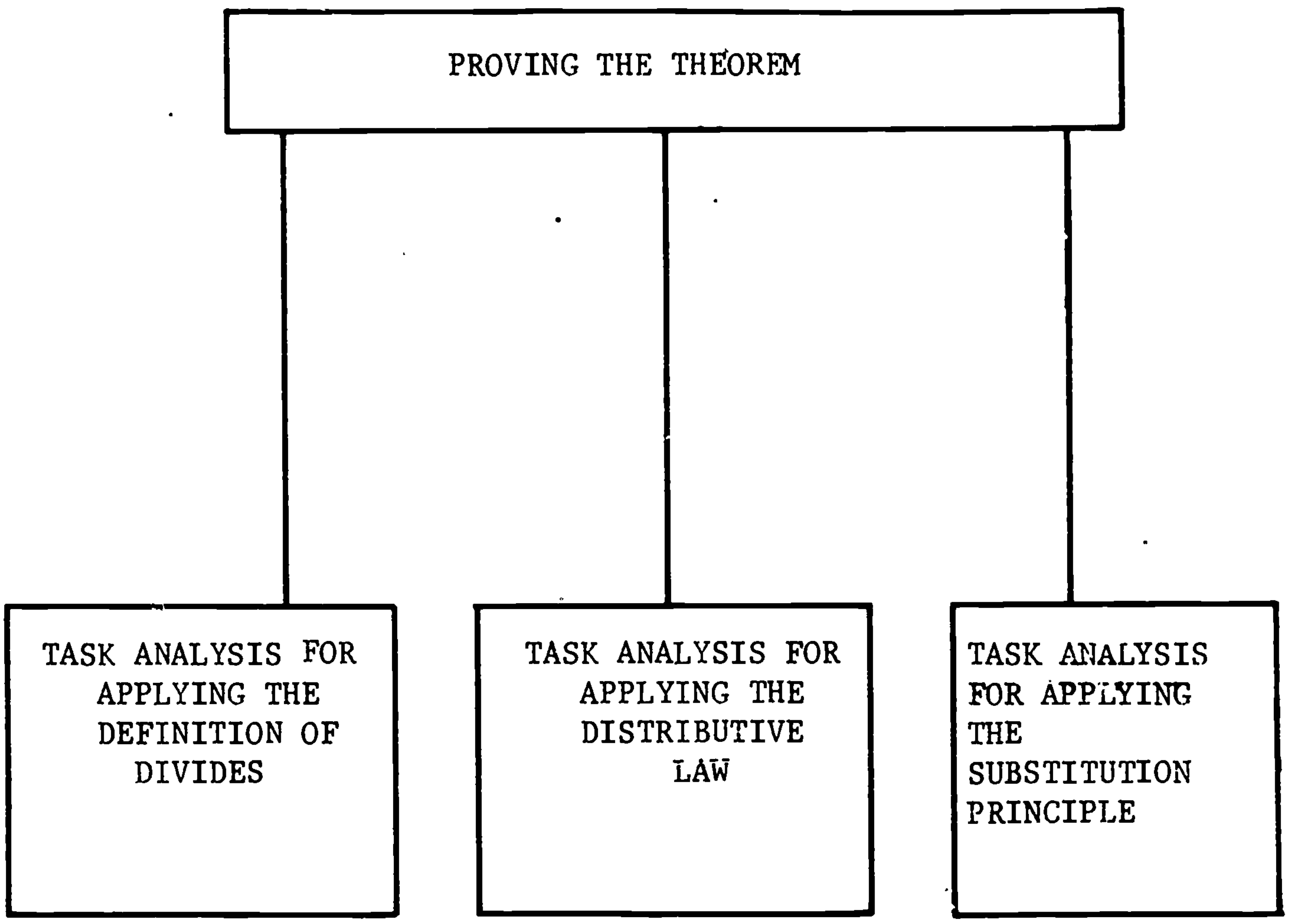


Figure 6. A New Task Analysis Model Containing Other Learning Hierarchies

If student behaviors are to be task analyzed to aid instruction, strategies must be included in the analysis. So the question becomes: how and where can strategies be inserted in the learning set hierarchy to accurately reflect their relationship to other behaviors?

The basic relationship which must be incorporated in the model is the fact that the strategy is applied to the subject matter; it gives order to otherwise unrelated facts. This suggests a further modification of the basic model as illustrated in Figure 7. Again we have a hierarchy of hierarchies, but this time it is a three-dimensional hierarchy. This model incorporates the following features of the behavior "proving a theorem":

(1) proving a theorem depends upon two things, a knowledge of subject matter and applying a strategy. This is depicted in the model by the fact that the planes containing the strategy and the behavioral hierarchies of content are below the plane for proving the theorem, and they are connected with bars to indicate the dependencies;

(2) a knowledge of subject matter requires less complex cognitive skills than the ability to organize that subject matter and apply a strategy to it, and proving a theorem is a more complex skill than the ability to apply a strategy. This is reflected in the model by the fact that the lowest plane in the hierarchy contains the behavioral hierarchies of content objectives, the middle plane contains an analysis of strategies, and the uppermost plane contains

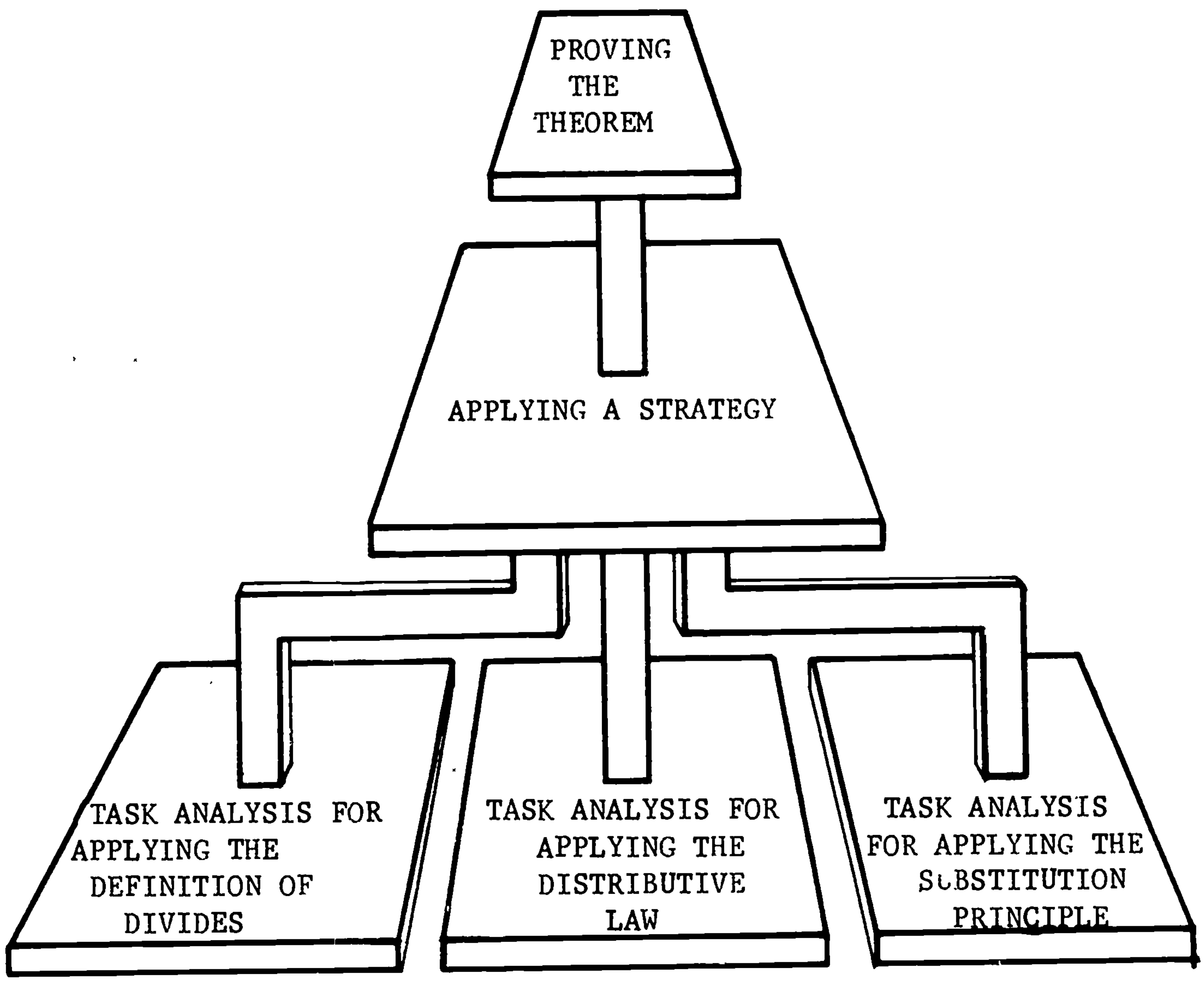


Figure 7. A Three-Dimensional Model for Task Analyzing the Behaviors Involved in Proving Mathematical Theorems

the proof of the theorem; and

(3) strategy and subject-matter content are combined to prove the theorem. The model depicts this in that the bars from each of the lower plane meet in the strategy plane, and a single bar extends upward to the top plane.

Utility is the ultimate test for any model. The investigator believes that this 3-dimensional model more accurately fits proof-making behaviors than does the traditional model, and that it is useful in preparing an instructional analysis for higher-order cognitive tasks. It illustrates that the task analysis consists of three parts: an analysis of subject-matter content, an analysis of the strategy to be employed, and an analysis of the proof itself into behavioral terms. Hence, this model was followed in task analyzing the behaviors in the unit.

First, two-column proofs were written for each of the six proofs. The degree of rigor to be employed in the proofs was a major consideration. Some mathematical programs, such as the one developed by the UICSM, are highly rigorous and require a sophisticated exposure to detailed logical principles. Since this unit was designed to be a first exposure to proof, an attempt was made to adopt a degree of rigor which was compatible with the abilities of sixth-grade students.

From the proofs, the prerequisite behaviors were identified. A task analysis was then written for each proof. The proofs of the first three theorems consist of direct demonstrations that the sum of A and B, the difference of A and B, and the sum of A, B, and C can each be expressed as a multiple of N. Three prerequisite behaviors are required: (1) applying the definition of divides; (2) applying the substitution principle and (3) applying the distributive law. The proofs and task analyses appear in Figures 8, 9, and 10.

The proofs of Theorems 4 and 5 are more complicated. The strategy is to employ an indirect argument. Instead of proving directly that $N \nmid (A + B)$, one shows that the other possibility, $N \mid (A + B)$, is false. And if a statement is false, its negation must be true (the Law of the Excluded Middle). In order to show that $N \mid (A + B)$ is false, one assumes that it is true, then uses the assumption to arrive at a contradiction. If an assumption leads to a contradiction, the assumption is false (the Law of Contradiction). Five prerequisite behaviors are required: (1) applying the definition of divides; (2) applying the substitution principle; (3) applying the subtraction (or addition) principle to solve an equation; (4) applying the Law of Contradiction; and (5) applying the distributive law. The proofs and task analyses appear in Figures 11 and 12.

Theorem 6 is an existence theorem. That is, the proof shows that another prime number must always exist. The proof does not tell us <u>what</u> prime number must exist, but merely guarantees that another one <u>exists</u>. The strategy is to construct a number and

Theorem 1: If $N|A$ and $N|B$, then $N|(A + B)$.

STATEMENT	REASON
1. $A = NR$ $B = NQ$	1. Definition of divides
2. $A + B = NR + NQ$	2. Substitution
3. $A + B = N(R + Q)$	3. Distributive law
4. Therefore, $N|(A + B)$	4. Definition of divides

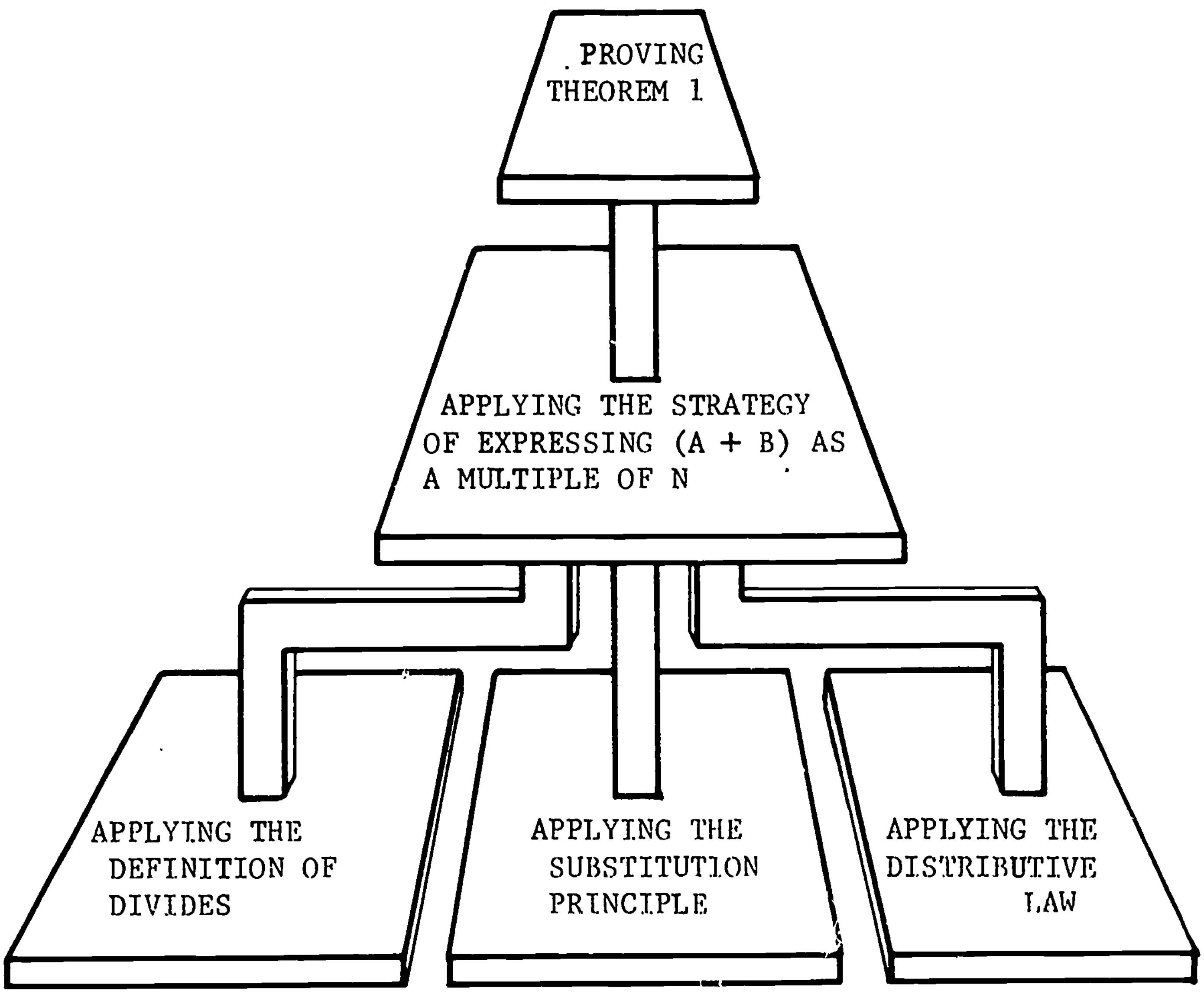


Figure 8. Proof and Task Analysis for Theorem 1

Theorem 2: If $N|A$ and $N|B$, then $N|(A - B)$.

STATEMENT	REASON
1. $A = NR$ $B = NS$	1. Definition of divides
2. $A - B = NR - NS$	2. Substitution
3. $A - B = N(R - S)$	3. Distributive law
4. Therefore, $N|(A - B)$	4. Definition of divides

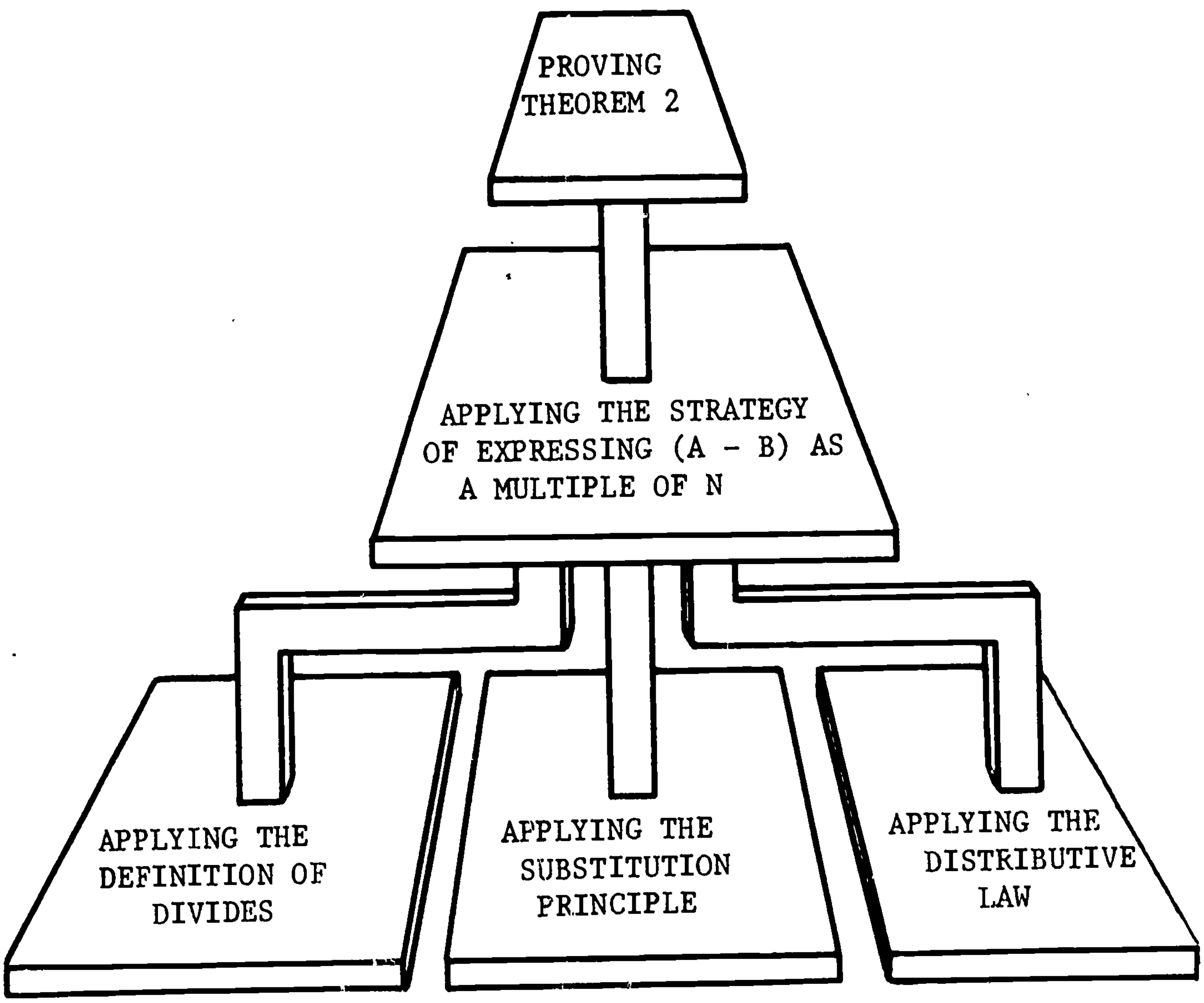


Figure 9. Proof and Task Analysis for Theorem 2

Theorem 3: If $N|A$, $N|B$, and $N|C$, then $N|(A + B + C)$.

STATEMENT	REASON
1. $A = NR$ $B = NS$ $C = NT$	1. Definition of divides
2. $A + B + C = NR + NS + NT$	2. Substitution
3. $A + B + C = N(R + S + T)$	3. Distributive law
4. Therefore, $N|(A + B + C)$	4. Definition of divides

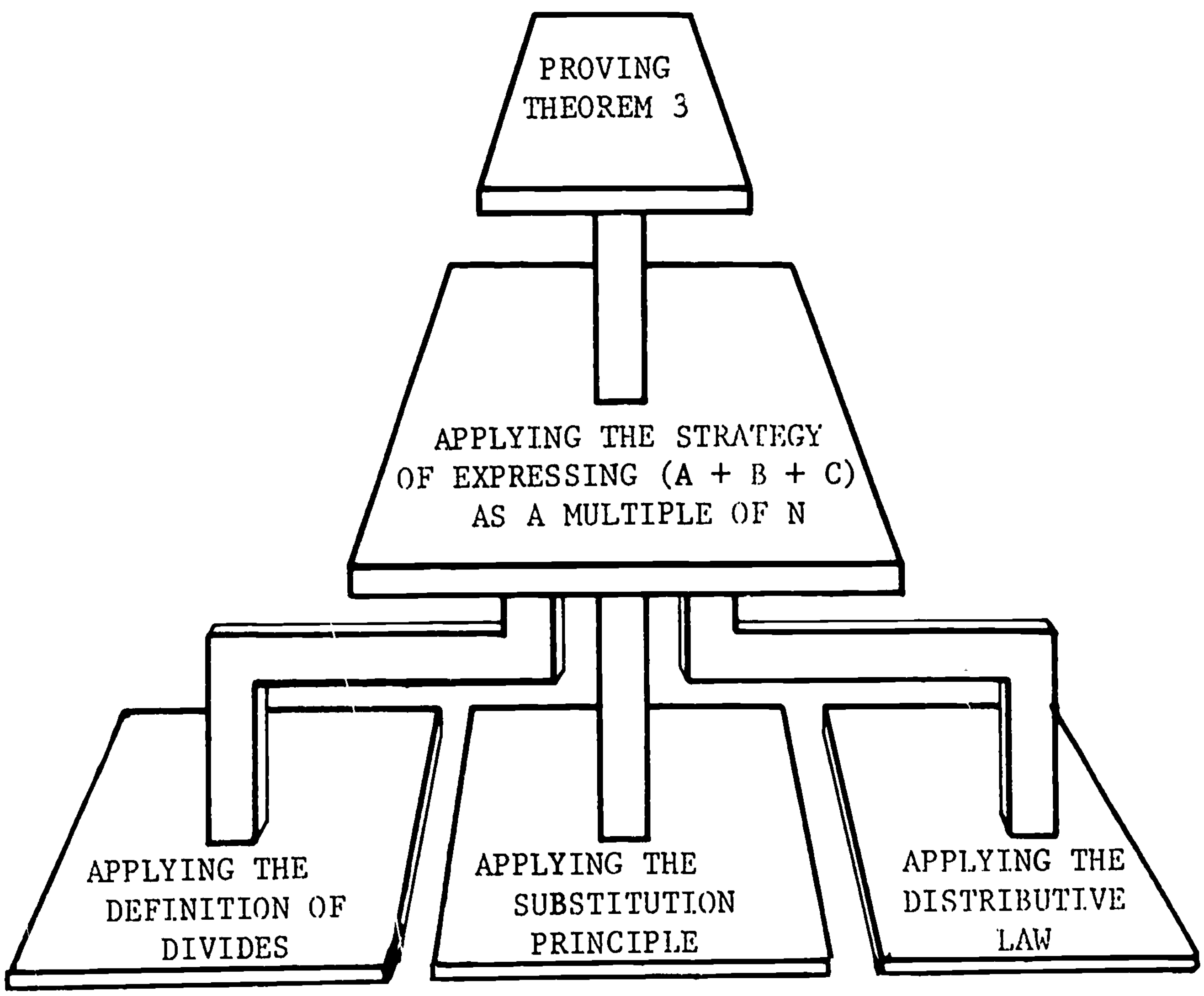


Figure 10. Proof and Task Analysis for Theorem 3

Theorem 4: If $N \nmid A$ and $N \mid B$, then $N \nmid (A + B)$.

STATEMENT	REASON
1. $B = NP$	1. Definition of divides
2. Assume $N \mid (A + B)$	2. Assumption
3. $A + B = NQ$	3. Definition of divides
4. $A + NP = NQ$	4. Substitution
5. $A = NQ - NP$	5. Subtraction principle
6. $A = N(Q - P)$	6. Distributive law
7. Hence $N \mid A$	7. Definition of divides
8. But $N \nmid A$	8. Given
9. Therefore $N \nmid (A + B)$	9. Law of Contradiction

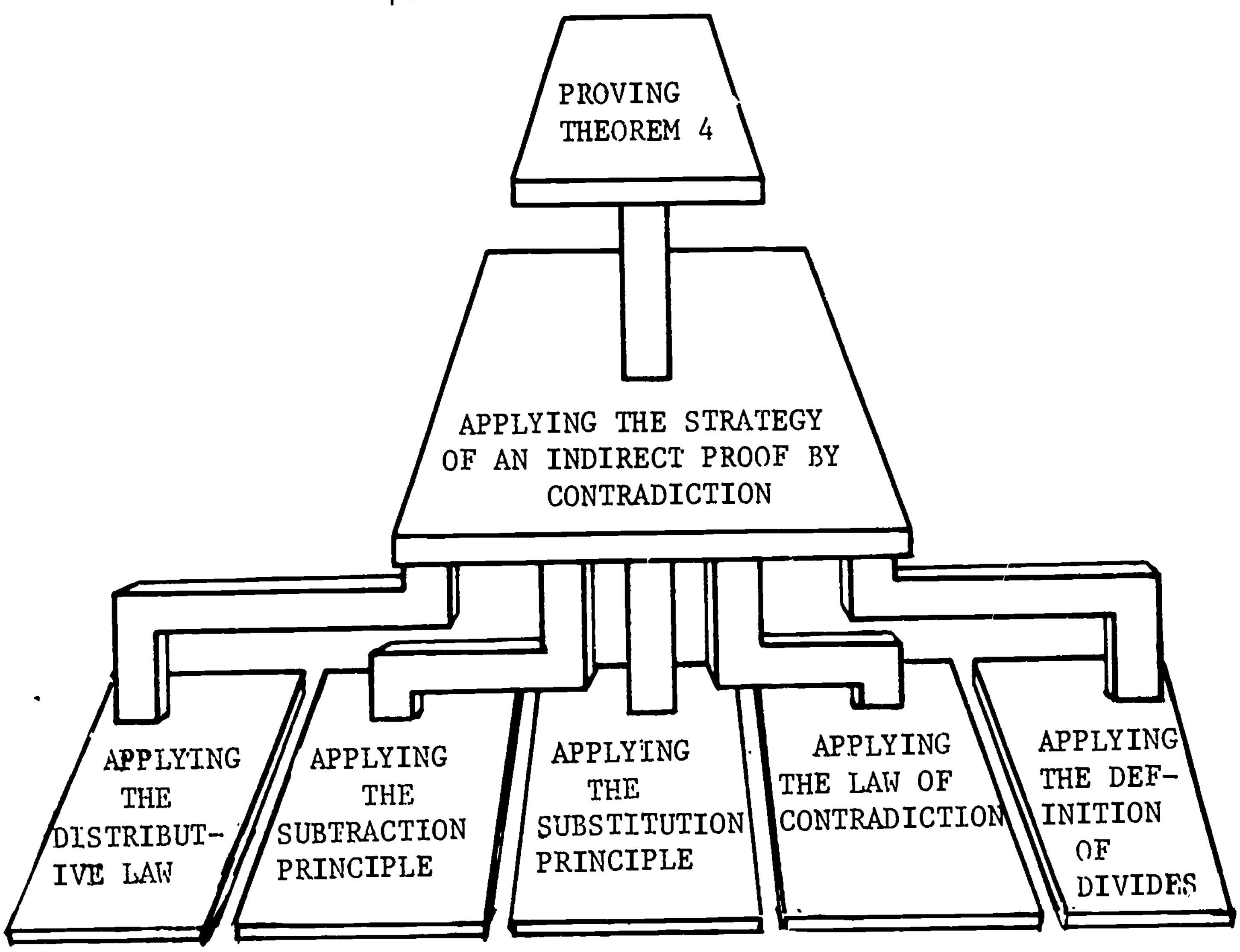


Figure 11. Proof and Task Analysis for Theorem 4

Theorem 5: If $N \nmid A$ and $N \mid B$, then $N \nmid (A - B)$.

STATEMENT	REASON
1. $B = NP$	1. Definition of divides
2. Assume $N \mid (A - B)$	2. Assumption
3. $A - B = NQ$	3. Definition of divides
4. $A - NP = NQ$	4. Substitution
5. $A = NQ + NP$	5. Addition principle
6. $A = N(Q + P)$	6. Distributive law
7. Hence $N \mid A$	7. Definition of divides
8. But $N \nmid A$	8. Given
9. Therefore, $N \nmid (A - B)$	9. Law of Contradiction

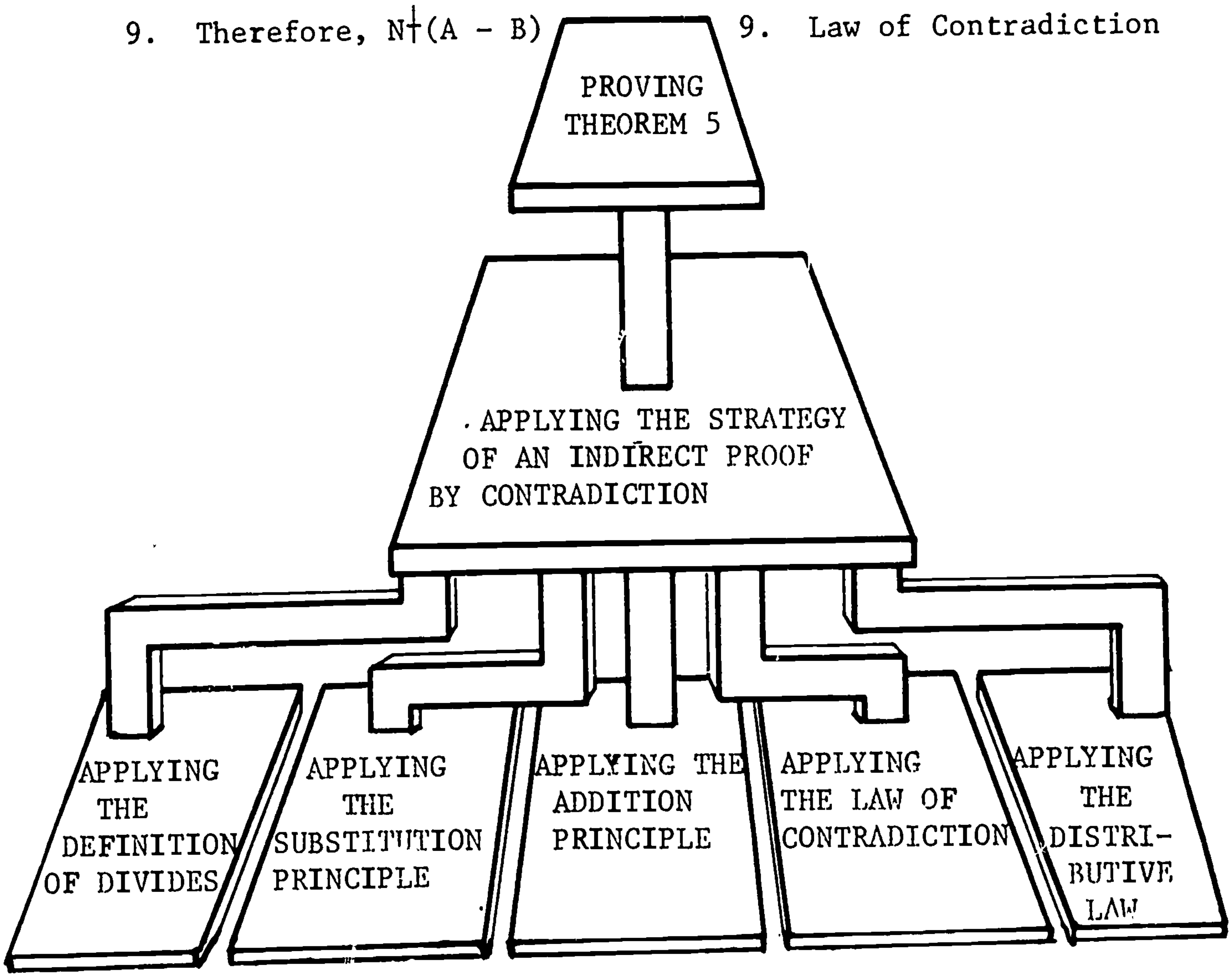


Figure 12. Proof and Task Analysis for Theorem 5

to show that none of the given prime numbers will divide it, thus implying the existence of another prime number which will divide it. Three prerequisites are needed: (1) applying the closure property of whole numbers; (2) applying the previously proven Theorem 4; and (3) applying the Fundamental Theorem of Arithmetic. The proof and corresponding task analysis appear in Figure 13.

These six task analyses were then used to plan the instructional unit.

INSTRUCTIONAL ANALYSIS (Step 2).

How does one go about teaching proof to sixth-grade students? Unfortunately, there is no theory of learning which can adequately explain how mathematics is learned. However, during the past decade several attempts have been made to study the processes involved in mathematical discovery. The following paragraphs discuss three of the more significant attempts.

The most complete study of the psychological problems underlying the learning of mathematics has been conducted by Piaget and his colleagues. The basic approach has been to take a number of age groups and give to a sample of children from each age group a particular kind of problem and observe the methods which the children employ to solve the problem. From the observed behaviors it is inferred that the children have various cognitive structures which are employed in tackling the problem. Four distinct stages of cognitive development have been identified. These stages are

Theorem 6: Given any set of prime numbers $\{P_1, P_2, \ldots, P_n\}$, there is always another prime number.

STATEMENT	REASON
1. $(P_1 x P_2 x \cdots x P_n) + 1$ is a whole number.	1. Closure
2. None of the numbers in the given set will divide it.	2. Theorem 4
3. If $(P_1 x P_2 x \ldots x P_n)$ is a prime, the result is established. If not, it is a product of prime numbers none of which are in the given set. Hence, in either case there is another prime.	3. Fundamental Theorem of Arithmetic

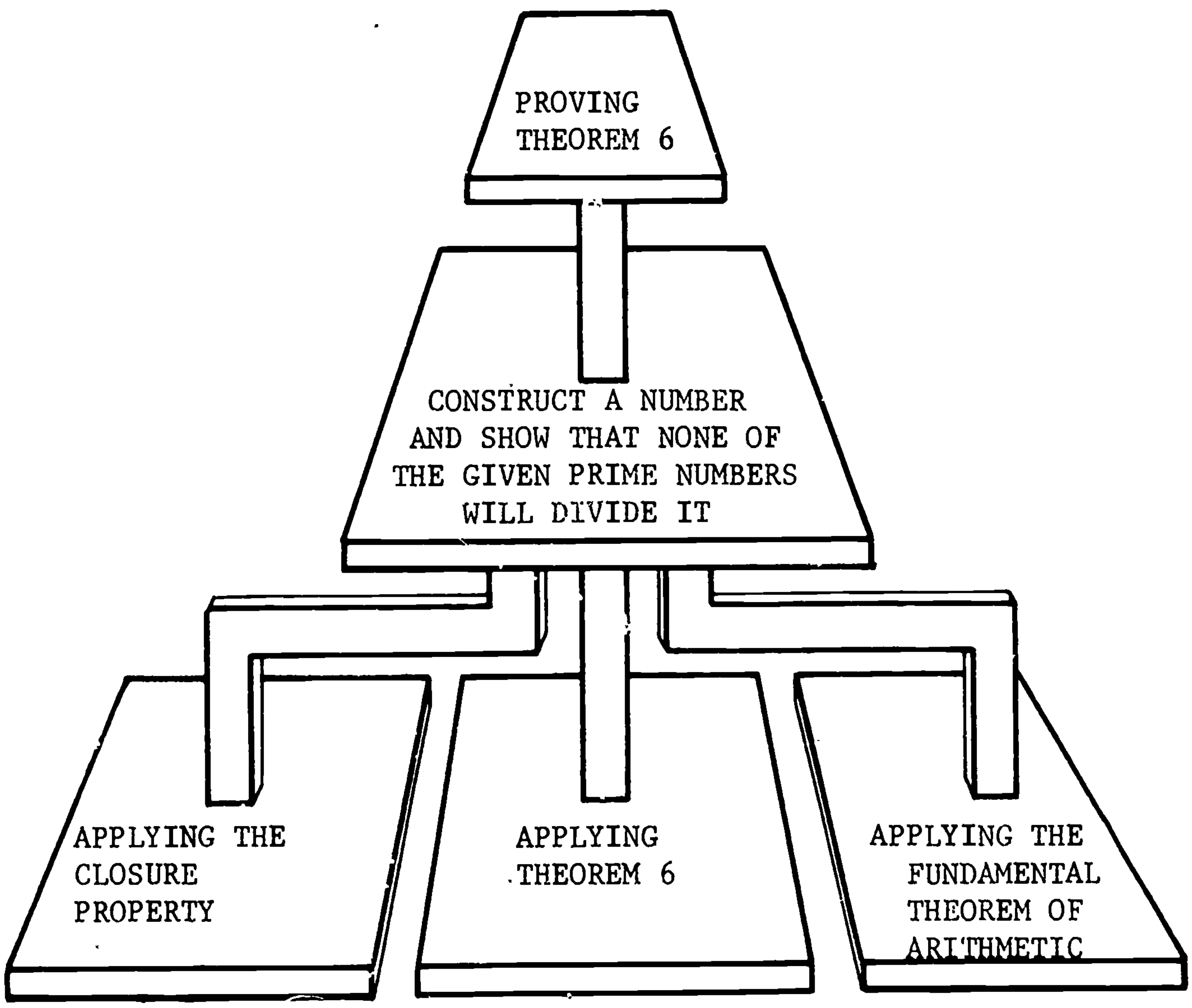


Figure 13. Proof and Task Analysis for Theorem 6

well known and need not be described here. For the purposes of the present study it is important to note that sixth-grade children are entering or are in the stage of formal thinking, suggesting that sixth graders could profit from an introduction to deductive reasoning.

Bruner (1966) has studied the process of mathematical thinking and has categorized it into three stages: the enactive, the iconic, and the symbolic. At first the child thinks in terms of actions. His problem-solving abilities are limited because if he cannot act out the solution, he cannot solve the problem. The child then passes to the iconic stage of cognitive development in which he can manipulate images. Whereas images are easier to manipulate than actions, there is a kind of permanence about them which makes it difficult to apply transformations to them. Since transformations are at the heart of mathematical thinking, Bruner is of the opinion that mathematical thinking does not occur before the child enters this last stage of development. Bruner, as does Piaget, agrees that the young adolescent is entering this final stage.

Dienes (1966) has also contributed to our understanding of mathematical learning. He claims that the energy in play can be directed so as to produce creative mathematical work. He distinguishes three types of play. Manipulative play involves playing with the actual objects we are studying. Representational play occurs when the objects represent other objects or ideas. And rule-bound play involves following rules and games. These types of activities lead to changing the rules of games, thus leading into playing around with mathematical structures.

What do these theories offer in the way of instructional advice? One theme runs through each of the theories: cognitive understanding progresses from the concrete to the representational to the symbolic. The implications for instruction seem clear: one should present ideas in as concrete a form as possible before proceeding to more abstract forms of the ideas. With respect to the theorems in this unit, this means introducing the theorems with specific numerical examples. It was therefore decided to present numerical examples of the theorems and provide the students with the opportunity to discover the generalizations of the theorems. Both Beberman (1964) and Davis (1965) have found this procedure to be highly successful.

This provides us with a general mode of presentation, but it does not answer the original question of how one teaches proof to elementary school children. How does the student gain an understanding of what constitutes a proof? Surely one doesn't read him the definition quoted above from the Twenty-Fourth Yearbook and expect him to understand it, for he probably has no conception of what is meant by "sequence," "establishing," "assumptions of logic," etc. And if he had not seen a proof before, he would have no idea of what "previously proved propositions" means (the definition is partly circular). Niven addresses himself to this problem when he states (1961, p. 7):

> "The nature and meaning of mathematical proof!" It is not possible here and now to give a precise description of what constitutes a proof, and herein lies one of the most puzzling bugbears for the beginning student of mathematics. If the nature of proof cannot be described or formulated in detail, how can anyone learn it? It is learned, to use an oversimplified analogy, in the same manner as a child learns to identify colors, namely, by observing someone else identify green things, blue things, etc., and then by imitating what he has observed. There may be failures at first caused by an inadequate understanding of the categories or patterns, but eventually the learner gets the knack. And so it is with the riddle of mathematical proof. Some of our discussions are intended to shed light on the patterns of proof techniques, and so to acquaint the reader with notions and methods of proof. Thus while we cannot give any sure-fire receipe for what is and what is not a valid proof, we do say some things about the matter, and hope that the reader, before he reaches the end of this book, will not only recognize valid proofs but will enjoy constructing some himself.

A similar viewpoint guided the development of this unit on proof. The purpose of the unit was to expose the students to several different proofs; it was hoped that they could understand and learn the proofs, but it was not expected that they would gain anything but a very brief introduction to what constitutes a mathematical proof. A mature understanding of proof will only come with time and repeated exposure to proof.

Pedagogical considerations led to the changing of one of the proofs. The classical proof of the third theorem involves the use of subscripts to denote an arbitrary set of prime numbers, $\{P_1, P_2, \ldots, P_n\}$. Whereas most sixth-grade students are familiar with the use of variables, as a rule they have not been exposed to the use of subscripts. From an instructional point of view, a thorough treatment of subscripts would be necessary in order that the students grasp the meaning of

$\{P_1, P_2, \ldots, P_n\}$. Therefore, the proof was made less general by considering the set to consist of consecutive prime numbers: $\{2, 3, 5, \ldots, P\}$, where P represents the largest prime number in the set. This change also allows a clearer pedagogical approach to the proof. Starting with the set $\{2,3\}$, the technique used in proving the general theorem can be employed to show that there is another prime number. Form $(2 \times 3) + 1$. Since $2 | (2 \times 3)$ and $2 \nmid 1$, the preceding theorem tells us that $2 \nmid \left((2 \times 3) + 1\right)$. Similarly, $3 \nmid \left((2 \times 3) + 1\right)$. The Fundamental Theorem of Arithmetic can now be applied to guarantee the existence of another prime number. The same procedure can be repeated for the sets $\{2, 3, 5\}$, $\{2, 3, 5, 7\}$, $\{2, 3, 5, 7, 11\}$, etc. The advantage of this approach lies in the fact that the students can inductively discover the proof of the general theorem by themselves. Furthermore, the students can actually calculate the numerical expressions in order to directly verify the procedure. Hence, these changes were made in the theorem and a new task analysis was constructed (Figure 14).

FORMATIVE PILOT STUDY #1 (Step 3)

In January of 1969, a two-week formative pilot study was conducted at Poynette, Wisconsin. Poynette is a rural community with a population of approximately 1000 persons. Six sixth-grade students (three boys and three girls) with an average IQ score of 116 participated in the class. In all, ten fifty-minute classes were held.

Theorem 6: Given any set of prime numbers {2,3,5,...,P}, there is always another prime number.

STATEMENT	REASON
1. (2x3x5x...xP) + 1 is a whole number.	1. Closure
2. None of the primes in the given set will divide it.	2. Theorem 4
3. If (2x3x5x...xP) + 1 is a prime, the result is established. If not, it is a product of primes. In either case, there is another prime number.	3. Fundamental Theorem of Arithemtic

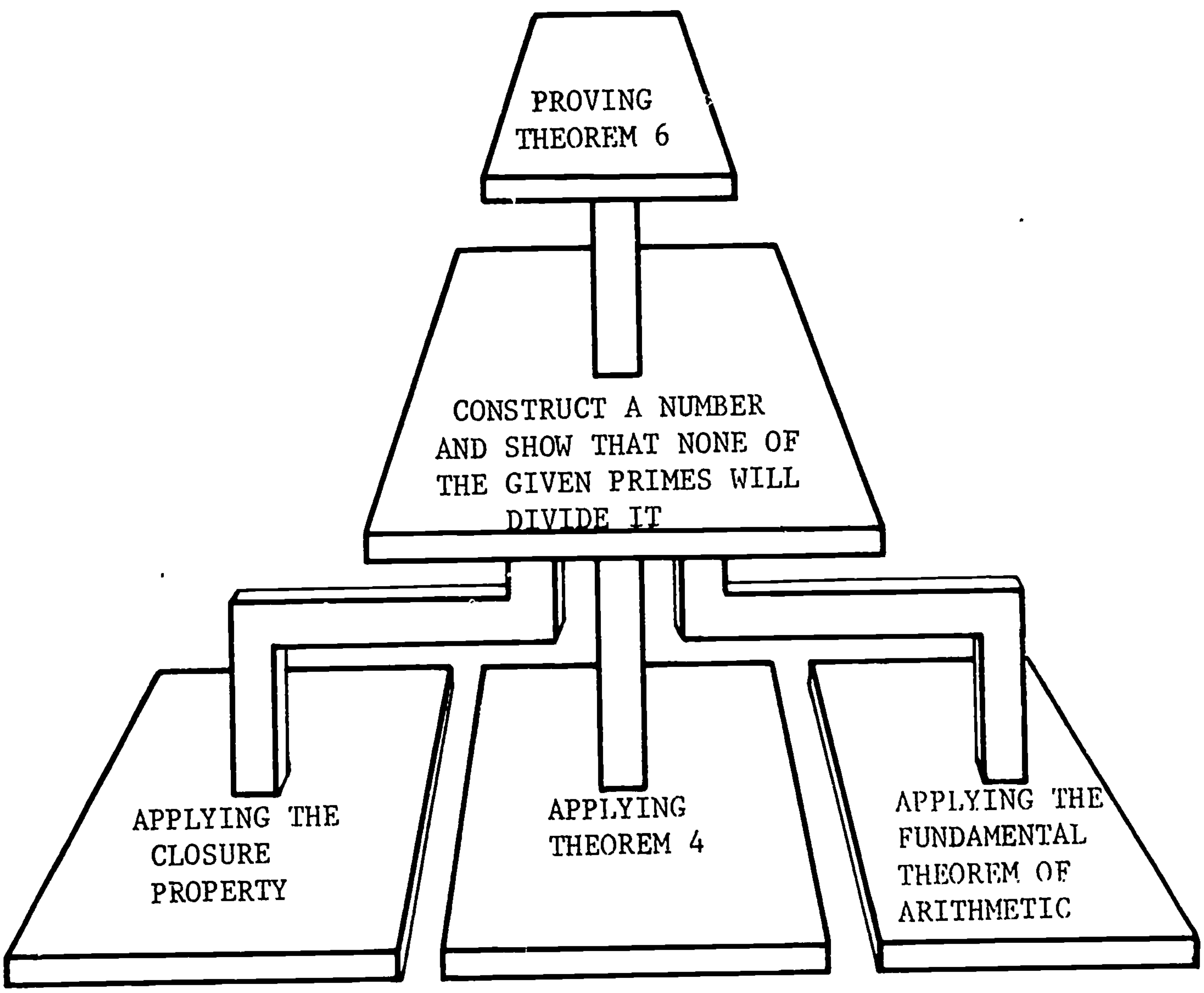


Figure 14. Revised Task Analysis for Theorem 6

The study was exploratory in nature and was conducted primarily to gather information which would aid in a more careful task and instructional analysis. An attempt was made to answer four basic questions:

(1) Was it possible to motivate the need for mathematical proof? In other words, was it possible to convince the students that certain kinds of mathematical statements require a proof?

(2) Were the theorems appropriate? That is, were they of interest to the students, could the students understand them, and could the students learn to prove them?

(3) Was the instructional plan of proceeding from numerical examples to the general theorem a valid procedure?

(4) How adequate was the task analysis? Were all the prerequisites listed?

Seven lessons were presented to the students:

(1) Three meanings of the word "proof." The main purpose of this lesson was to provide an overview for the unit by contrasting proof by authority, proof by empirical evidence, and proof by reasoning;

(2) The need for proof in mathematics. The purpose of this lesson was to motivate the need for proof by torpedoing inductive hypotheses;

(3) Prerequisites;

(4) Theorem 1;

(5) Theorem 2 and Theorem 3;

(6) Theorem 4; and

(7) Theorem 6.

A journal account of the daily activities appears in Appendix A.

The following results were obtained:

(1) The students were able to generalize theorems from particular numerical examples;

(2) The students were able to learn the proofs of the first three theorems;

(3) The students were unable to learn the proofs of Theorems 4 and 6. Two factors probably contributed to this failure: (a) inadequate time for instruction; and (b) inadequate analyses of the prerequisite concepts involved in the theorems;

(4) Once the students had learned the proof of Theorem 1, four of the six students were able to prove Theorem 2. After Theorem 2 was discussed, all six students were able to formulate and prove Theorem 3;

(5) Torpedoing the inductive hypotheses was a successful activity; and

(6) The students were interested in the theorems.

MATHEMATICAL REANALYSIS #1 (Step 4)

The results of the study indicated that two major changes had to be made:

(1) Despite the fact that lack of time had prevented a thorough presentation of Theorem 4 (eighth day of the journal), it was evident to the investigator that a more detailed task analysis of the theorem

was needed. In particular, the law of contradiction needed to be analyzed into its component concepts. This was done and the resulting task analysis is shown in Figure 15; and

(2) As described in the tenth day of the journal, the students were unable to complete the proof of Theorem 6 because they had been unable to apply The Fundamental Theorem of Arithmetic: they were unable to argue both cases of the "or" statement. In an attempt to eliminate this difficulty, the fundamental theorem was replaced by the following theorem: Every whole number greater than 1 is divisible by some prime number. Since this result was to be accepted without proof, it was called "Axiom 1." The new proof and task analysis are shown in Figure 16.

No other changes were made in the mathematical components of the unit.

INSTRUCTIONAL REANALYSIS #1 (Step 5)

Several aspects of the unit worked very well. The examples of inductive reasoning which were torpedoed seemed to convince the students that care must be exercised in accepting mathematical statements. The students understood the theorems and were able to discover generalizations from numerical examples. Hence the basic instructional approach worked quite well.

However, there were several problem areas:

(1) The lesson on motivation was too far removed from the presentation of the first proof. As described in the journal, the lesson of motivation was presented on the second day of instruction;

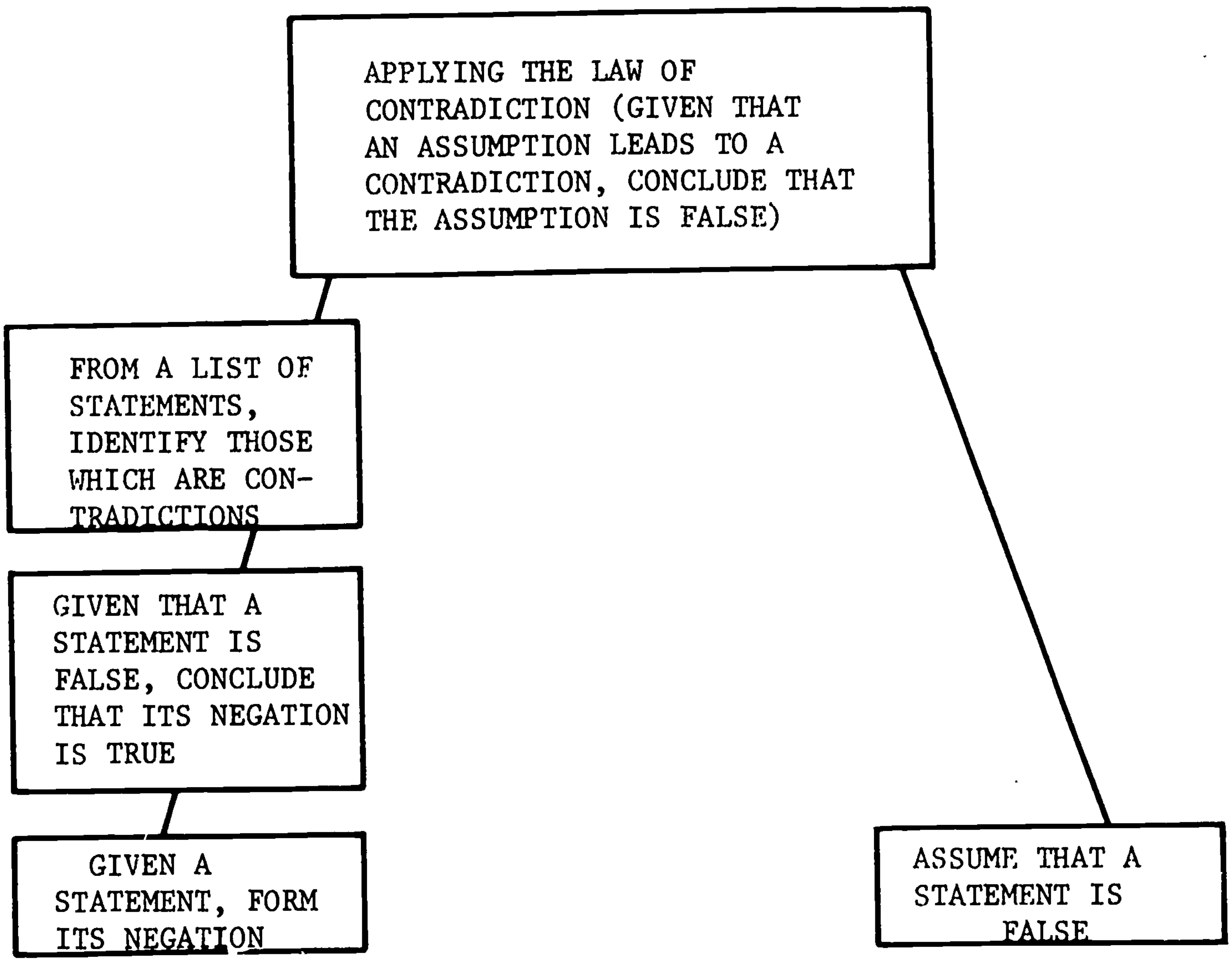


Figure 15. A Task Analysis for Applying the Law of Contradiction

Theorem 6: Given any set of prime numbers {2,3,5,...,P}, there is always another prime number.

STATEMENT	REASON
1. (2x3x5x...xP) + 1 is a whole number	1. Closure
2. None of the numbers in the given set will divide it	2. Theorem 4
3. Since some prime number must divide it, there is another prime number	3. Every whole number greater than 1 is divisible by some prime number

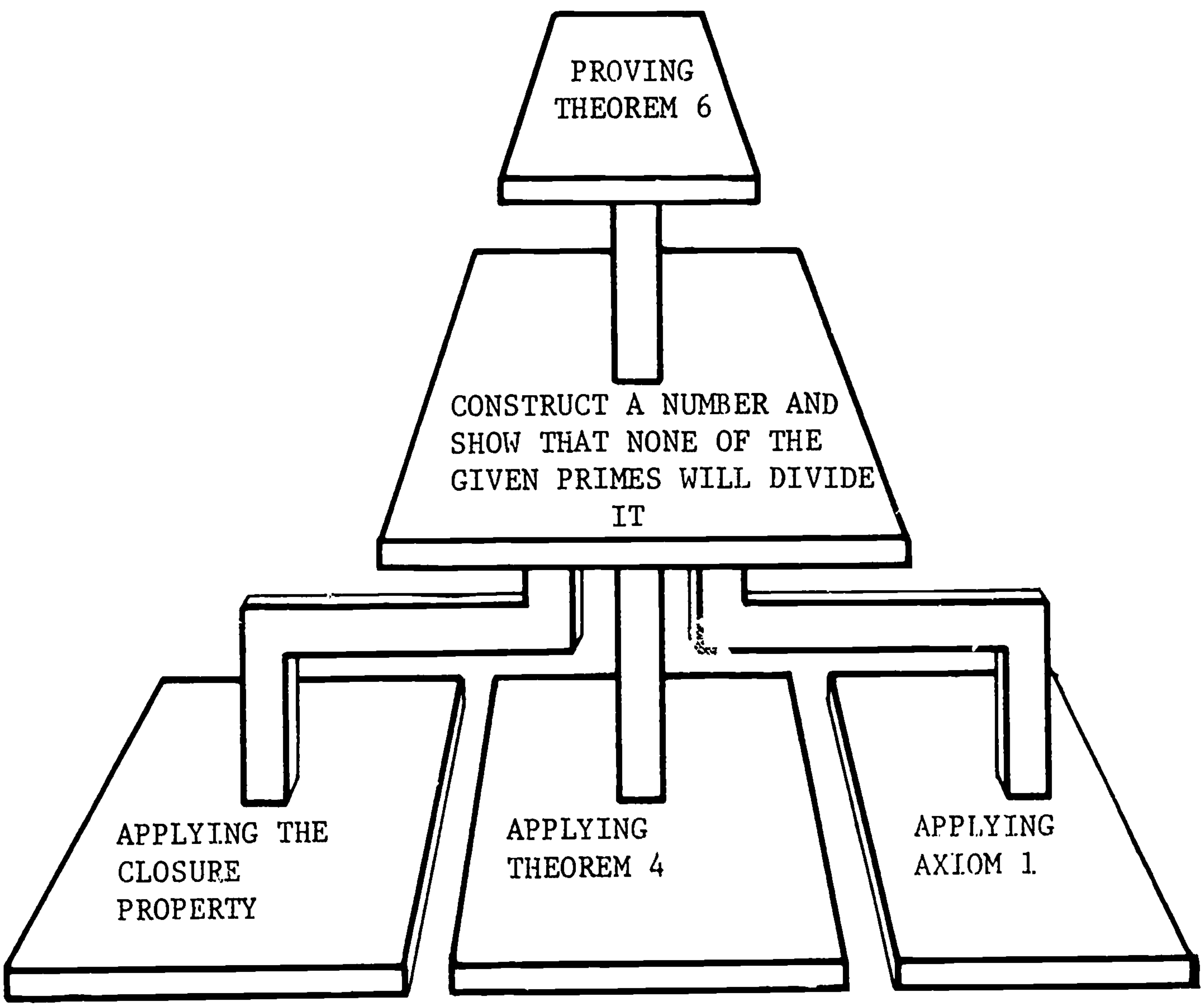


Figure 16. Second Revision of Task Analysis for Theorem 6

this was followed by two days of prerequisite skills; and the first proof was not presented until the fifth day of the unit. The effect of the torpedoing was somewhat diminished by the time the first proof was presented.

It was decided to adopt a different sequence: first the prerequisites, then the lesson on motivation, and then the first proof. In this way it was hoped that the motivation would be more immediately relevant to the proofs.

(2) The first proof was not presented until the fifth day of instruction. This was so because three lessons preceded it. The third lesson included prerequisites which were not necessary for the proof the Theorem 1. It was decided to omit these prerequisites until after the first three theorems had been proven.

(3) The first lesson, which was on the meanings of "prove," came before the students had any idea of what constituted a mathematical proof. It was decided that the lesson would be more meaningful if it were presented after the proofs of several theorems. This would allow for a better comparison of the different meanings of the word "prove." This would also move the first proof up one day in the sequence.

(4) Teaching and drilling on the prerequisites consumed too much class time. The prerequisites were reviewed to see if there were some which were not absolutely essential to the proofs of the theorems. Three changes were made in an attempt to conserve class time.

(a) The students had been given a list of whole numbers from 1 to 200 and were asked to use the sieve of Eratosthenes to find all of the prime numbers less than 200. This activity was very time consuming because several students made errors in applying the sieve. Since most errors were made on numbers greater than 100, it was decided to limit the exercise to all prime numbers less than 100.

(b) The proofs of the first five theorems required the following application of the left-hand distributive law:

$$NP + NQ = N(P + Q).$$

The instruction included a general discussion of the distributive law, including the right-hand distributive law and applying the law in the reverse order, i.e., $N(P + Q) = NP + NQ$. It was decided to include only the form which was used in the proofs.

(c) The proof of Theorem 6 requires the closure properties for whole numbers under addition and multiplication. A general discussion of closure under the four basic operations was presented. It was decided to restrict the discussion to those properties needed in the proof of Theorem 6.

(5) The lecture method was used to introduce new concepts. In the investigator's opinion there was a need to find other modes of presentation which would help minimize the role of the teacher. The

investigator created a sequence of illustrated stories which introduced new ideas and provided drill on prerequisite skills. It was envisioned that the stories would serve as a springboard for discussing the major ideas of the unit.

(6) Two activities were added to the unit in an attempt to create more student interest:

(a) The game of Prime was added to the unit (Holdan, 1969). It is similar in nature to Bingo and provides the students with drill on prime and composite numbers.

(b) An activity was designed which provided the students the opportunity of experimenting with a desk computer. The criteria for divisibility by 3 and by 9 have been verified with small numbers. The computer would permit the students to verify these criteria with very large numbers.

The revised unit consisted of twelve lessons.

I. Prerequisites for Theorem 1

A. Definition of divides

B. Distributive law

C. Substitution principle

II. The need for proof in mathematics

A. Partitioning circles

B. $N^2 - N + 11$

C. Limitations of inductive reasoning

III. Theorem 1

IV. Theorem 2

V. Theorem 3

VI. Activity with desk computer to verify divisibility facts

VII. The meanings of "proof," including a discussion on the nature of mathematical proof

VIII. Prerequisites for Theorems 4 and 5

A. Contradiction

B. Law of the Excluded Middle

C. Law of contradiction

D. Indirect proof

IX. Theorem 4

X. Theorem 5

XI. Prerequisites for Theorem 6

A. Sieve of Erathosthenes

B. Game of Prime

XII. Theorem 6

FORMATIVE PILOT STUDY #2 (Step 6)

In February of 1969 a second pilot study was conducted at Huegel Elementary School in Madison, Wisconsin. Huegel is a multiunit school which utilizes flexible scheduling and individualized instruction. Two fifth-grade and five sixth-grade students (four girls and three boys) volunteered to participate in a three-week study. The students were from an ungraded classroom and had an average IQ score of 115.

The main purpose of the study was to further investigate those components of the unit which had been unsuccessful in the first study. Two more proofs were also added to the unit:

$$\text{If } 3 \mid (A + B), \text{ then } 3 \mid (10A + B).$$

$$\text{If } 9 \mid (A + B), \text{ then } 9 \mid (10A + B).$$

These are, of course, the criteria for divisibility by 3 and by 9 for two-digit numbers. The plan was to teach the students to prove the first result, and then ask them to prove the second without further help. The task analyses for these theorems appear in Figures 17 and 18. A journal account of the study appears in Appendix B.

The main results were as follows:

(1) The students were able to learn the proofs of Theorems 1, 2, and 3;

(2) All of the students were able to prove Theorems 2 and 3 after having learned the proof of Theorem 1;

(3) Five of seven students were able to learn the proof of the divisibility criteria;

(4) When Theorems 1, 2, and 3 were combined on a test with the theorems on divisibility criteria, many errors were made;

(5) Five of the seven students were able to prove Theorem 6;

(6) None of the students were able to learn the proof of Theorem 4;

(7) As in the previous study, all of the students mastered the prerequisite skills.

If $3|(A + B)$, then $3|(10A + B)$.

STATEMENT	REASON
1. $A + B = 3N$	1. Definition of divides
2. $10A + B = (9 + 1)A + B$	2. $10 = 9 + 1$
3. $= 9A + A + B$	3. Distributive law
4. $= 9A + 3N$	4. Substitution
5. $= 3(3A + N)$	5. Distributive law
6. Therefore, $3|(10A + B)$	6. Definition of divides

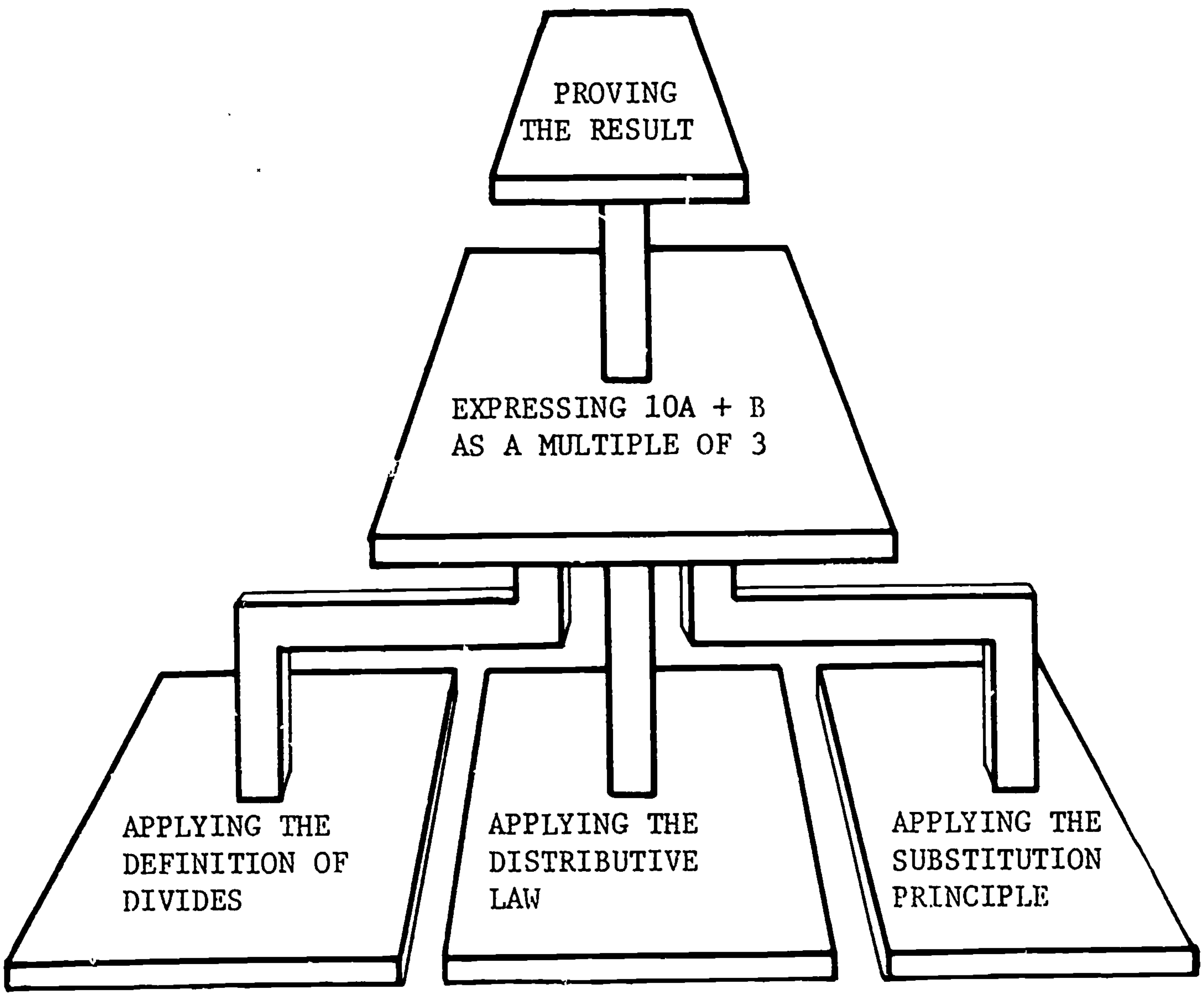


Figure 17. Proof and Task Analysis for the Criterion for Divisibility by 3

If $9|(A + B)$, then $9|(10A + B)$.

STATEMENT	REASON
1. $A + B = 9N$	1. Definition of divides
2. $10A + B = (9 + 1)A + B$	2. $10 = 9 + 1$
3. $= 9A + A + B$	3. Distributive law
4. $= 9A + 9N$	4. Substitution
5. $= 9(A + N)$	5. Distributive law
6. Therefore, $9|(10A + B)$	6. Definition of divides

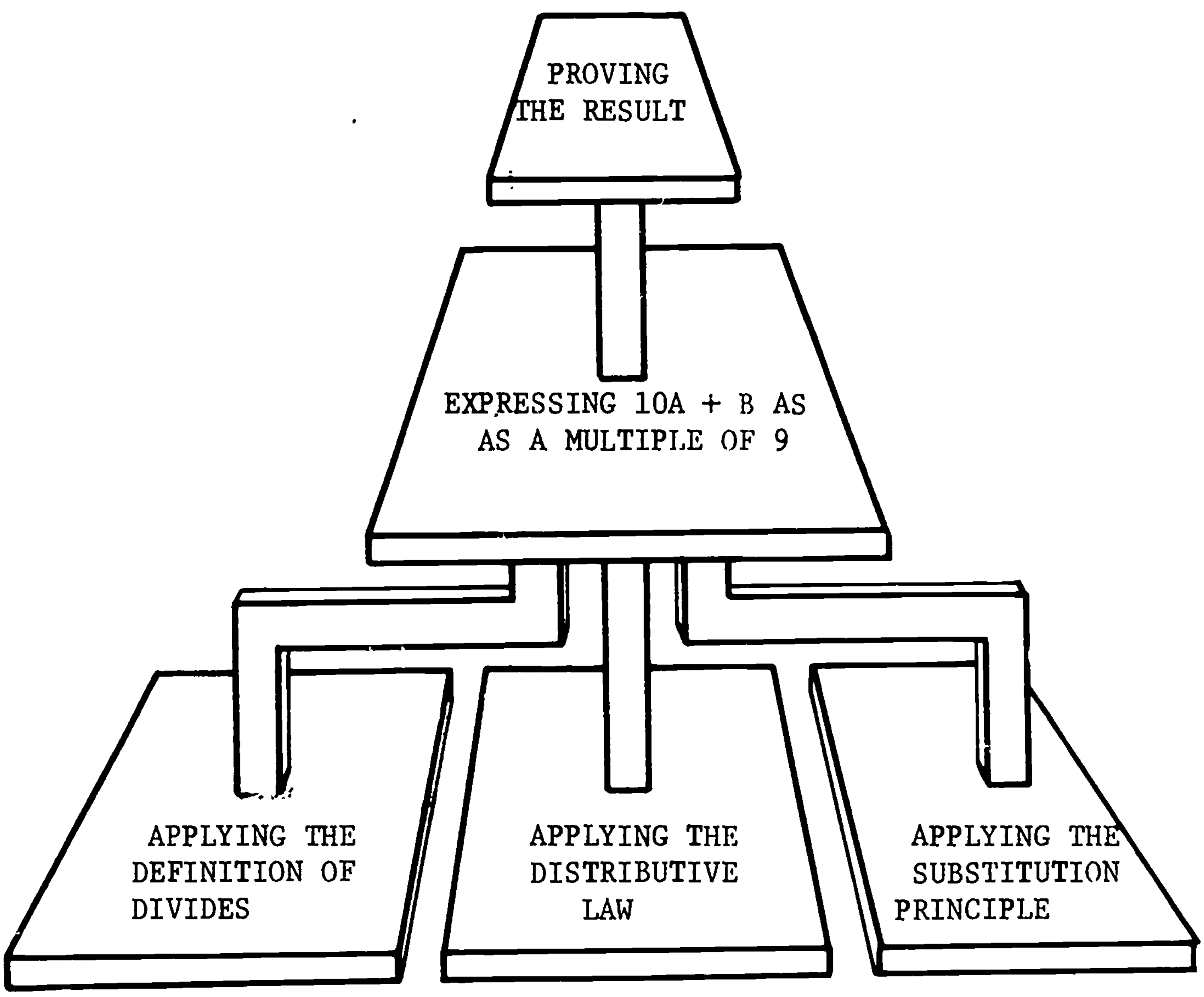


Figure 18. Proof and Task Analysis for the Criterion for Divisibility by 9

(8) The illustrated stories were enhusiastically received by the students and were helpful in introducing new concepts; and

(9) The game of Prime and the activity with the desk computer generated interest.

MATHEMATICAL REANALYSIS #2 (Step 7)

Despite the fact that the proof of Theorem 6 was presented the day before the posttest was administered, five of the seven students learned the proof. Hence the changes which were made in the proof proved to be successful.

The proof of Theorem 4 was again too difficult for the students. This was true in spite of the fact that they had demonstrated a mastery of the prerequisites. As a result of quizzes and student interviews, the author was convinced that the students understood the nature of an indirect proof and the Law of Contradiction. The problem appeared to be the length of the proof. It was considerably longer than the other proofs. A search was made for a simpler proof, and one was found. The proof employs Theorem 2 and is simpler than the previous one. It should have been discovered sooner, but the sequence of events in the planning stages of development had hidden it from view. Theorems 1, 4, and 6 had been task analyzed before the decision was made to add Theorems 2, 3, and 5 to the unit. Because sixth-grade students are not usaully exposed to formal instruction on negative numbers, Theorem 1 could not be directly applied in proving Theorem 4. Hence the long-winded proof for Theorem 4 became necessary. When Theorem 2 was added to the unit, the investigator had failed to see how it could be utilized in proving Theorem 4. A similar proof can be given for Theorem 5 (Theorem 1 is used instead of Theorem 2). The revised proofs and task analyses appear in Figures 19 and 20.

The proof eliminates the need for teaching the students how to solve equations. Furthermore, it serves to illustrate how previously proved results can be utilized in proving other results.

One other change was made as a result of the study. The proof of Theorem 6 requires the closure properties of the whole numbers under multiplication and addition. The students had a negative reaction to this terminology and suggested that the closure properties were "facts" about whole numbers under multiplication and division. It was decided to define "facts" to represent "the closure properties of whole numbers under multiplication and addition."

INSTRUCTIONAL REANALYSIS #2 (Step 8)

Several revisions were made in the instructional program.

(1) There was more discussion and fewer lectures than in the first study. This was due largely to the stories. After the students had read them, different students would be asked to summarize the mathematical content of the story. Hence the students, rather than the teacher, explained the concepts. Although it is difficult to assess how much the stories contributed to the overall learning which took place, they did aid in presenting new materials and were very popular with the students. The investigator was constantly

Theorem 4: If $N \nmid A$ and $N | B$, then $N \nmid (A + B)$.

STATEMENT	REASON
1. Assume $N \mid (A + B)$	1. Assumption
2. $N \mid B$	2. Given
3. $N \mid A$	3. Theorem 2
4. But $N \nmid A$	4. Given
5. Therefore, $N \nmid (A + B)$	5. Law of Contradiction

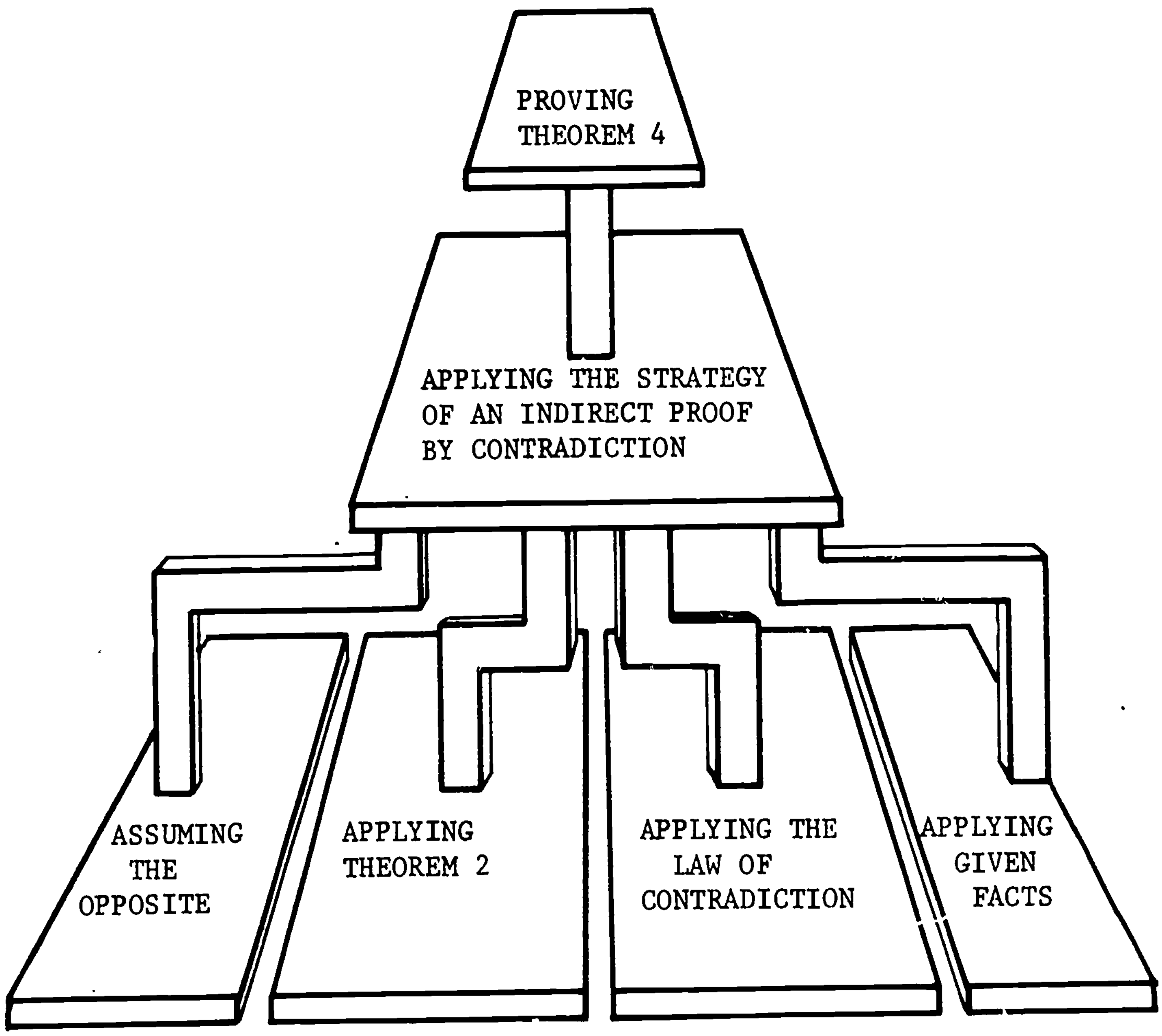


Figure 19. Revised Proof and Task Analysis for Theorem 4

Theorem 5: If $N \nmid A$ and $N \mid B$, then $N \nmid (A - B)$.

STATEMENT	REASON
1. Assume $N \mid (A - B)$	1. Assumption
2. $N \mid B$	2. Given
3. $N \mid A$	3. Theorem 1
4. But $N \nmid A$	4. Given
5. Therefore, $N \nmid (A - B)$	5. Law of Contradiction

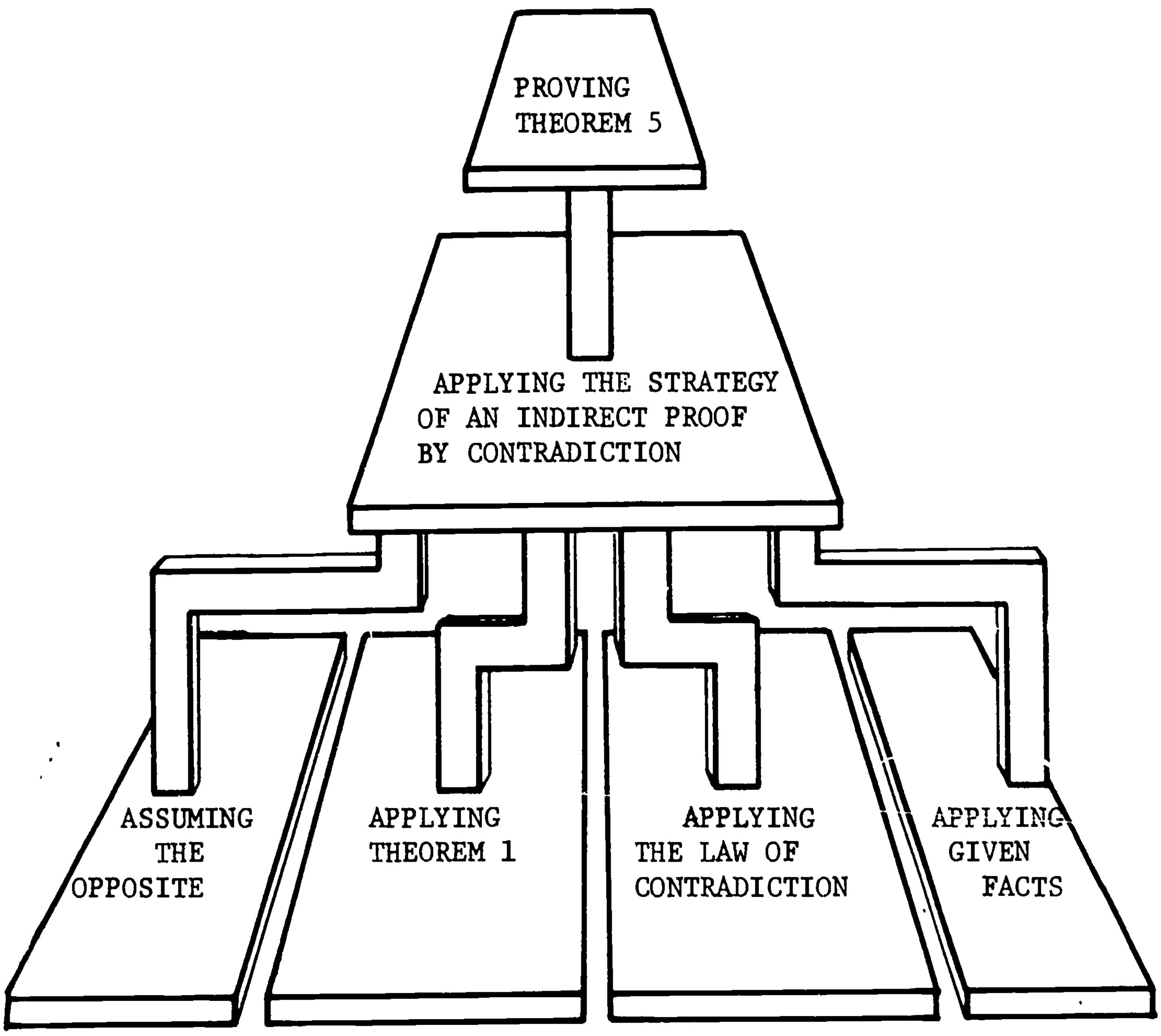


Figure 20. Revised Proof and Task Analysis for Theorem 5

implored to write new episodes. A decision was made to produce additional episodes to cover more of the basic ideas in the unit. The final product appears in Appendix C.

(2) The students had limited cognitive involvement with the proofs. They would watch an explanation of a proof on the board, and then try to reproduce it without looking at the board. The investigator felt that there should be more opportunities for the students to think about the proofs. As a result, several additional exercises were constructed:

(a) Proofs were written with a number of statements and reasons missing. The students would be required to fill in the missing steps;

(b) Proofs were written which contained errors. The students would be required to examine the proofs, find the errors, and made the necessary corrections; and

(c) Application sheets were prepared. The students would be given two divisibility facts and be asked to apply a particular theorem to these facts. For example, apply Theorem 2 to these facts:

$$N|(A + B) \text{ and } N|B.$$

The correct response is $N|A$. In addition to illustrating how the theorem can be applied, it was hoped that such exercises would help prepare the students for the proofs of Theorems 4 and 5;

(3) The students were able to learn the proofs of the divisibility criteria, but when all of the theorems were included on the same test, three of the students made errors. The reasons for one proof would be cited as reasons for another. For these reasons, the proofs of divisibility criteria were excluded from the unit;

(4) Two activities were used in an attempt to motivate the need for mathematical proof: evaluating the expression $N \;\; - N + 11$ and partioning the circle. The investigator felt that one activity would adequately illustrate the point. Since the circle is not directly related to the unit, it was excluded from the unit;

(5) When playing Prime, the investigator had to read the slips several times so that everyone understood the conditions. It was decided to use an overhead projector with transparencies for the next study. The condition could then be seen by everyone; and

(6) The students evaluated $(2 \times 3 \times \ldots \times 13) + 1 = 30{,}031$. Since their list of prime numbers contained only primes less than 10,000, the students were unable to determine if this number was prime. The University of Wisconsin Computing Center provided the investigator with a printout of all prime numbers less than 60,000. The number was not on the list, for $30{,}031 = 59 \times 509$. This illustrates that the technique of multiplying the given primes together and adding one does not always yield a prime number; it only guarantees the <u>existence</u> of another prime number. This list of prime numbers would be used in the next pilot study.

The students who had participated in the first two formative studies were average and above average in academic abilities and had a mean IQ score of 116. Two factors led to a decision to develop the unit for average and above-average students:

(1) In discussing the recommendations of the Cambridge Report, Allendorerfer comments (1965, p. 693):

> My third objection is that the report completely ignores the very substantial problem of what mathematics should be taught to the lower seven-eighths of the ability group, and in particular to the lower third.

The investigator is in partial agreement with this viewpoint and thinks there is a strong possibility that such a unit may be inappropriate for students with below-average academic abilities.

(2) The unit of instruction includes illustrated stories which the students are required to read, and below-average students quite frequently have reading problems.

Two other considerations led to a decision to limit the class size to ten.

(1) Keliher (1967, p. 21) reports research studies in which "small classes produced more educational creativity and promising new procedures, children were more likely to receive individual attention, and there was more variety in instructional methods."

(2) The teacher would be able to detect learning difficulties as they arose so that corrective measures could be taken immediately. The two pilot studies led the investigator to believe that ten students would provide the optimal classroom atmosphere for the unit.

FORMATIVE PILOT STUDY #3 (Step 9)

The goal of an instructional program is to produce students who can perform explicit behaviors. Many variables can influence the success in achieving this goal. A systems approach to instruction views the educative process in terms of these variables. If a desired goal is to be achieved, attempts must be made to control and integrate into an effective instructional program all of the components which might conceivably affect the attainment of that goal.

Romberg (1968, p. 1) has defined "system" in the following way:

> The word "system" as used in this paper refers to a "man-made controlled functional structure." A "man-made structure" means that the system has interdependent components which can be changed or manipulated. "Controlled" means that there is a feedback or monitoring procedure which can be used to manage the system, and "Functional" means that the system is goal oriented with a stated purpose or intent.

For the purposes of this study the instructional system was viewed in terms of five basic components: input, mechanism, feedback, resources, and output. The output consists of students with terminal behaviors. The input consists of all the variables which enter the classroom and affect the output. These variables are of two kinds: those which are directly controllable by the experimenter and those which are not. The experimenter can select the teacher, the students, and the materials used in the classroom. On the other hand, he does not necessarily have control over such factors as the attitudes of the community, the faculty, or the parents, any or all of which might affect the outcome of the instruction.

The mechanism is a set of instructional plans to be employed by the teacher. Feedback is an evaluation procedure which provides data for altering the instructional procedures, and resources consist of the basic hardware without which the experiment would not be possible: financial support for a classroom facility, secretarial and duplication assistance, etc.

Figure 21 illustrates how the components of the system relate to one another. The instructional system is viewed as an electrical machine. The teacher, the students, the instructional materials, and other variables affecting output are put into the machine, these interact with the instructional program, evaluation provides feedback which may result in new input, and the output is a group of students with terminal behaviors. And, of course, the machine will not operate unless it is plugged in, i.e., unless there are resources to support the system.

MECHANISM (Instructional program)

Content outline. Ten lessons were included in the unit for the third pilot study.

I. Prerequisites for the first three theorems

   A. Definition of divides

   B. The substitution principle

   C. The left-hand distributive law

II. The need for proof in mathematics

   A. An example of inductive reasoning which yields a false conclusion

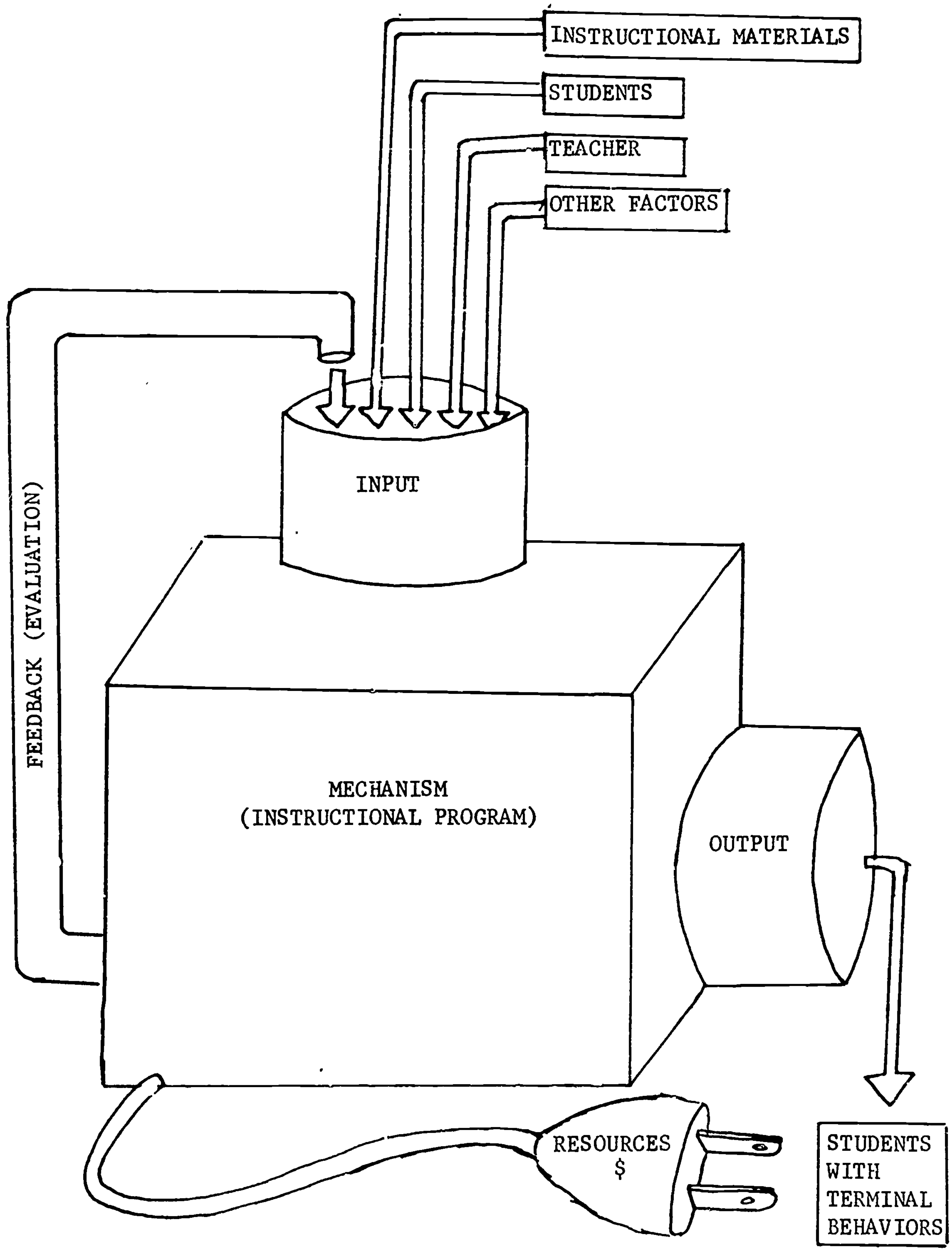


Figure 21. Instructional System

III. Theorem 1 and its proof

IV. Theorem 2 and its proof

V. Theorem 3 and its proof

VI. The meanings of "proof" and the nature of mathematical proof

VII. Prerequisites for Theorems 4 and 5

A. Contradictions

B. Law of contradiction

C. Indirect proof

D. Law of the Excluded Middle

VIII. Theorem 4 and its proof

IX. Theorem 5 and its proof

X. Theorem 6 and its proof

Lesson plans. Using the experiences of the first two studies as guidelines, the investigator prepared lesson plans. They were then given to the teacher to study. After reading the lesson plans, the teacher made a number of suggestions and requested that the plans be written in greater detail. She wanted to know exactly how to present each topic; this included specific examples to use, questions to ask, comments to make, etc. Her unfamiliarity with some of the basic concepts made it necessary to rewrite the lessons in greater detail, and additional exercises were prepared.

The format of the plans included the following:

(1) a brief statement of the purpose of the lesson;

(2) a statement of behavioral objectives;

(3) a list of the materials to be used in teaching the lesson;

(4) a description of the procedures to be used in teaching the lesson; and

(5) one copy of each of the printed material to be used in the lesson.

The lesson plans, together with a journal account of each lesson may be found in Appendix C.

Procedures. Plans were made to follow the same basic procedures as were developed in the previous studies. Student participation was to be encouraged and the teacher was to entice the students to contribute as much as possible to the development of the topics in the unit. The students were to be given the opportunity to discover the theorems from repeated numerical examples and to write the proof of the general theorem after being given a proof for a particular numerical example. In short, the students were to be encouraged to play an active role in all learning situations, thus minimizing the role of the teacher as a source of knowledge.

The results of the preceding pilot studies indicated that a criterion of (80/80) was feasible with a small group of students. Instruction was to aim at mastery: those students who failed to master a skill were to receive individual help until they reached criterion. Each mastery test was to be marked either "master" or "non-master." These students who failed to master the tests were to be given additional chances to become masters.

While the teacher conducted the class, the investigator was to be an observer and tape record each lesson. At the completion of each day's lessons, the teacher and investigator planned to analyze the lesson and make preparations for subsequent lessons. The prepared lesson plans were tentative in nature and could be changed in order to cope with learning problems encountered in the classroom.

Feasibility. Plans were also made for testing the feasibility of the unit. An attempt was made to answer three questions:

(1) Can the students display an understanding of the meanings of the theorems? Two types of items were developed for determining the students' understanding of the theorems. First, the students were to be asked to write numerical examples which illustrate the meaning of each of the first five theorems. For example, $3|27$ and $3|8$, hence $3|19$ illustrates Theorem 5. Second, the students were to be asked to apply the theorems to a given set of facts. For example, given that $3|111$ and $3|84$, when asked to apply Theorem 2 the student should conclude that $3|(111 - 84)$. These items were included on the mastery tests for the lessons containing the first five theorems.

(2) Can the students reproduce the proofs of the theorems? This was the majo. . -navioral objective of the unit. The mastery test for each theorem included a proof, and the posttest asked the student to prove all six theorems.

(3) Can the students display an understanding of the proofs? Two measures were to be used to determine this. First, for each theorem the students were to be tested on all of the prerequisites involved in the proof of the theorem. Second, each student was to be interviewed separately. He was expected to explain why each reason was used in the proof. He was to be asked to state the reasons in full when names or abbreviations were used as a reason. For example, if the student used Axiom 1 as a reason, he was to be asked to give a complete statement of the axiom.

The results obtained by these measures will be discussed in Chapter IV.

Mastery tests. A mastery test was written for each of the lessons which had explicit behavioral objectives. The tests appear in Appendix C with the lesson plans. An (80/80) criterion was established for each mastery test. When eight students mastered the tasks on the test, the class was to proceed to the next lesson. Those who did not master the concepts were to be given individual help.

Pretest-posttest. A pretest-posttest was written which contained both the proofs of the theorems and the prerequisites for the proofs. The prerequisites were included to help determine instructional strategy. That is, the performance of the students would indicate which of the prerequisites needed to be taught. The posttest would indicate how successful the unit of instruction was in teaching prerequisites.

The proofs were included on the pretest-posttest to measure the main behavioral objectives of the unit. A copy of the test appears in Appendix E. In addition to the proofs, it contains eight prerequisites:

(1) applying the left-hand distributive law;

(2) recognizing the validity of the closure property for the set of whole numbers under addition and multiplication;

(3) applying Axiom 1;

(4) applying the substitution principle;

(5) applying the definition of divides;

(6) applying the Law of Contradiction;

(7) identifying prime numbers; and

(8) recognizing that repeated instances of a given principle does not constitute proof of that principle.

INPUT

The teacher. A certified elementary school teacher, whose duties as an employee of the Wisconsin Research and Development Center for Cognitive Learning included teaching experimental classes, was assigned to teach the unit. She assisted the investigator in planning the study.

Her basic teaching philosophy is consistent with the procedures developed in the first two pilot studies by the investigator. It is her belief that the authoritarian role of the teacher should be minimized while encouraging children to take a more active role in learning.

Her preparation in mathematics included high school courses in Algebra (2 years), plane Geometry, solid Geometry and Trigonometry, and college courses in College Algebra, Analytic Geometry, two semesters of Calculus, and a modern math course for teachers. Hence, she was better prepared in mathematics than the average elementary school teacher. However, she had not been exposed to extensive experience with proof and was not very familiar with the deductive and axiomatic nature of mathematics. Therefore, it was necessary for the investigator to teach the unit on proof to the teacher.

Four weeks before the third pilot study was to be conducted, The investigator discussed the unit with the teacher. The first priority was to teach her the proofs of the six theorems. Three sixty-minute sessions were held during the next ten days. At the end of this time, the teacher was familiar with the proofs. The sequence of six theorems was used to illustrate the nature of mathematical proof to the teacher. Eight more sessions were held before the experiment began.

The students. The third formative pilot study was conducted at Lake Mills Middle School in Lake Mills, Wisconsin. Lake Mills is a rural community of about 3000 people and is located 25 miles from Madison, the capital of Wisconsin and the home of the University of Wisconsin. While originally being a farming community, the population of Lake Mills is now predominantly composed of salaried lower middle-class citizens.

The community is located on an Interstate highway and many of the citizens commute to Madison to work. In addition, the community is linked to Madison via television and radio. Hence, Madison has a strong influence upon the community. Except for a few migratory farm workers, there are no racial minority groups in Lake Mills.

The middle school contains sixth-, seventh-, and eighth-grade students. The students had a different teacher for each subject. Class periods were approximately 44 minutes in duration.

The three sixth-grade mathematics classes were taught by the same teacher. Since the classes met at different hours of the day, it was not possible to randomly select students from the total sixth-grade population of the school. The selection was further restricted by the fact that the unit was being developed for college-capable (average and above-average) students. One class was selected at random and the ten best mathematics students were selected from this class to be in the Experimental Group. Seven girls and three boys were selected to participate.

A Control Group was selected from the other two math classes by matching procedures. Sex, Henmon-Nelson IQ scores, grades in mathematics, and teacher appraisal were the variables considered. The teacher appraisal included such factors as work habits, attitudes towards mathematics, etc. The Experimental Group had a mean IQ score of 117.6 and the Control Group had a mean IQ score of 121.3.

To gather more data on the Experimental and Control Groups, the Mathematics and Reading sections of the Sequential Tests of Educational Progress (STEP form 4A) were administered after the study was completed. On the Mathematics section the mean percentile score was 83.9 for the Experimental Group and 86.2 for the Control Group. On the Reading section, the mean percentile score was 73.4 for the Experimental Group and 68.5 for the Control Group. A summary of the data may be found in Appendix F.

The results of the pretest also gives some information on the relative abilities of the two groups. The test consisted of twenty-five items, six proofs and nineteen prerequisites. There were five items on the distributive law, one item on closure, five items on substitution, five items on divisibility, one item on prime numbers (ten numbers were given and the students were to identify those which were primes) one item on Axiom 1 (the students were asked to give a prime divisor for each of seven numbers), and one on inductive reasoning.

The Experimental Group averaged 29% on the prerequisites and, as expected, 0% on the proofs. The Control Group averaged 30% on the prerequisites and 0% on the proofs. The results indicated that the students in the Experimental Group were fairly competent on Axiom 1, the closure property of whole numbers, and prime numbers. Table 1 summarizes the results of the pretest and indicates how similarly the two groups performed.

TABLE 1

Results of the Pretest

	Experimental Group	Control Group
Distributive law	5/50	1/50
Axiom 1	51/70	46/70
Closure	7/10	9/10
Substitution	25/50	29/50
Divisibility	12/50	13/50
Law of Contradiction	0/10	0/10
Prime numbers	86/100	79/100
Induction	0/10	0/10
Proofs	0/60	0/60

Note. The entry in each cell gives the ratio total number of correct responses to the total number of items.

Materials. The instructional materials for each lesson are listed below. The printed materials may be found with the lesson plans in Appendix C.

(1) The proof of each theorem written on a separate poster. Their function is to aid the teacher while discussing the proofs. They were to be placed where the students could see them much of the time;

(2) illustrated stories which present homework exercises and introduce new ideas;

(3) an Underwood-Olivetti 101 desk computer for verifying Axiom 1 and divisibility criteria;

(4) an overhead projector to aid in playing Prime; and

(5) a variety of practice and exercise sheets.

Other factors. Many other factors may influence what goes on in the classroom; no class exists in isolation. An attempt was made to create a positive school atmosphere for the experiment:

(1) A brief description of the purpose and nature of the experiment was mailed to each parent whose child was involved in the experiment;

(2) The principal and the regular sixth-grade math teacher were thoroughly briefed on the experiment; and

(3) Both the experimenter and the teacher made efforts to communicate with the staff. Frequent visits to the faculty lounge before and after class helped other faculty members and the custodial staff gain a better understanding of the experiment.

While is it not possible to measure the effect of these attempts, no negative responses were received from any of the faculty members.

FEEDBACK (EVALUATION)

Feedback was to come from three basic sources:

(1) The classroom observations of the teacher and investigator The small group of students were to be encouraged to express their opinions at all times so that learning difficulties could be

detected as they arose. While the students worked exercises at their desks, the size of the class would permit the teacher to circulate among the students and observe their performances. The lessons would also be recorded on tape. The lessons could, therefore, be evaluated by listening to the tapes;

(2) Student interview. The investigator was to interview the students in an attempt to determine how well they understood the proofs of the six theorems. The students were to be asked to explain each reason in the proof; and

(3) Mastery tests. The behavioral objectives for each lesson were to be measured by lesson mastery tests. These are included with the lesson plans in Appendix C.

Data from these sources were to aid the teacher and investigator in making decisions. If the (80/80) criterion was not met, or if there was evidence that a lesson had missed the mark, recycling would be necessary. Iteration was to continue until mastery was achieved.

OUTPUT

The investigator was fairly confident that the students would be able to learn most of the proofs as they were presented in each lesson. Of greater importance was whether or not they would be able to keep the various proofs straight when confronted with proving all of them at the same time. Hence the unit had one major behavioral objective:

Given the statements of the six theorems, the student will write a correct proof for each of them.

This objective was to be measured on a pre- and posttest which was written for the experimental aspect of the study.

The desired output also included features which would not be measured in behavioral terms. For example, it was hoped that the students will gain some understanding of mathematical proof. A major premise of this study is that an "understanding" of proof develops gradually as one is exposed to more and more examples of proof. Hence, it is not an all-or-nothing acquisition which is readily measured. The ultimate test for this objective (understanding of proof) will require a longitudinal study. This will be discussed in the concluding chapter of this paper.

RESOURCES

The resources for this study were provided by the Lake Mills Middle School and the Wisconsin Research and Developemtn Center for Cognitive Learning. The school provided the classroom. The Center provided secretarial and duplicating help, the services of the experimental teacher, and the transportation to and from Lake Mills.

TEACHING THE UNIT

A journal account of the third formative pilot study appears in Appendix C. Most of the lessons went as planned. When the teacher thought the students were adequately prepared, a mastery test was administered to measure the objectives of each lesson (Appendix D contains the results of the mastery test). The students failed to reach criterion on two occasions, and further instruction was required to reach criterion (see the journal account for Thursday, April 17 and Wednesday, April 30 in Appendix C).

Iteration (or recycling) was required on one other occasion. This was due to the fact that the need for mathematical proof was not properly motivated (see the journal account for Monday, April 21). As a result of the difficulties which were encountered with this lesson, changes were made in the daily plan of operations. The teacher role-played and practiced teaching each lesson to the investigator before presenting it to the students. In all, this meant that about two hours of preparation were required for each lesson. This procedure was helpful and the remainder of the lessons went much more smoothly.

In addition to the main objectives of the unit, the study examined two other aspects of the students' behaviors. First, the basic pedagogical approach to each of the theorems was to encourage the students to: (1) formulate the statement of each theorem from numerical examples; and (2) construct the proof of the general theorems from proofs for particular numerical examples. Classroom observations indicated that all of the students were able to perform these tasks. No objective data was collected on these tasks.

Secondly, three proofs were added to the original unit in order to determine if the students were able to transfer their knowledge of one proof to the proof of a similar theorem. After Theorem 1 had been presented, the students were given the opportunity to prove Theorems 2 and 3 without further instruction. The journal of Lessons 4 and 5 describes the procedures which were used. Eight students were able to prove Theorem 2 without any hints. The remaining two students wrote correct proofs when the teacher suggested that the proof was similar to the proof of Theorem 1. All ten students were then able to write correct proofs for Theorem 3.

After Theorem 4 was presented, the students were asked to write a proof for Theorem 5. The journal for Lesson 8 details the procedures which were followed. The proof is the same as the proof of Theorem 4, except that Theorem 1 must be applied instead of Theorem 2. Each student applied Theorem 2 instead of Theorem 1. Otherwise, all of the proofs were correct. When the teacher suggested that the students think through each step in the proof, two students made the necessary correction. Six more students made the correction when the teacher reviewed the strategy for Theorem 4. Two students had to be shown their mistakes.

One lesson was omitted from the third formative study. Since the students had been exposed to prime and composite numbers, and in the interest of conserving time, the lesson on prime numbers was not presented. The definitions of prime and composite numbers were given and the game of Prime was played, but the sieve of Eratosthenes was omitted.

The pretest, the instruction, and the posttest required eighteen days to complete. At the conclusion of the study the students were given a certificate (See Appendix G), letters were mailed to each of the students' parents informing them of their child's progress, and a formal report was given to the principal of the school.

Chapter IV

RESULTS

This chapter presents the results which are pertinent to determining the feasibility of the unit. The results are presented in three sections: prerequisites, proofs, and other behaviors. The validity and reliability of the pretest-posttest are discussed after the results for the tests are presented.

PREREQUISITES

The pretest-posttest results for each prerequisites skill are presented in the following pages. Tables are used to summarize the results. In the tables, "Ratio" means the ratio of the total number of correct responses to the total number of items on the test. For example, in Table 2 the ratio 5/50 for the Experimental Group on the pretest means that there were five correct responses out of a total of fifty total responses. The "Percent" column expresses each ratio in terms of the percent of correct responses. For example, 5/50 is 10 percent.

Distributive law. Only one student was able to apply the left-hand distributive law on the pretest. Hence, the Experimental Group responded correctly to five of the fifty items. On the posttest, the Experimental Group responded correctly to all fifty instances of the distributive law. On the pretest the students in the Control Group responded correctly to one item. On the posttest, one student answered all five items correctly, accounting for the only correct responses by the Control Group. Table 2 summarizes the results.

TABLE 2

Distributive Law

	Pretest		Posttest	
	Ratio	Percent	Ratio	Percent
Experimental Group	5/50	10	50/50	100
Control Group	1/50	2	5/50	10

Axiom 1. Axiom 1 states that every whole number greater than one is divisible by some prime number. The students were given seven whole numbers; for each they were to write a prime number which divided it. Approximately one month before the experiment, the students had been exposed to a unit on prime numbers. On the pretest, seventy-three percent of the items were answered correctly by the Experimental Group and sixty-six percent by the Control Group. On the posttest, the Experimental Group responded correctly to ninety-six percent of the items, and the Control Group responded correctly to seventy-three percent of the items. It should be recalled that the lesson on prime numbers was not presented to the Experimental Group during the experiment. The results are summarized in Table 3.

TABLE 3

Axiom 1

	Pretest		Posttest	
	Ratio	Percent	Ratio	Percent
Experimental Group	51/70	73	67/70	96
Control Group	46/70	66	51/70	73

Closure properties. The students in the Experimental Group answered seven of the ten items correctly on the pretest and all of the items correctly on the posttest. The students in the Control Group answered nine correctly on the pretest and seven correctly on the posttest. The results are summarized in Table 4.

TABLE 4

Closure

	Pretest		Posttest	
	Ratio	Percent	Ratio	Percent
Experimental Group	7/10	10	10/10	100
Control Group	9/10	90	7/10	70

Substitution. The students had some prior experience with substitution. On the pretest the students in the Experimental Group responded correctly to twenty-five of the fifty items; the students in the Control Group responded correctly to twenty-nine of the items. On the posttest the students in the Experimental Group responded correctly to forty-eight of the items and the students in the Control Group answered twenty-three correctly. Table 5 summarizes the results.

TABLE 5

Substitution

	Pretest		Posttest	
	Ratio	Percent	Ratio	Percent
Experimental Group	25/50	50	48/50	96
Control Group	29/50	58	23/50	46

Definition of Divides. This was one of the most crucial concepts in the unit. The two groups performed quite similarly on the pretest. The students in the Experimental Group answered twelve of the fifty items correctly and the students in the Control Group answered thirteen correctly. On t posttest the students in the Experimental Group answered forty-nine of the items correctly and the students in the Control Group answered twelve of them correctly. Table 6 summarizes the results and it appears on the following page.

TABLE 6

Definition of Divides

	Pretest		Posttest	
	Ratio	Percent	Ratio	Percent
Experimental Group	12/50	24	49/50	98
Control Group	13/50	26	12/50	24

Law of Contradiction. None of the students in either group were able to answer the item correctly on the pretest. Nine of ten students in the Experimental Group were able to do so on the posttest, but none of the students in the Control Group were able to answer it correctly. Table 7 summarizes the results.

TABLE 7

Law of Contradiction

	Pretest		Posttest	
	Ratio	Percent	Ratio	Percent
Experimental Group	0/10	0	9/10	90
Control Group	0/10	0	0/10	0

Prime numbers. Both groups were fairly proficient at identifying prime numbers. The students in the Experimental Group answered eighty-six of the 100 items correctly and the students in the Control Group answered seventy-nine correctly on the pretest. Although the lesson which had been prepared on prime numbers was not presented to the Experimental Group, these students answered ninty-five of the 100 items correctly on the posttest. The students in the Control Group responded correctly to sixty-nine of the items on the posttest. The results are shown in Table 8.

TABLE 8

Prime Numbers

	Pretest		Posttest	
	Ratio	Percent	Ratio	Percent
Experimental Group	86/100	86	95/100	95
Control Group	79/100	79	69/100	69

Inductive reasoning. None of the students were able to answer the item correctly on the pretest, but nine of the ten students in the Experimental Group answered it correctly on the posttest. Table 9 summarizes the results.

TABLE 9

Inductive Reasoning

	Pretest		Posttest	
	Ratio	Percent	Ratio	Percent
Experimental Group	0/10	0	9/10	90
Control Group	0/10	0	0/10	0

PROOF

As presented in the unit on proof developed in this study, a valid mathematical proof is comprised of a sequence of statements which establish the validity of the result. An acceptable reason must also be given for each statement. Hence, writing a mathematical proof is a complicated task. As a result, to score a proof as either "acceptable" or "not acceptable" can be misleading. For instance, one student may complete all but one step of a proof and another may not be able to write any of the proof. If the proofs were scored on an all-or-nothing basis, both students would receive the same score. In the process, much information is lost concerning the relative abilities of the two students. Therefore, Table 10 summarizes the results of the proofs in two ways. First, the ratios of the total number of correct proofs to the total number of proofs are given. Second, the ratios of the total number of correct steps to the total number of steps are given. For example, the proof of Theorem 1 is comprised of four statements and four reasons; hence, there are eight steps involved and the proof is scored on a basis of eight points.

None of the students were able to prove any of the theorems on the pretest. Three students in each group did make an attempt at proving the theorems. For Theorem 1, one student wrote, "It's true because you are adding them". Another student wrote, "It's true because you are distributing the N to a B and an A". A third student gave a numerical example as proof. The students in the

Control Group were also unable to prove any of the theorems on the posttest.

TABLE 10

Proofs

	Pretest		Posttest	
	Ratio: correct proofs	Ratio: correct steps	Ratio: correct proofs	Ratio: correct steps
Experimental Group	0/60	0/500	58/60	498/500
Control Group	0/60	0/500	0/60	0/500

The results reveal almost total mastery of the proofs by the students in the Experimental Group. Only one student made an error: she missed the proofs of Theorems 4 and 5 because she cited the wrong theorems as reasons. When asked to explain the proofs, she explained that she had confused the numbers of the theorems.

SUMMARY OF PRETEST-POSTTEST SCORES

Table 11 summarizes the results of the pretest-posttest. On the pretest, the students in the Experimental Group scored twenty-nine percent on the prerequisites and zero percent on the proofs. The students in the Control Group scored thirty percent on the prerequisites and zero percent on the proofs. Hence, the performance of the two groups was quite similar. On the posttest, the students in the Experimental Group scored ninety-six percent on the prerequisites and ninety-seven percent on the proofs. The students

TABLE 11

Summary of Pretest-posttest Results

	Pretest		Posttest	
	Prerequisites	Proofs	Prerequisites	Proofs
Experimental Group				
Mean number of correct responses	5.67	0	18.0	5.8
Percent of correct responses	29	0	96	97
Variance	2.67	0	.97	.40
Control Group				
Mean number of correct responses	5.75	0	6.00	0
Percent of correct responses	30	0	32	0
Variance	1.72	0	3.39	0

in the Control Group performed about the same as on the pretest. Their mean number of correct responses increased from 5.75 to 6.00, or from 30 to 32 percent. This increase was due to the performance of one student whose score increased from 6 to 11 (she had learned the distributive law).

The variances on the pretest were low because the scores were low. As would be expected under the conditions of mastery learning, the variance on the posttest for the Experimental Group decreased from 2.67 to .40. Table 11 appears on the preceding page.

ANALYSIS OF VARIANCE

The large gains made by the Experimental Group are clearly significant. Nonetheless, the data were subjected to an analysis of variance to verify what appears to be obvious (Finn, 1967). Since the students in the Control Group were unable to prove any of the theorems, the variance for that portion of the test is zero. Hence, the data on proofs were not compared. The data on prerequisites were subjected to several tests. There were no significant differences between the Experimental and Control Groups when compared on the pretest grand means for prerequisites (p less than 0.8374). Similarly, there were no significant differences between the two groups when compared on Henmon-Nelson IQ scores and Sequential Tests of Educational Progress (STEP) scores for mathematical and reading achievement (p less than 0.3222). These results indicate that the two groups of students were fairly similar in nature.

The two groups were also compared on the posttest grand means for prerequisites. An F ratio of 139.4438 was highly significant (p less than 0.0001). To take into account differences which existed between the two groups, two analysis of covariance tests were performed. With pretest scores eliminated, the F ratio increases to 177.8154. With IQ scores, math achievement scores, and reading achievement scores eliminated, the F ratio is 164.6214. In each case the p value is less than 0.0001. This is as expected, for the Control Group had higher grand means on the IQ, mathematics achievement, and pretest scores than the Experimental Group. Thus, when these measures are taken into consideration, the results become more significant.

Therefore, the analysis of variance confirms the fact that the results are highly significant

THE RELIABILITY AND VALIDITY OF THE PRETEST-POSTTEST

Most standardized tests can be classified as "norm-referenced" tests. A norm-referenced test is used to ascertain an individual's performance in relationship to the performances of other individuals on the same test. Such tests are used to make decisions about individuals.

The pretest-posttest used in this study is a "criterion-referenced" test. A criterion-referenced test is used to ascertain an individual's performance with respect to some criterion. Thus, the individual's performance is compared with some established criterion rather than with the performances of other individuals. Such test are useful in making decisions about instructional programs.

Popham and Husek (1969) have recently discussed the difficulties involved in applying the classical concepts of reliability and validity to criterion-referenced tests. Since the meaningfulness of a norm-referenced test score is basically dependent on the relative position of the score in comparison with other scores on the same test, test reliability is based on the variability of test scores. The greater the variability, the easier it is to rank individuals. On the other hand, variability is irrelevant with criterion-referenced tests. The meaningfulness of a criterion-referenced test score is not dependent on a comparison with other scores; it is dependent on a comparison with an established criterion. Popham and Husek (1969, p. 5) make the following comments concerning test reliability:

> ...although it may be obvious that a criterion-referenced test should be internally consistent, it is not obvious how to assess the internal consistency. The classical procedures are not appropriate. This is true because they are dependent on score variability. A criterion-referenced test should not be faulted if, when administered after instruction, everyone obtained a perfect score. Yet, that would lead to a zero internal consistency estimate, something measurement books don't recommend.

In order to examine the classical concept of reliability as it applies to a criterion-referenced test, Hoyt reliability coefficients were computed (Baker, 1966). These are presented in Table 12. Since all of the students in the Experimental Group had high posttest scores, and all of the students in the Control Group had low posttest scores, one would expect high measures of internal consistency if the scores for both groups are considered together. This did occur: the reliability for the test on prerequisites was 0.9599 and the reliability for the test on proofs was 0.9912. However, when the test scores for the two groups are considered separately, lower reliability coefficients are obtained. Of particular interest are the coefficients for the Experimental Group. On prerequisites, the pretest reliability was 0.7111 and the posttest reliability was 0.3589. On proofs, the pretest reliability was 0.0000 (there were no correct responses) and the posttest reliability was 0.6000.

These results confirm the viewpoint expressed by Popham and Husek; that is, a criterion-referenced test administered after an instructional treatment will have a low measure of internal consistency.

Popham and Husek (1969, p. 6) also point out that the validity of a criterion-referenced test should be based on the content of the items on the test:

> Criterion-referenced measures are validated primarily in terms of the adequacy with which they represent the criterion. Therefore, content validity approaches are more suited to such tests. A carefully made judgement, based on the test's apparent relevance to the behaviors legitimately inferable from those delimited by the criterion, is the general procedure for validating criterion-referenced measures.

The pretest-posttest used in this study consisted of twenty-five items, six proofs and nineteen prerequisites. The items are based upon the objectives of the instructional unit. In fact, instruction aimed at teaching the behaviors included on the test. Hence, the test has content validity.

TABLE 12

Hoyt Reliability Coefficients

	Pretest	Posttest
Experimental Prerequisites	0.7111	0.3589
Experimental Proofs	0.0000	0.6000
Control Prerequisites	-0.0879	0.7708
Control Proofs	0.0000	0.0000
Combined Prerequisites	0.5590	0.9599
Combined Proofs	0.0000	0.9912

Note: "Combined" means considering the Experimental Group and the Control Group together.

OTHER BEHAVIORS

Data were collected by several other means to help assess the feasibility of the unit of instruction.

The meanings of the theorems. In an attempt to determine if the students understood the meanings of the theorems, the students were asked to perform two tasks: (1) to give numerical examples to illustrate the meanings of the theorems; and (2) to apply the theorems to given divisibility facts. These items were included on the mastery tests for each of the first five theorems. These tasks were performed without error.

Understanding the proofs. Two measures were used to determine how well the students understood the proofs of the theorems:

(1) a knowledge of prerequisites; and (2) the ability to explain and defend the proofs in an interview situation. The results of the prerequisites were reported earlier in the chapter. The interviews were more revealing. Nine of ten students were able to satisfactorily explain and defend their proofs of Theorems 1, 2, and 3, seven were able to do so for Theorems 4 and 5, and eight were able to do so for Theorem 6.

CONCLUDING REMARKS

This chapter has reported those results which are directly related to determining the feasibility of the unit of instruction on proof. The development and testing of the unit required nine sequential steps, and many of the things which happened along the way are not reported in this chapter. In addition to interpreting the results reported in this chapter, many of these other results will be discussed in Chapter V.

Chapter V

SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS FOR FURTHER STUDY

SUMMARY

Deduction is at the very heart of the modern conception of mathematics, yet existing elementary school mathematics programs do not include a discussion of proof and deduction. Hoping to remedy this situation, The Cambridge Conference of School Mathematics recommended in 1963 that proof be presented in the elementary school curriculum. The findings of many psychologists suggest that sixth-grade students possess the cognitive structures necessary for a study of proof. These considerations suggest that the recommendations of the Cambridge Conderence be tested.

This study was designed to gather information concerning the ability of sixth-grade students to learn mathematical proofs. Its purpose was twofold: (1) to demonstrate that the curriculum development model advocated by Romberg and DeVault can be successfully applied to develop a unit on proof for use with average and above average sixth-grade students; and (2) to use this unit to test the feasibility of presenting selected proof material to sixth-grade students.

The study carried the development of the unit through the first two phases of the model. Nine sequential steps, including three pilot studies, were involved in developing the unit. The final unit contained six theorems.

An experiment was conducted in conjunction with the third pilot study. Ten average and above average students were selected from an intact classroom and a Control Group was selected by matching procedures. A Nonequivalent Control Group Design was used. Every student in the Experimental Group mastered all of the behavioral objectives of the unit. The results of the pretest-posttest are summarized in Table 13.

TABLE 13

Summary of Pretest - Posttest Results

	Pretest		Posttest	
	Prerequisites	Proofs	Prerequisites	Proofs
Experimental	29	0	96	97
Control	30	0	32	0

Note. - The entries in each give the percent of correct responses.

An analysis of variance was performed on the posttest scores on prerequisites and the results were highly significant. When pretest scores, Henmon-Nelson I. Q. scores, and Sequential Test of Educational Progress (STEP) scores on mathematical and reading achievement are used as covariates, the significance is increased.

LIMITATIONS OF THE STUDY

The results of the experiment must be approached with extreme caution. The failure to obtain a random sample, the rural characterestics of the community, the ideal conditions under which the unit was presented, and the existence of a Hawthorne effect combine to produce a highly unique set of circumstances. Hence, the major limitation of this study is its lack of generalizability.

The formative development of the unit is in the initial pilot phase. Further pilot examinations and validation are needed before the results can be safely generalized. The conclusions which follow should be interpreted with this limitation in mind.

CONCLUSIONS

This study had two major purposes. Correspondingly, the conclusions are reported in two sections, those related to the development of the unit, and those related to the experiment.

Conclusions Related to the Experiment

(1) The main purpose of the study was to test the feasibility of presenting proof materials to sixth-grade students. Three basic questions were asked:

(a) Can the students demonstrate an understanding of the theorems? The results as reported in Chapter IV indicate that the students understood the meanings of the six theorems.

(b) Can the students reproduce the proofs of the theorems? The results of the mastery tests and the posttest show that the students learned the proofs of the six theorems.

(c) Can the students demonstrate an understanding of the proofs of the theorems? The results showed that nine students understood the proofs of theorems 1, 2, and 3, that seven students understood the proofs of theorems 4 and 5, and that eight students understood the proof of theorem 6.

Therefore, the conclusion is that the feasibility of the unit is established for a particular group of average and above average students under somewhat ideal conditions.

(2) The results show that the Experimental and the Control Groups performed quite similarly on the pretest. Neither group was able to prove the theorems, and there was only 1% difference between their performances on the prerequisites. The difference on the pretest scores were non-significant. On the posttest, the Control Group showed very little gain: they were still unable to prove the theorems and there was a 2% increase in their performance on prerequisites. The Experimental Group answered 96% of the prerequisite items correctly and proved 97% of the theorems. Hence, the Experimental Group made gains which were both practi- and statistically significant.

The Control Group performed as expected. The increase in the variance from the pretest to the posttest for the Control Group can be readily explained. Several topics in the unit had been presented to all the sixth graders in the school earlier in the year: the distributive law, prime and composite numbers, substitution and evaluation. While the experiment was being conducted the Control Group attended their regular math class where they did not review these concepts. They studied operations with rational numbers. Hence, the passage of time would tend to increase the variance of a test on these concepts.

As expected, the variance for the Experimental Group decreased as a result of instruction. Teaching for mastery tends to decrease the variability of performances. Almost total mastery was achieved on both the proofs and the prerequisites.

Therefore, the second conclusion concerns the pretest-posttest gains made by the Experimental Group: what accounts for this gain? The very nature of the study would seem to rule out most sources of internal invalidity, for instruction aimed at teaching and measuring specific behaviors, and recycling was repeated until the students had mastered those behaviors.

Nevertheless, a Non-equivalent Control Group Design was used to rule out as many competing hypotheses as was possible. Campbell and Stanley (1968, p. 13) state that this design controls for the main effects of history, maturation, testing, instrumentation, selection, and mortality. Since matching procedures were used in selecting the Control Group, and since the pretest means of the two groups were nearly identical, main effects due to regression can be excluded as a source of internal invalidity.

Another competing hypothesis is that the learning resulted from activities which took place outside of the classroom. For instance, perhaps the actual classroom instruction was ineffective and the students learned the material by studying at home. This was not the case with at least four of the six theorems, for the students actually learned four of the proofs during the periods in which the proofs were presented.

Therefore, the conclusion is that the pretest-posttest gain was due to one of two factors: (a) the instructional treatment; or (b) a pretest-treatment interaction. The investigator has no way of determining between the two. However, given the size of the gain by the Experimental Group, the investigator is certain that a large proportion of the gain was due to instruction.

(3) The third conclusion is that mastery learning was a successful operational procedure. Each student attained mastery of every behavior skill. The small class size enabled the teacher to constantly monitor the progress of each student, and individual help was given when necessary. When the teacher was confident that the students had learned the behaviors of a lesson, a mastery test was administered and mastery was usually attained.

More importantly, the students reacted very positively to the idea of mastery learning. This was particularly true of three of the less gifted students who normally receive grades of B or C in mathematics. These students, realizing that it was possible to receive an A for the unit, were usually the first students to learn a proof.

(4) The illustrated stories served three functions:

(a) From the students' point of view they provided an interesting context for drill and practice exercises;

(b) From the teacher's point of view they provided an effective way for presenting new topics; and

(c) They were enthusiastically received by the students and contributed to a good classroom morale.

Therefore, the conclusion is that the illustrated stories made a valuable contribution to the overall success of the instructional unit.

(5) The students enjoyed the unit. However, a strong Hawthorne effect was very evident. A Hawthorne effect occurs when the students in an experiment are aware that they are participating in an experiment. As a result, the students might try harder than usual, thus increasing posttest results.

(6) The pedagogical procedure of proceeding from numerical examples to generalizations was successful in involving the students in the development of the theorems.

(7) The students were unable to prove Theorem 5 without hints. The conclusion is that the students were unable to transfer the strategy of Theorem 4 to the proof of Theorem 5.

(8) A new task analysis model was created and used to analyze

the mathematical content of the unit. It extends Gagne's two-dimensional model to three dimensions in order to include strategies (or plans) in the analysis. The model was extremely helpful in planning instructional procedures: it reminded the teacher that the plan for each proof had to be stressed. As a result, the reasoning behind each proof was clearly and repeatedly stated.

The value of any model lies in its utility. The conclusion is that the new model is an appropriate model for higher-order cognitive tasks.

(9) The final conclusion concerns the teacher: her training in mathematical proof was not adequate enough to permit her to comfortably teach the unit. In spite of the fact that the investigator considered the experimental teacher to be a very competent elementary school teacher, in spite of the fact that the teacher had taken more than two years of collegiate mathematics, and in spite of the fact that she had to spend a considerable amount of time with the investigator preparing for each lesson, she encountered difficulties in teaching the unit. These difficulties can probably be attributed to the fact that her training did not include an exposure to proof and the deductive nature of mathematics. In fact, this unit was her first real encounter with substantial proof materials.

Conclusions Related to the Development of the Unit

The other purpose of this study was to demonstrate that the curriculum development model advocated by Romberg and DeVault could be successfully applied to develop a unit on proof for use in the elementary school. Since the feasibility of the unit has been established, the effectiviness of the model has been demonstrated. Several observations are worthy of note.

The theoretical framework of the developmental procedures used in this study combine the ideas of two psychologists whose theories of learning are almost diametrically opposed: Bruner, who is a cognitive psychologist, and Gagne, who is associated with stimulus-response theory.

The mathematical phase of development involved Gagne's concept of task analysis. The iterative nature of the model, however, was guided by the spirit of Bruner's hypothesis: any subject can be taught effectively in some intellectually honest form to any child at any stage of development. The approach consisted of task analyzing the desired behaviors (Gagne), trying out the materials with students, and, if the materials were not appropriate, re-analyzing the tasks into simpler cognitive elements (Bruner).

At face value, Bruner's hypothesis is patently false. Yet the hypothesis may have functional validity as a principle of curriculum development. The modifications made in the proof of Theorem 6 illustrates its usefulness. The original theorem stated:

> Given any set of prime numbers $\{P_1, P_2, P_3, \ldots, P_n\}$, there is always another prime number.

First, the subscripts were eliminated to simplify the statement of the theorem:

Given any set of prime numbers $\{2, 3, 5, 7, \ldots, P\}$, there is always another prime number.

This simplication made the theorem mathematically less general: the P_i, $1 \leqslant i \leqslant n$, can represent any prime numbers in any order, but the primes in the second set are fixed. Nonetheless, the change made the theorem more accessible to the minds of the sixth graders. In addition, this made it possible to give simple numerical examples to illustrate the technique used in proving the theorem; that is, $2 + 1 = 3$, $(2 \times 3) + 1 = 7$, $(2 \times 3 \times 5) + 1 = 31$, etc.

With this change, the classical proof of the theorem was then presented. The students had great difficulty in applying the Fundamental Theorem of Arithmetic. A change was therefore made in the proof. Since the Fundamental Theorem had been accepted as a postulate, it was replaced with another assumed statement which was not an "or" statement. As a result of these changes, the statement of the theorem and the proof were made accessible to the students. In short, (as predicted by Bruner's hypothesis) there was a form of the proof which was appropriate for sixth-grade students. This example illustrates the basic philosophical approach to curriculum development employed in this study: the mathematics should be made to conform to the cognitive abilities of the students, and not vice versa.

Starting with an original unit which proved to be relatively ineffective, repeated revisions were made in both the mathematical and instructional components of the unit until a highly effective unit was produced. Hence, the conclusion is that the procedures outlined by Romberg and DeVault were successfully employed to develop a unit on proof which was effective with a particular group of average and above average sixth-grade students.

RECOMMENDATIONS FOR FURTHER STUDY

The study carried the development of a unit on proof into the Pilot Examination phase of the developmental model of Romberg and DeVault. The Analysis and Pilot Examination phases of curriculum development tend to be more exploratory than experimental in nature, the primary concern of the curriculum developer being to find materials and procedures which work in the classroom. The findings and conclusions made during these phases of development must be subjected to further examination. The results of the study suggest the following recommendations for further study:

(1) The formative development of the materials in the unit should be continued in an attempt to answer two questions:

(a) Will the unit be effective with other groups of average and above average students?

(b) Will other elementary school teachers encounter difficulties in learning and teaching the unit?

The teachers who are to teach the unit should attend a pre-service workshop in order to learn the content and intended pedagogy of the unit. Instructional materials would have to be written for this purpose. If the pilot studies indicate that the teachers are having difficulties teaching the unit, the materials might then be tried out at the junior high school level.

(2) The success of this study suggests that it might be profitable to continue experimentation with other proof materials at other grade levels. The present study used number theory as the vehicle for presenting proof. Other areas of mathematics might offer excellent opportunities for proof-making activities.

From the present vantage point, an integrated program of proof for grades 4 - 12 is a distinct possibility for the future. If proof materials can be developed which are appropriate for grades 4 - 9, studies could be conducted to determine the effect of such materials on the student's ability to prove results at the high school level. Sound decisions could then be made as to the feasibility of including proof materials in the elementary school mathematics curriculum.

(3) Other types of comparative studies can also be conducted. As discussed in Chapter II, there are two basic approaches to the teaching of proof in mathematics. The first, advocated by Suppes, begins with a study of logic. Once the principles of logic are learned, a study of mathematical proof is undertaken. The second approach is the one adopted for this study. Proof is taught using the actual content of the mathematics curriculum. If a comprehensive program on proof can be developed for grades 4 - 9 using both approaches, a comparative study could be conducted to determine which approach is more efficient.

(4) Bloom's concept of mastery learning was very effective. This suggests that an instructional system based upon mastery might in general be more effective in producing learning than a system which is based upon assessment of native ability. Therefore, mastery learning should be examined at each grade level in all subject matter areas.

(5) The model which was followed in developing the unit on proof needs to be applied in other subject matter areas.

CONCLUDING REMARKS

This study was undertaken in response to the recommendations of the Cambridge Conference on School Mathematics. It has but scratched the surface of a larger problem, that of determining the extent to which proof materials can be incorporated into the school mathematics curriculum.

The study was conducted under highly favorable circumstances and the results must be interpreted with this in mind. The findings have been interpreted as being partly positive and partly negative: the preliminary indications are that average and above average sixth-grade students (with IQ scores ranging from 107 to 135) are capable of learning to prove mathematical theorems, but that the task of teaching proof is, at present, well beyond the training of the typical elementary school teacher. Further study is needed to determine the accuracy of these indications.

APPENDIX A

JOURNAL FOR FIRST FORMATIVE PILOT STUDY

JOURNAL OF FIRST FORMATIVE STUDY

FIRST DAY. A pretest was administered to assess the students' knowledge of prerequisites needed in proving the theorems. The test consisted of twenty items on prerequisite behaviors. As anticipated, the results indicated that a lesson would have to be devoted to teaching prerequisites.

The remainder of this first meeting was devoted to a discussion of the meaning of the word "prove". An example of proof by authority and proof by empirical evidence was given. A list of ten statements was distributed to the students and they were asked to tell how they would prove each of them to a doubtful friend. Many answers were unrealistic. For example, taking the friend on a space ship to prove to him that the earth is spherical in shape. One of the items was on the closure of even integers under addition, and each student cited a numerical example as proof. Optical illusions and poor use of authority were also discussed.

SECOND DAY. The period was devoted to motivating the need for proof. The activity with the partitioning of circles into regions was very successful. Everyone expected the circle with six points on the circumference to yield 32 regions, and they were surprised to find only 30 points.

The activity using the expression $N^2 - N + 11$ was then conducted. Each student was asked to evaluate $N^2 - N + 11$ for each number from 0 to 10. Only two mistakes were made. The students guessed that $N = 11$ would also yield a prime, and the hypothesis was torpedoed.

These activities were followed by a discussion of inductive reasoning and it was stressed that the conclusions of such reasoning are only probable. The need for proof was stressed.

THIRD DAY. The lesson preprequisites was started. The distributive law was explained, a practice sheet was completed, and the students were shown how to employ the distributive law to perform rapid calculations. For example, 7 x 97 + 7 x 3 = 7 (97 + 3) = 7 (100) = 700, all of which can be done in one's head. The students were given a list of problems to evaluate in this way. Both forms of the distributive law were presented (right-hand and left-hand).

The remainder of the period was devoted to prime numbers. The students employed the sieve of Erathosthenes to find all the prime numbers less than 200. Three students made mistakes and had to begin all over again. The students were asked to learn all of the primes less than 50.

FOURTH DAY. The definition of divides was given and the students were given drill sheets with problems which required them to apply the definition in both ways: (1) given an equation, state a divisibility fact; and (2) given a divisibility fact, state an equation. Criteria for divisibility by 3 and by 9 were then discussed.

The Fundamental Theorem of Arithmetic was presented and the students were given practice at factoring whole numbers into prime factors.

The students were then shown how to solve equations using addition and subtraction. The example of a balance scale was used to illustrate both principles, and the students were given ten equations to solve.

FIFTH DAY. The proof of Theorem 1 was presented. Numerical examples were written on the board and the students generalized the statement, If $3|A$ and $3|B$, then $3|(A+B)$. A proof was written for it, it was left on the board, and the students were asked to write a proof for the

following: If $2|A$ and $2|B$, then $2|(A+B)$. By looking at the board, five of six students wrote a correct proof. A variety of different examples was then given and Theorem 1 was discovered by the students, i.e., If $N|A$ and $N|B$, then $N|(A+B)$.

Two students were able to write a proof for the theorem. The investigator then explained the proof in detail, stressing the strategy which was employed. A quiz was given and three students made errors. The students were asked to memorize the proof.

SIXTH DAY. The proof of Theorem 1 was reviewed, and a quiz was administered. All six students were able to prove the result. Numerical examples were given for Theorem 2, the students immediately formulated the theorem, and they were asked to prove it. Three persons wrote correct proofs. The others made the following kind of error:

$$A + B = NC - ND = N (C-D).$$

That is, a plus sign was inserted instead of a minus sign. The Theorem and its proof were explained in detail. The similarities and differences between it and the proof of Theorem 1 were pointed out. The students were given exercises in which they had to determine whether or not a given expression was divisible by a given number. For example, Is $(3 \times 4 \times 5) + (3 \times 7 \times 10)$ divisible by 3? Theorem 1 tells us that it is. The exercises included application of both Theorem 1 and Theorem 2. A test was then administered on the proofs of these theorems. Everyone wrote a perfect paper.

SEVENTH DAY. Three numerical examples of Theorem 3 were written on the board, the theorem was immediately stated, and the students were asked to write a proof for it. Five of the six students wrote a correct proof.

The students were then asked to give one numerical example for each of the first three theorems. They were all able to do so.

A review was conducted on solving equations. The law of the Excluded Middle was then discussed: If a statement is true, then its negation is false. Several examples were given and the students were asked to supply one example each (the law was not called by name). A list of statements containing examples of the Law of the Excluded Middle were distributed and the students were asked to identify the statements which were always true.

EIGHTH DAY. Indirect proof was discussed and then numerical examples were given for Theorem 4. The general theorem was then stated, and a proof was written on the board. There was a great deal of confusion. The students did not follow the proof. Because there was only two days remaining, it was decided to skip this theorem and present Theorem 6 on the following day.

NINTH DAY. The day was spent on Theorem 6. The expression (2 x 3 x ... x P) + 1 was evaluated for P = 2, 3, 5, and 7. The proof of Theorem 6 was presented and discussed. The students were told that they would be tested on the theorems on the following day. A review was made of Theorems 1, 2, 3 and 6.

TENTH DAY. A test containing prerequisites and theorems was administered on the final day of the study. The prerequisites were mastered, as were the proofs of Theorems 1, 2 and 3. No one could prove Theorem 4. Each student was able to form the correct expression for Theorem 6, concluded that none of the prime numbers in the set divided it, but they were unable to apply the Fundamental Theorem of Arithmetic.

APPENDIX B

JOURNAL FOR SECOND FORMATIVE PILOT STUDY

JOURNAL OF SECOND FORMATIVE STUDY

FIRST DAY: A pretest was administered. The results showed that the students were proficient at substitution, but weak in the other prerequisites. The distributive law and the definition of divides were presented.

SECOND DAY: The discussion of divisibility was continued with criteria for divisibility by 0, 2, 5, 3, and 9. The students were asked to determine if given numbers were divisible by particular numbers. Substitution and evaluation were presented. The students worked on drill sheets containing the prerequisites for the first theorem. A quiz was given and six of the seven students displayed a mastery of the skills.

THIRD DAY: The need for proof in mathematics was motivated by torpedoing inductive hypotheses. Both activities (the circles and $N^2 - N + 11$) were successful. Inductive reasoning was discussed. The illustrated stories were introduced.

FOURTH DAY: Beginning with numerical examples, the proof of theorem 1 was developed. The students were able to generalize from numerical examples. Each student learned the proof. The stories were enthusiastically received.

FIFTH DAY: Theorem 1 was reviewed and the students were asked to prove Theorem 2. Two students were able to do so. The first week's work was reviewed.

SIXTH DAY: The proofs Theorems 1 and 2 were reviewed and the students were asked to prove Theorem 3. Five students wrote correct proofs. Divisibility criteria were reviewed. The students requested additional stories.

SEVENTH DAY: The students were permitted to experiment with a desk computer. Each student verified criteria for divisibility by 3 and by 9 with large numbers. The first episode of a four part story was handed out.

EIGHTH DAY: The different meanings of "proof" were discussed. Optical illusions and poor choices of authorities illustrate the shortcomings of empirical and authoritative evidence. The nature of mathematical proof was discussed. The second episode of the story was distributed.

NINTH DAY: A proof for the criteria for divisibility by 3 was developed for two digit members. The students were then asked to prove the result for divisibility by 9. Only two students were able to do so. Many numerical examples were given and the students practiced writing the proofs. The third part of the story was handed out.

TENTH DAY: A review was made of the first five theorems, and then the students were asked to prove all five theorems. Four of the students got parts of the proofs confused. The most frequent error was giving incorrect reasons: the reasons for one proof would appear as reasons in another. The proofs of the divisibility criteria were much more difficult for the students to learn. The final episode in the story was distributed.

ELEVENTH DAY: The prerequisites for Theorems 4 and 5 were discussed. Drill sheets on contradictions and the law of the Excluded Middle were used, and examples of indirect reasoning in everyday situations were presented. The students requested more stories.

TWELFTH DAY: Theorem 4 was developed. The students expressed confusion. No one was able to learn the proof.

THIRTEENTH DAY: The students used the sieve of Eratosthenes to find all prime numbers less than 100.

FOURTEENTH DAY: The game of PRIME was played and the proofs of the theorems were reviewed.

FIFTEENTH DAY: The proof of theorem 6 was presented. Given the sets $\{2, 3\}$, Euclid's method was employed to establish the existence of another prime. The procedure was repeated with $\{2, 3, 5\}$, $\{2, 3, 4, 7\}$, $\{2, 3, 5, 7, 11\}$, and $\{2, 3, 4, 7, 11, 13\}$. The students seemed to follow the presentation.

SIXTEENTH DAY: The unit was reviewed and the nature of proof discussed. A final story was distributed.

SEVENTEENTH DAY: The students were tested on the theorems. Everyone could prove the first three theorems, five persons could prove Theorem 6, three persons proved the divisibility criteria, and no one could prove Theorem 4.

APPENDIX C

LESSON PLANS AND JOURNAL
FOR THIRD FORMATIVE STUDY

LESSON ONE

PREREQUISITES

The proofs of the first three theorems in the unit require three basic concepts: the distributive law, substitution of a known quantity into a given expression, and the definition of divides. A criterion of (80/80) must be reached before proceeding to Lesson Two.

A. DISTRIBUTIVE LAW

BEHAVIORAL OBJECTIVE:

1. When given a list of equations, the student can identify those which are examples of the distributive law.
2. The student can apply the distributive law to expressions of the form NP + NQ.

MATERIALS:

1. Introduction and first lesson of story
2. Three drill sheets
3. Mastery Test
4. Introduction, First Lesson, and parts one and two of "Long Live the King"

PROCEDURE: Begin by writing a numerical example on the board. Ask how many of them have heard of the distributive property. Pointing to the numerical example, explain the distributive law: "If the sum of two products have a common factor, then this common factor can be pulled out and multiplied by the sum of the remaining factors." Then say, "Let's see if this law really holds for this example." Write,

$$(3 \times 4) + (3 \times 5) \stackrel{?}{=} 3(4 + 5)$$

$$12 + 15 \stackrel{?}{=} 3(9)$$

$$27 \stackrel{?}{=} 27$$

Yes

Then erase the question marks.

Stress that the operations inside parentheses are always done first. Repeat with another numerical example. State that the distributive law holds for all numbers.

Then write an algebraic example on the board, such as NP + NQ = . "If N, P, and Q represent numbers, then NP means N times P; with letters, we can leave out the multiplication sign. Hence AB means 'A times B' and 5K means '5 times K'. This will not work, of course, with two numerals. 2 3 means 'twenty-three,' but 2 x 3 means 'six.'"

Ask for a volunteer to apply the distributive law to NP + NQ. If a correct response is given, explain how it fits the general form. If not, write the correct response, N(P + Q), on the board, explain, and give another example: 5A + 5B = ? Repeat giving examples and explaining until a student provides the correct response. Then pass out drill sheet #1. Do the first one together, then ask the students to complete the sheet. Correct and discuss in class. Students can correct their own papers.

To illustrate a computational use of the distributive law, ask the students to evaluate 7 (91) + 7 (9). (A fast way to do this is to apply the distributive law: 7 (91) + 7 (9) = 7 (91 + 9) = 7 (100) = 700). "Can anyone do it in his head?" If so, ask them to explain. If no response, say "Perhaps you could use the distributive law." If still no response, illustrate with another example.

$$8\ (994) + 8\ (6) = 8000$$

"Using the distributive law, I can do this in my head. Can you?" If no one can do so, explain that $8(999) + 8(1) = 8(999 + 1) = 8(1000) = 8000$, all of which can be done in one's head.

Distribute drill sheet #2. Do the first one together, then ask the students to write the remaining answers as rapidly as possible. Correct in class and discuss.

Hand out drill sheet #3. The students are to identify instances of the distributive law. When completed, discuss. Items 4 and 5 are true statements, but they are instances of the commutative property.

Distribute introduction and the first lesson episodes of the Emirp stories. Instruct the students to answer all questions in the story as they read it. When they have finished the stories, check the answers in class.

Then administer the mastery test.

B. DIVISIBILITY

BEHAVIORAL OBJECTIVES:

1. Given a divisibility fact of the form $N \mid A$, the student can apply the definition of divides and write an equation of the form

$$A = NP$$

2. Given an equation of the form $A = BC$, the student can apply the definition of divides and write the divisibility facts

$$B \mid A \text{ and } C \mid A$$

MATERIALS:

1. Two drill sheets
2. Mastery test

PROCEDURE: Begin the discussion by writing a numerical example such as 30 = 3 x 10 on the board. Explain that since 30 can be expressed as a product of the numbers 3 and 10, 3 and 10 are called factors of 30. "This equation expresses a multiplication fact...but it also gives us two division facts. We say that 3 and 10 divide 30." Write

30 = 3 x 10

3 is a factor of 30

10 is a factor of 30

also 3 divides 30 and 10 divides 30.

Explain that to say that "3 is a factor of 30" means the same thing as "3 divides 30." Then give another example.

45 = 9 x 5

Point out that 9 and 5 are factors of 45 and that 9 divides 45 and 5 divides 45. "In mathematics we often use symbols in place of words to make things easier to write. For example, "one plus two equals three" can be written as 1 + 2 = 3. To save chalk and time, we shall use a symbol for the word "divides." It will be a vertical line. This is not a fraction." Place a third example on the board.

100 = 2 x 50

"Can anyone give me two division facts expressed by this equation?" Discuss. Answer questions. Point out that for each factor there is a division fact. Then give the general definition: "If A, B, and C represent any 3 numbers, and if A = BC, we say that B is a factor of A, that C is a factor of A, and that $B|A$ and $C|A$."

Give several more numerical examples, asking students to supply the division statements and to identify the factors.

Leave a numerical example on the board, e.g., $24 = 4 \times 6$. $4|24$ and $6|24$. Then write an algebraic expression on the board underneath it.

$$A = TX$$

"If these letters represent numbers, can anyone give me two division facts expressed by this equation?" They should reply, "$T|A$ and $X|A$."

After questions have been answered, hand out a drill sheet #1 and correct them in class. Circulate while students are working on them, try to correct errors.

The reverse procedure should then be presented. Instead of starting with an equation ($A = BC$) and deriving division facts ($B|A$ and $C|A$), start with a division fact (such as $T|V$) and derive an equation ($V = TQ$).

"If we are told that $3|30$, then we know that there is some number, call it N, such that $30 = 3N$. We know, however, that $N = 10$ in this case. I'm going to write some division facts on the board. For each one you must write an equation; for example, $2|24$ means $24 = 2 \times 12$."

$$4|12$$

$$6|30$$

Ask for volunteers and discuss.

"If J and K are two numbers, and if we are told that $J|K$, then we know that there is some number, call it D, such that $K = JD$." Point out that we could have used any letter we wished for D. Encourage them to use different letters, so long as they have not been previously used in the problem.

"Who can come to the board and write an equation for this fact: $T|W$?"

If no one volunteers, write $W = TK$ and try another example. Point out that W must be the larger number since T divides it, so write W =   .

Then give examples involving the sum and difference of two numbers. Write, $6|(4 + 8)$ means $(4 + 8) = 6 \times 2$. "Remember, 4 + 8 represents <u>one</u> number, and not two. Who can write an equation for this division fact? $5|(7 + 8)$. The correct answer is $(7 + 8) = 5 \times 3$." Discuss, then proceed to algebraic examples. "Who can write an equation for $W|(A + B)$?" Explain that $(A + B)$ represent <u>one</u> number not two. The correct answer is $(A + B) = WT$, where T is arbitrary.

Then pass out a drill sheet (#2), work it in class, and correct it together. Finally, administer a mastery test.

C. <u>SUBSTITUTION</u>

Substitution is used in proving several of the theorems in the unit. It is also used when giving numerical instances of a given theorem.

<u>BEHAVIORAL OBJECTIVES</u>:

1. Given an equality such as $A = BK$ and an expression such as $A + 78$, the student can apply the substitution principle: $A + 78 = BK + 78$.

<u>MATERIALS:</u>

1. Drill sheet
2. Mastery test
3. Lesson mastery test

<u>PROCEDURE</u>: This skill is a relatively easy one to master. Begin by writing this example on the board:

If $M = BK$ and $N = BL$, then $M + N =$

Explain that since M and BK represent the same number, BK can replace M. And since N and BL represent the same number, BL can replace N. Hence, $M + N = BK + BL$.

Then write another example on the board:

If $A = NP$ and $B = NQ$, then $A - B =$

Ask for a volunteer to substitute into the given expression. Discuss. Be sure to point out that the minus sign must remain the same. Only A and B are changed. Hand out the drill sheets, do the first two together in class, then ask the students to complete the problems at their desks. Discuss. Then administer the mastery test on substitution.

Then review all of the prerequisites and administer the Lesson mastery test.

NOTE: The episodes of the story serve two basic functions: (1) to introduce new ideas and serve as a springboard for a discussion of these new ideas; and (2) to provide practice and drill on prerequisite skills. They may be inserted into the lesson whenever the teacher thinks they are most appropriate. Explain to the students that they are to read the stories with a pencil, and that they must do all of the exercises in each of the stories.

JOURNAL FOR LESSON ONE

Tuesday, April 15, 1969

A pretest was administered to the twenty students who were participating in the experiment. The following instructions were given:

"We are from the Research and Development Center of the University of Wisconsin. As you know, we are here to try out some new ideas in mathematics for sixth-graders. So that we will have a better idea of what sixth-graders know, we would like you to take this test. We do not expect you to do all of the problems. In fact, some of the problems will be new to you. Do the best you can on the problems you understand. If you have any questions, raise your hand and one of us will come to your desk. Are there any questions?"

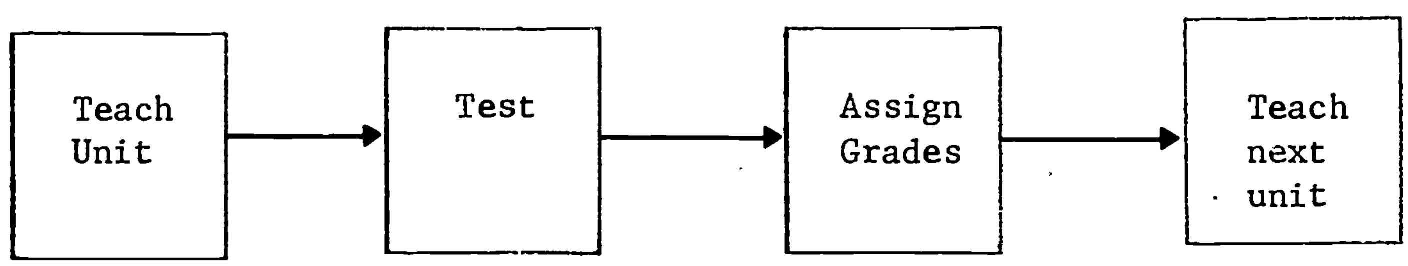
The test required 15 minutes.

When the testing was completed, the students in the control group returned to their regular classes. The investigator then gave a brief introduction to the experimental class:

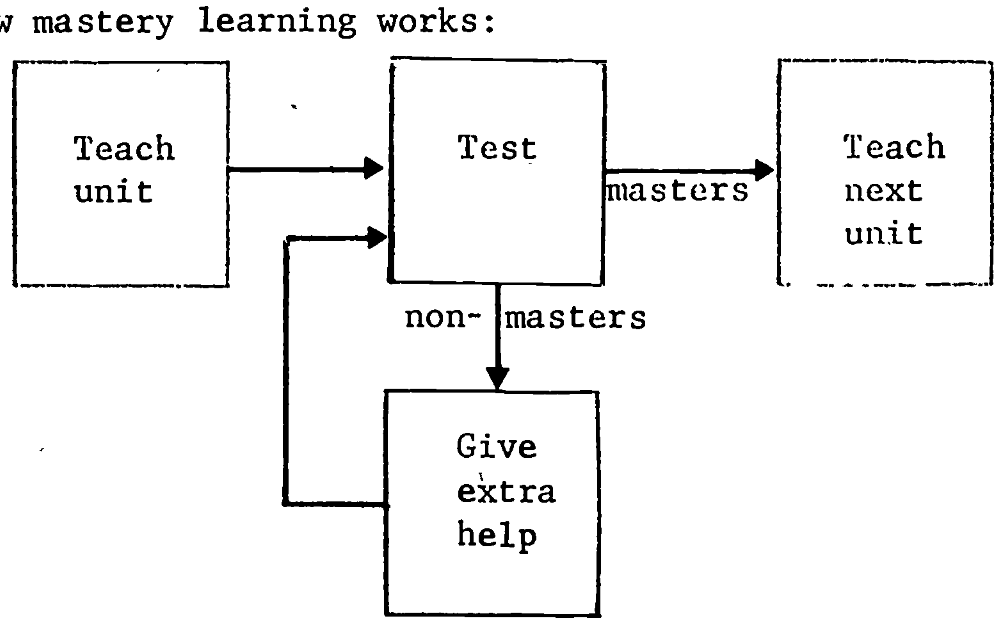
"The unit in which you are participating is on mathematical proof. We will learn to prove that certain mathematical statements are true. This unit will last four weeks, and at the end of that time we hope that you will be able to prove some mathematical statements.

Are you familiar with this set: $\{0, 1, 2, 3, \ldots\}$? It is the set of whole numbers. In this unit we will prove some interesting facts about this set. Whenever we speak of "number", we shall mean "whole number." Before we prove some results, it will be necessary for us to review a few ideas to make certain we understand them. Ideas like the distributive law, prime and composite numbers, and how to evaluate expressions. Once we have learned these, we will be ready to prove some things.

We will be using a new idea in this class. Usually, the following procedure is followed: The teacher presents a unit of study, she administers a test, assigns grades of A,B,C,D or F, and then proceeds to the next unit. This diagram shows the procedure:



We shall use a new idea called "mastery learning." This means that we want every student in this class to learn everything we teach. That is, you will master every idea, concept, and skill in this unit. If you are not able to do something at first, the teacher will give you extra help. We believe that if you really want to learn the ideas in this unit, we can give you enough help so that you can. This diagram shows how mastery learning works:



First the lesson is taught, then you are tested. If everyone masters the ideas, we go on to the next lesson. If all of you do not master the ideas (in this case you will be called non-masters), the teacher will give you extra help. Then you will be tested again. This will continue until you are all masters."

At this point a student asked the following:

Question: What if one of us just cannot learn something?

Answer: We will probably give two tests, and if only one or two of you are non-masters, we will go on. But if 3 or more of you are non-masters, we will continue to study the topic.

"If you successfully complete this unit, you will receive this certificate with your name on it. It says, "This is to certify that ________________ has completed an experimental unit on mathematical proof and has mastered the concepts therein!"

With 15 minutes remaining, the teacher began instruction. She stressed that she wants to know whenever anyone doesn't understand something, and encouraged them to ask questions. She wrote this example on the board.

$$(3 \times 4) + (3 \times 5) =$$

"Do you know what the distributive law is?" "Can you apply it to this sum?" One student said, "That is equal to (3 x 4) + (3 x 5)! Another said, "That is 27." The teacher explained the distributive law and wrote, $(3 \times 4) + (3 \times 5) = 3\ (4 + 5)$.

The equation was then verified:

$$12 + 15 = 3\ (9)$$

$$27 = 27$$

Another example was written on the board:

$$(2 \times 8) + (2 \times 2) =$$

A correct response was given immediately, and the equation was verified.

The teacher explained that if N and M represented numbers, NM meant N x M. She then gave two more examples on the board:

$$NM + NQ =$$

$$5A + 5B =$$

Correct responses were given. Drill sheet #1 was then passed out. The first two were done orally, and the students were asked to do the remaining problems. When finished, each student read an answer. All were correct.

With 4 minutes remaining in the class, the teacher gave each student the introduction to the story.

ANALYSIS: The lesson went as planned and no changes were made in the lesson.

Wednesday, April 16, 1969

The teacher began by discussing how the stories would be used. She then wrote the following expression on the board:

$$(4 \times 5) + (4 \times 9) =$$

"Can anyone apply the distributive law to this expression?" Every student raised his hand, and a correct response was given. The teacher restated the distributive law, pointing to the board as she spoke.

"If the sum of two products have a common factor (in this case 4), then this sum can be written as product of that common factor and the sum of the remaining factors (in this case 5 + 9)."

She then wrote 91 (9) + 9 (9) = on the board and asked, "Can anyone give me the answer to this problem?" One student volunteered and merely repeated the expression. "That's correct, but can you tell what number this expression represents? Can you do it in your head by using the distributive law?" Another student gave the correct response, 900. The teacher wrote 8 (75) + 8 (25) = on the board. Four students raised their hands; a correct response of 800 was given. After a correct response was given for 9 (88) + 9 (12), a student explained how the distributive law could be used to rapidly perform these special calculations. She passed out drill sheets and asked them to write down the numerical answers as rapidly as possible. Within one minute the students had completed the list. Each student supplied one answer orally as the class corrected the papers. As with all drill sheets, each student corrects his own paper.

The students were then given a list of equations and were asked to identify those which were examples of the distributive law. The first two were done orally in class, and the students completed the remaining one. In correcting the problems, each student supplied one answer. One mistake was made: an example of the commutative law was identified as the distributive law. The teacher pointed out that while the equation was true, it was not an instance of the distributive law.

Copies of the story involving the distributive law were then handed out, the students worked the problems in the story, and the problems were discussed. This took fifteen minutes.

A mastery test on the distributive law was administered. It required five minutes and all students were masters. The teacher introduced divisibility by writing 30 = 3 x 10 on the board. She pointed out that 3 and 10 are factors of 30 and that 3 divides 30 and 10 divides 30. The teacher wrote "45 =" on the board and asked for factors of it. 5 x 9 was suggested, and then she wrote:

5 divides 45

9 divides 45

She then introduced the use of the vertical line to represent "divides", explaining that it was easier to write a vertical line than to write "divides" each time. Asking the students to supply two divisibility statements for each equation, the teacher wrote the following equations on the board:

10 = 50 x 2

A = BC

90 = 45 x 2

A = TX

66 = 6 x 11

In each case correct responses were given by a student.

ANALYSIS: The day's lesson went as planned. There were three mistakes on the Mastery Test (out of a total of 200 responses).

Reading the story took fifteen minutes; some students are much faster readers than others. It was decided to assign all future stories as homework so as to conserve class time.

Thursday, April 17, 1969

The teacher began by writing 24 on the board and asking for factors of 24. 6 x 4 was offered, and then she asked for two divisibility facts.

The students responded with $6 \mid 24$ and $4 \mid 24$. The same procedure was followed with $A = BC$. Drill sheet #1 was handed out, the first problem was done orally, and the students worked the remaining problems. In correcting the papers, each student read an answer. All were correct.

The teacher then wrote $3 \mid 30$ on the board and asked if anyone could give an equation to represent this divisibility fact. A student responded correctly, $30 = 3 \times 10$. Then $10 \mid 50$, $3 \mid 12$ and $6 \mid 24$ were written successively on the board and correct responses were given by the students.

When the teacher wrote $A \mid T$ and asked for an equation, two students raised their hands. One said, "$A = TN$." This was corrected by another student. The teacher explained that since the unit only discusses whole numbers, that the T must be the larger number, hence T is equal to the product of two smaller numbers. This was followed by $P \mid N$. A student gave an incorrect response: $P = NA$. Again the teacher explained. Another example was given, $T \mid X$. This time a correct response was tiven: $X = TN$. One student asked if we could use a number, such as, $X = 2T$. The teacher explained why this was not permissible.

A sum was then given, $4 \mid (6 + 2)$. She stressed that (6 + 2) should be thought of as <u>one</u> number, not two. A correct response was given. $8 \mid (9 + 7)$, $A \mid (B + C)$, $L \mid (M + N)$ and $S \mid$ ([illegible] · B) were all answered correctly. One student said that he did not understand what the last example meant, so the teacher illustrated it with several numerical examples.

Drill sheet #2 was then handed out, done in class and corrected. The mastery test on divisibility was then administered. It took 17 minutes.

With 6 minutes remaining in the period, the teacher began substitution.

The first example was: If A = 3, then A + 6 = ? 9 was given as the answer, and the teacher explained that to do the problem the 3 is substituted for the A. She then introduced the square notation: N^2 means N x N. Four examples were given, and then three subs itution problems were written on the board:

If N = 3, then $N^2 - N + 1$

If N = 5, $N^2 - N + 11$

If A = MK and B = MQ, A + B =

The students as a group supplied the correct answers. One student observed that in the last example the substitution did not get us anywhere. Another asked why 25 - 5 + 11 wasn't 25 - 16. The teacher explained that this would be true if parentheses were around (5 + 11). The period ended with the teacher distributing Part One of "Long Live the King".

ANALYSIS: Six of the ten students were masters on the mastery test for divisibility. Since a criteria of (80/80) has been set on all prerequisite skills, recycling was necessary. Of the 25 incorrect responses, 22 were made in writing divisibility facts for a given equation. It also appeared that there was some interference. In class the students had drilled on these two skills separately, the two kinds of problems were not given at the same time. The test, to the contrary, mixed the two kinds of problems together. The first three problems on the Mastery Test were equations, and 29 of the 30 responses were correct. Items 4, 5, and 6 were divisibility facts. Item 7 was another equation, and from this point on these of the students missed this kind of item. It thus appears that putting both kinds of items together caused confusion for three of the students.

It was decided to alter Friday's lesson and present more work on divisibility. A drill sheet was prepared as a homework assignment. One other change was made. When given that A = MK and B = MQ and asked to substitute into the expression A + B, a student observed that the resulting expression MK + MQ was more complicated than the original expression. The teacher did not respond. It was therefore decided to explain why such a substitution might be desirable. That is, that it would be used later when proving a theorem, and that this substitution permitted us to apply the distributive law.

Friday, April 18, 1969

The teacher began by reading the answers for the story. On yesterday's Mastery Test, the students had difficulty writing divisibility facts from a given equation. Items 7,9,11,13,15,17, and 20 were discussed, and the teacher asked the students to give answers for the following problems which she wrote on the board.

$$
\begin{aligned}
G &= VX \\
X &| M \\
(A + B) &= NJ \\
(M - Q) &= TF \\
3 &| 6 \\
3 &| A \\
(P + Q) &= FG
\end{aligned}
$$

For X|M, one student said "XM = S," and then corrected herself. Since only 6 of 10 students had mastered the skills on divisibility, the teacher explained that another test would be administered on Monday. It was stressed that she wanted everyone to be masters.

The teacher then continued the previous day's lesson on substitution. She explained that if A = MN and B = MP, then (A + B) = MN + MP. The resulting expression is more complex than (A + B), but it was pointed out that the distributive law could be applied to MN + MP and that this was going to help us prove some theorems later in the unit. The drill sheet on substitution was then passed out. The teacher helped students as they worked the problems at their desks. One student was slow in completing the sheet. The teacher read the answers.

The substitution Mastery Test was then administered.

A practice sheet on all prerequisites was then distributed to the class and the students worked the problems at their desks. Two students had difficulty and the teacher explained each type of problem to these students.

The teacher then reviewed each of the prerequisites on the board.

With five minutes remaining in the period, Part Two of "Long Live the King" was handed out and the students read for the remainder of the period.

ANALYSIS: The lesson went as planned.

Monday, April 21, 1969

The previous day's story was discussed, and after a brief review of the prerequisites, the Mastery Test was administered. It took twelve minutes of class time.

ANALYSIS: Eight of ten passed the Mastery Test. Two students missed one problem each, six students had a perfect paper, another student missed five problems (4 of them on divisibility). It was decided that the two non-masters would come in during their lunch hour (preceding the class) for extra help.

DISTRIBUTIVE LAW DRILL SHEET #1

1. AB + AC =

2. 5T + 5R =

3. MR + MT =

4. 3W + 3D =

5. NP + NQ =

6. AP + AQ =

7. 8Q + 8W =

8. 6F + 6T =

9. 2D + 2X =

10. 13A + 13F =

11. CX + CY =

12. DM + DK =

DISTRIBUTIVE DRILL SHEET #2

1. 4 (7) + 4 (3) =

2. 5 (99) + 5 (1) =

3. 8 (93) + 8 (7) =

4. 6 (999) + 6 (1) =

5. 2 (95) + 2 (5) =

6. 7 (98) + 7 (2) =

7. 9 (997) + 9 (3) =

8. 3 (92) + 3 (8) =

9. 4 (97) + 4 (3) =

10. 8 (998) + 8 (2) =

DRILL SHEET #3

LIST OF EQUATIONS: Identify Instances of the Distributive Law (write Yes or No)

1. AB + AC = A (B + C)
2. 5X + 5Y = 5 (A + B)
3. 2A + 2T = 2 (A + T)
4. AB = BA
5. A + B = B + A
6. MN + MR = N (M + R)
7. AB + AT = T (A + B)
8. XA + XM = X (A + M)
9. 3U + 3V = 10
10. 2A = 3 (A + B)

DISTRIBUTIVE LAW MASTERY TEST

Which of the following are instances of the distributive law? Write yes or no.

1. AB + AC = 10
2. AB + AC = A (B + C)
3. NP + NQ = N (P + Q)
4. N + M = M + N
5. 10 x 5 = 5 x 10
6. 3K + 3G = 3 (K + G)
7. 15T + 15R = 15 (A + B)
8. 2M = 2K
9. (3 x 5) + (3 x 7) = 3 (5 + 7)
10. TP + TK = T (P + K)

Apply the distributive law to each of the following:

11. AB + AC =
12. NP + NQ =
13. 30V + 30K =
14. 7 (92) + 7 (8) =
15. CD + CX =
16. VE + VR =
17. MT + MR =
18. 52K + 52V =
19. 10T + 10G =
20. NA + NB =

DIVISIBILITY DRILL SHEET #1

For each of these equations, give two divisibility facts.

1. $30 = 5 \times 6$

2. $100 = 4 \times 25$

3. $A = XY$

4. $W = RT$

5. $A = VW$

6. $B = UV$

7. $M = KT$

8. $101 = 1 \times 101$

9. $32 = 2 \times 16$

10. $K = VT$

DIVISIBILITY DRILL SHEET #2

Write an equation to express the following facts:

1. $6|60$
2. $A|T$
3. $B|C$
4. $12|24$
5. $M|R$
6. $N|(A + B)$
7. $T|(A - B)$
8. $W|3K$
9. $V|W$
10. $2|T$

DIVISIBILITY MASTERY TEST

For each equation, write 2 divisibility facts. For each divisibility fact, write an equation.

1. $A = TW$
2. $B = RV$
3. $40 = 5 \times 8$
4. $3|15$
5. $4|16$
6. $T|R$
7. $A + B = NX$
8. $K|V$
9. $A - B = MY$
10. $N|(A + B)$
11. $A + B = MN$
12. $M|A$
13. $C + D = TV$
14. $T|(A - B)$
15. $A = CD$
16. $F|R$
17. $M = FT$
18. $3|9$
19. $2|16$
20. $V = 2W$

SUBSTITUTION DRILL SHEET

1. If K = 6, then M + K =

2. If A = NP and B = NQ, then A + B =

3. If K = T, then M + K =

4. If M = XT and N = XR, then M + N =

5. If A = MK, D = MR, B = ML, then A + B + D =

6. If C = NP and D = NQ, then C - D =

7. If K = 7B, then M + K =

8. If M = 5T, then N - M =

9. If KP = 2, then M - KP =

10. If V = NK and W = NR, then V - W =

SUBSTITUTION MASTERY TEST

1. If C = NP and D = NQ, then C + D =

2. If T = 5, then M + T =

3. If A = 5Q, then 7Q + A =

4. If F = MP and G = MQ, then F - G =

5. If NQ = 6, then P + NQ =

6. If AB = 2, then AB + C =

7. If A = 6, B = 7, then A + B =

8. If A = NP, B = NQ, C = NR, then A + B + C =

9. If K = 3, then M + K =

10. If N = AB and M = AC then N + M =

PRACTICE SHEET ON PREREQUISITES

APPLY THE DISTRIBUTIVE LAW:

1. 3W + 3V

2. XT + XR

WHICH OF THE FOLLOWING ARE INSTANCES OF THE DISTRIBUTIVE LAW:

3. AB = BA

4. AC + AD = A(C + D)

SUBSTITUTE:

5. If A = 10, then 4 + 10 =

6. If T = MK and S = ML, then T - S =

WRITE AN EQUATION FOR EACH OF THESE DIVISIBILITY FACTS:

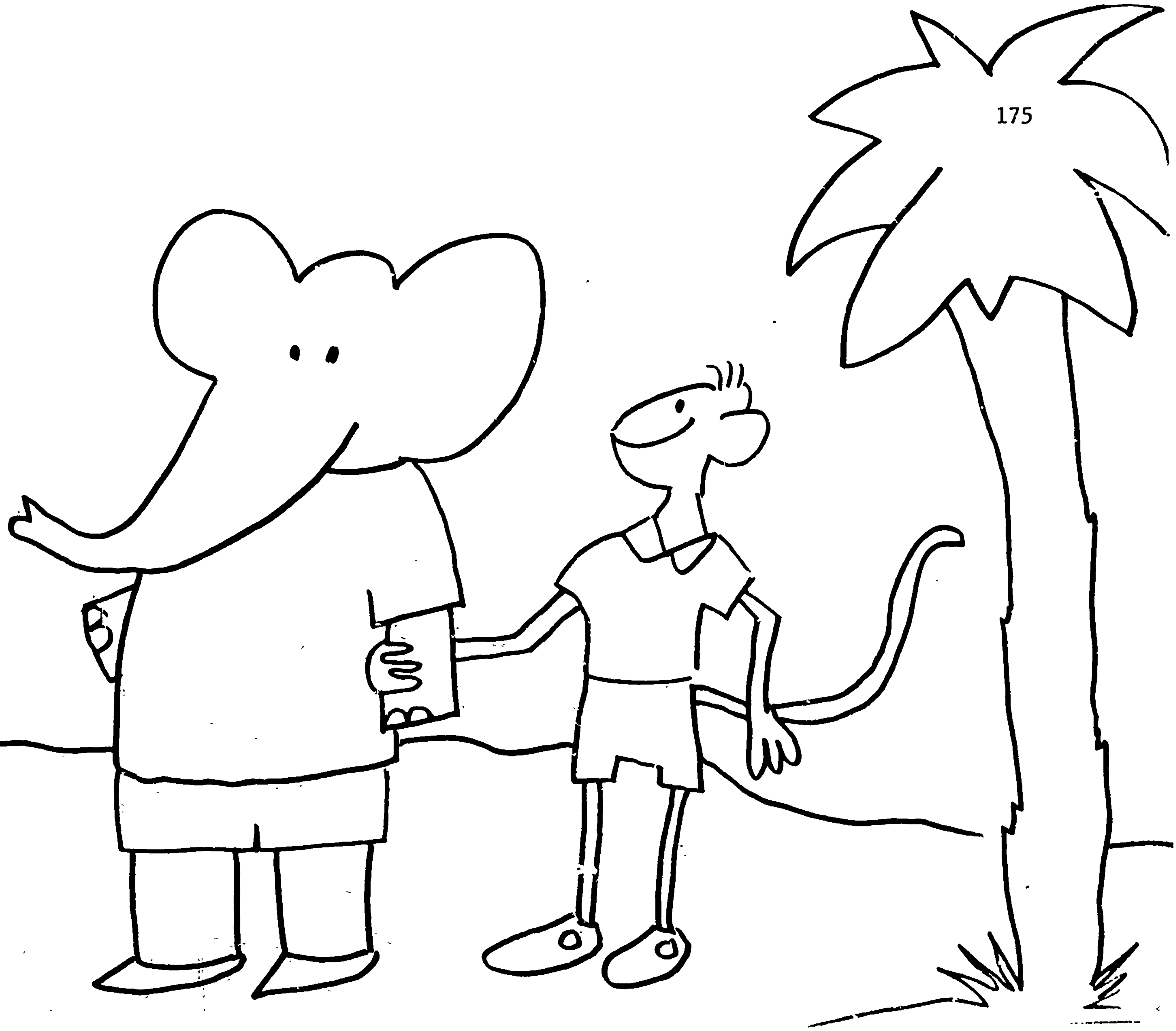
7. $A \mid M$

8. $2 \mid (A + B)$

WRITE TWO DIVISIBILITY FACTS FOR EACH OF THE FOLLOWING EQUATIONS:

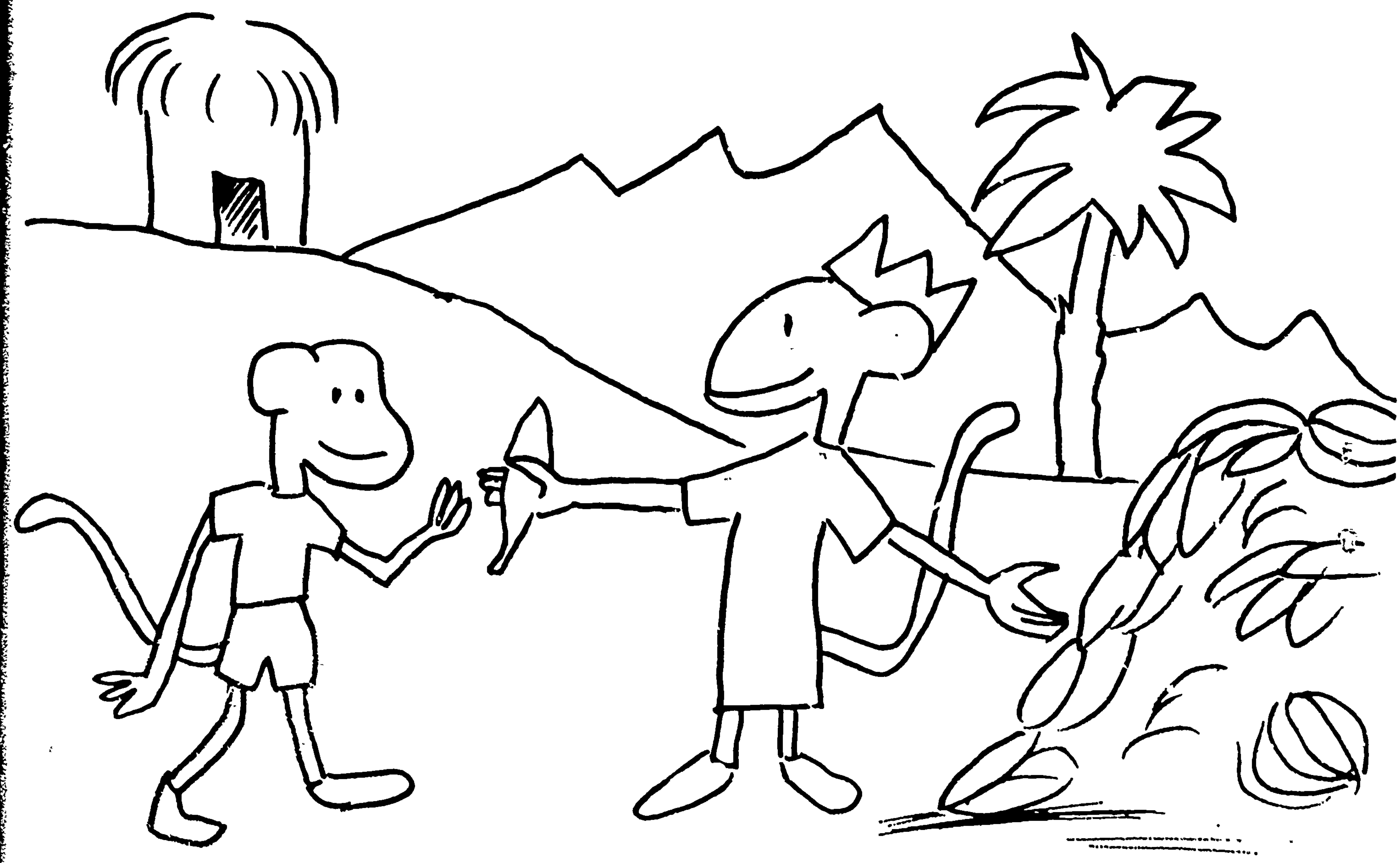
9. A = TF

10. (A + B) = XJ



Meet Harvey the elephant and Sidney the monkey. They are very intelligent animals. The other animals discovered this last Bananaday. In case you did not know, Bananaday is the third day of the week on the animal's calendar: Sunday, Monday, Bananaday, Wednesday, Thursday, Friday, and Saturday. Bananaday is the day of the week that all the monkeys of Monkeyville gather bananas. They always put all of the bananas in one big pile, and then the King,

King Herman,the King of Monkeyville, passes the bananas out to the monkeys. They line up in a long line, and as they walk past Herman he gives a banana to each monkey. When all the monkeys have a banana, they go through the line again. This continues until all the bananas are gone. This takes a lot of time. It is also unfair, because the monkeys at the end of the line often get one less banana than the other monkeys.



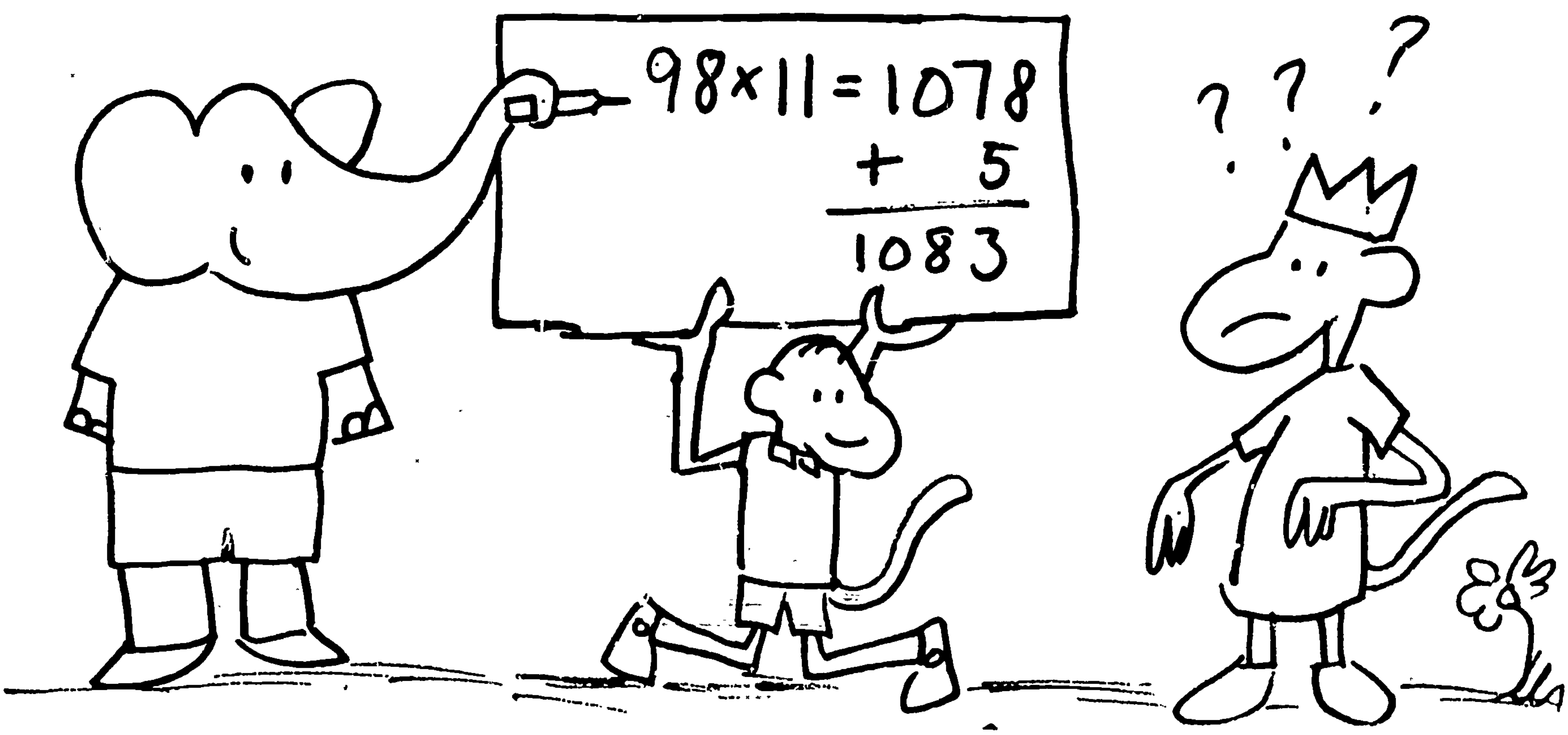
Last Bananaday, Sidney said, "Excuse me, your highness, but Harvey and I have thought of a faster way to hand out bananas". "You have?" asked the King.

"Count the bananas," said Harvey. "Then divide by 98, the number of monkeys in Monkeyville", added Sidney.

"And this will tell you how many bananas to give to each monkey," said Harvey.

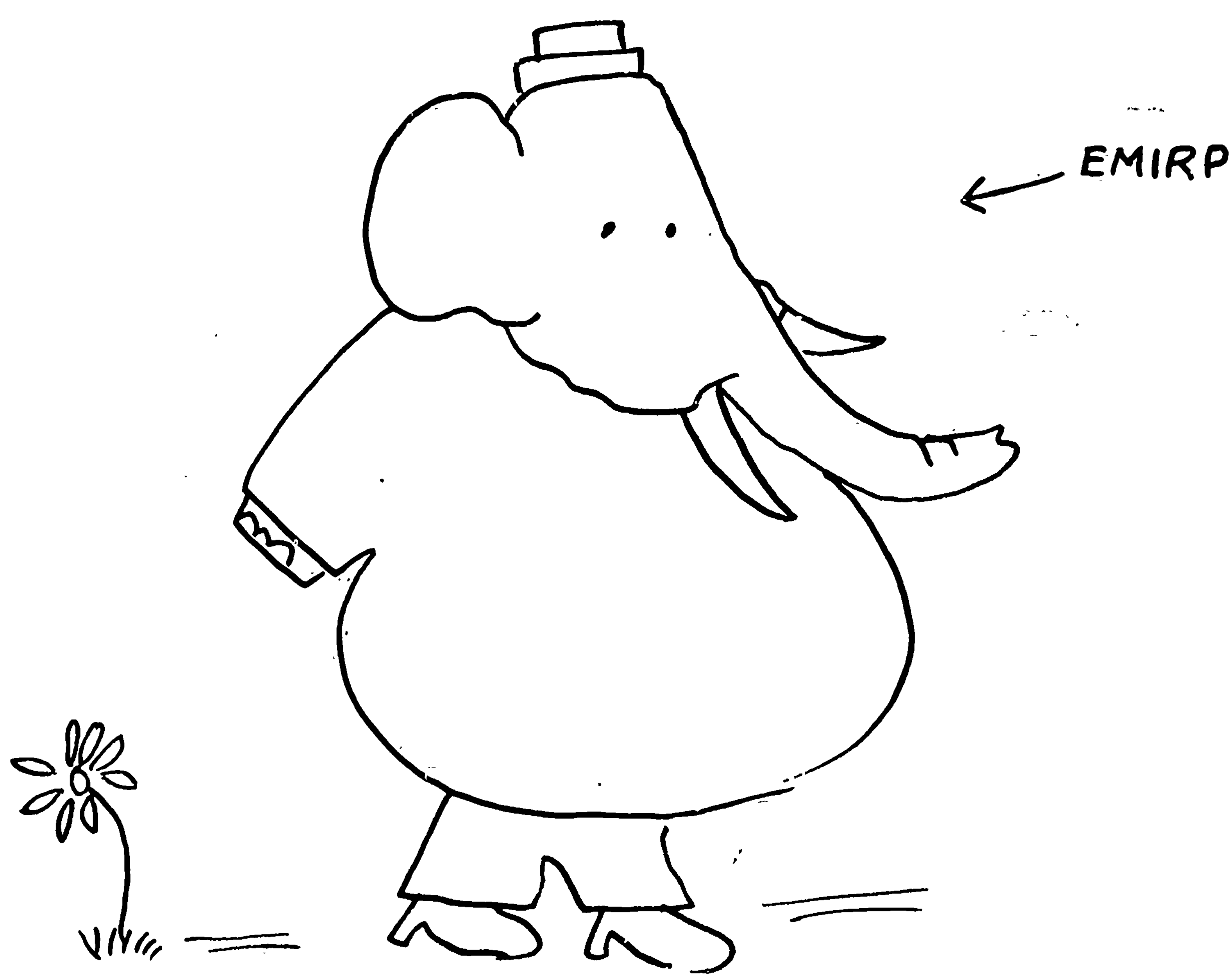
"My word," said the King. He was confused, for he could only count to 20 and could not divide at all.

"There are 1,083 bananas in this pile. So if you give each monkey 11 bananas, there will be 5 bananas left over. Then, you can save these 5 for visitors who visit you from time to time."



Herman did not believe it. "This sounds like a lot of monkey business," he said as he started handing out bananas one at a time. Well, you can imagine the King's surprise when he discovered that our two young friends were right. Sidney was immediately made the official banana counter for all of Monkeyville. This made all the monkeys happy because now they did not have to stand in line for hours to get their weekly supply of bananas.

The King also decided to send Sidney and Harvey to school. Since animals do not have schools, the King made arrangements for them to study with Mr. EMIRP, the wisest elephant in the jungle.

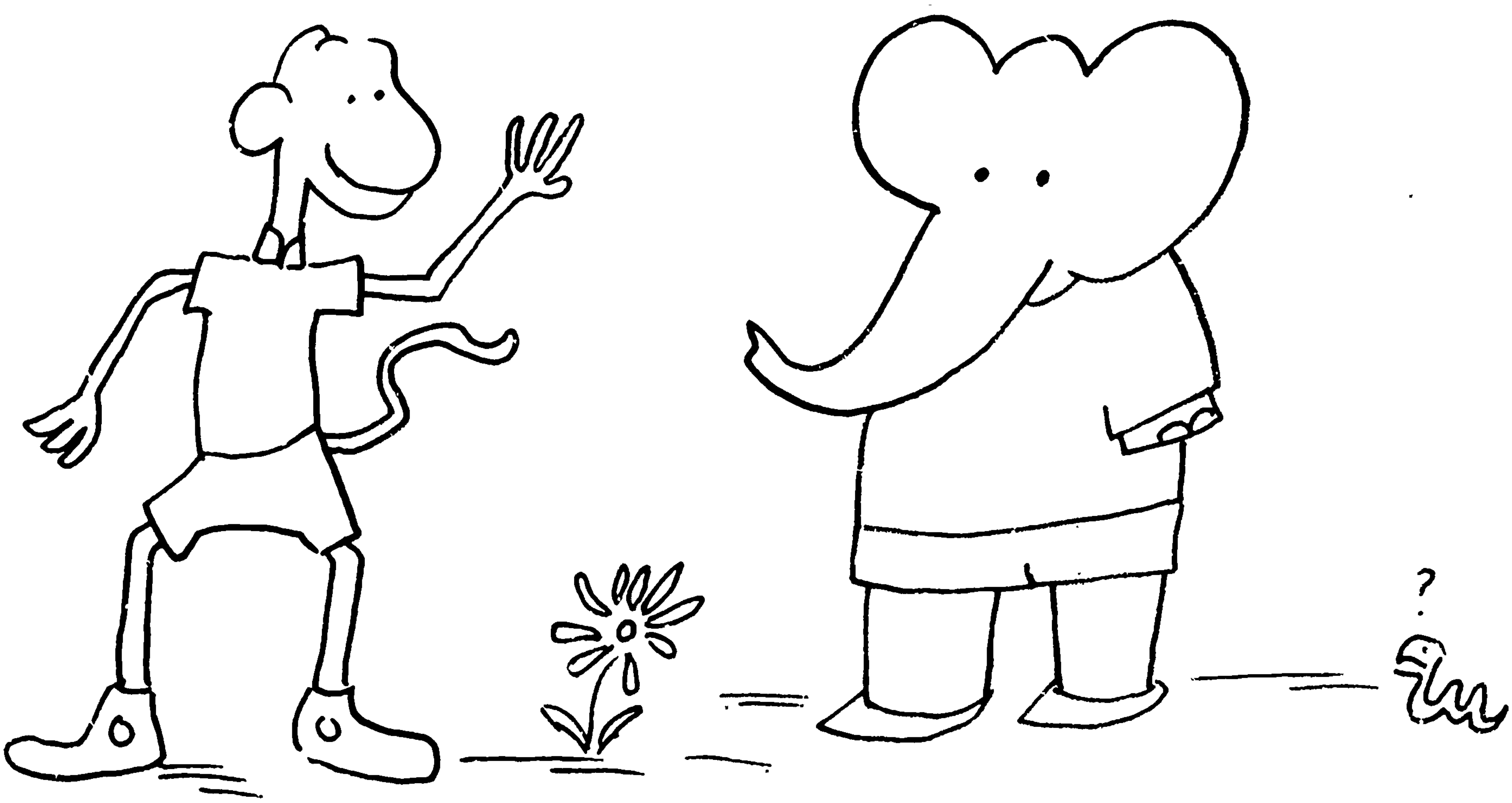


You are invited to tag along with Sidney and Harvey as they study mathematics with the great Mr. EMIRP.

There is only one catch---you must do all of the problems that Mr. EMIRP gives to Sidney and Harvey.

Our two friends are about to go for their first mathematics lesson.

Are you ready?



Technical Report No. 111

A FORMATIVE DEVELOPMENT OF A UNIT ON PROOF FOR USE IN THE ELEMENTARY SCHOOL PART II

Report from the Project on Analysis of Mathematics Instruction

By Irvin L. King

Thomas A. Romberg, Assistant Professor of Curriculum and Instruction and of Mathematics
Chairman of the Examining Committee and Principal Investigator

Wisconsin Research and Development
Center for Cognitive Learning
The University of Wisconsin

January 1970

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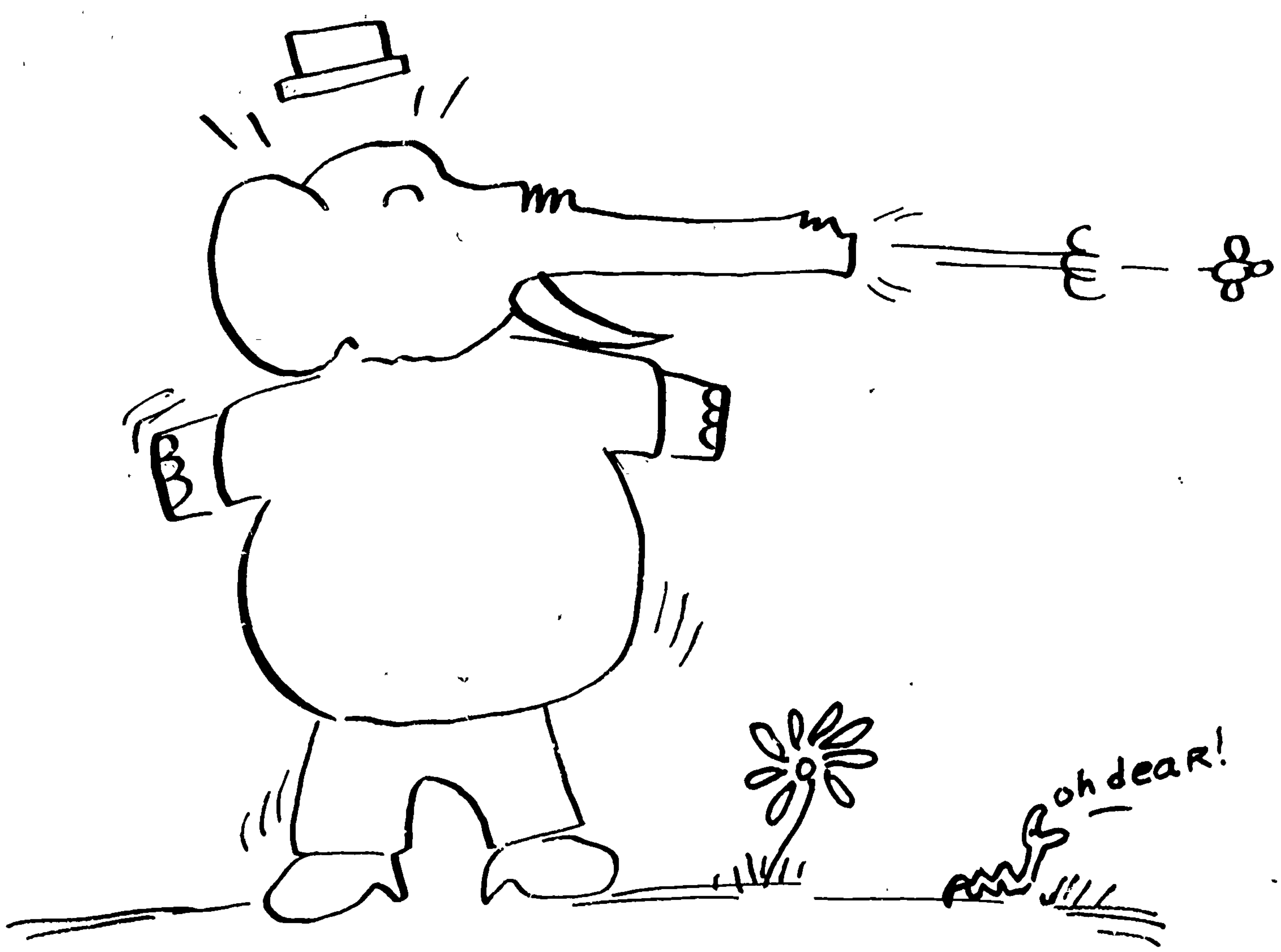
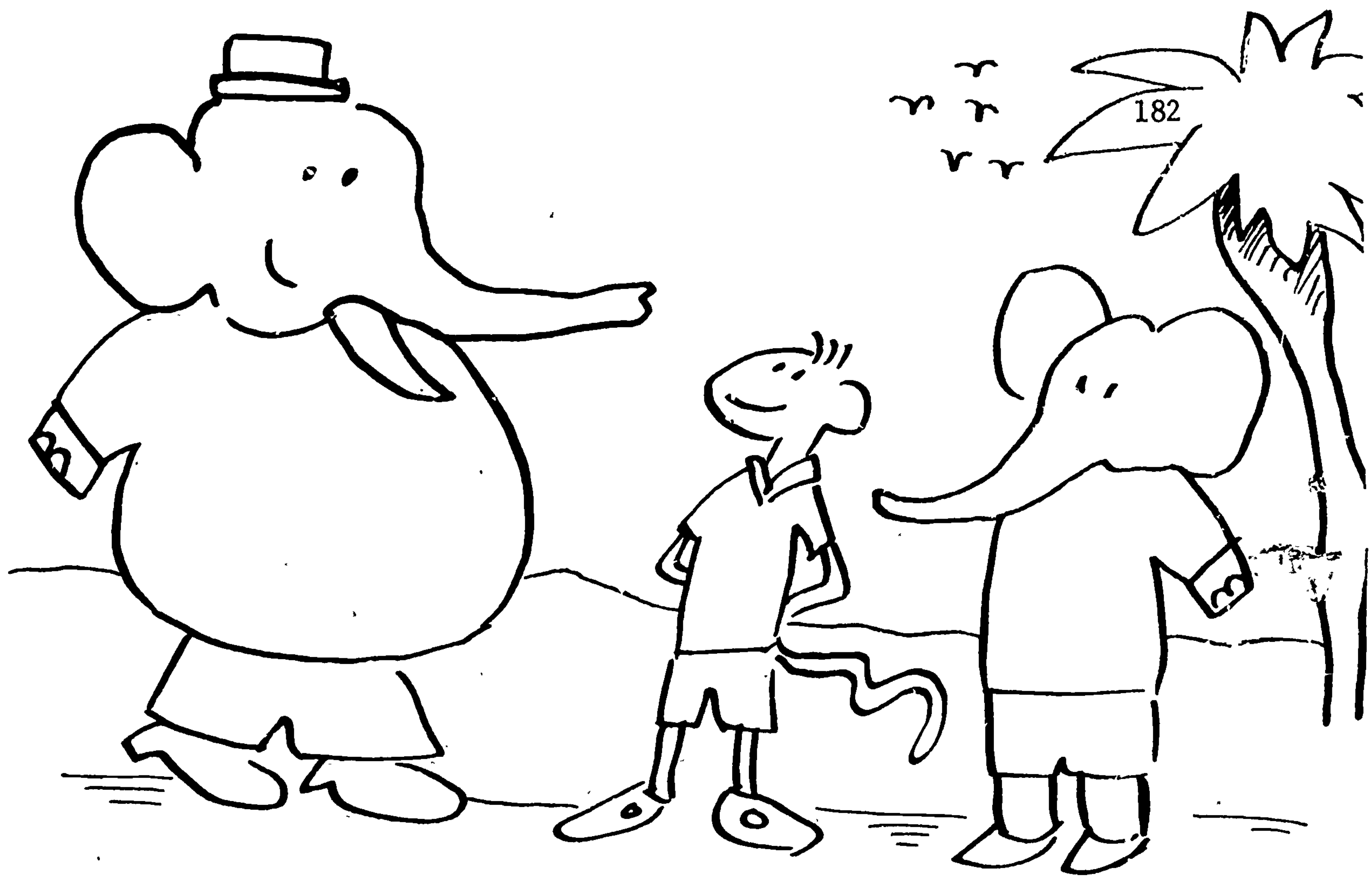
THE FIRST LESSON

When Harvey and Sidney arrived for their first lesson, they found Mr. EMIRP leaning against a tree, fast asleep. They had heard so much about the great Mr. EMIRP that they did not dare awaken him. They did not have to wait long, however, for Mr. EMIRP was breathing so heavily that he sucked a bee up his trunk. He exploded with a sneeze which sent the bee sailing at least a mile into the jungle. As he was wiping off the end of his trunk, he saw Harvey and Sidney standing nearby.





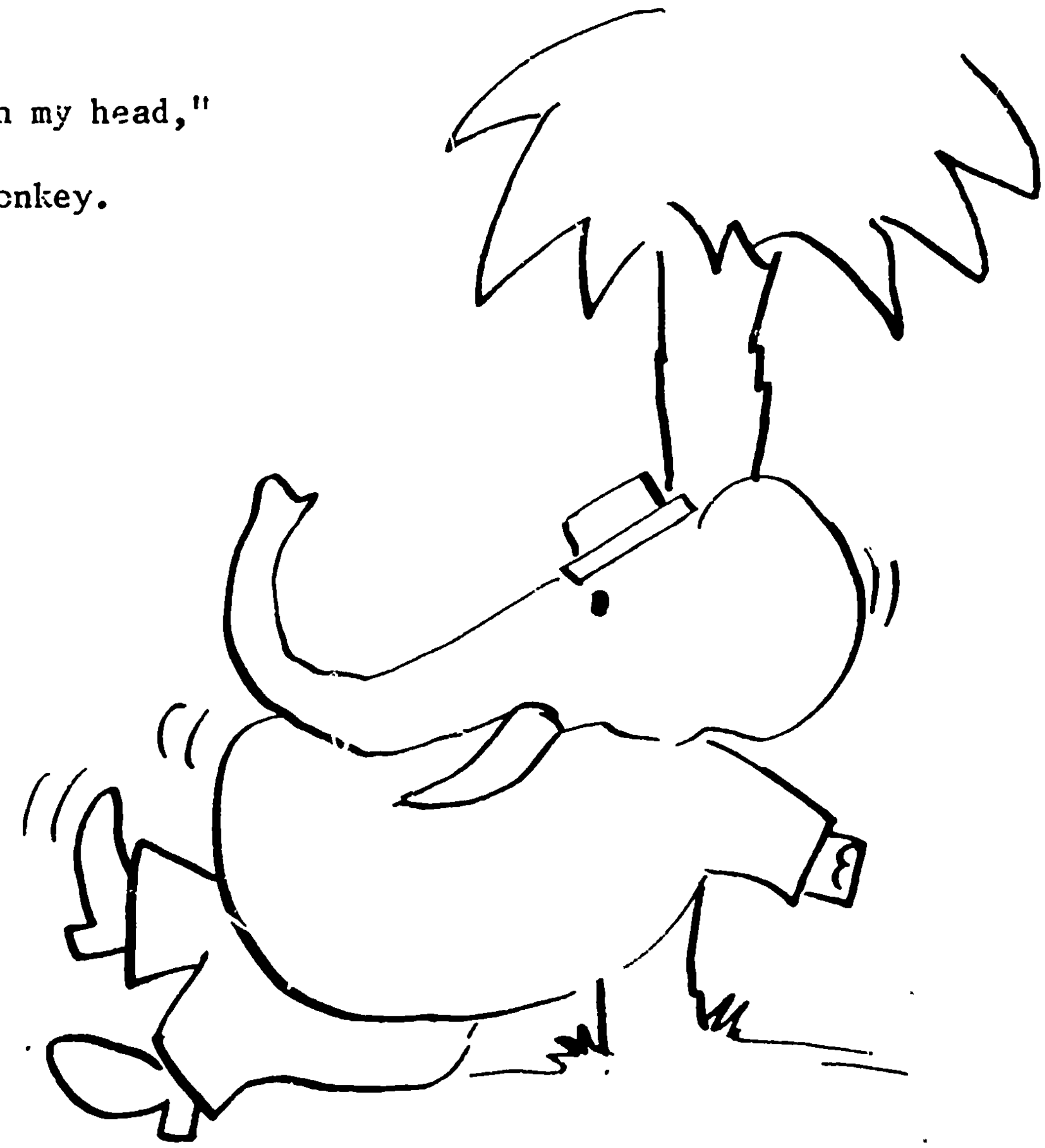
"Well, well," said EMIRP, "you two must be Harvey and Sidney. I've heard good things about you." "Thank you, Sir," the boys replied, "We have come for our first lesson." "Oh yes," said EMIRP, still somewhat sleepy. He decided to give the boys a difficult problem so that he could catch a few more winks of sleep while they were working on it. "Your first problem," he said, "is to multiply 59x509."

Now EMIRP was almost asleep when Sidney shouted out, "The answer is 30,031."

Mr. EMIRP was so surprised that he fell flat on the seat of his pants.

"Holy elephant feathers! That's right," he said, "Tell me. How did you do it so quickly?"

"I just did it in my head," said the young monkey.



The great EMIRP was certainly impressed. He realized that he was not dealing with ordinary animals. He dusted off the seat of his pants and said, "O.K. Let's get down to business. Today we shall study the DISTRIBUTIVE property."

"The what?" said Harvey.

"The DISTRIBUTIVE property," answered Mr. EMIRP. "It's very simple if you know what a factor is."

"Isn't that something a farmer uses to plow his fields?" asked Sidney.

"No, no," said EMIRP. "You're thinking of TRACTOR". Then he explained what a factor is. "If A, B, and C represent any three counting numbers, and if A = BC, then we say that B is a factor of A, and that C is a factor of A. For example, 10 = 2x5, so 2 and 5 are factors of 10."

"Oh," said Harvey. "Since 100 = 4x25, 4 and 25 are factors of 100."

"Beautiful," said Mr. Emirp. "Can you answer these questions?"

1. is 3 a factor of 15?
2. is 7 a factor of 35?
3. is 10 a factor of 82?

Harvey went to work on the problem, but a butterfly had caught Sidney's attention.

"For each of the following, give two factors" said EMIRP.

4. 32 5. 50 6. 100 7. 34 8. 75

"Notice that 30 = 2 x 15 tells us that both 2 and 15 are factors of 30," said EMIRP. "And 30 = 2 x 3 x 5 tells us that 2, 3, and 5 are factors of 30. Try the following problems: What are the factors of each of the following numbers:

9. 35 = 5 x 7, so the factors of 35 are ...

10. 100 = 2 x 2 x 5 x 5, so the factors of 100 are...

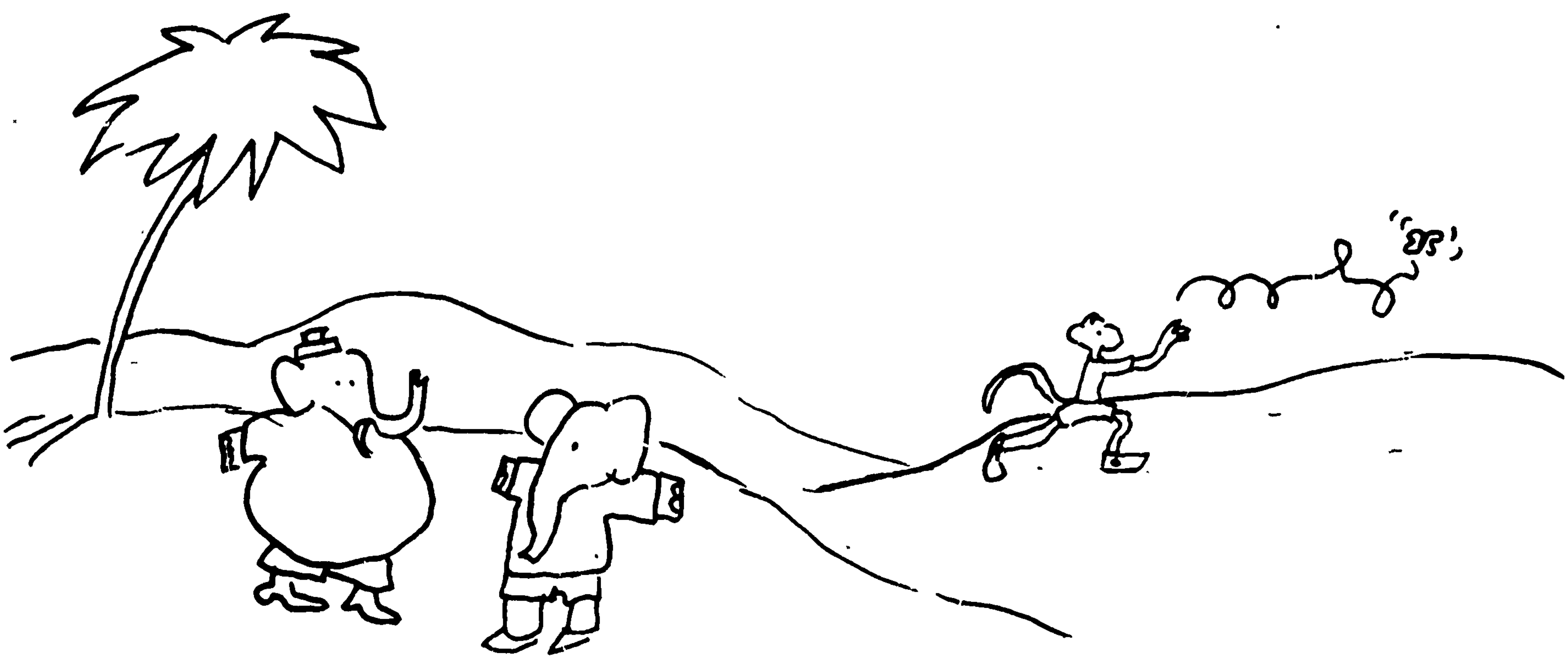
11. 34 = 2 x 17, so the factors of 34 are ...

12. A = 3B, so the factors of A are ...

13. T = XY, so the factors of T are ...

14. (A + B) = GH, so the factors of (A + B) are ...

15. (T - G) = 3AB, so the factors of (T - G) are ..."



"Gosh," said Harvey. "These are sure difficult."

"Yes, but you must learn to work with letters of the alphabet if you want to learn algebra."

Harvey did his best. He only missed one problem.

"If you know what a factor is, we can look at the DISTRIBUTIVE property," said EMIRP. He explained it to Harvey. "Now, $(3 \times 6) + (3 \times 1) = 3(6 + 1)$," said the wise Mr. EMIRP.

"It does?" asked Harvey.

"Yes. You can check it out. On one side of the equation we have $(3 \times 6) + (3 \times 1)$. That's 18 + 3, or 21. And on the other side we have 3(6 + 1) which is 3(7) which is also 21." "Well I'll be a monkey's uncle," said Harvey.

Meanwhile, the bee who had been sneezed by EMIRP had crashed into a tree and was knocked unconscious. A worm came by, picked up a leaf,

and fanned the air. Soon the bee regained his senses. "Revenge," said the bee; and he flew off like a bullet without even thanking the worm who had been a good samaritan. There was just one thing on the bee's mind. "Now then," said EMIRP," apply the distributive property to the following sums:

16. 3A + 3B =

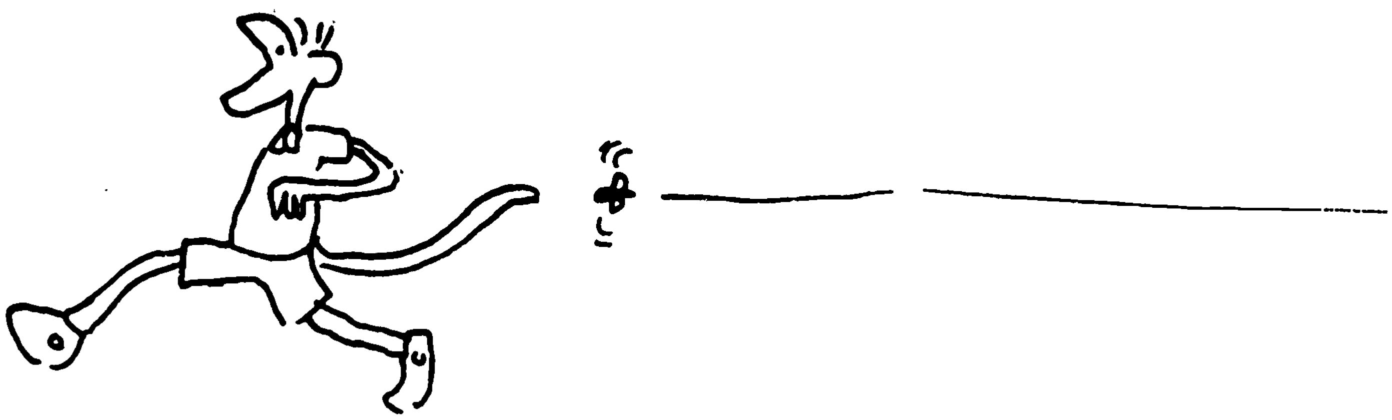
17. 7T + 7X =

18. AB + AC =

19. DEF + DT =

20. ABC + AX =

"HELP!" shouted Sidney. The bee had seen him chasing the butterfly. Now what do you think was the last thing the bee saw before he was sucked up into EMIRP's trunk? Sidney, of course. Because Sidney was the last thing he saw, he thought that Sidney had given him the free ride. "Revenge," said the bee as he went for Sidney.



Harvey and Mr. EMIRP ran safely into the house, but poor Sidney was not so lucky. Just as he went through the door the bee stung him on the tip of his tail.

"There's only one thing we can do," said EMIRP, "and that is to soak your tail in hot honey."

As Sidney soaked his tail, Mr. EMIRP continued the lesson. "If A, B and C represent any three counting numbers, then the distributive law states that

$$AB + AC = A(B + C).$$"

Then EMIRP turned to Sidney and asked, "Well Sidney, did you learn anything today?"

"Oh yes," said Sidney sadly. "I learned that a bee can sting." And that is the end of the tale.

LONG LIVE THE KING: PART ONE

Our three friends were about to begin their next math lesson when a Wierdo bird came with a message. "Help! We need the great EMIRP to help us."

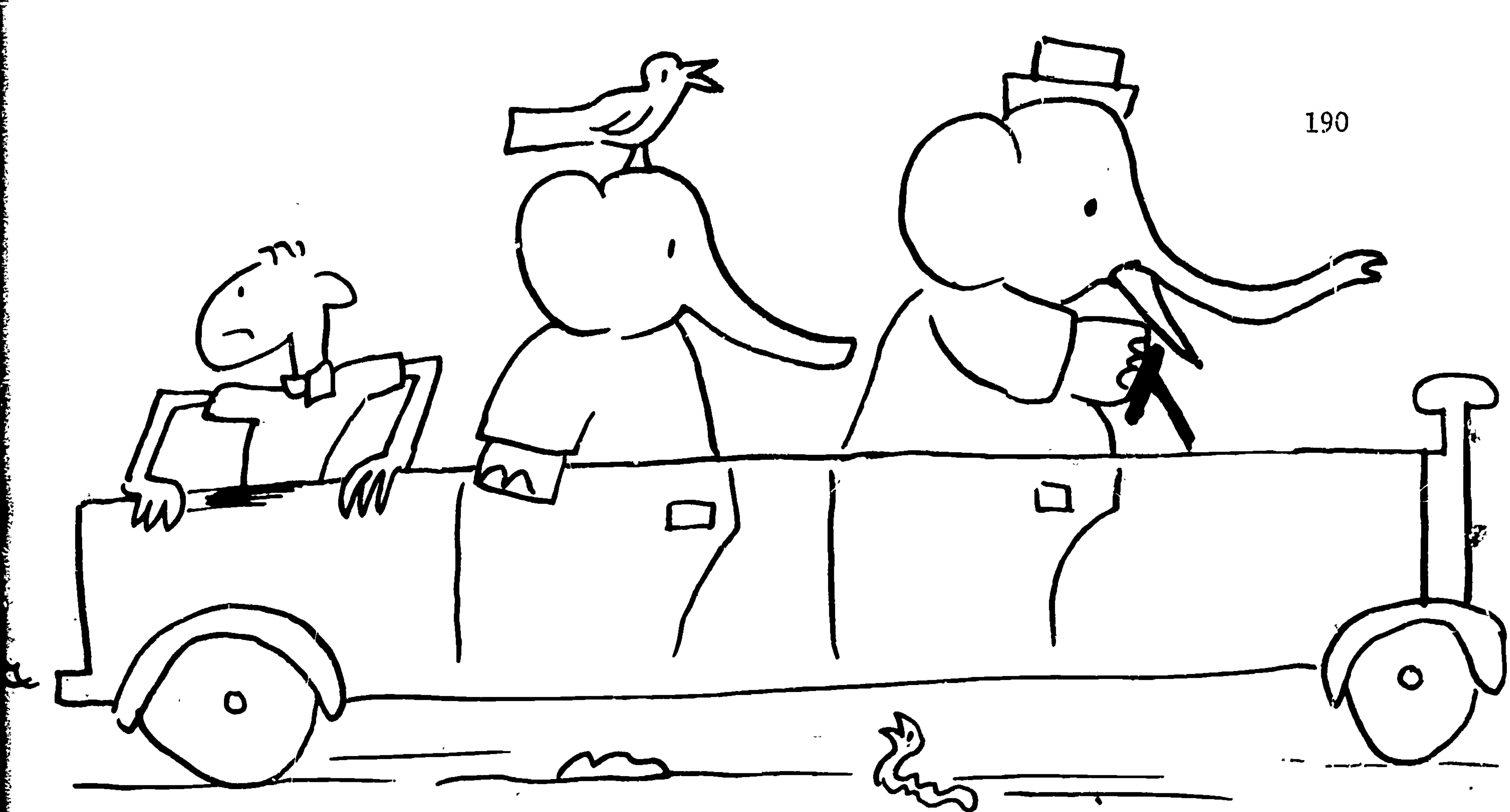
"What's wrong?" asked EMIRP.

"King Herman was on an expedition looking for new banana trees. They were high on the mountain, and Herman fell over a cliff. He tumbled several thousand feet and landed on a ledge."

"Great gobs of elephant tusk! Is he O.K.?" asked Harvey.

"We don't know. We cannot reach him."

"Let's go," said EMIRP, and they all jumped into Mr. EMIRP's car, a 1933 Monkeymobile, and sped to the scene of the tragedy.



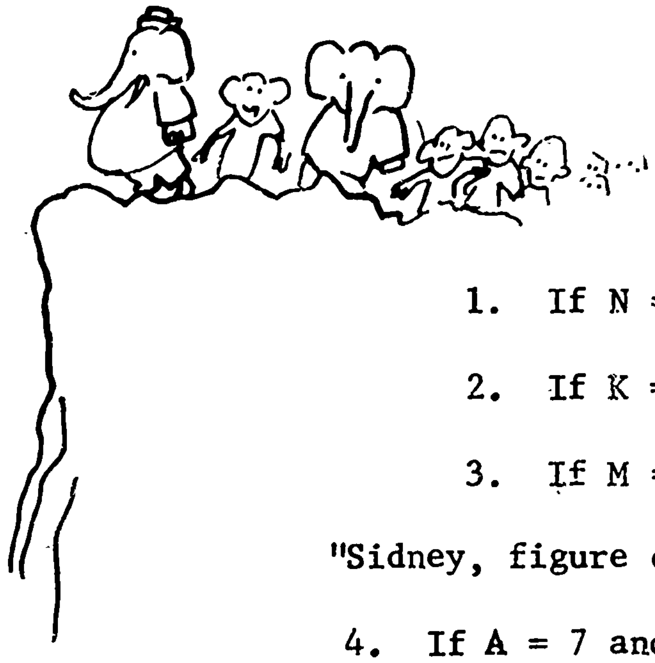
EMIRP went into action immediately. "Bird, fly down to that ledge and see if the King is O.K.!" The bird flew away.

"Gee," said an old monkey, "Why didn't I think of that?"

"Brains," replied EMIRP, pointing to his head.

The bird soon returned. He was frantic. "You must hurry. The King is O.K. but the ledge under him is very weak and could break at any minute. And if it breaks,....." The bird broke out in tears at the thought of it.

EMIRP went into action. "Now lets see. That ledge is made of limestone and appears to be about 3 inches thick.



"Herman weighs about 20 pounds, so ... quick, Harvey! Solve these problems for me:

1. If N = 7, what is 7 x N?

2. If K = 5, what is (8 x K) + 10?

3. If M = 8, what is (M x M) - 5?

"Sidney, figure out these problems:

4. If A = 7 and B = 6, what is A + B?

5. If T = 16 and S = 30, what is T + S - 3?

6. If H = 0 and G = 13, what is H + G + 3?

They gave the answers to EMIRP in a flash. His computer-like mind went to work, and he said, "If my calculations are correct, we have exactly 5 minutes and 31 seconds before the ledge breaks."

"Mr. EMIRP," yelled the monkey, "couldn't we tie ropes together to make a long rope, then lower it to the King.

EMIRP said, "It's about 3,001 feet. Harvey,

7. $NA + NB =$

Sidney,

8. $5A + 5B =$

They handed him the answers. "No. That will take more than 6 minutes."

"There is only one chance.

9. What prime will divide 42?

10. What prime will divide 100?

"Harvey, apply the distributive law to these expressions:

11. $3A + 3B =$

12. $NP + NQ =$

13. $AX + AY =$."

As it was a matter of life and death, they had the answers in no time.

"Hum-m-m," said EMIRP. "Just as I expected. Bird, how many birds live nearby?"

"Oh, gosh. Lets see...5,981 in the last census."

An old mother monkey shouted, "Hurry EMIRP. There is only 3 minutes left."

"Get the volley ball net from my car. Harvey, are these numbers divisible by 3:

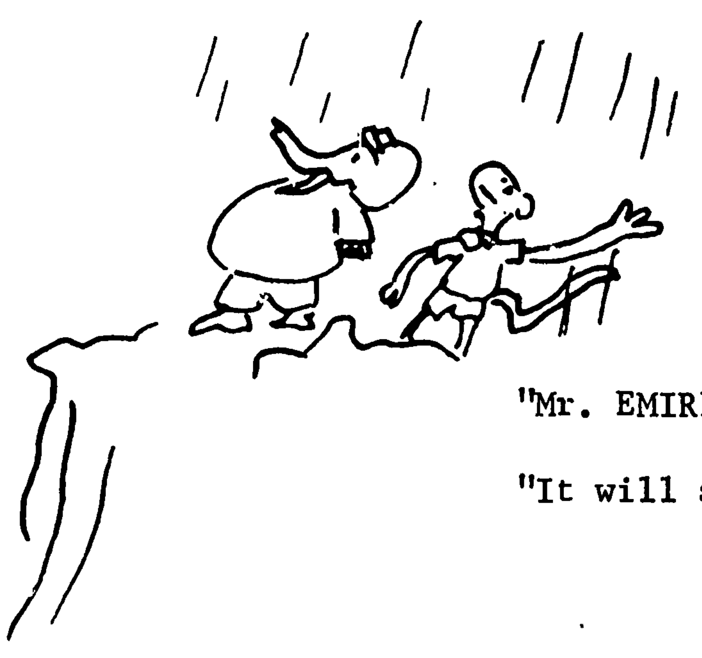
14. 1,000,000,000,000,000,000,000,000,000,000,000,011

15. 100,000,101,030

16. 213,100,701,501

Harvey and Sidney gave him the answers almost immediately.

EMIRP's mind spun like a computer. "That's it. Call all the birds here at once." Just then it started to rain. "Oh no. The water will affect the limestone and the birds' wings and ...." Quickly, EMIRP's mind recalculated all the figures.



"Mr. EMIRP, we have 1 minute left."

"It will still work," shouted EMIRP.

"All you birds grab ahold of the volley ball net. Fly down to the ledge and let Herman climb into the net. Hurry!" As the birds flew down for the rescue, EMIRP said, "I didn't have time to count the birds. If there are 5,953, the King will be saved. If fewer ...."

Were Sidney's and Harvey's answers correct?

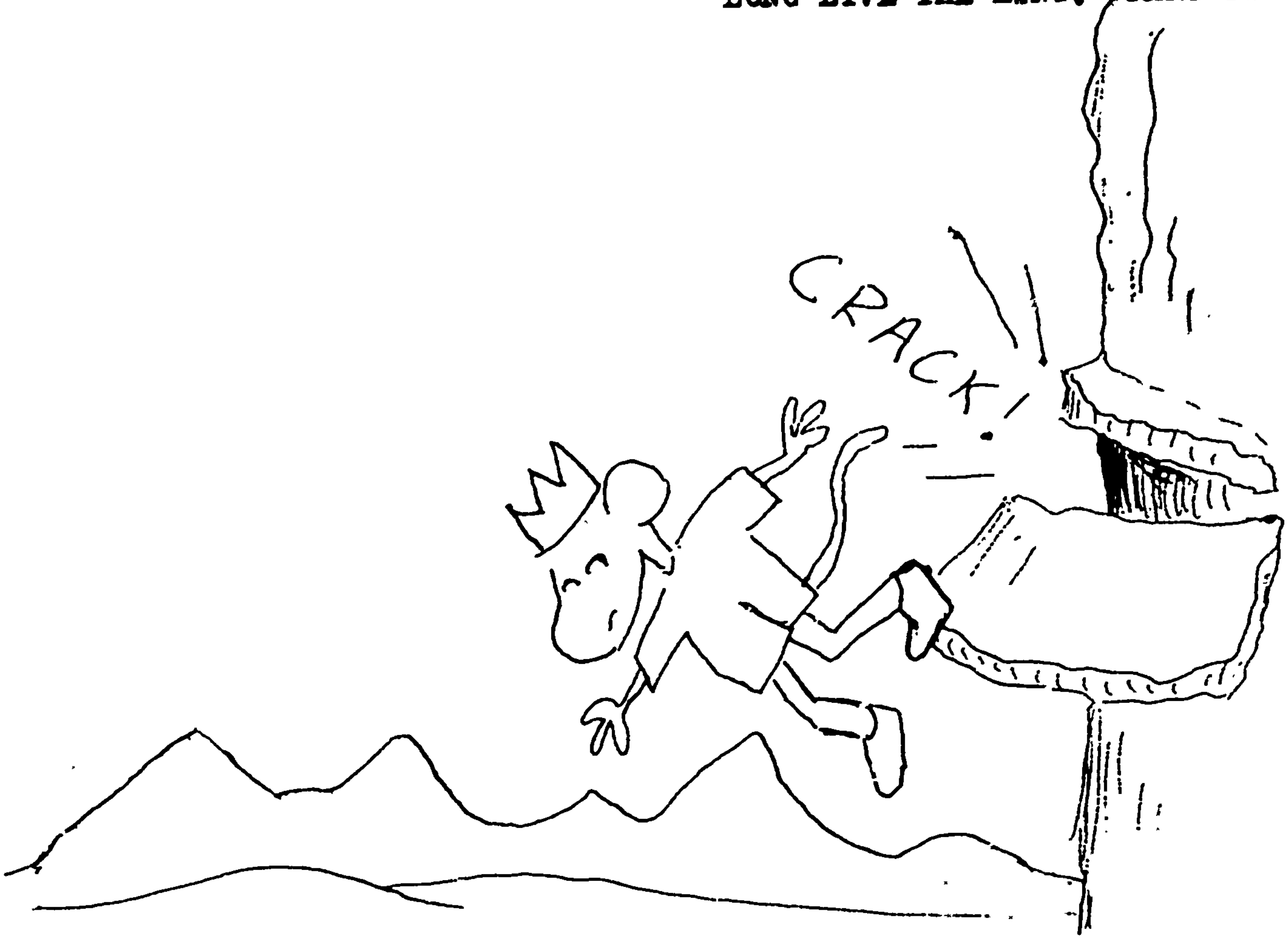
Did EMIRP make a mistake in his calculations?

Is there enough time left?

Are there at least 5,953 birds on the volley ball net?

We'll find out in the next episode.

LONG LIVE THE KING: PART TWO



Just before the birds reached the ledge, it broke. Hearing the ledge crack, everyone closed his eyes. They did not want to see their beloved King fall into the Hississippi River and drown.

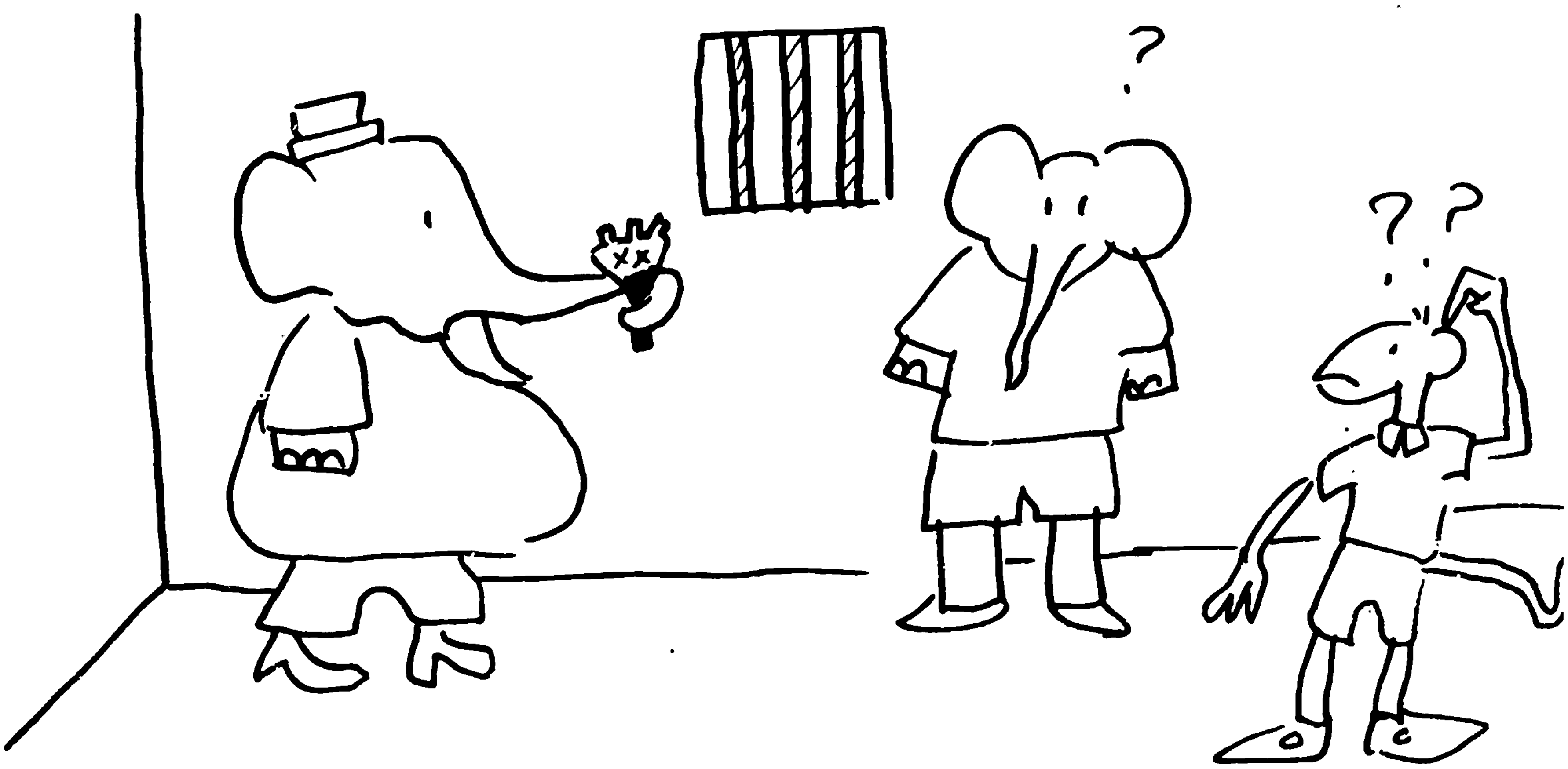
EMIRP had failed. And to make matters worse, Gronk was now King of Monkeyville. He stood before the monkeys and said, "EMIRP has deliberately killed our dear sweet King. These three must be punished."

Inside the Monkeyville jail, EMIRP and the boys discussed their plight. "Let's recalculate our figures." They did, and they were all O.K. "That's strange," said EMIRP, "mathematics never lies. I suspect foul play."

He reached into his pocket and pulled out a strange looking gadget.

"What is that?" asked Harvey.

"It's an emergency jail-breaker. I always carry it with me in the event of just such an emergency.



All I have to do is set these 3 dials at the proper numbers, then pull this plug. Quick, if you can give me the answers to these problems, I can set the dials:"

Apply the distributive law:

1. $7A + 7F =$

2. $TB + TV =$

3. $5R + 5W =$

4. $XY + XC =$

5. $JK + JM =$

6. TV ÷ TB =

Are these numbers divisible by 9?

7. 2,346

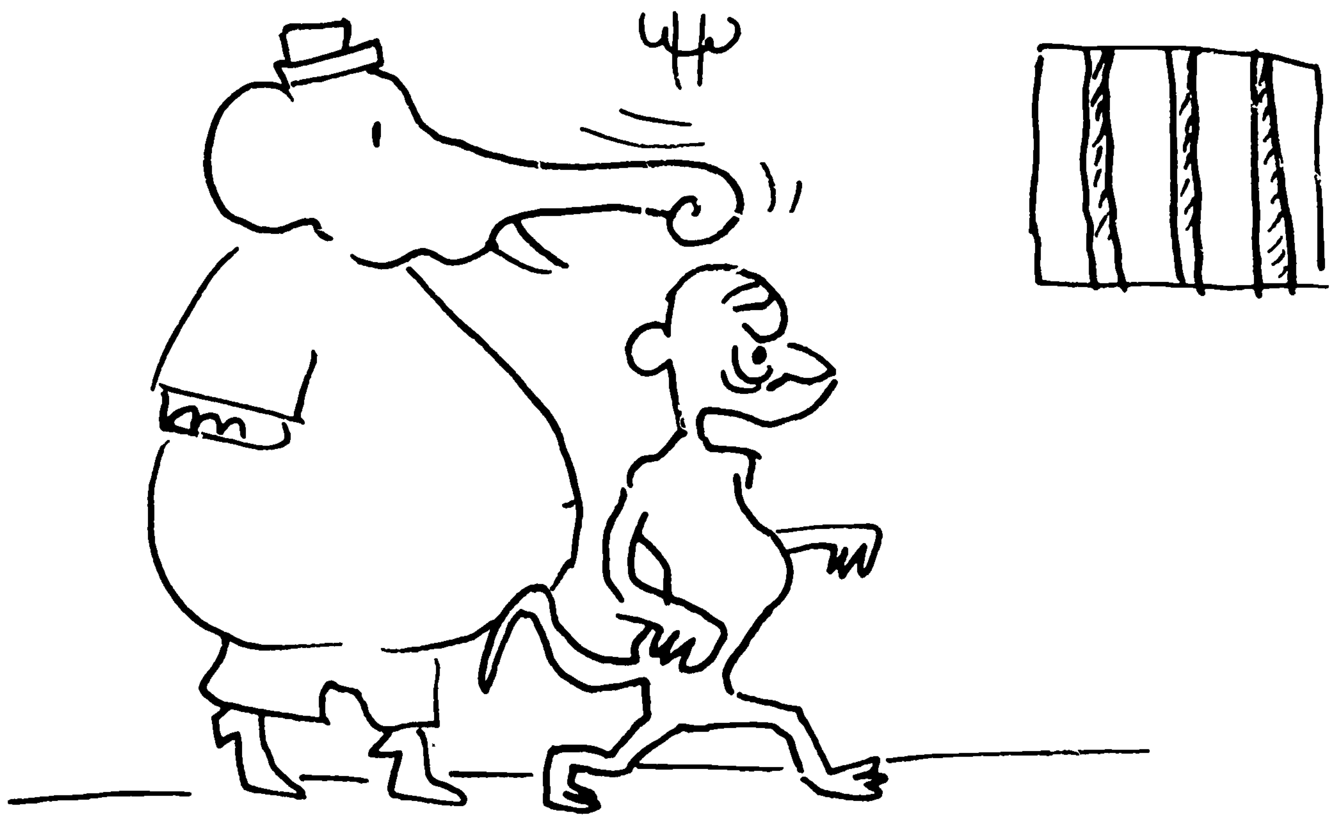
8. 11,000,000,000,000,000,000,000,000,000,000,007

Harvey and Sidney gave EMIRP the answers, and EMIRP set the dials. He pulled the plug and put it under the covers of the bed. The gadget made a wierd high-pitched noise. The guard came running to see what was happening. "What's going on in here?" he asked.

"There is a ghost in our bed," replied EMIRP.

The guard took his key and entered the cell.

"There is no such thing as a ghost," he said as he tore the blankets off the bed. Whap! EMIRP bopped the guard over the head.



In no time our three friends had made their escape.

"Now we must find out what really happened," said EMIRP. They fled into the jungle. When they got back to EMIRP's house, someone was waiting for them.

"Hello," said a large bald-headed eagle. "Are you Mr. EMIRP?"

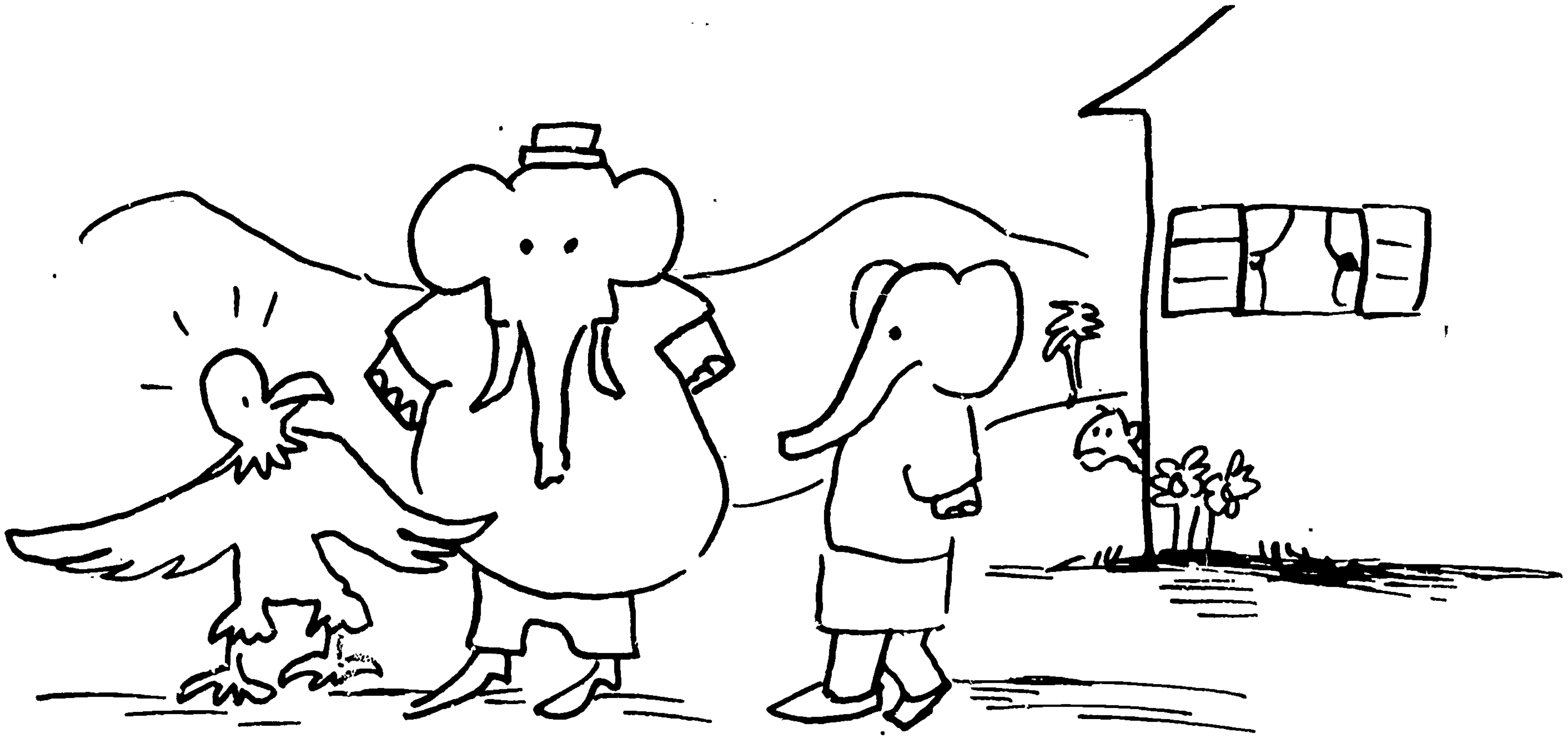
"Why yes," replied EMIRP.

"Then come with me," ordered the eagle.

"Are you the police?" asked EMIRP.

"Oh no," said the eagle. "I have just come from King Herman. He wants to see you. Follow me."

King Herman? But he is dead. Or is he? Is this some sort of trap? We'll find out in our next episode.



LESSON TWO

THE NEED FOR PROOF IN MATHEMATICS

The purpose of this lesson is to motivate the need for proof. The point will be made that particular instances of a general theorem do not constitute a proof, and that some other type of proof is required.

BEHAVIORAL OBJECTIVES: None

MATERIALS:

1. Blank tables to evaluate $N^2-N + 11$
2. Desk Computer
3. List of Prime Numbers
4. Instructions for Programming Computer

PROCEDURE: The students have studied prime and composite numbers earlier in the year. A brief review should prepare them for this lesson.

Ask if anyone can give a correct definition for "prime number." After a response is given, write the definition on the board:

A prime number is any whole number which has exactly two divisors, 1 and itself.

Note that this definition would not be quite correct if the students had studied both positive and negative integers for, in that case, we would have to consider negative divisors. The definition would then be:

A prime number is any whole number which has exactly two positive divisors, 1 and itself.

Do not discuss this with the students unless the question is raised by the students.

Ask for a definition of composite number, then write it on the board: A composite number is any whole number which has more than two divisors.

It should be pointed out to the students that 1 is neither a prime nor a composite according to our definition because it has only one divisor, namely itself.

Distribute the tables for evaluating the expression $N^2 - N + 11$. "Mr. Emirp wanted to find lots of prime numbers. He claimed that the expression $N^2 - N + 11$ always yielded a prime number when it was evaluated for any whole number. For example, if $N = 0$, then $0^2 - 0 + 11 = 11$, and 11 is a prime number." Write the evaluation on the board and ask the students to verify that 11 is prime by checking to see if it is on the list of prime numbers.

The sheets have room to evaluate $N^2 - N + 11$ for all whole numbers through 10. Assign each student the task of evaluating $N^2 - N + 11$ for one of the numbers from 1 to 10. Ask each to read his answer. Write his response on the board and ask the students to check each against the list of primes. Make certain that their calculations are correct. The answers should be 11, 13, 17, 23, 31, 41, 53, 67, 83, and 91. Then say, "Well, I guess that proves it. It works for all these numbers, so it must work for all numbers. Right?"

If no scepticism is voiced, say "Let's check it out for one more number just to be certain. Evaluate $N^2 - N + 11$ for 11." This will give the composite number 121. Point out that it is composite because 11 divides it. Then say, "When it was pointed out to Mr. Emirp that N^2-N + 11 will not always give a prime, Sidney claimed that N^2-N + 17 will always give a prime. To help us see if he is correct, we will use a computer." Punch in the program which will evaluate N^2-N + 17 for N = 0, 1, . . . 16, and get the answers. Write them on the board, and have the students check them out against a list of prime numbers. Then say, "Well, we get a prime everytime, so I guess this proves that Sidney was correct." Try to get the students to agree. Whether or not scepticism is voiced, suggest that it be evaluated for N = 17. Hopefully one of the students will make the suggestion. 17^2-17 + 17 = 17^2 = 289, which is clearly composite because 17 divides it. "N^2-N + 17 doesn't always yield a prime, but when I pointed this out to Sidney, Harvey said that N^2-N + 41 always gives a prime." By this time it is hoped that a student suggest trying 41 immediately. Program the computer to evaluate N^2-N + 41 for N = 0, 1, . . . 40. It will again yield primes. "Well, that proves it. It works for 41 numbers, so it works for all numbers. Right?" By this time the students should realize that giving particular instances of a theorem does not prove it. N = 41 will not give a prime. The point should then be made that we can never be sure of generalizing from repeated instances of an event. Stress that while such reasoning gives us hunches as to what might be true, the conclusions are only probable, and not certain. In mathematics probable conclusions are not always good enough. We demand proof. Just because something works for 100 or 200 cases is not proof that it will always work.

JOURNAL OF LESSON TWO

Monday, April 21, 1969

With twenty minutes remaining in the period, the teacher began lesson two. She asked for a definition of prime and composite number. "Any number which has only two divisors," was the response. The teacher agreed that the student had the right idea and wrote the precise definition on the board. The teacher handed the students a list of all prime numbers less than 10,000. The students expressed surprise at the size of the list. "All these are primes?" The teacher asked, "How many primes are there?" "They just keep going," was the reply. The teacher then presented the lesson designed to motivate the need for proof. She evaluated $N^2-N + 11$ for $N = 0$. They asked each student to evaluate it for a different number, thus exhausting the numbers from 1 to 10. The answers were then checked with the computer. They were all correct.

When all the answers were verified, the teacher said, "So this will always give us a prime number. Right?" [She did not say, "Since it works for N = 0, 1, 2, . . . 10, this proves that it works for all numbers. Right?"] The students agreed, but one student suggested trying it with some more numbers. The teacher suggested N = 11. It was then concluded that it did not always yield a prime number. [The point was not made that 10 cases did not prove that it always works.]

The teacher then hypothesized that $N^2-N + 17$ would always yield a prime. She programmed the computer and read off the answers for N = 0, 1, 2, . . . 16. The students verified that the answers were prime. Before the teacher could say anything, several students said, "Try it for 17."

The teacher said, "It works for all these. Does that prove it works for all numbers?" "No, try it for 17!" was the response from almost everyone. "If it works for 17, will you believe me?" Some said "yes" and others "no." When N = 17 produced a composite number, two girls immediately shouted that it was not prime. "Are you sure? Everybody check it because these girls were trying to say that." It was concluded that 289 was composite. Two girls laughed, and another girl admonished them, saying: "Don't laugh at her. She probably knew it wasn't prime." [Again the point was not made that 17 instances did not prove the result. Instead of letting the students form the hypothesis, the teacher did. When she was wrong several students thought she had made a mistake.]

When N^2-N + 41 was suggested, the teacher said, "This is the one that will work?" [Instead of leaving it an open question, the teacher stated that it would work.] "Try 41," was the immediate response. While she was programming the computer, one student worked out the answer. Some students thought it would give a prime [the teacher had said it would], and others thought otherwise. 1681 was read as the answer by the student who had calculated it and this was confirmed by the computer. The students then tried to explain the procedure. "If you had 199, it would work all the way up to 199, but would not work for 199." [The students attention was focused on the form of the expression and not on inductive reasoning which leads to a mistake.] "It always works for a double number." The discussion drifted and the teacher did not explain that N^2-N + C did not always yield primes for all numbers less than C. A student asked, "Why did we do this?" The teacher said that we were going to prove things. [Again, the point was not made.]

The students were then given a practice sheet on divisibility, Part one of "Long Live the King," and were asked to study for the mastery test which would be given on Monday. The students were enthusiastic about the Emirp story.

ANALYSIS: The motivation for the need for proof fell short of the mark. Rather than having the students formulate the hypothesis and then torpedoing the hypothesis, the teacher would say, "I know this will always give a prime." At several points in the lesson some of the students thought that the teacher was actually making unintentional mistakes. At the end of the activity a student asked, "Why did we do this?" The teacher did not adequately explain.

Two factors probably contributed to the failure of this lesson. First, the teacher had never taught the lesson before. We had gone over it on two separate occasions, but both times the investigator had played the role of the teacher.

Second, a lesson of this type is difficult to teach. It requires that the teacher finesse the students into formulating a hypothesis, and then torpedoing that hypothesis. In the process students will make many replies, some of which are pertinent to the objective of the lessons, and some of which are irrelevant. The teacher must be able to distinguish between the two in order to get the point of the lesson across. For example, at one point in the lesson the teacher let the students drift into a discussion of $N^2-N + C$, C a constant. Several erroneous notions were suggested without correction (such as $N^2-N + P$ will yield primes for all integers less than P for any prime numbers P). It was never pointed out that $P - P = 0$. The attention of the students was thus diverted from the objective to be learned.

In my opinion these difficulties stem from the fact that the teacher is presenting ideas which are at the very frontier of her knowledge. She does not have an overview of proof, she does not see the large picture. To correct this weak lesson we decided to present another discussion of inductive reasoning in an effort to motivate the need for proving mathematical statements. A sheet with examples of inductive reasoning was prepared for this purpose. We decided to incorporate the motivation for proof as an introduction to theorem 1 (Tuesday's lesson).

To prevent recurrences of this type of ineffective lesson, it was decided that the teacher would teach each lesson to the investigator prior to entering the classroom. In this role-playing activity the investigator would ask all sorts of questions in an attempt to prepare the teacher for the types of questions the students might ask.

Time was also a factor. This lesson was started half way through the period. The teacher felt rushed. It was decided to avoid starting a new lesson or activity unless there was adequate time to complete it.

Directions for programming the EPIC 3000 to evaluate $N^2 - N + 11$

MANUAL	punch 11
	store c
LEARN	punch 0
	enter
	rpt
	rpt
	X
	I
	-
	rec c
	+
	print
AUTO	punch 1
	enter
	punch 2
	enter
	.
	.
	.
	punch 10
	enter
	punch 11
	enter

The same program will work for $N^2 - N + 17$ and $N^2 - N + 41$ if 17 or 41 is entered instead of 11.

LESSON THREE

THEOREM 1: IF $N \mid A$ AND $N \mid B$, THEN $N \mid (A + B)$

The purpose of this lesson is to teach the students to write a proof for this theorem.

BEHAVIORAL OBJECTIVES:

1. The student can write a proof for Theorem 1.
2. The student can give numerical examples to illustrate Theorem 1.
3. The student can apply the theorem to given divisibility facts.

MATERIALS:

1. Six incomplete or incorrect proofs
2. Poster board with proof written on it
3. Sheets with examples of inductive reasoning
4. Sheets with applications
5. Blank sheets for proving Theorem 1.
6. Mastery Test
7. Parts 3 and 4 of "Long Live the King"

PROCEDURE: Distribute the sheets with examples of inductive reasoning. The purpose of this activity is to stress that repeated occurrences of an event do not constitute proof. Items 2 and 7 are reasonable, whereas the others are not. The point should be made that while the results of inductive reasoning may be reasonable, the examples do not prove the result. Discuss the examples orally. Some students might argue that Items 2 and 7 are proven by the large number of instances. For item 2, ask "Isn't it possible that some day a monkey will catch smallpox after he has been vaccinated?" For item 7, ask "Isn't it possible that the die is <u>not</u> loaded and that it came up the same way 100 times purely by chance? It could happen, you know."

When all items have been discussed, start the discussion of Lesson Three.

"We are now going to prove some facts about division." Write

$2|4$ and $2|6$. Does $2|(4 + 6)$?

$2|6$ and $2|10$. Does $2|(6 + 10)$?

Give four of five such examples. If no one generalizes and suggests that this will always work, ask if anyone sees a pattern. Then write the statement:

IF $2|A$ AND $2|B$, THEN $2|(A + B)$.

"Do these five examples prove that this will always be true?" Hopefully they will respond "no". If they say "yes", call their attention to the previous discussion on inductive reasoning. Then stress that we want to write a proof for this fact. "Let us look at a numerical example to see why it is true. We will prove:

IF $2|6$ AND $2|8$ THEN $2|(6 + 8)$

We already know that this is true, and we really don't have to prove it, but it will show us how to prove it for any number. We will make statements (write "statement" on the board), and for each statement we will give a reason (write "reason" on the board).

IF $2|6$ AND $2|8$, THEN $2|(6+8)$

STATEMENT	REASON
1. $6 = 2 \times 3$ $8 = 2 \times 4$	1. Definition of Divides
2. $(6+8) = (2 \times 3) + (2 \times 4)$	2. Substitution
3. $(6+8) = 2(3+4)$	3. Distributive Law
4. $2|(6+8)$	4. Definition of Divides

As you write this on the board, explain each step: "$2|6$ means $6 = 2 \times 3$ and $2|8$ means $8 = 2 \times 4$. So lets begin by writing these equations. We want to prove something about $(6 + 8)$, and so let us write $(6 + 8)$. But $6 = 2 \times 3$, so lets substitute and $8 = 2 \times 4$, so we can substitute. But look, now we can apply the distributive law. What does this last equation say? It says that $2|(6 + 8)$, which is what we wanted to prove." Show that it will also work for another example (without writing a formal proof).

$$(22 + 10) = (2 \times 11) + (2 \times 5) = 2\ (11 + 5)$$

"Which shows that $2|(22 + 10)$." Then write the general proof (leave the other proof on the board):

IF $2|A$ AND $2|B$, THEN $2|(A+B)$

STATEMENT	REASON
1. $A = 2P$ $B = 2Q$	1. Definition of divides
2. $(A + B) = 2P + 2Q$	2. Substitution
3. $(A + B) = 2\ (P + Q)$	3. Distributive law
4. $2\ |\ (A + B)$	4. Definition of divides

Ask the students to supply the statements and reasons.

Repeat this procedure with 3 as a divisor. Give four or five numerical examples, then proceed with a proof for:

IF $3|A$ AND $3|B$, $3|(A + B)$.

Again, the students should supply the steps in the proof.

Then generalize "This works for 2 and 3, will it work for any number?" Write some examples on the board:

$5|10$ AND $5|15$ DOES $5|(10 + 15)$?

$7|7$ AND $7|14$ DOES $7|(7 + 14)$?

Write the theorem on the board:

IF $N|A$ AND $N|B$, THEN $N|(A + B)$.

Before proving it, illustrate it with examples. Ask each student to make up one example and read it to the class. Then write the proof. Ask the students to help you construct a proof for it. Proceed as in the numerical case, making an effort to get the students to supply as much of the proof as possible. When it is completed, show several instances of the theorem with specific numbers. Write N = 3, A = 6, and B = 9. Erase every N in the proof and replace it with 3. Repeat for A and B. Replace the N, A, and B and repeat with another set of numbers. Then ask a student to come to the board and make up some numbers which will work. Have him substitute these values into the proof. You may wish to have several other students do the same thing. With each numerical example point out the reasoning of the proof.

Then distribute practice sheet #1 which contains an error in the steps of the proof. Ask the students to find the error if they can. Discuss. Practice sheet #2 has errors in the reasons. Do the same with it.

Then have the students work on the third practice sheet. When completed, quiz the students on the proof. Discuss the proof once more. Give one numerical example which illustrates the theorem and ask each student to make up an example and write it on a piece of paper. Have each student read his answer aloud.

Point out that the "plan" of the proof is to write the sum of A and B as a multiple of N. Pass out the drill sheet on application; do one example on the board, then have the students do the remainder at their desks. Circulate about the room, try to catch any errors that are made.

Then administer the mastery test.

JOURNAL FOR LESSON THREE

Tuesday, April 22, 1969

The two students who did not master the mastery test arrived ten minutes early and the teacher discussed their mistakes with them and gave them some additional problems to work.

In preparation for the lesson, two hours were spent with the teacher. On Monday afternoon the overall objectives of the unit were discussed: how the unit fits into the scheme of proof and the details of how the lesson were to be taught. The investigator acted as the teacher in presenting the ideas to the teacher. On Tuesday morning another session was held, and the teacher presented the materi to the investigator as she planned to do in the classroom. The investigator made inappropriate answers to her questions.

This preparation apparently paid dividends, for the lesson went smoothly and as planned. After passing back the mastery tests and going over the problems in the story, she passed out the sheet with examples of inductive reasoning. After two items were discussed one student remarked, "These are just like the problems we had the other day with $N^2 - N + 11$." This was somewhat gratifying for this activity was designed to make up for the poor lesson on Monday.

The teacher completed the proof for theorem 1 and had the students copy it on a sheet of paper. The students were asked to learn the proof. As the bell ended the period the teacher handed out Part Three of "Long Live the King."

ANALYSIS: The lesson went as planned and no revisions were made for the next day's lesson.

Wednesday, April 23, 1969

The teacher went over Part Three of the story, then distributed the first four incorrect proofs. Every student was able to spot the errors. Every student wrote a correct proof for Theorem 1 on the first quiz. The teacher distributed the two other incorrect proofs and the students made the necessary corrections. One of the students who had not mastered all of the prerequisites was the first to find the errors each time. He had memorized the proof the night before.

The students were given the application problems for homework, along with the final episode of Long Live the King.

ANALYSIS: A good lesson. It was decided that the mastery test should be administered at the first part of the next day's lesson.

No changes were made in Lesson 4.

Thursday, April 24, 1969

Because of an accident to a student on the baseball field, the principal was not in his office to ring the bell for classes to begin. Class started a few minutes late.

The teacher discussed the final episode of "Long Live the King." Then application sheets for Theorem 1 were discussed. The teacher went over the proof of the theorem, gave several examples to illustrate the theorem, and then administered the mastery test.

ANALYSIS: Every student received a perfect score on the Mastery Test for Theorem 1. No changes were made.

INDUCTIVE REASONING

1. Harvey visited his uncle in Los Angeles for two weeks. Each day he was there it rained.

   Does this prove that it rains every day in Los Angeles?

   Is it reasonable to conclude that it rains every day there?

2. Two million monkeys were vaccinated against smallpox. None of them got the disease.

   Does this prove that a monkey who has been vaccinated against smallpox will not get the disease?

   Is it reasonable to conclude that this is the case?

3. Harvey's little brother is learning to add whole numbers. He failed two tests on addition.

   Does this prove that he will never learn to add?

   Is it reasonable to conclude so?

4. Sid had a teacher who gave long math assignments.

   Does this prove that all math teachers give long assignments?

   Is it reasonable to conclude so?

5. The Monkeyville baseball team won its first ten games this year.

   Does this prove that they will win all their games this year?

   Is it reasonable to conclude that they will?

6. Mr. Emirp rolled a die two times and it turned up "2" each time.

   Does this prove that the die is loaded and will turn up "2" each time it is rolled?

   Is it reasonable to conclude that this is the case?

7. Mr. Emirp rolled a die 100 times and it turned up "2" each time.

   Does this prove that the die is loaded?

   Is it reasonable to conclude that it is?

8. $N^2 - N + 11$ gives a prime number for $N = 0$, $N = 1$, up to $N = 10$.

   Does this prove that it works for all numbers?

9. $N^2 - N + 17$ gives a prime number for all the numbers from 1 to 16.

   Does this prove that it will do the same for all numbers?

10. What about $N^2 - N + 41$?

THEOREM ONE APPLICATION SHEET

Theorem 1 says that if $N|A$ and $N|B$, then $N|(A + B)$. Apply this theorem to the following facts:

1. $3|12$ and $3|15$, so

2. $N|T$ and $N|R$, so

3. $713|M$ and $713|N$, so

4. $(A + B)|P$ and $(A + B)|Q$, so

5. $P|K$ and $P|L$, so

6. $7|(14 + 7)$ and $7|7$, so

   Remember $(14 + 7)$ represents <u>one</u> number, not two.

7. $A|(P + Q)$ and $A|R$, so

8. $2,375|A$ and $2,375|B$, so

9. $M|V$ and $M|W$, so

10. $V|G$ and $V|H$, so

PRACTICE SHEET # ONE

(1)

What is wrong with this proof?

THEOREM 1: IF $N|A$ AND $N|B$, THEN $N|(A + B)$.

STATEMENT	REASON
1. $A = NP$ $B = NQ$	1. Definition of divides
2. $(A + B) = NP + NQ$	2. Substitution
3. $= P\ (N + Q)$	3. Distributive Law
4. $N|(A + B)$	4. Definition of divides

(2)

Fill in the reasons:

THEOREM 1: IF $N|A$ AND $N|B$, THEN $N|(A + B)$.

STATEMENT	REASON
1. $A = NP$ $B = NQ$	1.
2. $(A + B) = NP + NQ$	2.
3. $= N\ (P + Q)$	3.
4. $N|(A + B)$	4.

PRACTICE SHEET # TWO

(3)

What is wrong with this proof?

THEOREM 1: IF $N|A$ AND $N|B$, THEN $N|(A + B)$.

STATEMENT	REASON
1. $A = NP$	1. Definition of divides
$B = NQ$	
2. $(A + B) = NP + NQ$	2. Distributive Law
3. $= N(P + Q)$	3. Substitution
4. $N\|(A + B)$	4. Definition of divides

(4)

Fill in the steps for this proof:

THEOREM 1: IF $N|A$ AND $N|B$, THEN $N|(A + B)$.

STATEMENT	REASON
1.	1. Definition of divides
2.	2. Substitution
3.	3. Distributive Law
4.	4. Definition of divides

PRACTICE SHEET # THREE

(5)

What is wrong with this theorem?

THEOREM 1: IF $N|A$ AND $N|B$, THEN $N|(A + B)$.

STATEMENT	REASON
1. $A = NB$ $B = NC$	1. Definition of divides
2. $A + B = NB + NC$	2. Substitution
3. $= N(B + C)$	3. Distributive law
4. $N|(B + C)$	4. Definition of divides

(6)

Find the errors in this proof:

THEOREM 1: IF $N|A$ AND $N|B$, THEN $N|(A + B)$.

STATEMENT	REASON
1. $A = NP$ $B = NQ$	1. Distributive law
2. $P + Q = NP + NQ$	2. Substitution
3. $= N(P + Q)$	3. Distributive law
4. $N|(P + Q)$	4. Definition of divides

THEOREM ONE: IF $N|A$ AND $N|B$, THEN $N|(A + B)$.

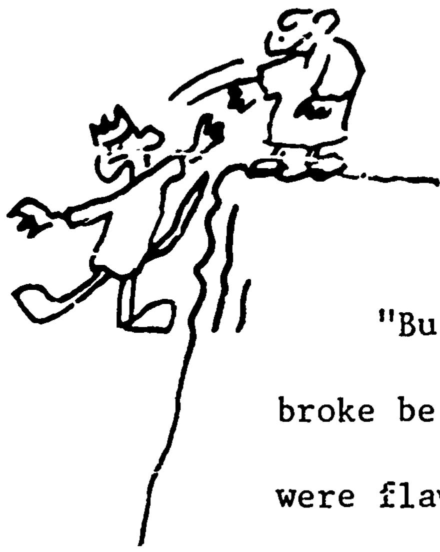
STATEMENT	REASON

Harvey, Sidney, and EMIRP followed the bald-headed eagle. He led them through dense jungle, and finally they arrived at a remote spot on the Hississippi River. "Quick! Inside this cave," said the eagle.

There, sitting on the floor, was King Herman. "Your royal highness. How glad we are to see you," said EMIRP. "All of Monkeyville thinks you are dead."

"Sit down, my dear friends," said the King, "and I will tell you all about it. I did not slip over the cliff. I was pushed by Gronk. He tried to kill me so that he could be King."

"That explains why he put us in jail -- to keep us from discovering the truth and to throw suspicion on someone else," said EMIRP.

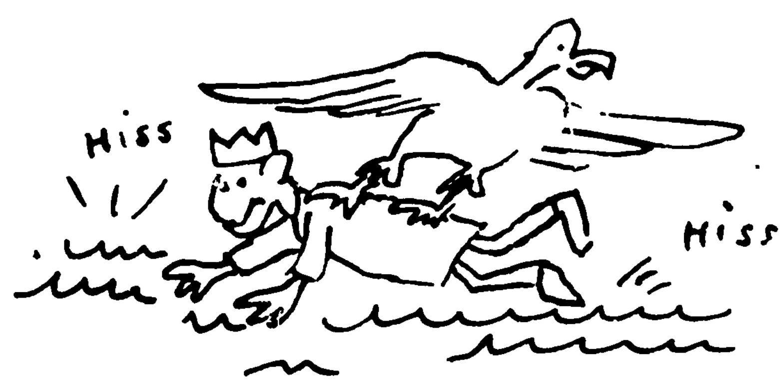


"But I still cannot understand why the ledge broke before the birds reached you. My calculations were flawless," said EMIRP.

"You're right," said the King. "When Gronk saw that you were on the scene, he knew you would save me. So he had his friends the vultures drop heavy rocks on the ledge."

"But we thought you had fallen into the Hississippi River and drowned," said Albert.

"I saw the ledge break," interrupted the eagle. "I flew as fast as I could and grabbed Herman just before he reached the deadly water."



"Amazing!" said Mr. EMIRP.

"EMIRP, you must help me get rid of Gronk."

"Yes," said the eagle. "Gronk and his gorillas have taken over. He forces the monkeys to gather bananas 7 days a week, instead of just on Bananaday."

"What does he plan to do with all those bananas?" asked Harvey.

"Sell them to Cuba. He plans to save

up enough money to live the rest of his life in complete luxury.

"But don't the monkeys complain?"

"Those who complain are put in a concentration camp," replied the bald-headed eagle. "Will you help us?"

"It is late. Let me sleep on it. In the morning I'll let you know what I plan to do," said EMIRP.

Before retiring for the evening, EMIRP gave some problems to Harvey and Sidney. He did not want them to get behind in their studies.

1. $(3 \times 94) + (3 \times 6) =$
2. $(7 \times 998) + (7 \times 2) =$
3. If $P = 6$, then $(2 \times 3 \times P) + 1 =$
4. If $K = 2$, then $(K \times K) - K + 11 =$
5. $AB + AC =$
6. $3T + 3V =$
7. IF $A=3$, then $2 \times 3 \times A =$
8. $AX + AY =$
9. If $A = 15$ and $B = 6$, then $A + B =$
10. If $M = T$ and $N = 6$, then $M + N =$
11. IF $T+S=K$ AND $K=10$, Then $T+S=$

When the others awoke in the morning, EMIRP was gone. They found a note which said:

> Dear Friends,
>
> I have gone to Monkeyville. Do nothing until you hear from me.
>
> Signed,
>
> Emirp

That evening as all the monkeys lined up to get their weekly supply of bananas, EMIRP stepped in front of the crowd. Gronk was about to have him arrested when EMIRP spoke.

"Dear friends," he shouted. "Under the rule of King Gronk our village has prospered. Things are better than they have ever been. To properly honor our great King, I proclaim tomorrow a holiday. We shall celebrate it every year. It will be called G-Day. G for Gronk."

"Yeah," said Gronk's gorillas.

"We will have a big celebration. And I shall personally build a monument to honor our great King."

Gronk was quite happy. "Yes," he said, "it shall be as EMIRP says."

Has EMIRP gone mad?

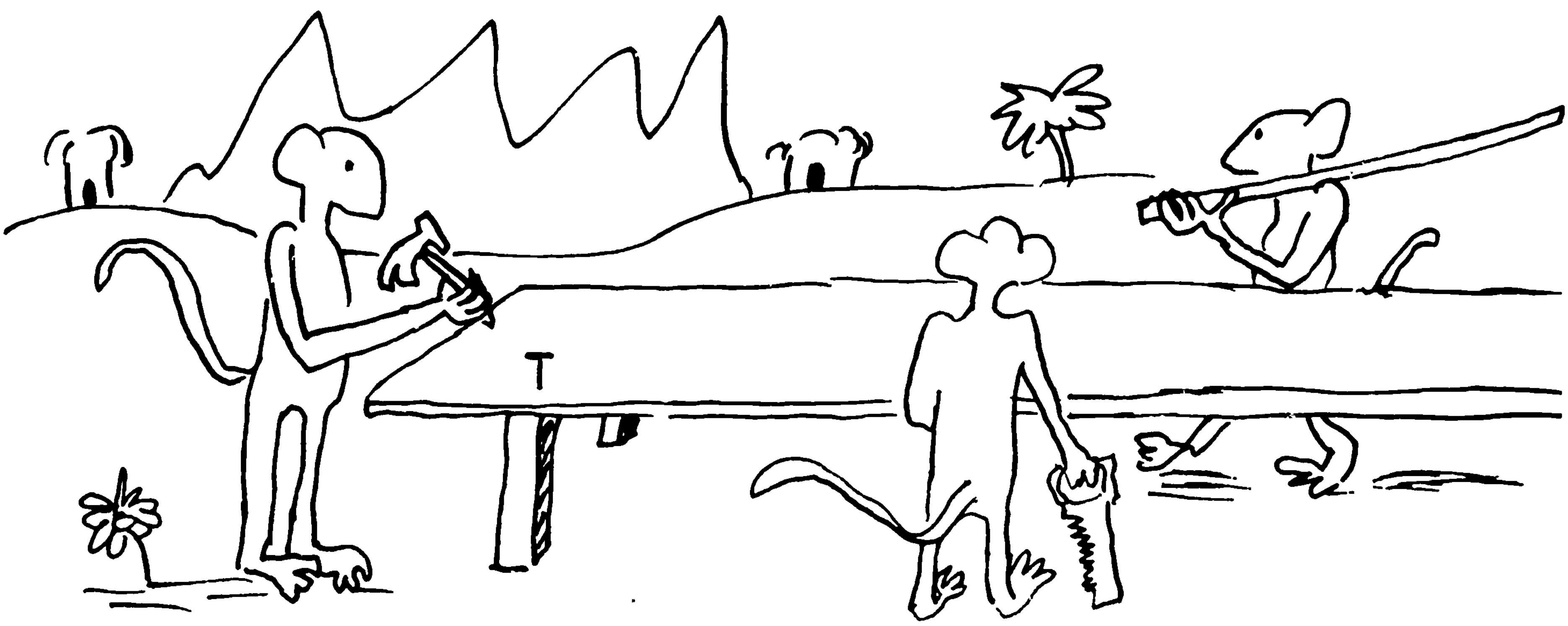
How could he praise such a scoundrel as Gronk?

Has he betrayed Herman?

We'll find out in the next episode.

LONG LIVE THE KING: FINAL EPISODE

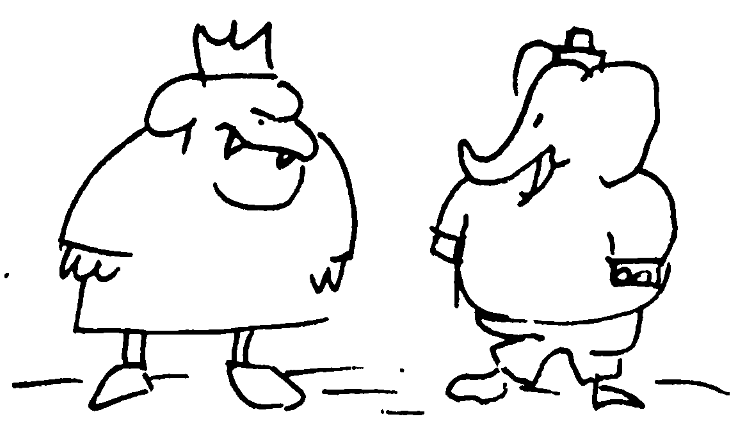
EMIRP made plans for the big celebration. A large banquet table was made for Gronk and the gorillas. Flags and banners were



made. Food was prepared. EMIRP had the entire village working.

He then went to Gronk and said, "I am going to leave now so that I can build the monument in privacy. I don't want anyone to see it so that it will be a big surprise."

"Oh goody, goody gumdrop," said Gronk. He liked surprises.



When EMIRP got to his house, he began to work immediately. "Lets see," he said to himself. "What will I need?" It became obvious that he would have to work the following problems:

1. If 3 is a factor of A + B, then A + B =
2. 3A + 3B + 3C =
3. AB + AC + AD =
4. If T = 6, then T + 11 =
5. If K = 2, then V + K =
6. Is 1,000,000,000,000,000,000,001 divisible by 3?
7. Is 360, 405, 117, 081, 333 divisible by 9?
8. If N = 5, then (N x N) + 11 =
9. WHAT PRIME NUMBER WILL DIVIDE 35?
10. What two prime numbers will divide 100?
11. IS 47 A COMPOSITE NUMBER?

At last, G-Day arrived. There was a parade, sack races, and a banana pie-eating contest. When it came time to eat, Gronk and his gorillas were seated as guests of honor at the banquet table. "Oh great King, we have prepared a meal fit for a King," said EMIRP. A huge bowl of kool-aid was on the table, and all of Gronk's gorillas were given a large glass of kool-aid. "I propose a toast," said EMIRP as he raised his

glass. "To Gronk. May he get everything that's coming to him." The King and his gorillas drank the kool-aid.

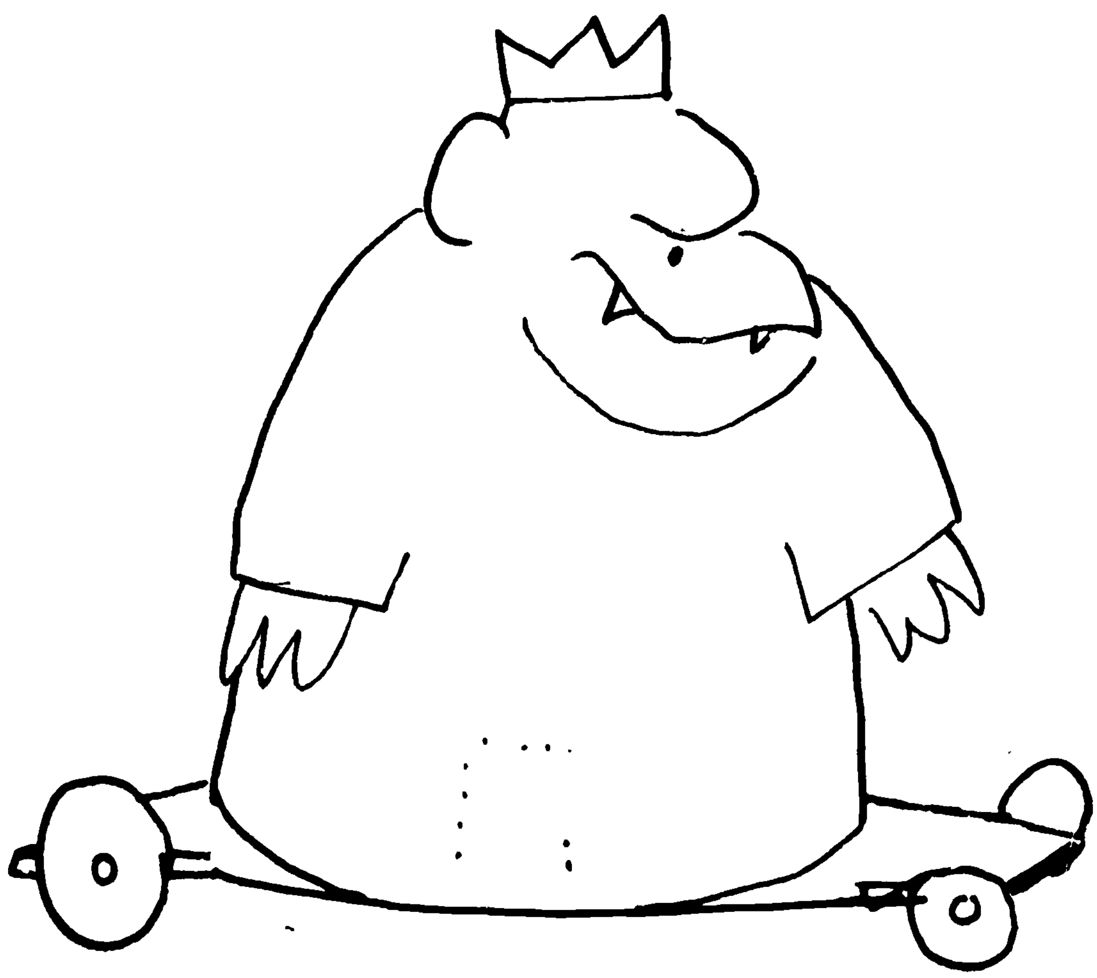
"And I propose a toast," said Gronk. Their glasses were filled again. "To Mr. EMIRP. To have recognized how great I am, he must be the wisest animal in the jungle." Again Gronk and his gorillas drank the kool-aid. "That's very good kool-aid," said Gronk. "What flavor is that?"

"Mostly cherry," answered EMIRP, "but its a special mixture for the occasion. A very special mixture."

The meal was fanastic. Banana roast, fried bananas, whipped bananas, banana salad, banana soup, banana cream pie, and banana ice cream sherbert.

After the splendid dinner, EMIRP stood up and said, "And now for the biggest surprise of all. Bring it in, boys!"

A group of monkeys wheeled in a large wooden statue of Gronk. It was mounted on wheels.



It was the proudest moment in Gronk's evil life.

"You are a true genius, EMIRP," said Gronk. "You shall be rewarded for this."

"I know," replied EMIRP. He then spoke in a loud voice so that everyone could hear. "I would like to dedicate this monument to the happiness of Monkeyville. I have written a special poem for this occasion." He read,

"Violets are blue,
Roses are red,
Gronk is a crook,
and Herman's not dead."

At that moment a secret door opened in the belly of the statue, and out hopped King Herman. They monkeys could not believe it. Tears came to their eyes. How happy they were. He had been a kind and loving King. Gronk was a demon. Then cheers broke out. "Hurray for Herman!

Long live our real King!"

"Quick," said Gronk to his gorillas. "Arrest that imposter!"

But the gorillas could not move. It was as if their legs had been frozen.

"Sorry bout that," said EMIRP. "But I spiked the kool-aid with special pills which make it impossible to walk."

"Hurray for EMIRP!" shouted the crowd.

Then King Herman spoke. "Take those devils to jail."

"It will be our pleasure," shouted the crowd.

That night they celebrated by burning the statue of Gronk. Happiness returned to Monkeyville. The monkeys were freed from the concentration camp and Gronk and his gorillas were given a 99-year sentence in jail.

We'll see what happened to Sidney and Harvey in our next story.

LESSON FOUR

THEOREM 2: If $N|A$ and $N|B$, then $N|(A-B)$.

The purpose of this lesson is twofold: (1) to see if the students can write a proof of this theorem without help. This should be a transfer test, for they have not been taught that multiplication distributes over subtraction. It should also indicate whether or not the student understood the strategy of the proof of Theorem 1. If they did, they should be able to prove this result; (2) to teach the students (if necessary) how to prove this result.

BEHAVIORAL OBJECTIVES:

(1) The student can write a proof for Theorem 2.

(2) The student can give numerical examples to illustrate Theorem 2.

(3) The student can apply the theorem to given divisibility facts.

MATERIALS:

(1) Sheets with the statement of Theorem 2 on them.

(2) Poster board with the proof of Theorem 2 on it.

(3) Four incorrect or incomplete proofs.

(4) Application drill sheets.

(5) Mastery test.

PROCEDURES:

Begin by writing some numerical examples on the board.

$7|28$ and $7|14$, does $7|(28 - 14)$?

$2|16$ and $2|4$, does $2|(16 - 4)$?

$10|100$ and $10|60$, does $10|(100 - 60)$?

For each example ask the students the question. When three or four examples have been given, ask, "Can anyone generalize from these examples and state a theorem which these examples illustrate?" If no correct response is given, say, "Can anyone state a theorem about these facts which is similar to Theorem 1?" Point to the poster board with Theorem 1 written on it. The students should have no difficulty in forming the theorem, but if they do, state it for them and give more numerical examples.

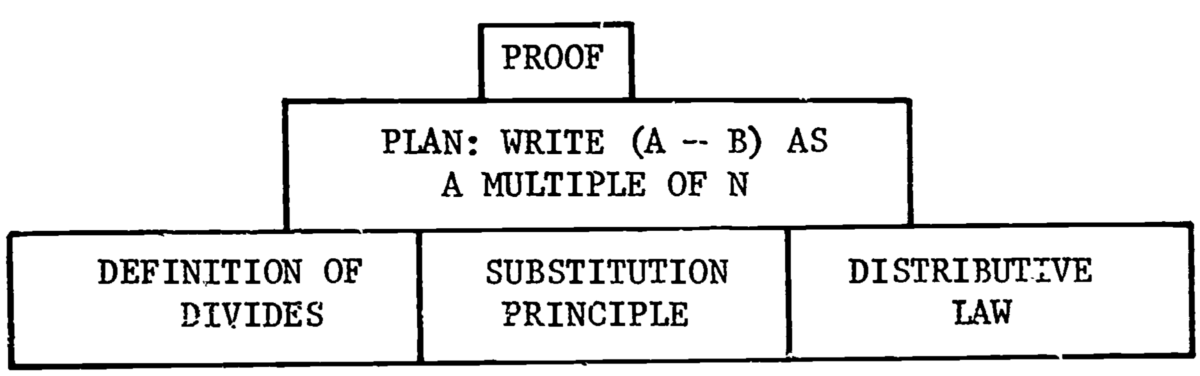
Then hand out a sheet with the theorem written on it. "Here is the theorem. Can any of you write a proof for it?" Make sure the poster board with the proof of Theorem 1 is facing the board and that all other materials are covered. The object of this activity is to see whether or not the students can write this proof without any help. If some students do write a valid proof, ask them to turn it face down. If not all the students write a valid proof, give a hint. "How does this theorem differ from Theorem 1? The only difference is a minus instead of a plus sign. In Theorem 2 we want to prove something about (A-B) instead of (A + B)." If any students write a valid proof, ask them to turn it over.

If some students have still not written a proof, give one last hint. Turn the poster with the proof of Theorem 1 on it so that it can be seen by the class. "The proof for Theorem 2 is very similar to the proof for Theorem 1. See if you can use this proof to guide you." Give them a chance to write the proof.

Then place the poster with the proof on the chalkboard tray and go over the proof. Place it beside the proof of Theorem 1 so that the similarities and differences can be compared. Stress the following points: (1) the only difference is the minus sign; since we want to prove something about (A - B), we begin with that expression; (2) the plan is to rewrite (A - B) as a multiple of N.

Then pass out the sheets which have Theorem 2 on it, have the students put all other materials away, and quiz them on the proof of the theorem. The poster board with the proof written on it will be on the side wall. Tell them they can look at it if necessary, but they should try to write the proof without looking. In this way it will be obvious which students do not know the proof, for they will have to turn their heads to see the poster. Go over the proof again, then do work sheets #3 and #4. Repeat the quiz. Then do work sheets #5 and #6.

The proof should now be analyzed in terms of the strategy employed. Use the word "plan" when discussing the strategy to the students. Draw the following diagram on the board:



"To prove Theorem 2 we used three basic things: the definition of divides, the substitution principle, and the distributive law. These were ideas we used, but HOW we used them was the crucial thing. We had to apply each of them in the proper order. The proof itself consists of the reasoning procedures we used in writing down these facts in the correct order. We have followed a PLAN. The plan is to write (A – B) as a multiple of N. As we write each proof in this unit try to think of the plan we are using. This might help you learn to prove the theorems."

Finally administer the Mastery Test. This lesson should take two days, so hand out Parts Three and Four of "Long Live the King" for homework.

JOURNAL FOR LESSON FOUR

Thursday, April 24, 1969

After the Mastery Test for Theorem 1 was administered, the teacher proceeded to see if the students could prove Theorem 2 without any further instruction. She began by writing three examples on the board:

$$3|12 \text{ and } 3|6. \text{ Does } 3|(12 - 6)?$$

$$7|28 \text{ and } 7|7. \text{ Does } 7|(28 - 7)?$$

$$10|50 \text{ and } 10|10. \text{ Does } 10|(50 - 10)?$$

The students responded with "yes", and one student asked, "But what if the little number is first?" The teacher then made the restriction that the first number must be at least as large as the second number. When she asked for someone to state the theorem which these examples illustrated, four students raised their hands. A student stated the theorem correctly.

The students were asked to prove theorem 2: if $N|A$ and $N|B$, then $N|(A - B)$. Eight of the ten students did so without any hints. The other two wrote a correct proof when the teacher suggested that the proof was similar to the previous proof.

The students then wo the four proofs which were either incorrect or incomplete. All students were able to catch the errors. The period ended.

ANALYSIS: Because of the ease with which the students wrote the proof for Theorem 2, it was decided to administer the Mastery Test for Theorem 2 without much further study. It was also decided that only two (instead of four) of the incorrect proofs for Theorem 3 be given to the students. We expected the students to learn the proof for Theorem 3 without difficulty.

It also appeared that several students were becoming tired of working with proofs and drill sheets. We decided to bring the computer to class on Tuesday (there would be no school Monday because of a field trip). This activity would provide a change of pace and could be used to illustrate divisibility facts.

Friday, April 25, 1969

The teacher reviewed the proof of Theorem 2 and distributed the application sheets. Special note was made of the fact that A + B - B was equal to A. After having each student give an example of the theorem, the Mastery Test was administered.

ANALYSIS: Two persons wrote incorrect proofs of theorem 2 on the Mastery Test. Both had put in plus signs instead of minus signs. Special attention would be given to this problem when the tests were handed back to the students.

APPLY THEOREM 2 TO THE FOLLOWING FACTS:

1. If $3 | 15$ and $3 | 6$ then

2. If $10 | 40$ and $10 | 20$ then

3. If $3 | T$ and $3 | W$ then

4. If $M | P$ and $M | Q$ then

*5. If $N | (A + B)$ and $N | B$ then

6. If $T | (C + D)$ and $T | D$ then

7. If $W | A$ and $W | B$ then

8. If $7 | 35$ and $7 | 14$ then

9. If $A | F$ and $A | G$ then

10. If $G | (A + B)$ and $G | B$, then

* What is $A + B - B$? Can you simplify it?

(1)

What is wrong with this proof?

THEOREM 2: If $N|A$ and $N|B$, then $N|(A - B)$.

STATEMENT	REASON
1. $A = NP$ $B = NQ$	1. Definition of divides
2. $(A + B) = NP + NQ$	2. Substitution
3. $= N\ (P + Q)$	3. Distributive Law
4. $N|(A - B)$	4. Definition of divides

(2)

What is wrong with this proof?

THEOREM 2: If $N|A$ and $N|B$, then $N|(A - B)$.

STATEMENT	REASON
1. $A = NP$ $B = NQ$	1. Substitution
2. $(A - B) = NP - NQ$	2. Distributive Law
3. $= N\ (P - Q)$	3. Substitution
4. $N|(A - B)$	4. Definition of divides

(3)
Fill in the reasons:

THEOREM 2: If $N|A$ and $N|B$, then $N|(A - B)$.

STATEMENT	REASON
1. $A = NP$ $B = NQ$	1.
2. $(A - B) = NP - NQ$	2.
3. $= N(P - Q)$	3.
4. $N\|(A - B)$	4.

(4)
What is wrong with this proof?

THEOREM 2: If $N|A$ and $N|B$, then $N|(A - B)$.

STATEMENT	REASON
1. $A = NB$ $B = NQ$	1. Definition of Divides
2. $(A - B) = NB - NQ$	2. Substitution
3. $= N(B - Q)$	3. Distributive Law
4. $N\|(A - B)$	4. Definition of Divides

THEOREM 2: If $N|A$ and $N|B$, then $N|(A - B)$.

STATEMENT	REASON

LESSON FOUR MASTERY TEST

1. Prove Theorem 2: If $N|A$ and $N|B$, then $N|(A - B)$.

STATEMENT	REASON

2-3. Give two numerical examples which illustrate Theorem 2.

Apply theorem 2 to the following facts:

4. If $K|E$ and $K|M$, then

5. If $A|R$ and $A|T$, then

LESSON FIVE

THEOREM 3: IF $N|A$, $N|B$ AND $N|C$, THEN $N|(A + B + C)$

The purpose of this lesson is to: (1) see if the students can write a proof for this theorem without instruction; (2) teach the proof to the students.

BEHAVIORAL OBJECTIVES:

1. The student can write a proof for Theorem 3.
2. The student can give numerical examples which illustrate Theorem 3.
3. Given three divisibility facts, the student can apply Theorem 3.

MATERIALS:

1. Sheets with the statement of Theorem 3 on them
2. Poster with proof of Theorem 3 on it
3. Two incorrect proofs of Theorem 3
4. Application drill sheets
5. Mastery Test for first three theorems

PROCEDURE: The procedures should be almost identical with the procedures of Lesson Four. Begin with numerical examples of Theorem 3.

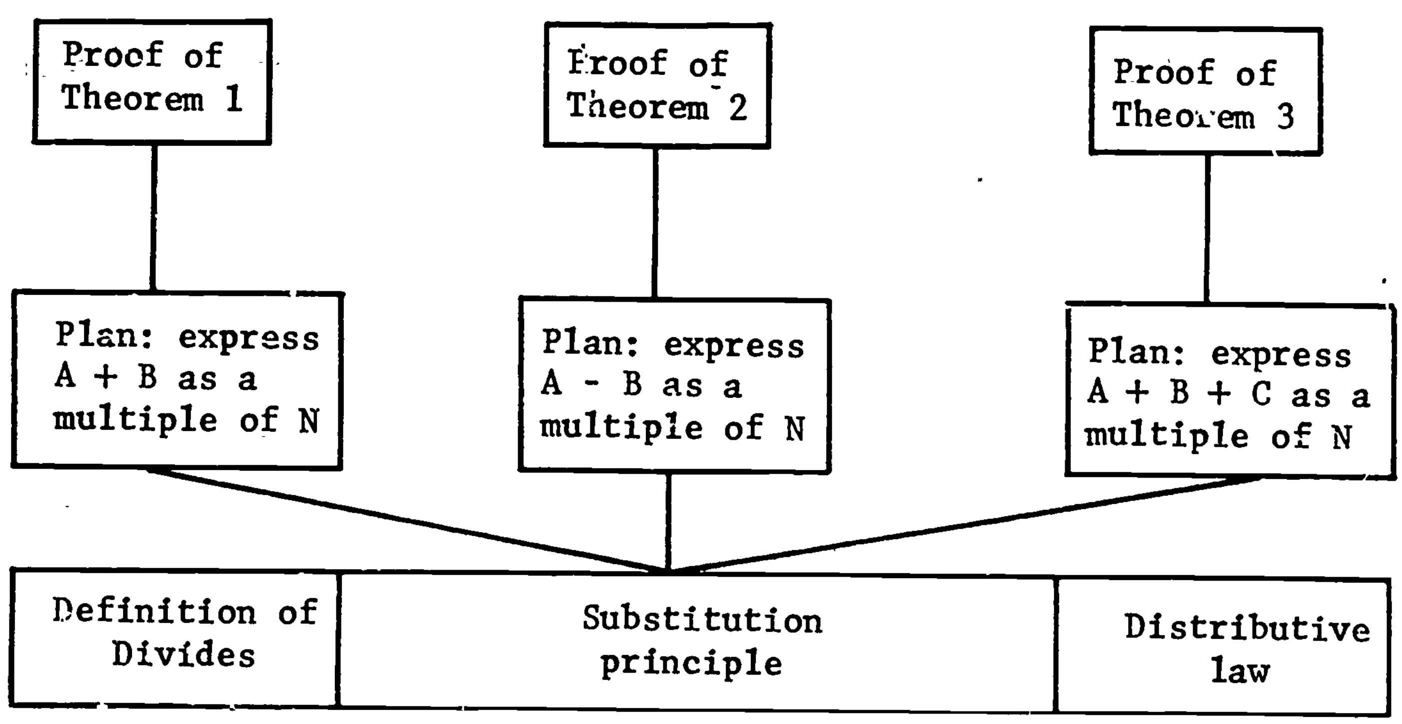
$3|6$, $3|3$ and $3|12$. Does $3|(6 + 3 + 12)$?

Then ask for the theorem. First ask them to write a proof without further comment. If anyone is unable to do so, give several hints.

Ask how the theorem differs from Theorem 1. And finally, suggest that the proof is very similar to the proof of Theorem 1.

If all the students write a valid proof, administer the Mastery Test on all three theorems. Otherwise, explain the proof on the poster, hand out and discuss both the incorrect proofs and the application sheets. Then administer the Mastery T.st.

Then discuss the similarities of the first three proofs. Ask the students to compare them. It might be helpful to draw the following diagram on the board:



JOURNAL OF LESSON FIVE

Friday, April 25, 1969

After the Lesson Four Mastery Test was given, one numerical instance of Theorem 3 was written on the board, a student immediately generalized, and everyone was able to write a correct proof.

The students were then asked to prove all three theorems. Everyone wrote perfect papers.

The Story on Indirect Reasoning was handed out for the next lesson.

ANALYSIS: Good lesson. The teacher skipped the sheets with incorrect proofs for Theorem 3, as well as the application sheets, for it was clear that the students understood the theorem.

It was observed that three of the students wrote down the reasons first, and then went back to fill in the statements. We decided to ask all of the students how they learned the proofs.

Theorem 3 states that if $N|A$, $N|B$ and $N|C$, then $N|(A + B + C)$.

Apply this theorem to the following facts:

1. $7|7$, $7|14$ and $7|21$, so
2. $23|A$, $23|B$, and $23|C$, so
3. $T|R$, $T|S$ and $T|W$, so
4. $(A + B)|F$, $(A + B)|G$, and $(A + B)|H$, so
5. $145|W$, $145|X$ and $145|Y$ so

THEOREM 3: IF $N|A$, $N|B$ and $N|C$, then $N|(A + B + C)$.

STATEMENT	REASON

1. Fill in the statements:

Theorem 3: IF $N|A$, $N|B$ and $N|C$, then $N|(A + B + C)$.

STATEMENT	REASON
1.	1. Definition of divides
2.	2. Substitution
3.	3. Distributive law
4.	4. Definition of divides

2. What is wrong with this proof?

Theorem 3: IF $N|A$, $N|B$ and $N|C$, then $N|(A + B + C)$.

STATEMENT	REASON
1. $A = NP$ $B = NQ$ $C = NA$	1. Definition of divides
2. $(A + B + C) = NP + NQ + NA$	2. Substitution
3. $= N(P + Q + A)$	3. Distributive law
4. $N|(A + B + C)$	4. Definition of divide

MASTERY TEST ON FIRST THREE THEOREMS

Theorem 1. If $N|A$ and $N|B$, then $N|(A + B)$.

STATEMENT	REASON

Theorem 2. If $N|A$ and $N|B$, then $N|(A - B)$.

STATEMENT	REASON

Theorem 3. If $N|A$, $N|B$, and $N|C$, then $N|(A + B + C)$.

STATEMENT	REASON

Activity for Tuesday, April 29, 1969

This activity was prepared to give the students an opportunity to verify both the theorems in the unit and divisibility criteria with large numbers. Since the class has not met since last Friday, it will also be interesting to see if the students will still remember the proofs of Theorems 1, 2, and 3.

BEHAVIORAL OBJECTIVES: None

MATERIALS:

1. Olivetti-Underwood programma 101 desk computer
2. Quiz sheets for first three theorems

PROCEDURE: Divide the students into two equal groups. Each group will spend half the period with the investigator and half the preiod with the teacher.

The teacher will quiz the students on the first three theorems and discuss the plan for each proof.

The investigator will use the desk computer to have the students verify the following:

1. Axiom 1: every whole number greater than 1 is divisible by some prime number.
2. Divisibility criteria for 3 and 9.
3. Theorems 1, 2, and 3.

The idea is to encourage the students to make up very large numbers for verification purposes. For example: 110,000,001 is divisible by 3. Each student will be permitted to use the computer for each of these activities.

The proof of Theorem 6 involves the expression $(2 \times 3 \times 5 \times \ldots \times p) + 1$. This will give a prime number for every prime from 2 through 11. When $p = 13$, $(2 \times 3 \times \ldots \times p) + 1 = 30{,}031$. The computer will factor this number: $30{,}031 = 59 \times 509$. The students will be asked to record this factorization on their list of prime numbers. "We will have use for this number at a later date."

The groups will change activities midway through the period.

JOURNAL OF ACTIVITY

Tuesday, April 29, 1969

The students took a field trip to Milwaukee on Monday. Hence there was a three-day interval since their last encounter with the proofs.

The students were divided into two groups of five students. The investigator worked with one group at the computer, and the teacher worked with the other group.

A program was prepared for the Olivetti-Underwood programma 101 desk computer. Given any whole number, the computer would print out the smallest prime number which divides it. In anticipation of Theorem 6, the whole number 30,031 was factored by the computer. The students were asked to write the factorization down on their list of prime numbers. Axiom 1 was then stated: every whole number greater than 1 is divisible by some prime number. To verify the axiom, each student was permitted to enter any whole number into the programmed computer.

Each student was then permitted to verify the criteria for divisibility by 3 and by 9. They were instructed to think up large numbers whose digits summed to a multiple of 3, enter the number into the computer, and observe the print out to see if if was in fact divisible by 3. The same procedure was followed for 9. While one group was working with the computer, the teacher tested the other group on Theorems 1, 2, and 3. Halfway through the period the groups switched activities.

ANALYSIS: All ten students were able to recall the three proofs.

The activity with the computer was highly successful. The students enjoyed the activity and displayed an understanding of the divisibility criteria.

LESSON SIX

THE LAW OF CONTRADICTION

The purpose of this lesson is to prepare the students for the proofs of Theorem 5 and Theorem 6.

BEHAVIORAL OBJECTIVES:

1. The student can write the opposite of a given statement.
2. Given a problem in which an assumption leads to a contradiction, the student can apply the law of contradiction to conclude that the assumption is false.
3. Given that a statement is false, the student can apply a fact of logic to conclude that the negation of the statement is true.

MATERIALS:

1. Story of indirect reasoning
2. Sheets with examples of indirect reasoning
3. Sheets with statements and room for students to write the opposite of these statements
4. Story on the Law of Contradiction
5. Law of Contradiction exercise sheet
6. Law of Contradiction Mastery Test

PROCEDURE: First discuss indirect reasoning in the Emirp Story. The lights went out, and they listed four possibilities (there could be more, such as faulty wiring in the wall, etc., but they just considered these four).

1. Power failure
2. Fuse
3. Lamp
4. Light bulb

"What did they do? They listed four things it could be, and then ruled out three of them so it had to be the light bulb. This is called indirect reasoning. They did not prove that it was the light bulb directly, but indirectly. They ruled the other possibilites out."

"The same thing happened with the phone call, it could have been anyone of five Johns, but they were able to rule out four of the Johns. That left only John Bear."

Distribute the sheets with examples of indirect reasoning. Discuss the first one, then ask them to do the rest at their desks. The point which must be made is that all but one possibility have been ruled out in each case. Distribute the sheets with five instances of indirect reasoning. For item 1, write the following on the board:

In Chicago $\nwarrow$ false

In Africa $\nearrow$ true

"Since Emirp proved that he was not in Chicago, he could not have killed Mac the Fork. Note that the reasoning is indirect. He proved that he was in Africa, which meant that he was not in Chicago."

Do the same for the other four items.

Then write a statement and its opposite on the board:

Harvey is tall. Harvey is not tall.

Pass out sheets with statements and ask the students to write the opposite of each. Circulate around the room as they work the problem, correct any errors. Then point out that if a statement is true, its opposite is false. For each of the five statements on the sheet, ask, "If this statement is true (or false), what can you say about its opposite?"

Then define a contradiction: a contradiction is an assertion that both a statement and its opposite are true at the same time. This cannot be the case:if one is true, the other is false.

Hand out the story about Sidney and Riley. After they have read the story, discuss it. Ask if they can come to some conclusion. They should conclude that Riley was wrong. Outline the argument on the board.

1. We <u>know</u> the score was 112-0 after five innings.

2. We <u>assumed</u> that the Mets scored four more runs.

3. ∴ We conclude that the final score was 116-0.

But the score was <u>not</u> 116-0.

Point out that since our assumption in (2) led to a false conclusion, it must be false. Use this example to explain the Law of Contradiction: If an assumption leads to a false statement, then the assumption is false.

Distribute exercises on the Law of Contradiction and discuss them orally. Then administer the Mastery Test.

If more than two persons are not masters at the 80% level, additional exercises will have to be constructed. This, however, is not anticipated.

JOURNAL OF LESSON SIX

Wednesday, April 30, 1969

After discussing the story, the teacher distributed and discussed the examples of indirect reasoning. For each example she called on a different student, and each one gave a correct response. One student raised a question about the story on indirect reasoning: "Emirp eliminated only four Johns, but there are a lot of Johns in the world." Another responded by saying it had to be a John who knew Emirp, hence that narrowed it to just five Johns.

A statement and its opposite were written on the board. It was pointed out that if a statement is true, then its opposite must be false. Sheets were then distributed and the students were asked to write the opposites of five statements. Two person wrote "thin" and "skinny" as the opposite of "fat." These were discussed and it was decided that "not fat" was a more precise opposite.

The story of the baseball game was distributed, the students read it, and then it was discussed. Two students wrote that Riley was wrong, five wrote that the Mets scored more than four runs, and three said that the score was not 116-0. The teacher wrote the reasoning on the board, but failed to point out the Law of Contradiction.

The exercises on the Law of Contradiction were then discussed. but the law was not stated explicity. One example caused difficulty, #4. The students wanted to know why the assumption led to the contradiction. They were not willing to accept it as given.

The Mastery Test was then administered, and corrected immediately. Four persons missed item 6, the statement of the Law of Contradiction. It had not been stated explicity, and so the teacher presented it to the class at this time.

ANALYSIS: Except for the failure to explicity state the law of contradiction, the lesson went as planned. We were satisfied that the students understood the law by the end of the discussion.

INDIRECT REASONING

1. Mac the Fork was murdered in Chicago the night of September 3, 1932. The police suspected Emirp of the crime. But Emirp had an alibi: he proved that he was in Africa on the night of September 3, 1932. What did the police have to conclude?

2. Either $N \mid (A + B)$ or $N \nmid (A + B)$. Sidney discovered that $N \mid (A + B)$ was not correct. What can he conclude?

3. The mechanic looked at Emirp's Monkeymobile and said, "Your car will not run. It is either the fuel system or the electrical system which has gone on the blink." He checked the fuel system and everything was O.K. What could he conclude?

4. Mrs. Gornowicz gave a mastery test and nine of the ten papers had names on them. But one did not have a name on it. How did she determine whose paper it was?

5. Harvey had a pink rash all over his head and trunk. The doctor said, "It is either measles, chicken pox, or you are allergic to something." Tests showed that it was not measles or chicken pox. What could the doctor conclude?

Law of Contradiction

1. We know that $A = B$. We do not know if $C = 3$. So we assume that $C = 3$. This leads us to conclude that $A \neq B$. What can you conclude?

2. We know that $N|A$. We do not know whether or not $N|B$. So we assume that $N|B$ and discover that this leads us to conclude that $N \nmid A$. What can we conclude?

3. We know that $A + B = NP$. We do not know whether or not $N = 7$. So we assume that $N = 7$ and find out that $A + B \neq NP$. What can you conclude?

4. We know that John Dog is home. But we do not know if he is eating. We assume that he is eating and this leads us to conclude that he is not home. What can you conclude?

5. We know that Harvey is an elephant. We assume that $A = B$ and this leads us to conclude that Harvey is not an elephant. What can we conclude?

6. We know that $N|A$. We assume that $N|K$ and find that $N \nmid A$. What can we conclude?

7. We know that Emirp is a fat elephant. We assume that he is a fast runner and this leads us to conclude that he is not fat. What can we conclude?

LAW OF CONTRADICTION MASTERY TEST

Write the opposite of each of the following statements:

1. Elephants are big.

2. Monkeys eat bananas.

3. Lake Mills is a large town.

4. Sidney can run fast.

5. Lions eat lettuce.

6. An assumption leads to a contradiction. What can we conclude?

7. We know that $A = B$. We assume that $F|A$. This leads us to conclude that $A \neq B$. What can you conclude?

8. We know that $3 + 3 = 6$. We assume that $N|(Z + W)$ and this leads us to conclude that $3 + 3 \neq 6$. What can you conclude?

9. If the statement "John is a dog" is true, what can you say about the statement "John is not a dog"?

10. If the statement "$N|(A + B)$" is false, what can you say about "$N\nmid(A + B)$"?

Statement	Opposite
1. $A = 3$	1.
2. $N \dagger (A + B)$	2.
3. $2 + 2 = 6$	3.
4. The cat is fat.	4.
5. Jack is strong.	5.

INDIRECT REASONING

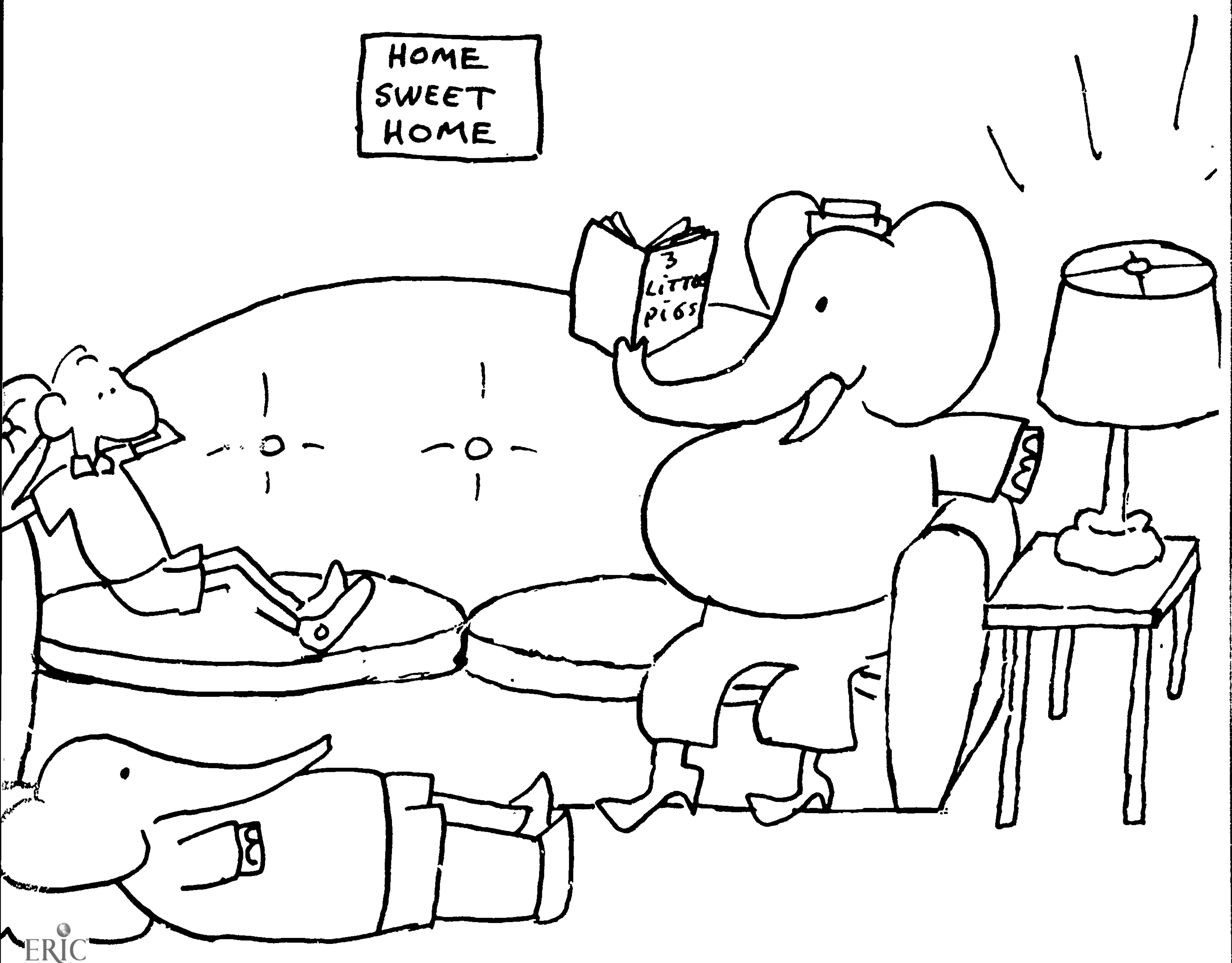
It was Bananaday evening and Harvey and Sidney were staying overnight with Mr. Emirp. They were sitting on the couch. Emirp was reading the story of The Three Little Pigs: "Not by the hair of my chinny-chin-chin, will I open the door so that you can come in."

Suddenly the lamp went out, and the house was completely dark.

"Oh, oh," said Sidney. "What's wrong?"

"I don't know," said Emirp. "But we can figure it out if we are clever enough. What could possibly have happened to cause this light to go out?"

"Maybe there has been a power failure and the lights are out all over town," suggested Harvey.



"Or maybe a fuse blew in the house," said Sidney.

"It could even be that something is wrong with this lamp," added Harvey.

"I'll bet it's the light bulb," said Sidney.

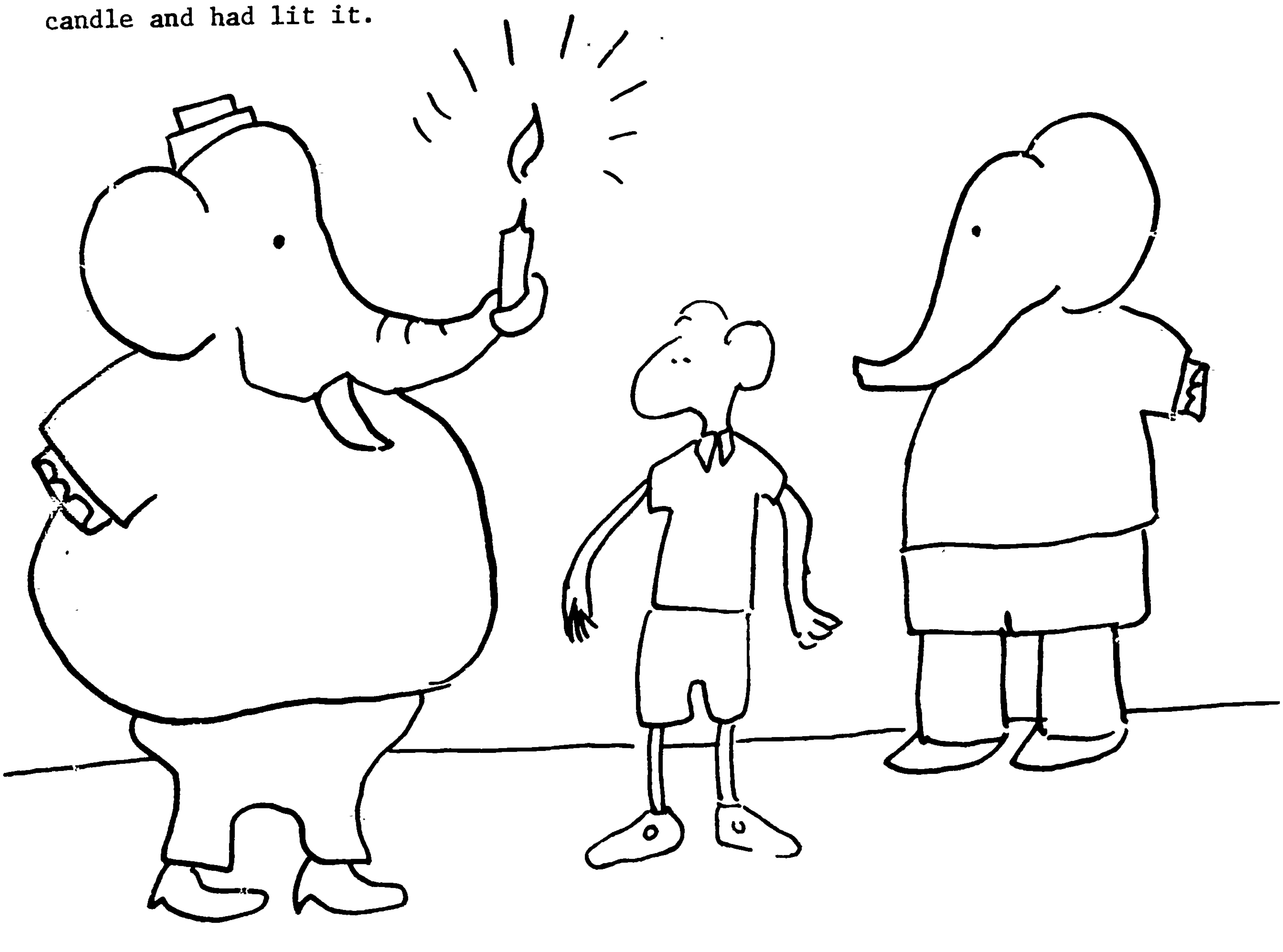
"Very good," said Emirp. "We have four possibilities:

1. A power failure.
2. A fuse.
3. The lamp.
4. The light bulb.

But which one is it?"

Harvey looked out the window. "Look! The lights are still on in the other houses. So it is not a power failure."

"Very good observation," said Emirp. By this time he had found a candle and had lit it.



"Let's check the fuses," he said. They went to the fuse box. All the fuses were good. "It is not a fuse," said Emirp.

"Then it is either the lamp or the light bulb," said Sidney. "How can we tell which one is bad?"

They all thought. Then Harvey spoke. "I know. Get another light bulb and see if it works in the lamp. If it lights up, then the old bulb is bad. If it does not light up, then it must be the lamp."

"Good idea," said Emirp. He got a new bulb and tried it in the lamp. "It works," said Emirp.

"So it must be the old light bulb," said Sidney.

"Yes. But do you realize what we have done?" asked Emirp excitedly.

"No," answered Harvey and Sidney together.

"We have used indirect reasoning to prove that the light bulb had burned out. We did not show that the bulb was bad directly, but we eliminated all the other possibilities: it wasn't the lamp, or the fuse, or a power failure, so it had to be the bulb."

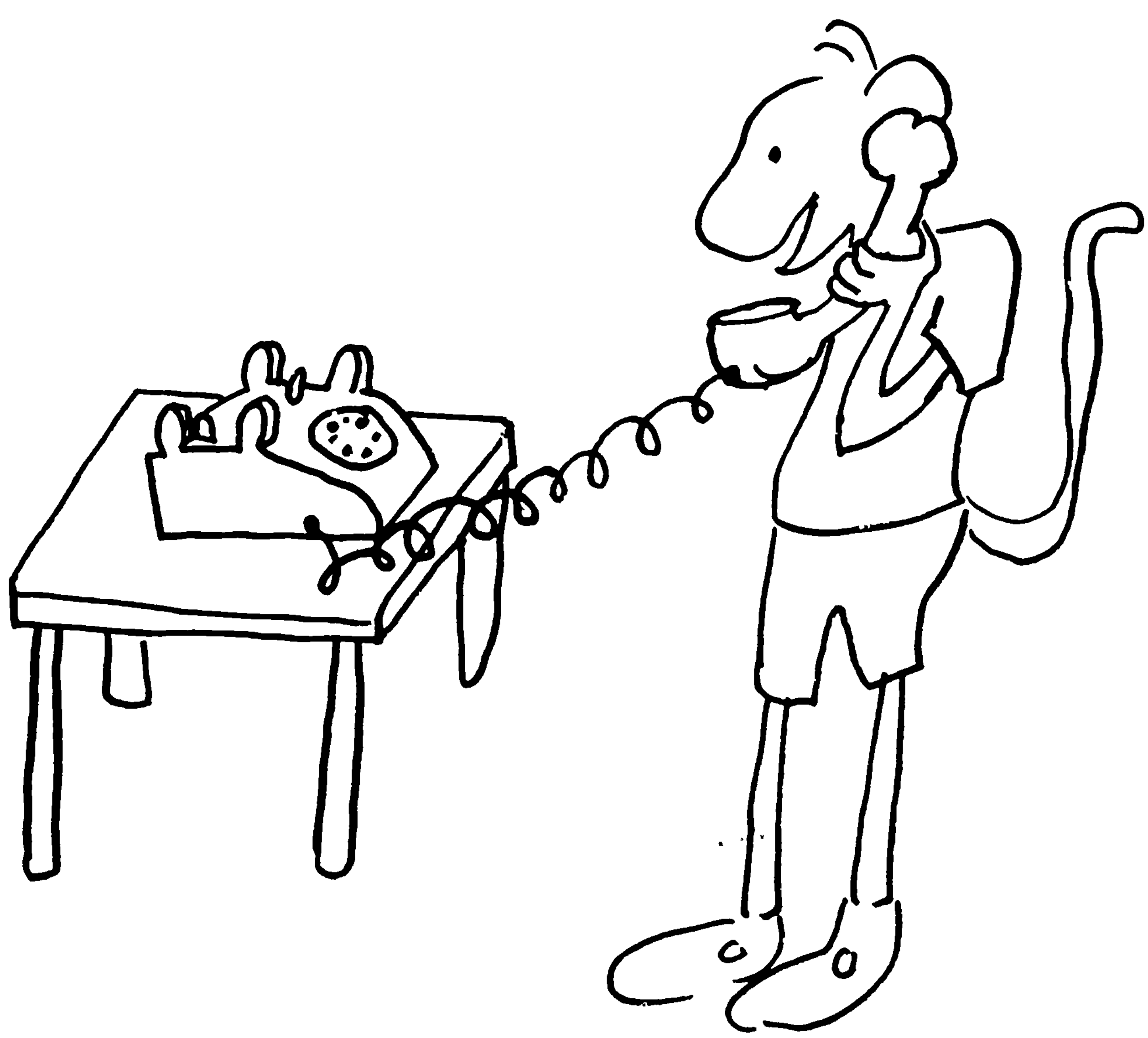
"Well I'll be a monkey's uncle," said Sidney.

"And I'll be an elephant's uncle," said Harvey.

They sat back down on the couch and Emirp continued the story. "Not by the hair of my . . . "

The telephone rang. "Rin-n-n-g-g-g."

"I'll get it," said Sidney.



"Hello. This is the home of the Great Emirp."

"Hello," said a voice on the other end of the line.

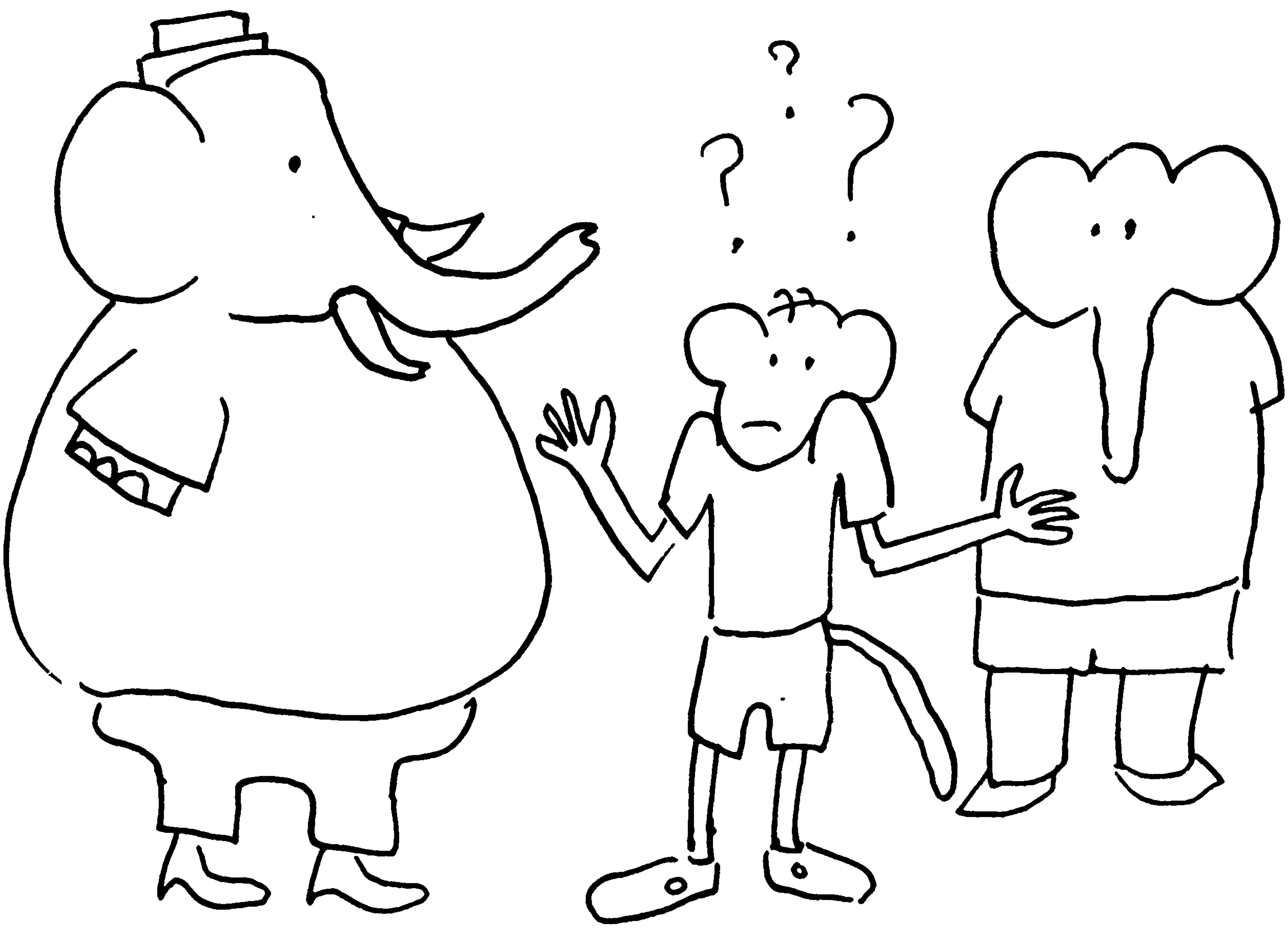
"This is John. Will you give Emirp a message for me. Tell him that I'll see him tomorrow at McDonald's for lunch. 12:00 sharp. Good bye." And the voice hung up.

"Who was it?" asked Emirp.

"John said that he would meet you at McDonald's for lunch tomorrow," said Sidney.

"John who?" asked Emirp.

"I don't know. He hung up before I could ask," said Sidney.



"Gosh, what will I do? I don't know which John it was. I know five Johns," said Emirp.

Then Harvey got a brilliant idea. "Maybe we could determine who it was by indirect reasoning. You know. Maybe we could rule out four of your friends named John. Then it would have to be the other John."

"Well, there is John Bear, John Horse, John Elephant, John Dog, and John-John."

"It can't be John-John," said Sidney. "He is in Greece."

Then Emirp said, "And it can't be John Dog, because he doesn't like hamburgers."

"It can't be John Elephant," said Harvey. "He is on a vacation in Biami Beach."

"And John Horse is in the horsepital with a broken leg," said Emirp.

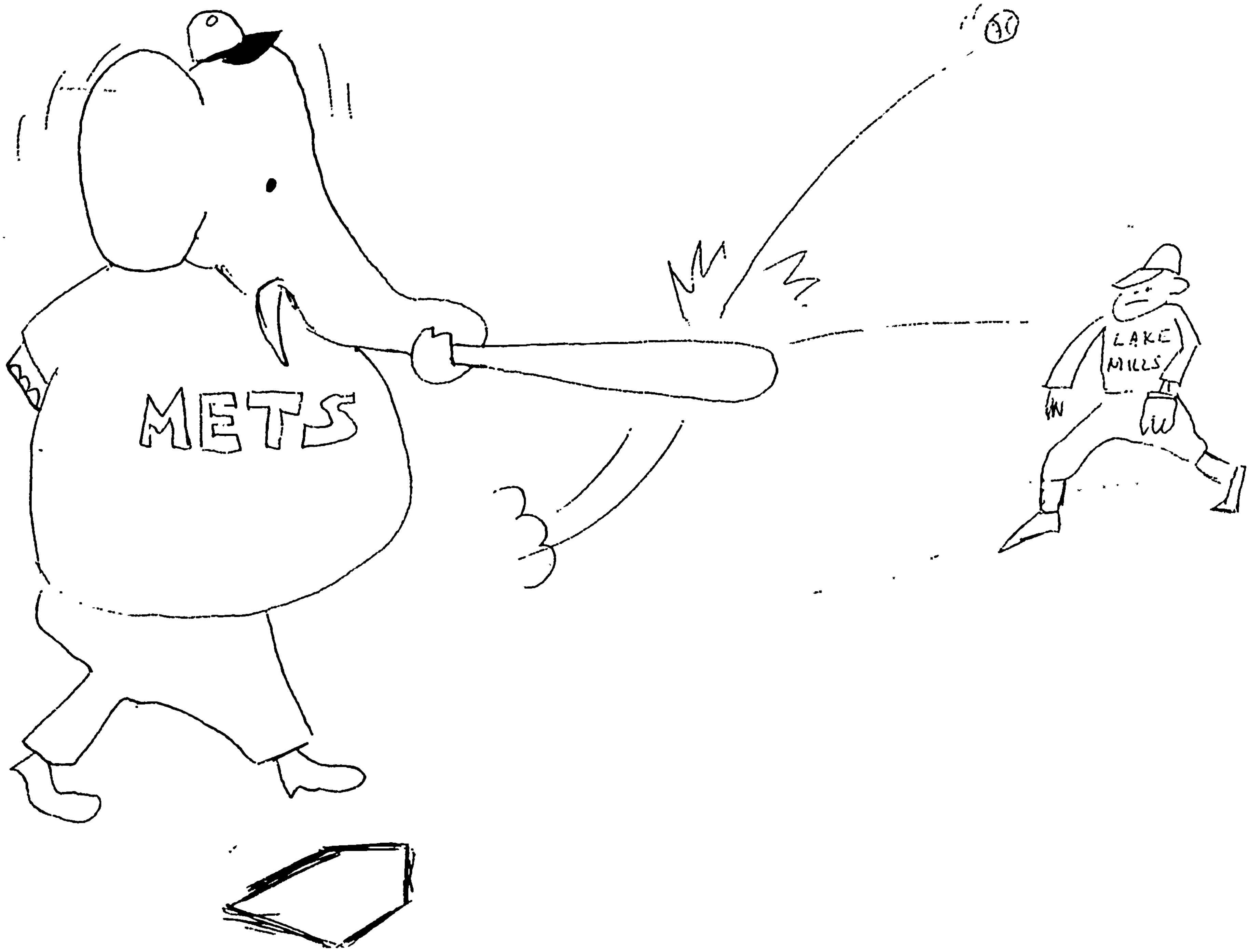
"So it must be John Bear," they all shouted at once. "Indirect reasoning triumphs again."

They finished the story of The Three Little Pigs, had a good night's sleep, and the next day Emirp had lunch at McDonald's with John Bear.

Which just goes to show that indirect reasoning can be very helpful.

THE LAW OF CONTRADICTION

The Lake Mills Monsters were playing the Monkeyville Mets in baseball. Sidney and his little brother Riley went to the stadium to see the game.



The slaughter started in the first inning when Emirp hit a grand-slam home run. At the end of five innings the score was 112-0 in favor of the Mets. Suddenly Riley said, "Hey, Sidney. Don't forget that you have to go to the Royal Palace to help the King with his bookkeeping this afternoon."

"I almost forgot," admitted Sidney. "Keep track of how many more runs we score," he said as he left the stadium.

That night at the supper table Riley said, "We scored 4 more runs after you left. Emirp hit another home run." Sidney then reasoned as follows:

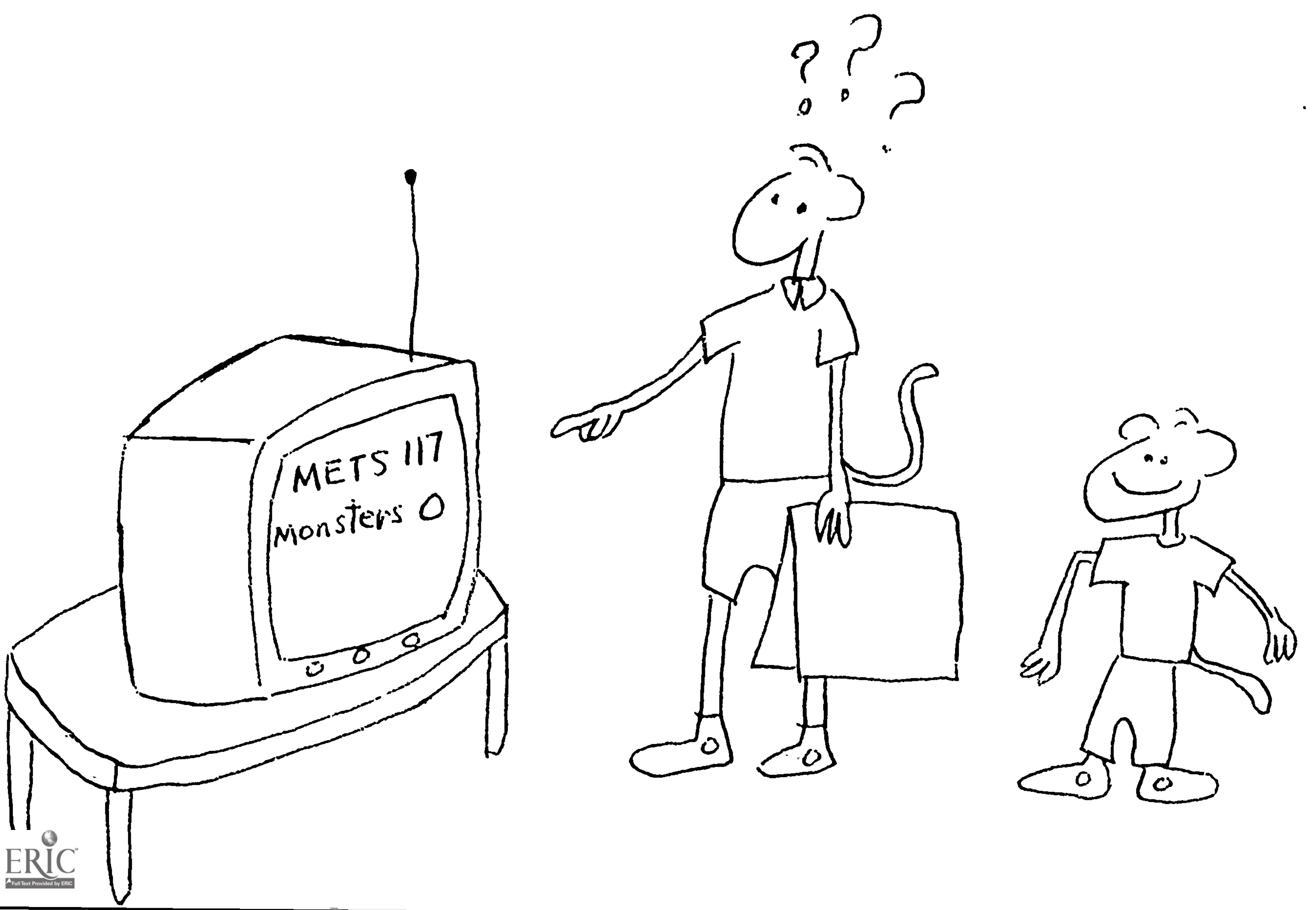
(1) I know that the score was 112-0 after five innings

(2) I assume that we scored 4 more runs

(3) $\therefore$ The final score was 116-0

Just then the newspaper monkey delivered the evening paper. Sidney read the sports page and it said that the final score was 116-0. He went to the T.V. set and waited for the sports news. "Yes sir," said the announcer, "the Monkeyville Mets demolished the Lake Mills Monsters today by a score of 117-0. After five innings the score was 112-0."

"Hum-m-m-m," said Sidney. "I know the following:



(1) After five innings the score was 112-0

(2) I assumed that we scored 4 more runs because Riley said so

(3) I therefore concluded that the final score was 116-0.

But I now find out that the final score was 117-0 and not 116-0."

What conclusion can Sidney make?

Can you spot a contradiction?

LESSON SEVEN

THEOREM 4: If $N \nmid A$ and $N|B$, then $N \nmid (A + B)$

The purpose of this lesson is teach the students to write the proof of this theorem.

BEHAVIORAL OBJECTIVES:

1. Given appropriate divisibility facts, the student can apply Theorem 4.
2. The student can give numerical examples to illustrate Theorem 4.
3. The student can write a proof for Theorem 4.

MATERIALS:

1. Proof of Theorem 4 on poster
2. Theorem 4 application sheet
3. Six incorrect proofs of Theorem 4
4. Sheets with statement of Theorem 4 (for writing proof)
5. Story of "Sleeping Beauty" for review purposes

PROCEDURE: Write the theorem on the board:

Theorem 4: If $N \nmid A$ and $N|B$, then $N \nmid (A + B)$

To determine how well the students understand the use of letters, ask each student to write a numerical example to illustrate Theorem 4. Record how many are able to do so. You might have to review what a numerical example is by giving one for Theorem 1.

Write several examples on the board, then explain that the proof will be an indirect proof. "We will show that when we assume that $N|(A + B)$, we get a false statement. This will mean that $N|(A + B)$, our assumption, is false." Write the proof using a numerical example. Pose the question:

$$4 \nmid 15 \text{ and } 4|16, \text{ does } 4|(15 + 16)?$$

"We know that 4 will not divide 31, but even if we didn't we could prove that it didn't. We are doing this so that we can see how to prove theorem 4."

STATEMENT	REASON
1. $4|(15 + 16)$	1. Assume
2. $4|16$	2. Given
3. $4|(15 + 16 - 16)$ or $4|15$	3. Theorem 2
4. But $4 \nmid 15$	4. Given
5. $4 \nmid (15 + 16)$	5. Law of Contradiction

Stress that the proof is indirect, and that we use the Law of Contradiction. The strategy is to assume the opposite of what we are trying to prove and derive a false statement.

Beside this proof write the proof for the general theorem. Elicit reasons and statements from the student. Distribute sheets #1 and #2 and let the students search for the errors in the proofs. Discuss. Then pass out sheets with theorem 4 on it and quiz them on the proof. Go over the proof again. Repeat with exercise sheets #3 and #4. Quiz. Repeat with application sheet.

Stress that the strategy or plan of proof is to assume $N|(A + B)$ and derive a contradiction. Then administer the Mastery Test.

JOURNAL OF LESSON SEVEN

Wednesday, April 30, 1969

With fifteen minutes remaining in the period, the teacher started Lesson Seven. Theorem 4 was written on the board, and all ten students were able to write a numerical instance of the theorem. The following proof was then written on the board.

IF $4 \nmid 15$ AND $4 \mid 16$, THEN $4 \nmid (15 + 16)$

STATEMENT	REASON
1. $4 \mid (15 + 16)$	1. Assumption
2. $4 \mid 16$	2. Given
3. $4 \mid (15 + 16 - 16)$ or $4 \mid 15$	3. Theorem 2
4. But $4 \nmid 15$	4. Given
5. $4 \nmid (15 + 16)$	5. Law of Contradiction

The proof was completed as the period ended. Some of the students were confused. "But we already know that four will not divide 31," was one response uttered as the students left the room.

ANALYSIS: The main point of confusion appeared to be that the students knew that the assumption "$4 \mid (15 + 16)$" was false to start with.

It was decided to prove a result which was not immediately evident:

IF $7 \nmid 77{,}775$ AND $7 \mid 42$, THEN $7 \nmid (77{,}775 + 42)$

Thursday, May 1, 1969

The mastery test on the law of contradiction was handed back and discussed. The following was then written on the board:

$$7 \nmid 77{,}775 \text{ AND } 7 \mid 42, \text{ DOES } 7 \mid (77{,}775 + 42)?$$

STATEMENT	REASON

On the next panel the general theorem was written

$$\text{IF } N \nmid A \text{ AND } N \mid B, \text{ THEN } N \nmid (A + B)$$

STATEMENT	REASON

The teacher then explained that she would show that $7 \nmid (77{,}775 + 42)$ without adding the numbers and dividing by 7. "This will show us how we can prove it for any number N, A and B." She wrote each statement on the board, and students provided each reason. The presentation was clear and the teacher explained the Law of Contradiction quite clearly. The students then volunteered to supply both statements and the reasons for theorem 4. No mistakes were made.

The students were then asked to copy the proofs. They were then given two incorrect proofs to correct. Everyone did so. They were then quizzed and only one student needed help. The students were asked how they learned the proof so rapidly. All said that they memorized the reasons first.

The application sheet was distributed and completed without error. The teacher did not work an example; only one student needed help in getting started.

The Mastery Test was then given and "The Sleeping Beauty" was distributed.

ANALYSIS: The presentation worked very well. Each student had a perfect score on the Mastery Test.

It was decided not to use the other exercise sheets. The students would be given the chance to prove theorem 5 on Friday. If time remained, the class could play the game of PRIME.

SHEET #1

(1)

THEOREM 4: IF $N \nmid A$ AND $N \mid B$, THEN $N \nmid (A + B)$

STATEMENT	REASON
1. $N \mid (A + B)$	1. Assume
2. $N \mid B$	2. Given
3. $N \mid A$	3. Theorem 2
4. But $N \nmid A$	4. Given
5. $N \nmid (A + B)$	5. Law of Contradiction

(2)

THEOREM 4: IF $N \nmid A$ AND $N \mid B$, THEN $N \nmid (A + B)$

STATEMENT	REASON
1. $N \mid (A + B)$	1. Assume
2. $N \mid B$	2. Theorem 2
3. $N \mid A$	3. Given
4. $N \nmid A$	4. Theorem 2
5. $N \nmid (A + B)$	5. Law of Contradiction

SHEET #2

(3)

THEOREM 4: IF $N \nmid A$ AND $N | B$, THEN $N \nmid (A + B)$

STATEMENT	REASON
1. $N \| (A + B)$	1.
2.	2. Given
3. $N \| A$	3.
4.	4. Given
5. $N \nmid (A + B)$	5.

(4)

THEOREM 4: IF $N \nmid A$ AND $N | B$, THEN $N \nmid (A + B)$.

STATEMENT	REASON
1.	1. Assume
2. $N \| B$	2.
3.	3. Theorem 2
4. $N \nmid A$	4.
5.	5. Law of Contradiction

SHEET #3

(5)

THEOREM 4: IF $N \nmid A$ AND $N \mid B$, THEN $N \nmid (A+B)$

STATEMENT	REASON
1.	1. Assume
2.	2. Given
3.	3. Theorem 2
4.	4. Given
5.	5. Law of contradiction

SHEET #4

(6)

THEOREM 4: IF $N \nmid A$ AND $N | B$, THEN $N \nmid (A + B)$

STATEMENT	REASON
1. $N \| (A + B)$	1.
2. $N \| B$	2.
3. $N \| A$	3.
4. But $N \nmid A$	4.
5. $N \nmid (A + B)$	5.

THEOREM 4: IF $N \nmid A$ AND $N | B$, THEN $N \nmid (A + B)$

STATEMENT	REASON
1.	1.
2.	2.
3.	3.
4.	4.
5.	5.

APPLY THEOREM 4 TO THE FOLLOWING DIVISION FACTS:

1. $3 | 6$ and $3 \nmid 8$ so

2. If $A \nmid G$ and $A | H$ then

3. $2 | (2 \times 3 \times 5)$ and $2 \nmid 1$, so

4. $3 | (2 \times 3 \times 4)$ and $3 \nmid 1$, so

5. $7 | (2 \times 3 \times 5 \times 7)$ and $7 \nmid 1$, so

6. $2 | 4$ and $2 \nmid 7$, so

7. $W \nmid M$ and $W | K$, so

8. $P | (P \times Q)$ and $P \nmid 1$, so

9. $M | (M \times N)$ and $M \nmid 1$, so

10. $11 | (2 \times 3 \times 5 \times 7 \times 11)$ and $11 \nmid 1$, so

LESSON SEVEN MASTERY TEST

1. Write a proof for Theorem 4:

   If $N \nmid A$ and $N | B$, then $N \nmid (A + B)$.

STATEMENT	REASON

2. Give two numerical examples which illustrate the meaning of Theorem 4:

Apply Theorem 4 to the following divisibility facts:

3. $3 \nmid 10$ and $3 | 6$, therefore . . .

4. If $T | W$ and $T \nmid C$, then . . .

THE SLEEPING BEAUTY

"Look at this proof!" said Emirp to Harvey and Sidney." Can you give the reasons for each step?"

IF $N|A$ and $N|B$, then $N|(A+B)$.

STATEMENT	REASON
1. $A = NP$ $B = NQ$	1.
2. $A+B = NP + NQ$	2.
3. $= N(P+Q)$	3.
4. $\therefore N\|(A+B)$	4.

While they thought about it, Emirp made an onion sandwich. Then Harvey and Sidney handed him their reasons. "Very Good, Harvey. You have them correct. But Sidney has made some errors. Can you find them?"

IF $N|A$ and $N|B$, then $N|(A+B)$.

STATEMENT	REASON
1. $A = NP$ $B = NQ$	1. definition of divides
2. $A+B = NP+NQ$	2. distributive law
3. $= N(P+Q)$	3. SUBSTITUTION
4. $\therefore N\|(A+B)$	4. definition of prime

Just then there was a great deal of excitement outside. "Emirp, Emirp," shouted a handsome young elephant.

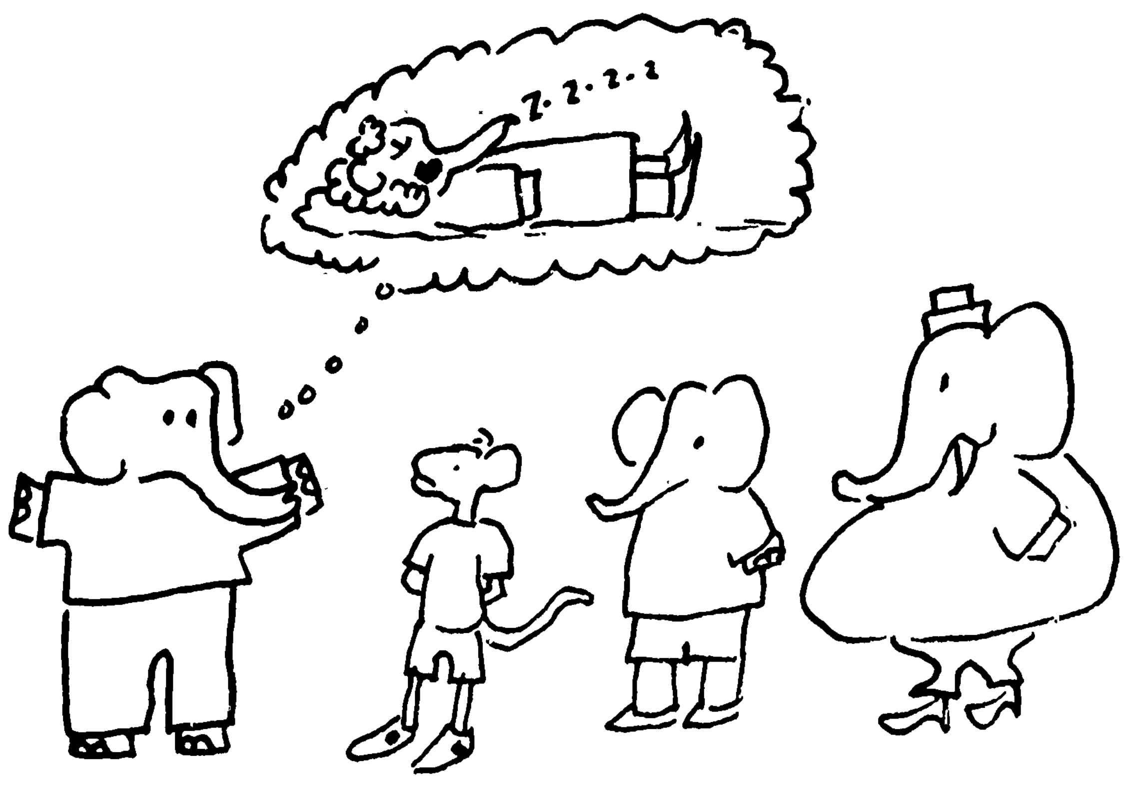
"What's going on" asked Emirp.

"Well, I was walking through a deserted part of the jungle when I saw a small cottage. I looked inside, and there was the most beautiful female elephant in the world."

"Wunnerful, Wunnerful," said Emirp.

"Not so," said the elephant."She is fast asleep in a deep trance. Nothing will wake her up. What does this mean, oh great wise one?"

Emirp thought, then he said, "I once read a story about a beautiful princess. A wicked witch put a spell on her. She pricked her finger with a needle and went into a deep sleep. Nothing could awaken her.

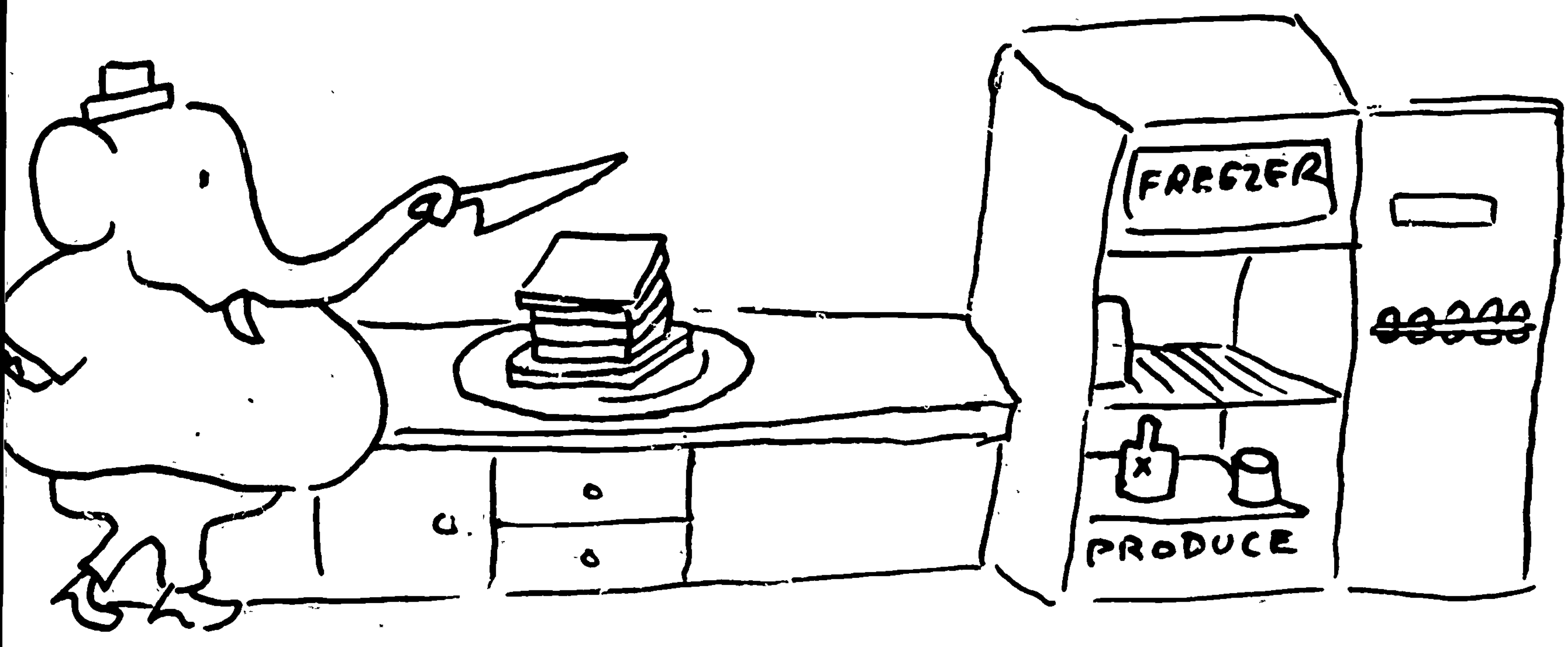


Then one day a prince came by and kissed her. The princess woke up. They were married, and they lived happily ever after."

The news spread through the jungle like wildfire. All the unmarried male elephants raced to the cottage. Each hoped that his kiss would awaken the beautiful elephant.

But Emirp and the boys continued the math lesson. "O.K." said Emirp, "write a proof for this theorem: if N/A, N/B, and N/C, then N/(A+B+C)."

While the boys were working on the proof, Emirp went to the refrigerator and made a garlic and limberger cheese sandwich.



This is what Harvey wrote:

IF $N|A$, $N|B$ and $N|C$, Then $N|(A+B+C)$.

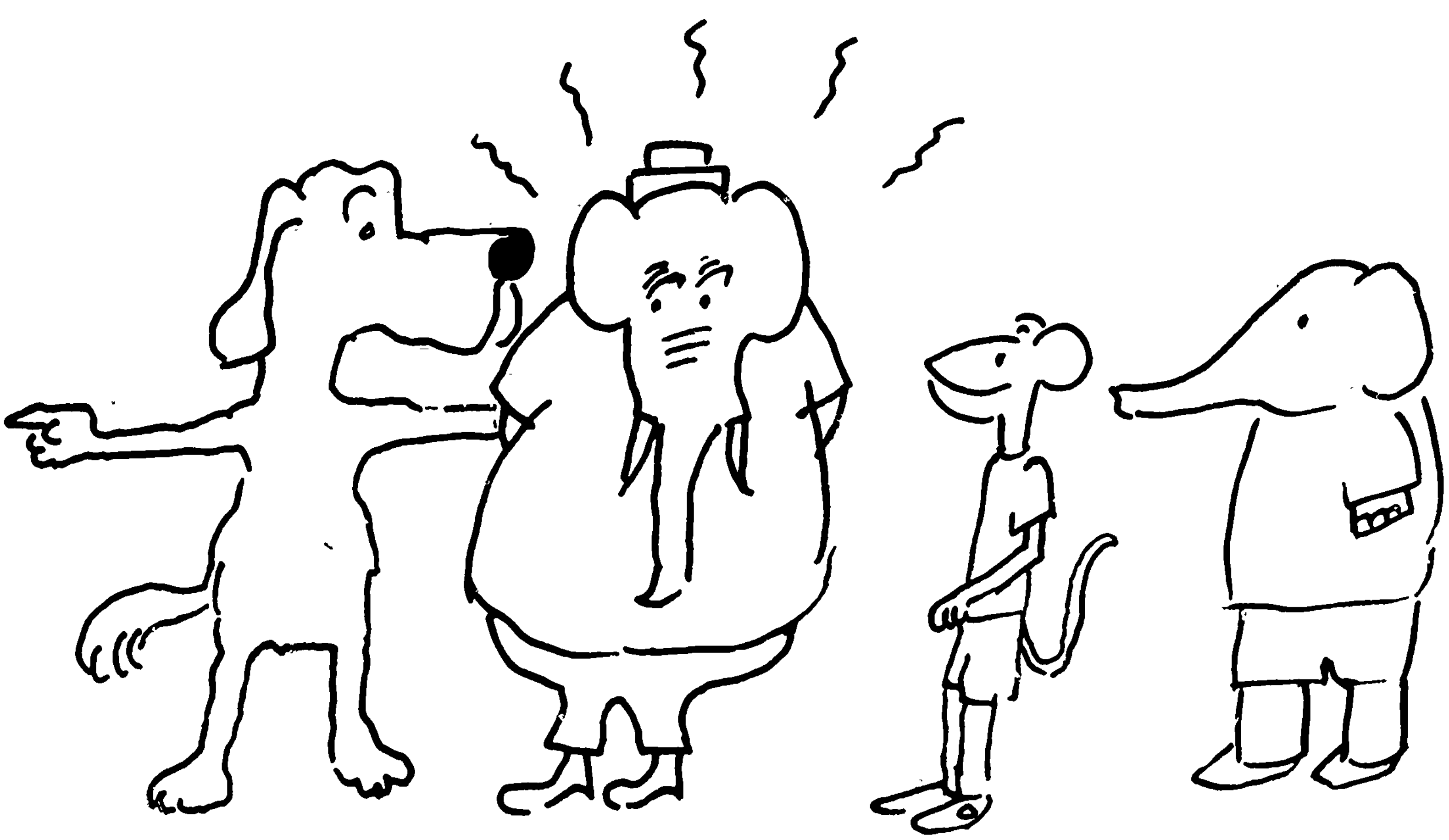
STATEMENT	REASON
1. $A=NR$ $B=NT$ $C=NA$	1. Definition of divides
2. $A+B+C=NR+NT+NA$	2. Substitution
3. $=N(R+T+A)$	3. DISTRIBUTIVE LAW
4. $\therefore N\|(A+B+C)$	4. Definition of divides

Can you find a mistake?

After Mr. Emirp corrected the proofs, someone knocked on the door. It was John Dog.

"Have you heard the news? "asked John Dog. "Every elephant in the jungle has kissed the Sleeping Beauty, but she still sleeps. You must come and try to save her."

Emirp blushed:"oh, I'm too bashful".



"Nonsense, think of the girl. She may sleep forever if you don't try. You are her last hope. Besides, you could use a wife."

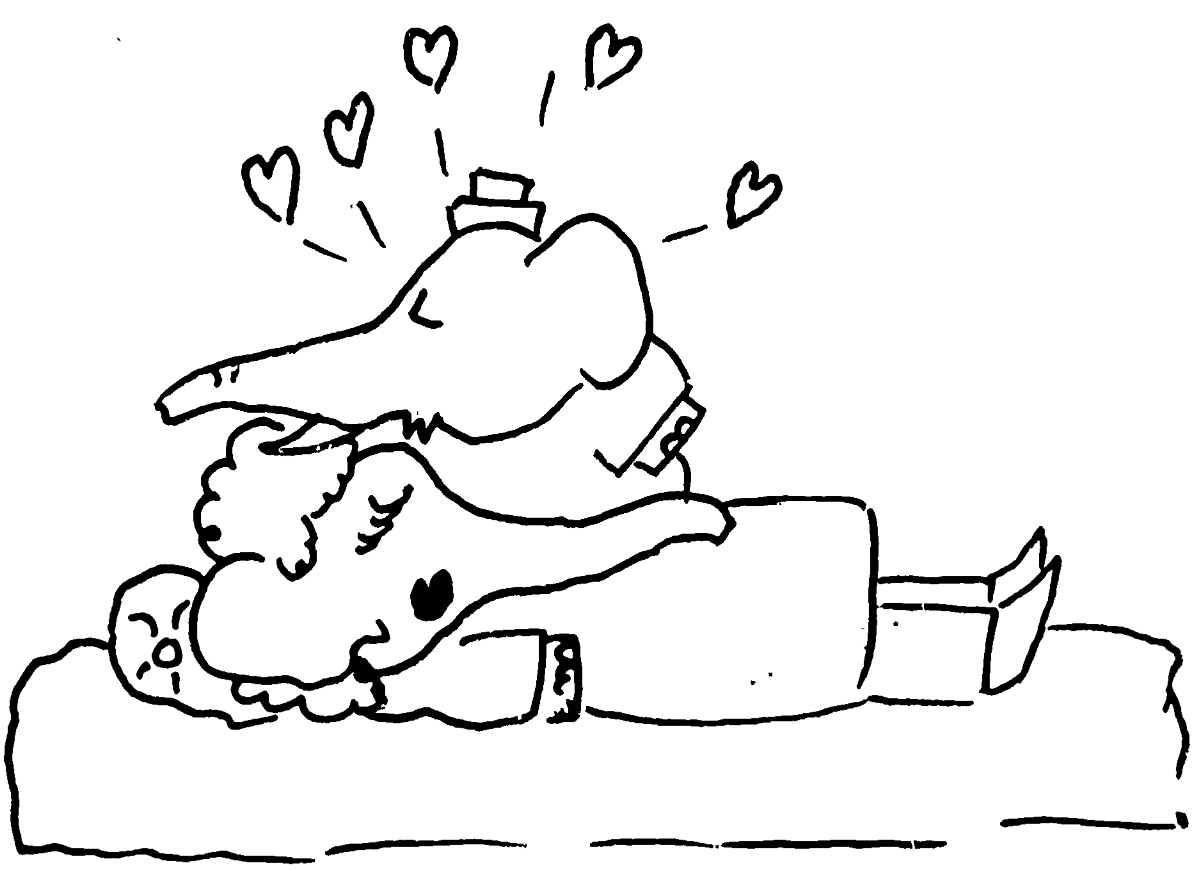
"Well," said Emirp, "I'll go if Sidney can find the errors in this proof."

IF $N|A$ and $N|B$, Then $N|(A \cdot B)$.

STATEMENT	REASON
1. $A = NV$ $B = NW$	1. definition of divides
2. $A+B = NV + NW$	2. substitution
3. $= N(V+W)$	3. DISTRIBUTIVE LAW
4. $\therefore N|(A-B)$	4. definition of divides

Sidney spotted the errors. Can you?

An hour later they all arrived at the cottage deep in the jungle. Hundreds of animals were waiting to see if Emirp could break the evil spell and awaken the Sleeping Beauty. He bent over and kissed her.



The Sleeping Beauty woke up.

"Hurray for Emirp!" shouted the animals. John Dog stepped between them and said,

"This shy fellow is the one who broke the evil spell. Now you can marry him and live happily ever after."

"Oh no," said the Beauty. "That guy has the worst smelling mouth in the world. When he kissed me I nearly died. No wonder I woke up." She stormed out of the cottage.

Now, what do you think is the moral of this story?

A. One must be careful when writing mathematical proofs.

OR

B. If you eat onion, garlic, and limberger cheese sandwiches, be sure to brush you teeth before you kiss a Sleeping Beauty.

LESSON EIGHT

THEOREM 5: IF $N \nmid A$ AND $N|B$, THEN $N \nmid (A - B)$.

The purpose is (1) to see if anyone can prove this theorem without instruction; (2) to teach the proof of the theorem.

BEHAVIORAL OBJECTIVE:

1. The student can write a proof for Theorem 5.
2. Given appropriate division facts, the student can apply Theorem 5.
3. The student can give numerical examples to illustrate Theorem 5.

MATERIALS:

1. Practice sheet
2. Mastery test for theorem 4 and theorem 5

PROCEDURE: Review the proof of theorem 4, being sure to stress the strategy involved in eliminating the B from A + B, i.e., subtract B. Then write theorem 5 on the board and give one numerical example. Pass out quiz sheets and ask them to prove it. Remind them that they may use anything they know to be true. This proof requires the application of Theorem 1 (instead of Theorem 2 which is used in proving Theorem 4; otherwise the proofs are identical).

After they have had an opportunity to write the proof without any help, put the poster with the proof of Theorem 4 on the board.

Suggest that the proof is very similar.

Finally, direct their attention to how we got rid of the B in A + B. Say, "In this case, Theorem 2 worked, didn't it?"

Collect the papers as correct proofs are written. Use the posters to compare the two proofs. Have them copy it once, then administer the Mastery Test (for Theorem 4 and Theorem 5).

JOURNAL ON LESSON EIGHT

Friday, May 2, 1969

The story of "The Sleeping Beauty" was discussed. Then the teacher reviewed the proof of Theorem 4 using the poster. She then wrote Theorem 5 on the board and asked each student to write a numerical example to illustrate Theorem 5. Everyone was able to do so, but two students subtracted the larger number. The teacher explained that the examples were correct if we were dealing with negative numbers.

The students were then given sheets of paper and asked to write a proof for Theorem 5. Everyone did the same thing: Theorem 2 was applied instead of Theorem 1. No other mistakes were made. The teacher said, "Everyone has one mistake. The proof is very similar to the proof of Theorem 4. Think through each step of the proof and see if you can't find your mistake." Two students made the correction.

A second hint was then given: "In Theorem 4 you had $N \mid (A + B)$ and you wanted to get rid of the B and so you subtracted using Theorem 2." Six more students then corrected their proofs. Two students had to have it explained to them.

The teacher then wrote the proof on the board and explained how this proof differed from the proof of Theorem 4.

The Mastery Test was administered.

<u>ANALYSIS</u>: The lesson went as planned and all students wrote correct proofs on the Mastery Test.

The students were not able to prove Theorem 5 without help. They had been able to write the proof of Theorems 2 and 3 without help, but the reasons are the same as the reasons for Theorem 1. One of the reasons in the proof of Theorem 5 is different from the corresponding reason in the proof of Theorem 4. Since the reasons remained the same for the other theorems, they may have reasoned that they should remain the same in this case.

Each person had correct statements.

PRACTICE SHEET

THEOREM 5: IF $N \nmid A$ AND $N | B$, THEN $N \nmid (A - B)$

STATEMENT	REASON
1.	1.
2.	2.
3.	3.
4.	4.
5.	5.

MASTERY TEST

THEOREM 4: IF $N \nmid A$ AND $N|B$, THEN $N \nmid (A + B)$.

STATEMENT	REASON

THEOREM 5: IF $N \nmid A$ AND $N|B$, THEN $N \nmid (A - B)$.

STATEMENT	REASON

ACTIVITY: The game of PRIME can be played any time before the lesson on Theorem 6.

The game is very similar to BINGO. Each student is given a sheet of paper with a 5 x 5 array of numbers on it. For example,

P	R	I	M	E
77	51	30	38	20
19	50	23	71	74
21	81	85	93	13
51	47	46	64	99
99	55	96	83	31

Each column of the array is identified with a letter of the word "PRIME." The teacher has 60 small transparent slips (12 for each letter) from which she randomly draws a transparency. She then reads it and places it on an overhead projector so that each student can see it. For example: P: a two-digit prime number.

The student looks in the P column, and if he has a two-digit prime number in the P column, he can cross it out. As in BINGO, the first person who crosses out an entire column, row, or diagonal is the winner.

The following are the descriptions used on the transparent slips:

I: A TWO-DIGIT NUMBER WITH A COMPOSITE NUMBER IN THE ONES PLACE

R: A PRIME NUMBER LESS THAN 20

E: A NUMBER LESS THAN 2

I: A COMPOSITE NUMBER

M: THE SMALLEST PRIME NUMBER

R: AN EVEN PRIME NUMBER

I: A TWO-DIGIT COMPOSITE WITH BOTH OF ITS DIGITS PRIME NUMBERS

R: A PRIME NUMBER GREATER THAN 80

E: A TWO-DIGIT NUMBER WITH A PRIME IN THE TENS PLACE

E: A TWO-DIGIT NUMBER WITH A PRIME NUMBER IN THE ONES PLACE

E: A COMPOSITE NUMBER DIVISIBLE BY 3

R: A TWO-DIGIT NUMBER WITH A COMPOSITE NUMBER IN THE TENS PLACE

M: A TWO-DIGIT NUMBER WITH A PRIME NUMBER IN THE ONES PLACE

P: A TWO-DIGIT NUMBER, THE SUM OF WHOSE DIGITS IS A COMPOSITE NUMBER

I: A NUMBER LESS THAN 2

P: A TWO-DIGIT NUMBER IF EITHER OF ITS DIGITS IS A PRIME NUMBER

P: A TWO-DIGIT NUMBER, THE SUM OF WHOSE DIGITS IS A PRIME NUMBER

R: A TWO-DIGIT COMPOSITE NUMBER WITH BOTH OF ITS DIGITS PRIME NUMBERS

R: AN ODD PRIME NUMBER

I: A TWO-DIGIT NUMBER IF EITHER OF ITS DIGITS IS A PRIME

R: A PRIME NUMBER BETWEEN 30 AND 60

E. A SINGLE-DIGIT PRIME NUMBER

M: A NUMBER LESS THAN 2

M: A PRIME NUMBER

M: A PRIME NUMBER LESS THAN 24

P: A TWO-DIGIT COMPOSITE NUMBER

I: A PRIME NUMBER BETWEEN 50 AND 60

I: A PRIME NUMBER

I: AN ODD PRIME NUMBER

P: A NUMBER LESS THAN 2

R: A COMPOSITE NUMBER

E: A PRIME NUMBER BETWEEN 90 AND 100

I: A TWO-DIGIT NUMBER WITH A COMPOSITE NUMBER IN THE TENS PLACE

M: A TWO-DIGIT PRIME NUMBER

P: THE SMALLEST PRIME

I: AN EVEN PRIME NUMBER

P: A PRIME NUMBER

M: A PRIME NUMBER GREATER THAN 85

M: A PRIME NUMBER BETWEEN 60 AND 80

M: A TWO-DIGIT COMPOSITE NUMBER

P: A PRIME NUMBER GREATER THAN 70

M: A TWO-DIGIT NUMBER, THE SUM OF WHOSE DIGITS IS A COMPOSITE NUMBER

R: A TWO-DIGIT NUMBER WITH A COMPOSITE NUMBER IN THE ONES PLACE

R: A NUMBER LESS THAN 2

E: AN EVEN PRIME NUMBER

E: A TWO-DIGIT COMPOSITE NUMBER

E: A TWO-DIGIT NUMBER, THE SUM OF WHOSE DIGITS IS PRIME

P: A COMPOSITE NUMBER

P: A TWO-DIGIT PRIME NUMBER

E: THE LARGEST PRIME NUMBER LESS THAN 100

P: A SINGLE-DIGIT PRIME NUMBER

R: A TWO-DIGIT NUMBER, THE SUM OF WHOSE DIGITS IS A COMPOSITE NUMBER

M: A SINGLE-DIGIT PRIME NUMBER

I: ANY PRIME NUMBER

M: A TWO-DIGIT NUMBER WITH A COMPOSITE IN THE TENS PLACE

E: A PRIME LESS THAN 22

E: A COMPOSITE NUMBER

I: A TWO-DIGIT NUMBER, THE SUM OF WHOSE DIGITS IS EVEN

2	151	353	577	811	1049	1297	1559
3	157	359	587	821	1051	1301	1567
5	163	367	593	823	1061	1303	1571
7	167	373	599	827	1063	1307	1579
11	173	379	601	829	1069	1319	1583
13	179	383	607	839	1087	1321	1597
17	181	389	613	853	1091	1327	1601
19	191	397	617	857	1093	1361	1607
23	193	401	619	859	1097	1367	1609
29	197	409	631	863	1103	1373	1613
31	199	419	641	877	1109	1381	1619
37	211	421	643	881	1117	1399	1621
41	223	431	647	883	1123	1409	1627
43	227	433	653	887	1129	1423	1637
47	229	439	659	907	1151	1427	1657
53	233	443	661	911	1153	1429	1663
59	239	449	673	919	1163	1433	1667
61	241	457	677	929	1171	1439	1669
67	251	461	683	937	1181	1447	1693
71	257	463	691	941	1187	1451	1697
73	263	467	701	947	1193	1453	1699
79	269	479	709	953	1201	1459	1709
83	271	487	719	967	1213	1471	1721
89	277	491	727	971	1217	1481	1723
97	281	499	733	977	1223	1483	1733
101	283	503	739	983	1229	1487	1741
103	293	509	743	991	1231	1489	1747
107	307	521	751	997	1237	1493	1753
109	311	523	757	1009	1249	1499	1759
113	313	541	761	1013	1259	1511	1777
127	317	547	769	1019	1277	1523	1783
131	331	557	773	1021	1279	1531	1787
137	337	563	787	1031	1283	1543	1789
139	347	569	797	1033	1289	1549	1801
149	349	571	809	1039	1291	1553	1811

1823	2131	2437	2749	3083	3433	3733	4073
1831	2137	2441	2753	3089	3449	3739	4079
1847	2141	2447	2767	3109	3457	3761	4091
1861	2143	2459	2777	3119	3461	3767	4093
1867	2153	2467	2789	3121	3463	3769	4099
1871	2161	2473	2791	3137	3467	3779	4111
1873	2179	2477	2797	3163	3469	3793	4127
1877	2203	2503	2801	3167	3491	3797	4129
1879	2207	2521	2803	3169	3499	3803	4133
1889	2213	2531	2819	3181	3511	3821	4139
1901	2221	2539	2833	3187	3517	3823	4153
1907	2237	2543	2837	3191	3527	3833	4157
1913	2239	2549	2843	3203	3529	3847	4159
1931	2243	2551	2851	3209	3533	3851	4177
1933	2251	2557	2857	3217	3539	3853	4201
1949	2267	2579	2861	3221	3541	3863	4211
1951	2269	2591	2879	3229	3547	3877	4217
1973	2273	2593	2887	3251	3557	3881	4219
1979	2281	2609	2897	3253	3559	3889	4229
1987	2287	2617	2903	3257	3571	3907	4231
1993	2293	2621	2909	3259	3581	3911	4241
1997	2297	2633	2917	3271	3583	3917	4243
1999	2309	2647	2927	3299	3593	3919	4253
2003	2311	2657	2939	3301	3607	3923	4259
2011	2333	2659	2953	3307	3613	3929	4261
2017	2339	2663	2957	3313	3617	3931	4271
2027	2341	2671	2963	3319	3623	3943	4273
2029	2347	2677	2969	3323	3631	3947	4283
2039	2351	2683	2971	3329	3637	3967	4289
2053	2357	2687	2999	3331	3643	3989	4297
2063	2371	2689	3001	3343	3659	4001	4327
2069	2377	2693	3011	3347	3671	4003	4337
2081	2381	2699	3019	3359	3673	4007	4339
2083	2383	2707	3023	3361	3677	4013	4349
2087	2389	2711	3037	3371	3691	4019	4357
2089	2393	2713	3041	3373	3697	4021	4363
2099	2399	2719	3049	3389	3701	4027	4373
2111	2411	2729	3061	3391	3709	4049	4391
2113	2417	2731	3067	3407	3719	4051	4397
2129	2423	2741	3079	3413	3727	4057	4409

4421	4759	5099	5449	5801	6143	6481	6841	7211	7561	7907	8273	8647	8971	9337	9677
4423	4783	5101	5471	5807	6151	6491	6857	7213	7573	7919	8287	8663	8999	9341	9679
4441	4787	5107	5477	5813	6163	6521	6863	7219	7577	7927	8291	8669	9001	9343	9689
4447	4789	5113	5479	5821	6173	6529	6869	7229	7583	7933	8293	8677	9007	9349	9697
4451	4793	5119	5483	5827	6197	6547	6871	7237	7589	7937	8297	8681	9011	9371	9719
4457	4799	5147	5501	5839	6199	6551	6883	7243	7591	7949	8311	8689	9013	9377	9721
4463	4801	5153	5503	5843	6203	6553	6899	7247	7603	7951	8317	8693	9029	9391	9733
4481	4813	5167	5507	5849	6211	6563	6907	7253	7607	7963	8329	8699	9041	9397	9739
4483	4817	5171	5519	5851	6217	6569	6911	7283	7621	7993	8353	8707	9043	9403	9743
4493	4831	5179	5521	5857	6221	6571	6917	7297	7639	8009	8363	8713	9049	9413	9749
4507	4861	5189	5527	5861	6229	6577	6947	7307	7643	8011	8369	8719	9059	9419	9767
4513	4871	5197	5531	5867	6247	6581	6949	7309	7649	8017	8377	8731	9067	9421	9769
4517	4877	5209	5557	5869	6257	6599	6959	7321	7669	8039	8387	8737	9091	9431	9781
4519	4889	5227	5563	5879	6263	6607	6961	7331	7673	8053	8389	8741	9103	9433	9787
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4549	4919	5237	5581	5903	6277	6653	6977	7351	7691	8081	8429	8761	9133	9461	9811
4561	4931	5261	5591	5923	6287	6659	6983	7369	7699	8087	8431	8779	9137	9463	9817
4567	4933	5273	5623	5927	6299	6661	6991	7393	7703	8089	8443	8783	9151	9467	9829
4583	4937	5279	5639	5939	6301	6673	6997	7411	7717	8093	8447	8803	9157	9473	9833
4591	4943	5281	5641	5953	6311	6679	7001	7417	7723	8101	8461	8807	9161	9479	9839
4597	4951	5297	5647	5981	6317	6689	7013	7433	7727	8111	8467	8819	9173	9491	9851
4603	4957	5303	5651	5987	6323	6691	7019	7451	7741	8117	8501	8821	9181	9497	9857
4621	4967	5309	5653	6007	6329	6701	7027	7457	7753	8123	8513	8831	9187	9511	9859
4637	4969	5323	5657	6011	6337	6703	7039	7459	7757	8147	8521	8837	9199	9521	9871
4639	4973	5333	5659	6029	6343	6709	7043	7477	7759	8161	8527	8839	9203	9533	9883
4643	4987	5347	5669	6037	6353	6719	7057	7481	7789	8167	8537	8849	9209	9539	9887
4649	4993	5351	5683	6043	6359	6733	7069	7487	7793	8171	8539	8861	9221	9547	9901
4651	4999	5381	5689	6047	6361	6737	7079	7489	7817	8179	8543	8863	9227	9551	9907
4657	5003	5387	5693	6053	6367	6761	7103	7499	7823	8191	8563	8867	9239	9587	9923
4663	5009	5393	5701	6067	6373	6763	7109	7507	7829	8209	8573	8887	9241	9601	9929
4673	5011	5399	5711	6073	6379	6779	7121	7517	7841	8219	8581	8893	9257	9613	9931
4679	5021	5407	5717	6079	6389	6781	7127	7523	7853	8221	8597	8923	9277	9619	9941
4691	5023	5413	5737	6089	6397	6791	7129	7529	7867	8231	8599	8929	9281	9623	9949
4703	5039	5417	5741	6091	6421	6793	7151	7537	7873	8233	8609	8933	9283	9629	9967
4721	5051	5419	5743	6101	6427	6803	7159	7541	7877	8237	8623	8941	9293	9631	9973
4723	5059	5431	5749	6113	6449	6823	7177	7547	7879	8243	8627	8951	9311	9643	
4729	5077	5437	5779	6121	6451	6827	7187	7549	7883	8263	8629	8963	9319	9649	
4733	5081	5441	5783	6131	6469	6829	7193	7559	7901	8269	8641	8969	9323	9661	
4751	5087	5443	5791	6133	6473	6833	7207								

JOURNAL ON THE GAME OF PRIME

Friday, May 2, 1969

With twenty minutes remaining in the period, the class played PRIME. The teacher explained how the game was played and drew several slips to illustrate the procedure. She reviewed all ways of winning: diagonally, vertically, and horizontally. The slips were placed on the overhead projector so the entire class could see them. Three games were completed.

ANALYSIS: Because of time considerations, and because the students had studied prime and composite numbers earlier in the year, it had been decided that we would not teach the lesson on prime numbers. Hence this activity helped bridge the gap created by the exclusion of the lesson on prime numbers. It was also highly motivational.

LESSON NINE

THE MEANINGS OF "PROOF"

The purpose of this lesson is to teach that there are different meanings of the word "proof," and that there are different ways to prove statements.

BEHAVIORAL OBJECTIVES:

1. Given a list of propositions, the student can identify an appropriate way to verify each proposition.
2. From a list of examples, the student can identify those which make a poor choice of authority.
3. The student can give an example of how our senses are unreliable.

MATERIALS:

1. "Seeing is Believing" story (which contains optical illusions).
2. List of propositions
3. Examples of authority
4. Mastery Test
5. Chart with the goals of the unit

PROCEDURES: Hand out the story and let the students read it. Discuss it in class and write the three basic ways of "proving" propositions on the board. "We have now proven five different theorems by using careful reasoning. In mathematics we try to prove many things by using logic and reasoning. We also prove things outside the math classroom. That is, the word "prove" has different meanings. We have seen in the story that we can prove things by reasoning, by using our senses, and by consulting a reliable authority."

"<u>SENSES</u>: In everyday situations we often use one or more of our five senses to prove things. For example, suppose John comes into the classroom and says, "Teacher, Billie is writing on the wall in the room across the hall." The teacher insists that Billie would never do a thing like that. "O.K.", says John, "I'll prove it to you. Come with me!" They walk across the hall and <u>see</u> Billie writing all over the wall."

"In this case John proved his statement by relying upon his and the teacher's ability to <u>see</u>."

"There are four other senses. Can you name them?" Write them on the board, then ask the students to provide a hypothetical story (like the one just given) to illustrate how each of the remaining four sense could be used to prove something. Then use the optical illusions to point out that our senses can deceive us.

<u>AUTHORITY</u>: It is impossible to directly verify every fact we need to know in order to get along in this world. It is therefore necessary to rely upon authorities for much of what we know. For example, give an historical example. Columbus sailed to America in 1492.

Then ask the class to name some different authorities. Make sure that the following are listed: dictionaries, history book, math books, (other books), encyclopedias, experts, other people, newspapers and

and television. The computer is also a good example.

Distribute sheet with examples of using authorities and discuss. The point should be made that one must be careful in selecting an authority. Authorities can be wrong, just as our senses can.

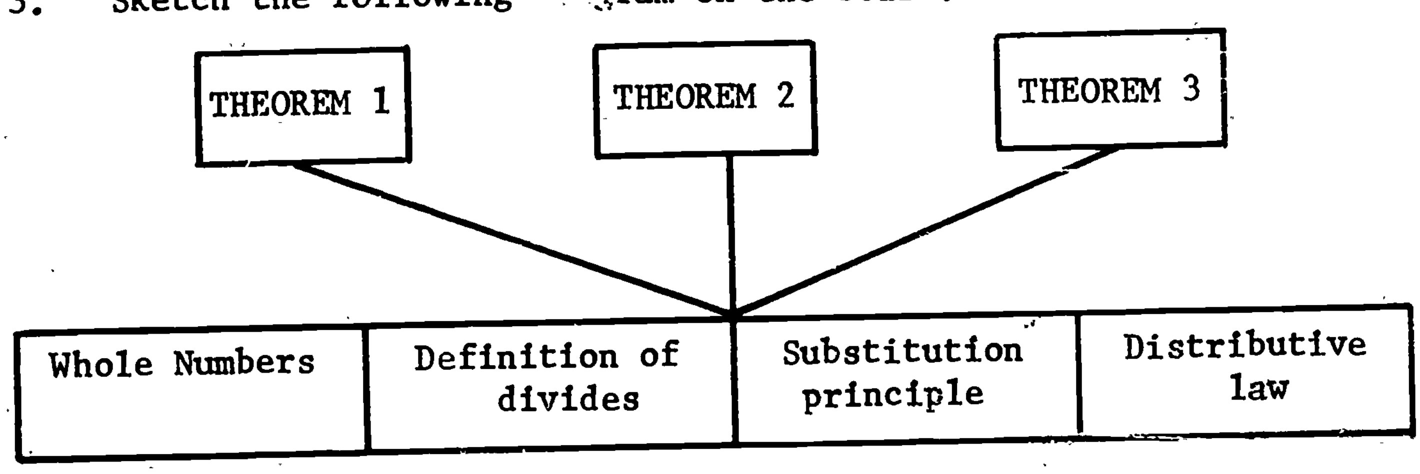
REASONING: "We also use reasoning to prove things in everyday situations. For instance, suppose I am in an office building and I see people entering the building with wet umbrellas. I conclude that it is raining outside." Then write an analysis of the reasoning involved on the board:

1. I know that if people enter the building with wet umbrellas, then it is raining outside.
2. I see people entering the building with wet umbrellas.
3. I conclude that it is raining outside.

We often use careful reasoning without realizing it."

Also point out that the proofs of the five theorems are examples of reasoning. Mathematical statements need a logical proof (stress).

"In order to prove something, we must have something with which to work. We started with an understanding of the set of whole numbers, a definition of divides, the substitution principle, and the distributive law. With these we were able to prove Theorems 1, 2, and 3." Sketch the following diagram on the board.



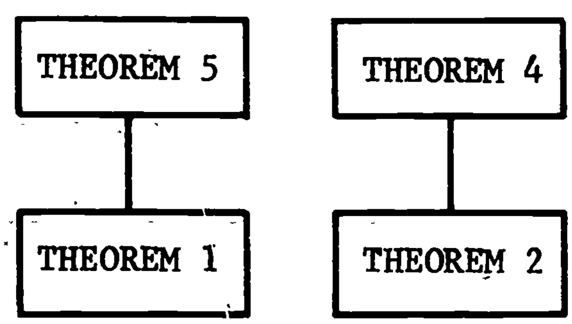
Then present a brief discussion of the nature of mathematical proof. Explain that we may use the following things to help us prove a mathematical theorem:

1. definitions

2. principles, laws, or axioms which we accept to be true without proof

3. previously proved theorems

The proofs of Theorems 4 and 5 illustrate how a previously proved theorem can be used to prove another theorem. Draw a box for Theorem 4 above the box for Theorem 2 and a box for Theorem 5 above the box for Theorem 1 in the diagram as follows:



Stress that we apply logical reasoning to these things in order to derive a proof for the theorem. It is not expected that the students will have a complete understanding of proof, for this will only come as a result of continued experiences with proof.

Then distribute the sheets with statements to prove on them. Let them answer each question, then discuss in class. Make certain that the responses are reasonable. For example, to prove that the earth is shaped like a sphere, it would not be reasonable to say that you could get in a space craft and look for yourself. A more reasonable way would be to rely upon an authority.

Administer Mastery Test.

Hand out story on prime numbers for the next lesson.

JOURNAL FOR LESSON NINE

Monday, May 5, 1969

The chart with the goals of the unit was put on the board and discussed. The stories were distributed and the students read them at their desks. The teacher asked for the three ways of knowing that something is true and wrote them on the board as the students responded. The presentation went as planned except for one example. The example of reasoning caused some confusion and many remarks. In response to the statement, "If people enter the building with wet umbrellas, then it is raining", the students made the following kinds of remarks:

"It might be snowing."

"It may have just stopped raining."

"Someone might dump water from a window."

ANALYSIS: The above example of everyday reasoning was a poor choice.

All students were masters on the Mastery Test, but one item caused some confusion: item 9 which states that you can believe everything you see. It was decided that it should be discussed in greater detail at the beginning of the next lesson.

LIST OF PROPOSITIONS

How would you prove each of the following statements to a friend?

1. There are more than 5 million people living in New York City.
2. There is a rose bush in Mrs. Gornowicz's yard.
3. The Amazon is the widest river in the world.
4. The earth is shaped like a sphere.
5. There really is a place called Japan.
6. 10,001 is a prime number.
7. The correct spelling is "parallel."
8. If you add two odd integers, the sum is even.
9. Gasoline burns.
10. Aristotle lived in Greece more than 2000 years ago.

EXAMPLES OF AUTHORITY

Which of the following are examples of using appropriate authorities?

1. George was reading his mother's high school history book, the one she had used in high school. It said that there are 48 states in the United States. George told his teacher that there are 48 states because the history book said so.

2. Jim claims that men have flown to Mars. He shows the class a book which was written by a man who believes in flying saucers. The book states that 3 men and a woman flew to Mars on a flying saucer to visit the little green people who lived there.

3. Howard says that the President of the United States is a communist. To prove it he shows the class a book written by a man who calls many people communists.

4. Jeannette says that the United States declared its independence in 1776. To prove it she got out her history book.

5. The Rhinos were having a war with the Hippos. Harvey asked a Hippo who started the war.

6. Horace did not believe that Oscar was 7 years old. To prove it, they went into the house and asked Oscar's mother.

MASTERY TEST

How would you prove each of the following statements?

1. At this very minute it is snowing outside.

2. George Washington was the first president of the USA.

3. If you add an odd integer to an even integer, the sum is odd.

4. The Chicago Cubs lost two games yesterday.

5. The earth is about 93 million miles from the sun.

Which of the following are examples of using appropriate authority?

6. Sidney could not add 4 1/3 + 5 11/16. So he asked a drunk old monkey how to do it.

7. John looked up the correct spelling of "dog" in the dictionary.

8. Stella's father cannot read. She asked him who would make the best president of the USA.

9. One thing is certain: you can believe everything you see. T F

10. Give an example to support your answer to problem 9.

SEEING IS BELIEVING?

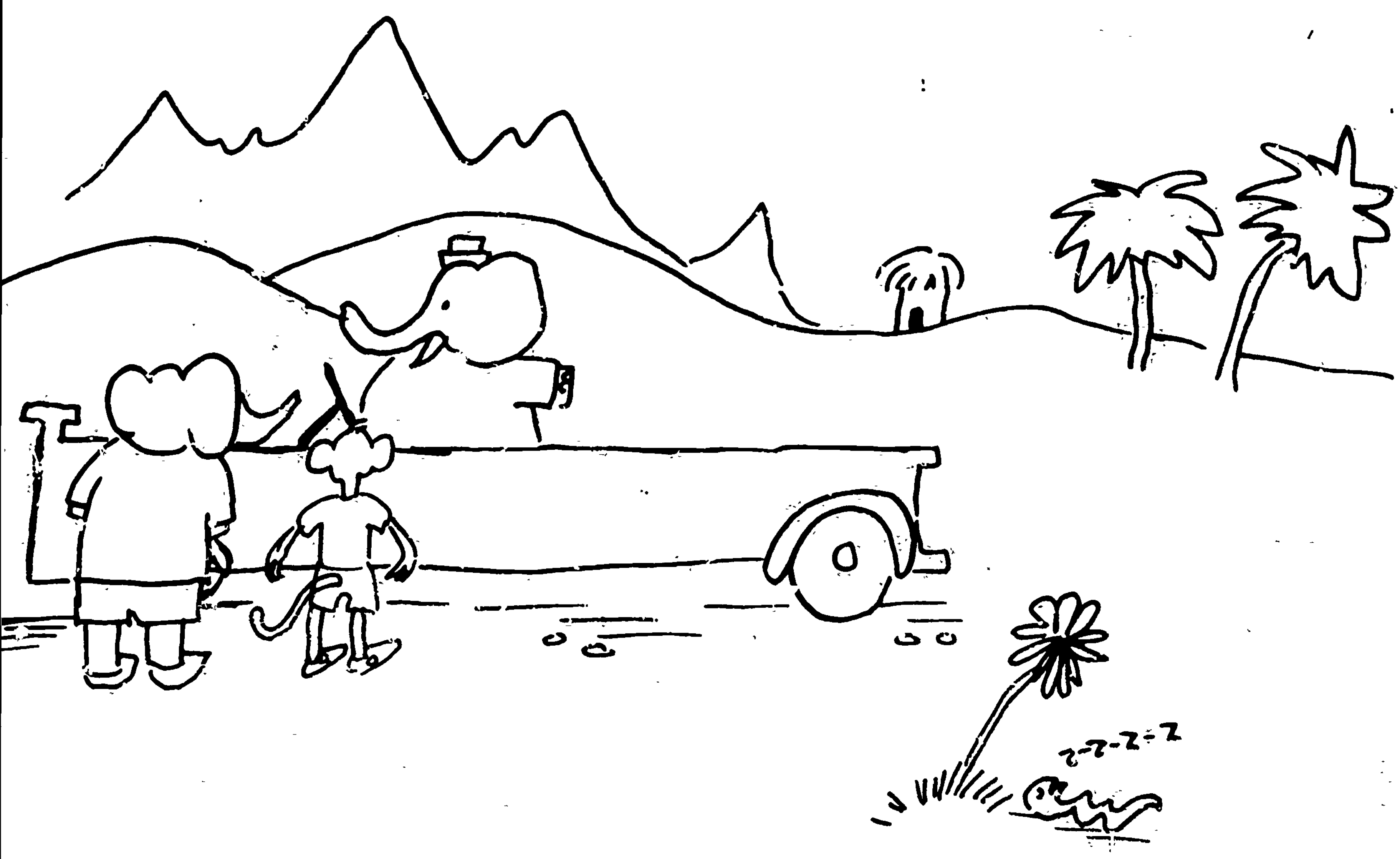
Emirp was just getting into his car, a 1933 Monkeymobile, when Sidney and Harvey arrived.

"No lesson today. My dog has been missing for a week. The birds tell me that he is in Rhinoville. I'm afraid that the Rhinos have dog-napped him. So I must go there and rescue him."

"Let us go with you. You might need help," said Sidney.

"O. K. But it may be a dangerous trip." They all jumped into the car, and down the road they flew.

"Mr. Emirp, how do you know when something is really true?" asked Harvey.



"Well, it depends on what you are talking about," said Emirp.

"Ellic Elephant says that the earth is round like a ball. But it looks flat to me."

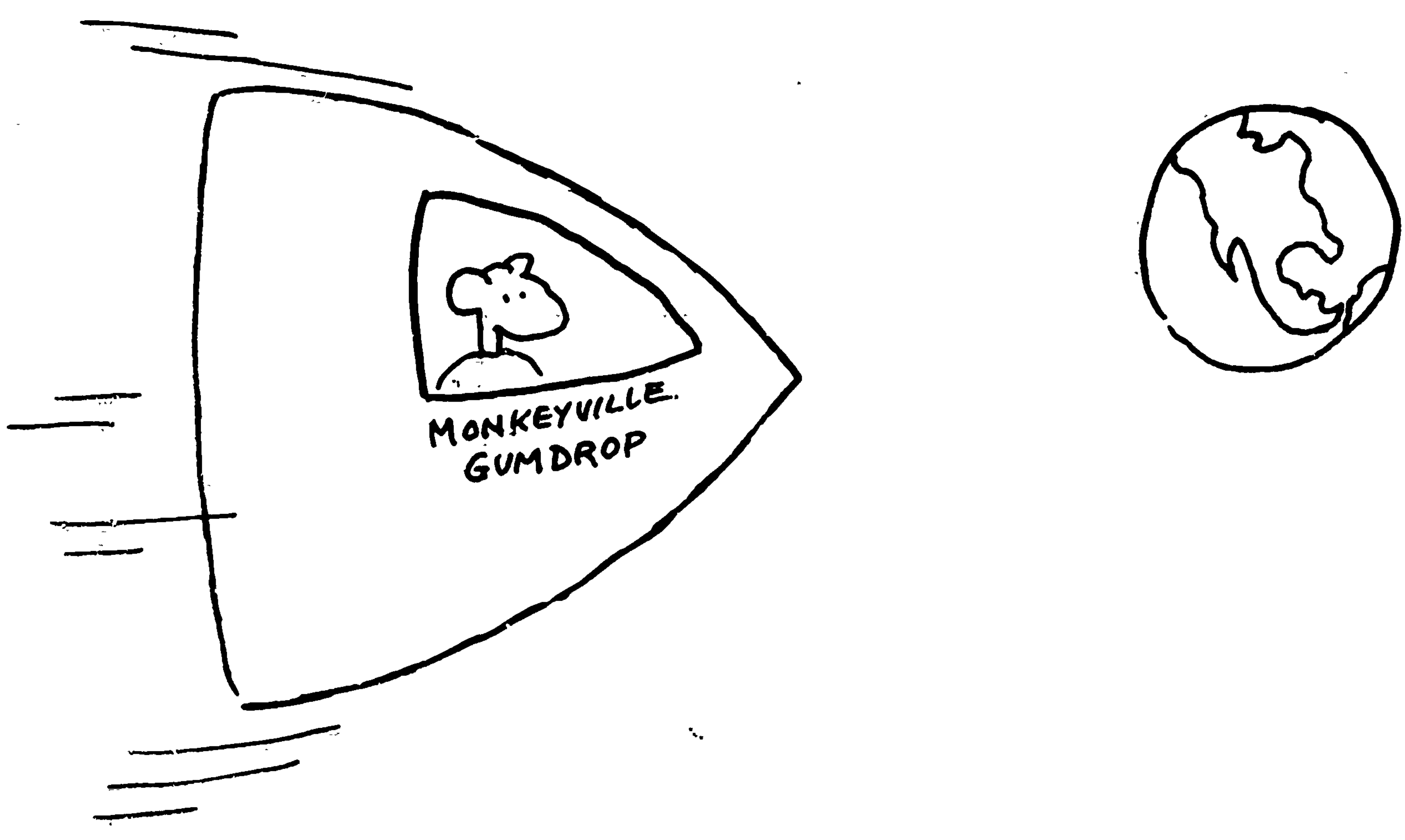
"There are many things," said Emirp, "which we cannot verify by ourselves. We must accept the word of some authority, some person who is an expert."

Sidney said, "I know the earth is round because I heard the astronauts on T.V. They are authorities, because they have seen the earth from space."

"Right. Also, how do you know that New York City is really there if you haven't seen it, or that George Monkeyton was the first King of Monkeyville?" asked Emirp.

"The Atlas shows New York City on the map, so I believe it really exists," said Harvey.

"And our history book says that George Monkeyton was the first King of Monkeyville. It is an authority, so I believe it," answered Sidney.

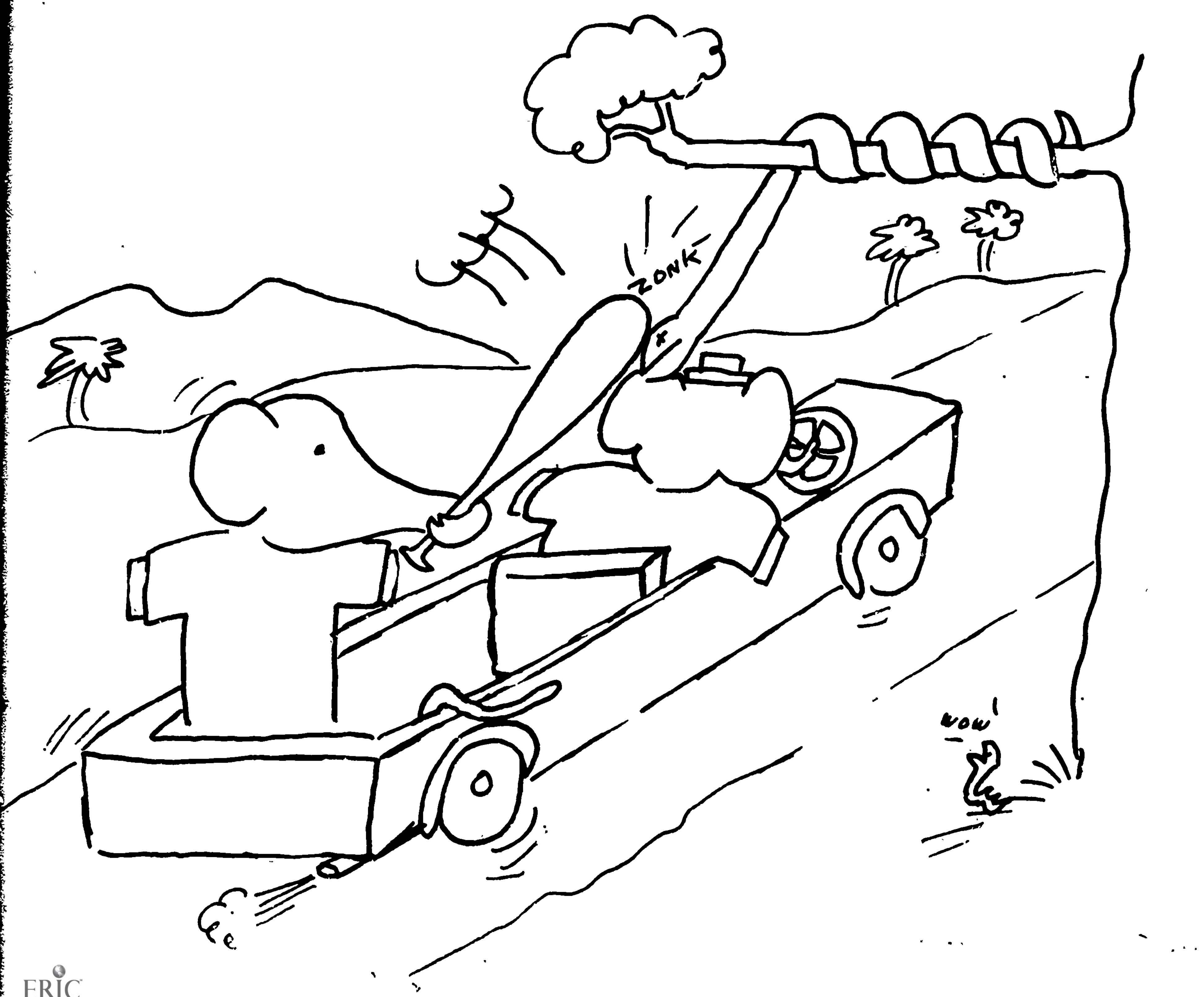


"So you see, there are many authorities to tell what is true."

A large tree was growing beside the road. Its limbs stretched out over the road. A huge snake was coiled around a limb.

"Look!" said Sidney. "There's a snake hanging from that tree. It must be hungry."

As they drove under the tree, Sidney ducked and closed his eyes. But Harvey was not afraid. He grabbed a baseball bat and clobbered the snake.



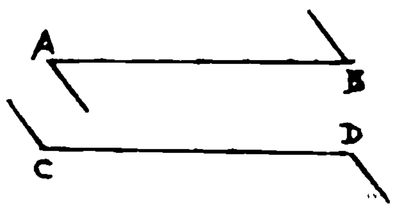
"That's another way we have of knowing things," said Emirp. "We perceive things with our five senses: hearing, seeing, smelling, tasting, and touching."

"Righto. That's why we knew that snake was there. Seeing is believing," said Sidney.

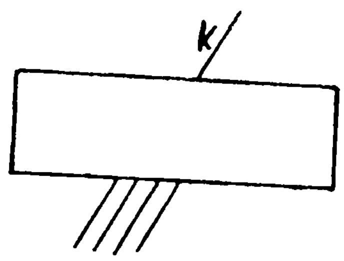
"We do know that many things are true because of our senses," said Emirp.

"But our eyes can play tricks on us." He pulled a sheet of paper out of the glove box.

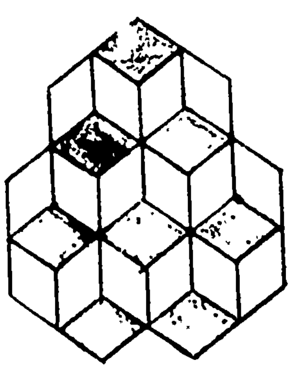
"Look at these," he said.



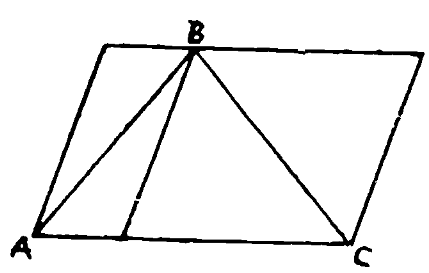
1. Which line is longest, AB or CD ?



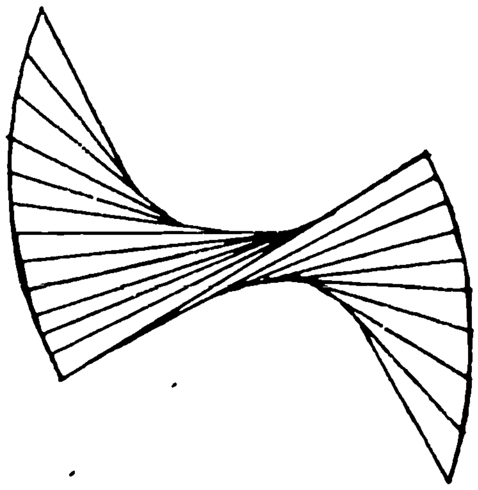
2. Which of the four lines below the rectangle is the continuation of K?



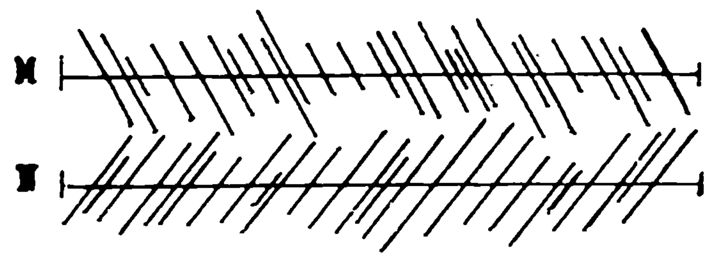
3. How many cubes are in this figure?



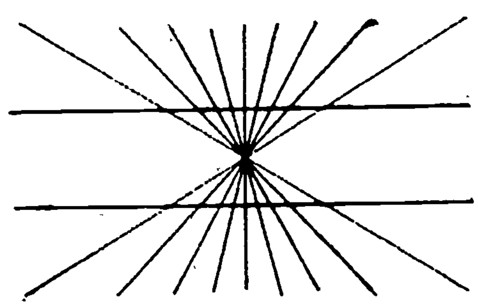
4. Which line segment is longest, AB or BC ?



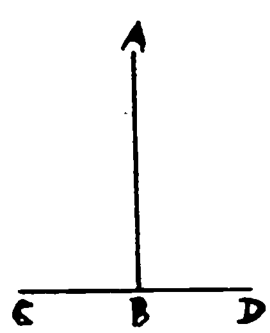
5. How many curved lines are used in making this picture?



6. Are the line segments M and N always the same distance apart?



7. Are the heavy lines straight or are they curved?



8. Which line segment is the longest, AB or CD ?

"You are right," said Sid. "You cannot believe everything you see."

The road ended. The jungle was so dense that a car could not go between the trees and brush. "We must walk," said Emirp.

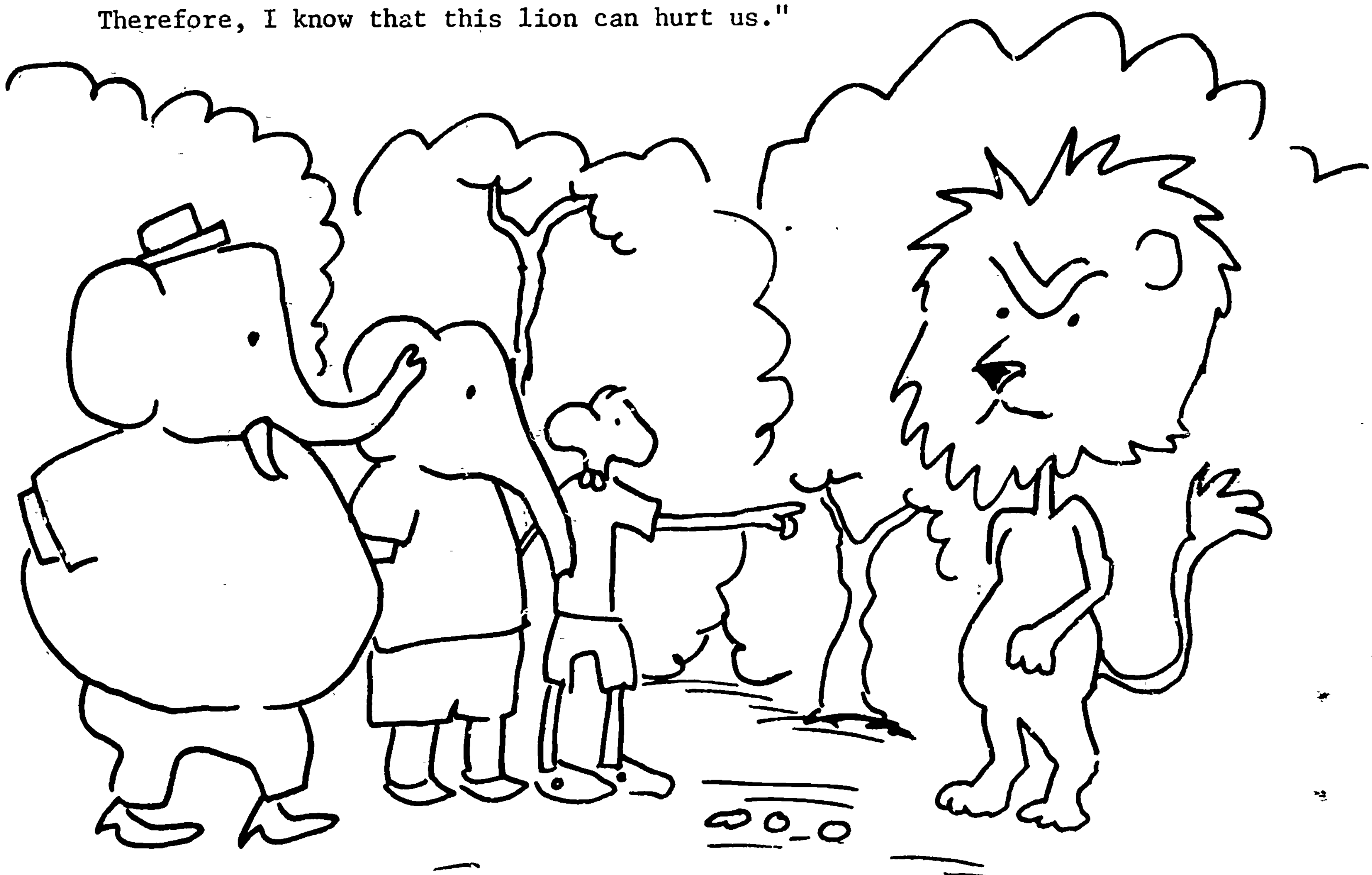
Thus far our friends have concluded that there are two ways to verify statements: by authority and by our senses.

"Is there any other way we can prove things?" asked Harvey.

"Yes," said Emirp, "by reasoning." Just then a lion jumped in front of them. "For example," said Emirp calmly. "See this lion. I know two things about lions, and therefore I can tell you what is about to happen.

1. Animals with sharp teeth can hurt you.
2. This lion has sharp teeth.

Therefore, I know that this lion can hurt us."

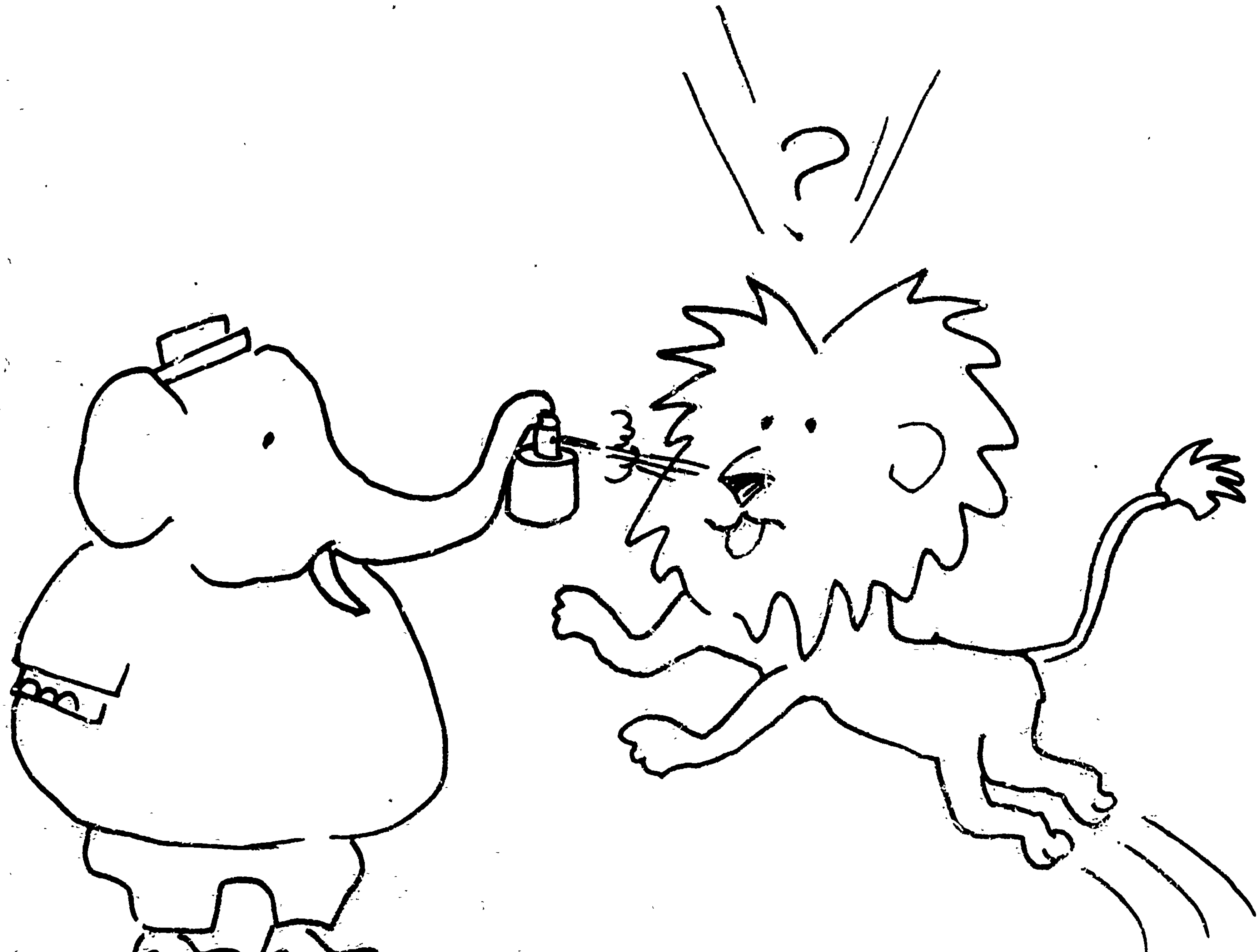


Just as the lion jumped at Sidney, Emirp reached into his coat and pulled out a spray can. Quick as a wink he sprayed a mist in the lions face.

"Yeowee," howled the lion. He ran crying through the jungle.

"What is in that can?" asked Sid.

"Tear gas for lions," answered Emirp. "I always carry it just in case of an emergency."



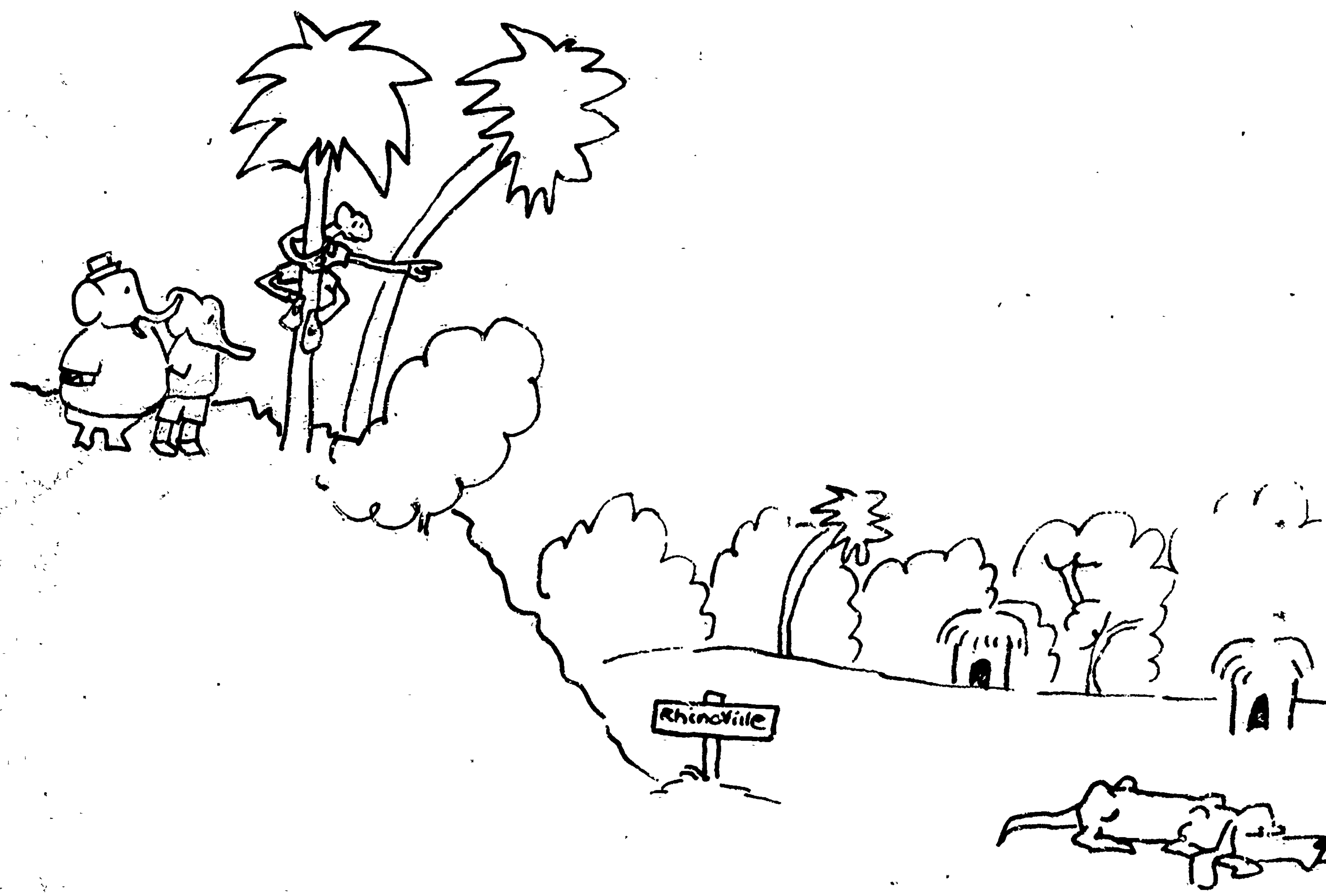
They continued through the jungle.

"Quiet. We are near Rhinoville. If they hear us, it might be the end of us."

They hid behind some trees and looked down at the village. "Look!" whispered Harv. "There's Dog."

Emirp took a whistle out of his coat." This is a whistle that a dog can hear, but the Rhinos cannot hear. I'll blow it and Dog will come to us." He blew the whistle. Harvey and Sidney could not hear a thing, but Dog's ears perked up. He came running towards Emirp.

"Quick, come with us!" said Emirp to Dog. "We have come to rescue you."



"Rescue me?" said Dog. "Didn't you get my note?"

"What note?"

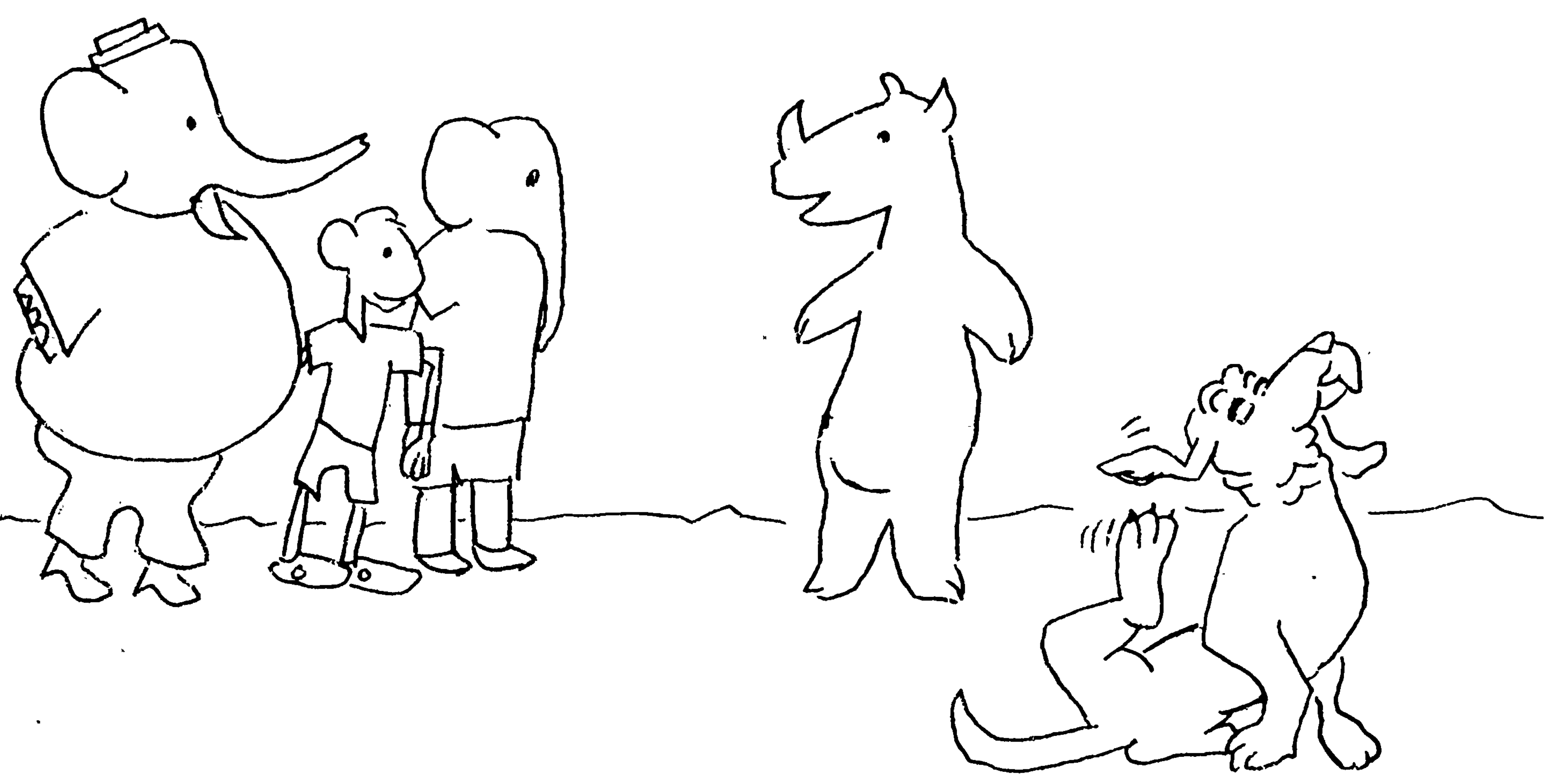
"The one I left on the table before I left."

"Oh. I'll bet I threw it away without realizing it was a note from you."

"Well, I am here to visit my cousin, Doggie Dog. We are having a great time, and the Rhinos are so nice."

They walked down into the village. The Rhinos asked them to stay for supper, and they all had a good time.

And so Harvey and Sidney learned that there are three ways to prove that something is true: by authority, by our senses, and by reasoning.



LESSON TEN

THEOREM 6: GIVEN ANY SET OF PRIME NUMBERS $\{2,3,5,...P\}$, THERE IS ALWAYS ANOTHER PRIME.

The purpose of this lesson is to prove this theorem.

BEHAVIORAL OBJECTIVE:

1. The student can write the proof of theorem 6.

MATERIALS:

1. Story on Prime Numbers
2. List of prime numbers less than 60,000
3. Two sheets with incorrect proofs
4. Practice sheet

PROCEDURE:

Pass out the computer print out of all prime numbers less than 60,000. "Here is a list of prime numbers less than 60,000. They were formed by one of the University of Wisconsin computers. It took 14 seconds to do. How long would it take a man to find all of the primes less than 60,000?"

"Suppose we started the computer and asked it to find prime numbers. If we didn't stop it, how long would it keep going?" The answer is forever, of course, since there are infinitely many primes.

"Today we are going to prove that there are infinitely many primes. We shall show that for any set of primes, there is always another one. For example, suppose someone said that $\{2,3\}$ was the set of all prime numbers. Even if we didn't know any more primes, we can show that there has to be another prime."

Then proceed to show how to get another prime. Write (and explain):

$(2 \times 3) + 1$ is a whole number

$2 \nmid 1$ and $2 \mid (2 \times 3)$ so $2 \nmid (2 \times 3 + 1)$

$3 \nmid 1$ and $3 \mid (2 \times 3)$ so $3 \nmid (2 \times 3 + 1)$

(because of theorem 4)

Since some prime number must divide $(2 \times 3) + 1$, there is another prime besides 2 and 3.

"We know, of course, that 7 is prime. But even if we did not know this, our reasoning would tell us that there is some prime besides 2 and 3."

Do the same for $\{2,3,5\}$, asking the students for each step.

$(2 \times 3 \times 5) + 1$ is a whole number

$2 \mid (2 \times 3 \times 5)$ and $2 \nmid 1$ so $2 \nmid (2 \times 3 \times 5 + 1)$

$3 \mid (2 \times 3 \times 5)$ and $3 \nmid 1$ so $3 \nmid (2 \times 3 \times 5 + 1)$

$5 \mid (2 \times 3 \times 5)$ and $5 \nmid 1$ so $5 \nmid (2 \times 3 \times 5 + 1)$

Some prime must divide it, so there is another prime.

"We know 31 is prime, but even if we didn't, we know there is some prime which divides it." By this time the students should see the procedure. If you think it necessary, it can be repeated for the set $\{2,3,5,7\}$, and even $\{2,3,5,7,11\}$ if necessary.

Now consider the set $\{2,3,5,7,11,13\}$, and ask, "How can we show that there has to be another prime?" Have the students supply each step. This time, number the steps, and give a reason for each statement. When this proof is complete, the general theorem will follow by replacing

13 by P. Explain the meaning of the three dots.

1. (2 x 3 x ... x 13) + 1 is a whole number	1. fact

Ask, "Will any of the numbers in the set divide (2 x 3 x ... x 13) + 1?" Point out that Theorem 4 tells us that none will divide it." Instead of writing it out for each of the numbers in the set, we will just write:

2. None of the numbers in the set will divide it.	2. Theorem 4
3. ∴ There is another prime number	3. Axiom 1

Then evaluate (2 x 3 x ... x 13) + 1 = 30,031. This number was factored by the desk computer and the students have it written on the list of primes. Explain that this procedure always gives us a prime, but as in this case the number itself is not prime. 30,031 = 59 x 509.

Present the proof for the set $\{2,3, \ldots, P\}$. Explain that the three dots represent all the numbers between 3 and P. For example, if P = 59,999 (the largest prime on the print out), then the three dots would represent over 100 pages of prime numbers.

Ask a student to come to the board and, by making appropriate changes, write a correct proof for the general theorem. Merely replace P by 13.

Distribute a sheet with theorem 6 on it and ask the students to copy the proof. Then pass out sheet #1. Then quiz. Sheet # 2. Circulate about the room correcting any errors.

Point out that the plan of the proof is to construct a number which is not divisible by any of the primes in the set. Then administer the mastery test.

JOURNAL ON LESSON TEN

Tuesday, May 6, 1969

The lesson went almost precisely as planned. When shown the list of all the prime numbers less than 60,000, many comments were made.

After the first example with $\{2,3\}$, the students all appeared to understand the procedure, for they were all anxious to volunteer to do it for the other examples given, namely for $\{2,3,5\}$ and $\{2,3,\ldots,13\}$.

The period ended after the students had copied the proof once.

ANALYSIS: A good presentation. We decided to continue with the lesson as it was originally planned.

Wednesday, May 7, 1969

The teacher began by reviewing the proof of Theorem 6 on the board. The rest of the lesson went as planned.

ANALYSIS: All the students were able to prove the theorem on the Mastery Test.

With two days remaining, it was decided that Thursday would be spent in a general review of the unit. The students wanted to spend the entire period having a party. Since the class period follows their lunch hour, a compromise was reached: After eating their lunches, the students would come to class early for cupcakes and kool-aid. Enough class time would be allotted for finishing the snacks.

PRACTICE SHEET

GIVEN ANY SET OF PRIME NUMBERS $\{2, 3, \cdots, P\}$, THERE IS ANOTHER PRIME NUMBER.

STATEMENT	REASON
1.	1.
2.	2.
3.	3.

SHEET # 1

THEOREM 6: GIVEN ANY SET OF PRIME NUMBERS $\{2, 3, \cdots, P\}$, THERE IS ANOTHER PRIME.

STATEMENT	REASON
1. $(2 \times 3 \times \cdots P) + 1$ is a whole number	1. Fact
2. None of the numbers in the set will divide it.	2. Axiom 1
3. ∴ There must be another prime	3. Theorem 4

THEOREM 6: GIVEN ANY SET OF PRIME NUMBERS $\{2, 3, \cdots, P\}$, THERE IS ANOTHER PRIME.

STATEMENT	REASON
1. $(2 + 3 + \cdots + P) + 1$ is a whole number	1. Fact
2. None of the numbers in the set will divide it.	2. Axiom 1
3. ∴ There must another prime	3. Theorem 4

SHEET # 2

THEOREM 6: GIVEN ANY SET OF PRIME NUMBERS {2, 3, ⋯, P}, THERE IS ANOTHER PRIME NUMBER.

STATEMENT	REASON
1. $(2 \times 3 \times \cdots \times 1) + P$ is a whole number	1. Fact
2. None of the numbers in the set will divide it	2. Axiom 1
3. ∴ There is another prime	3. Theorem 1

THEOREM 6: GIVEN ANY SET OF PRIME NUMBERS {2, 3, ⋯, P}, THERE IS ANOTHER PRIME NUMBER.

STATEMENT	REASON
1.	1. Fact
2.	2. Theorem 4
3.	3. Axiom 1

LESSON TEN MASTERY TEST

1. Given any set of prime numbers {2, 3, 5, ···, P}, there is always another prime number.

STATEMENT	REASON

2. State Axiom 1:

Replace the dots by the numbers and signs they represent:

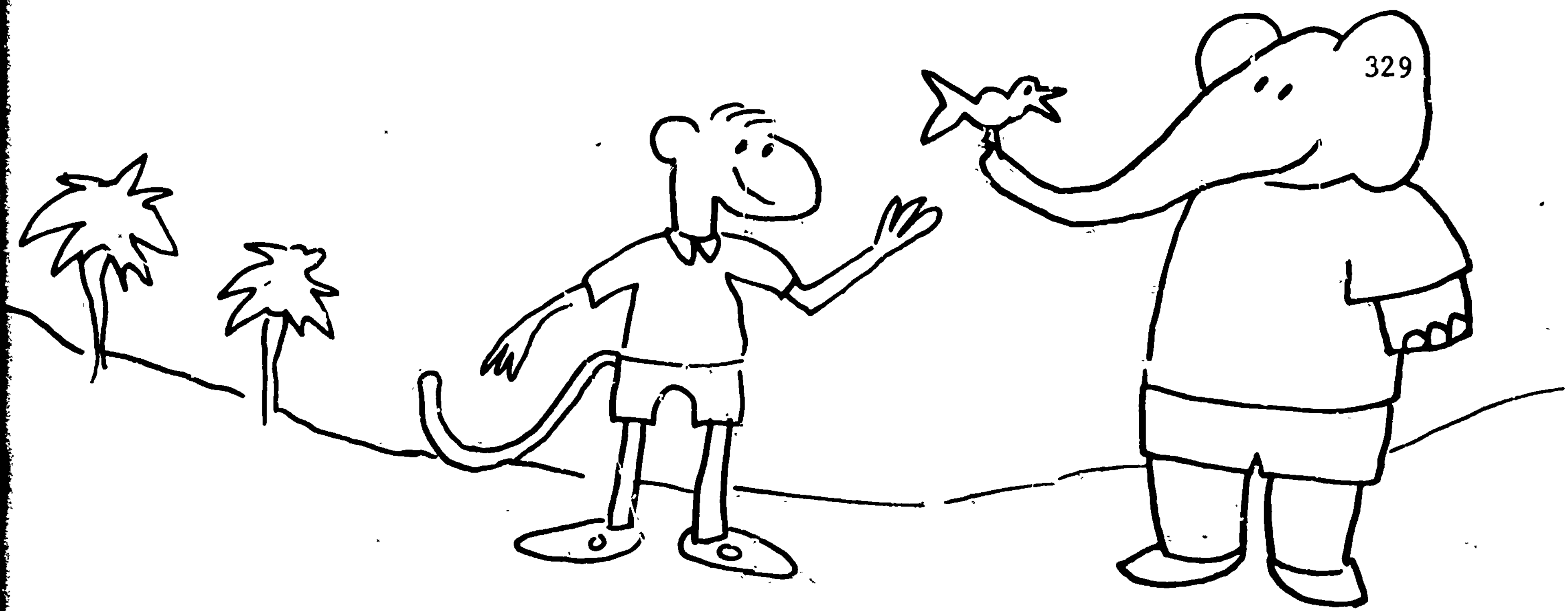
3. 1 + 2 + 3 + ··· + 8 =

4. (evens) 6 + 8 + 10 + ··· + 16 =

5. (odds) 1 + 3 + 5 + ··· + 11 =

6. (primes) 2 + 3 + 5 + ··· + 13 =

7. (primes) 2 x 3 x 5 x ··· x 17 =



Harvey and Sidney knocked loudly on the door, but Mr. EMIRP did not answer. "Gosh," said Harvey, "I wonder where he could be?"

Hearing this, a bird flew down out of a tree and landed on Harvey's trunk. "Excuse me," said the bird in a squeaky voice, "but Mr. EMIRP has gone fishing today."

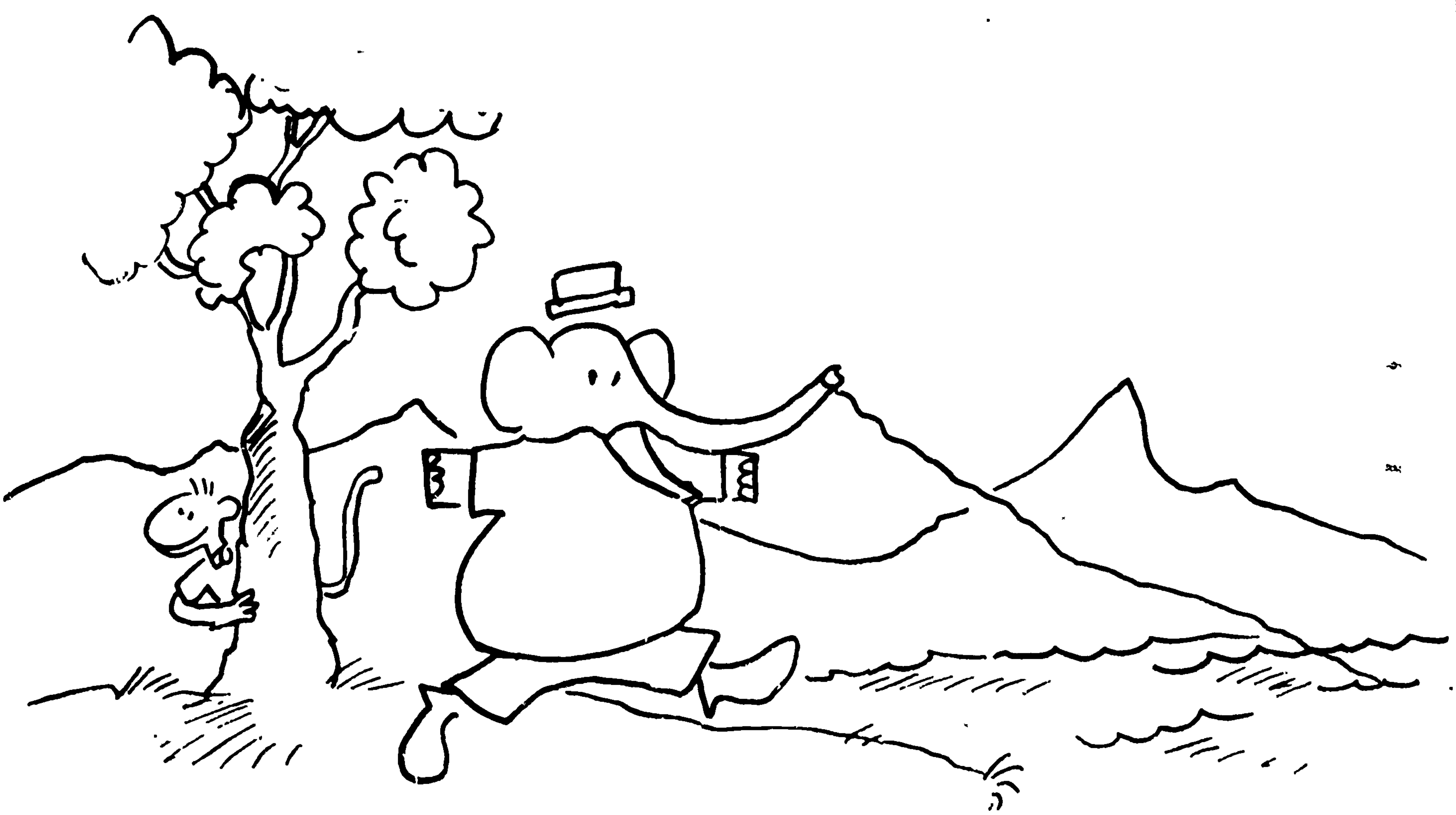
"That's too bad. We were supposed to study math with him today. Guess we'll have to forget it for today," Sidney said happily, for he was not in the mood for studying.

"Nonsense," said the bird, "I'll take you to EMIRP."

And so the boys followed the bird through the jungle. They soon found EMIRP sitting in the shade of a tree, fast asleep. He had the fishing line tied to the end of his trunk. If he got a bite, the line would tug his trunk and he would wake up to haul in his catch. Sidney sneaked behind EMIRP and gave the fishing line a big tug. Thinking he had a fish, EMIRP jumped to his feet. "A fish, a fish. I've caught a fish," he shouted as he reeled in the line by wrapping it around his trunk. You can imagine his disappointment when he discovered that there was no fish on the line.

As he sadly turned around he noticed Harvey and Sidney. "Golly," said EMIRP, "I really had a whopper, but he was so strong that he broke the line."

Sidney grinned from ear to ear, but Harvey did not think it was very funny.



"We have come for our math lesson," said Harvey.

"Good," said EMIRP as he untangled the fishing line from his trunk. "Do you know what a prime number is?"

"Is it something like prime ribs?" asked Harvey.

"No," chuckled Mr. EMIRP, "A prime number is **any whole number which has exactly two divisors,** 1 and itself."

"Oh. Then 11 is a prime number," said **Sidney.**

"That's right," said EMIRP. He had a large bag of peanuts, and he gave one to **Sidney** for saying something intelligent.

"And 5, 7, 11, and 13 are primes," shouted Harvey, hoping to get 4 peanuts. EMIRP tossed him one. "2 is a factor of 24, so 24 is an un-prime," said Sidney.

"Well," Mr. EMIRP quickly added, "any **whole number which has more than two divisors** is called a composite number." "Oh. Then 6, 8, 10, 40, and 52 are composite," said Sidney.

"All even numbers are composite," said Harvey with a big smile, hoping to get another peanut.

EMIRP frowned. Harvey thought. Sidney thought and thought. "Not quite," said Sidney, "there is one even **number** which is not a composite number."

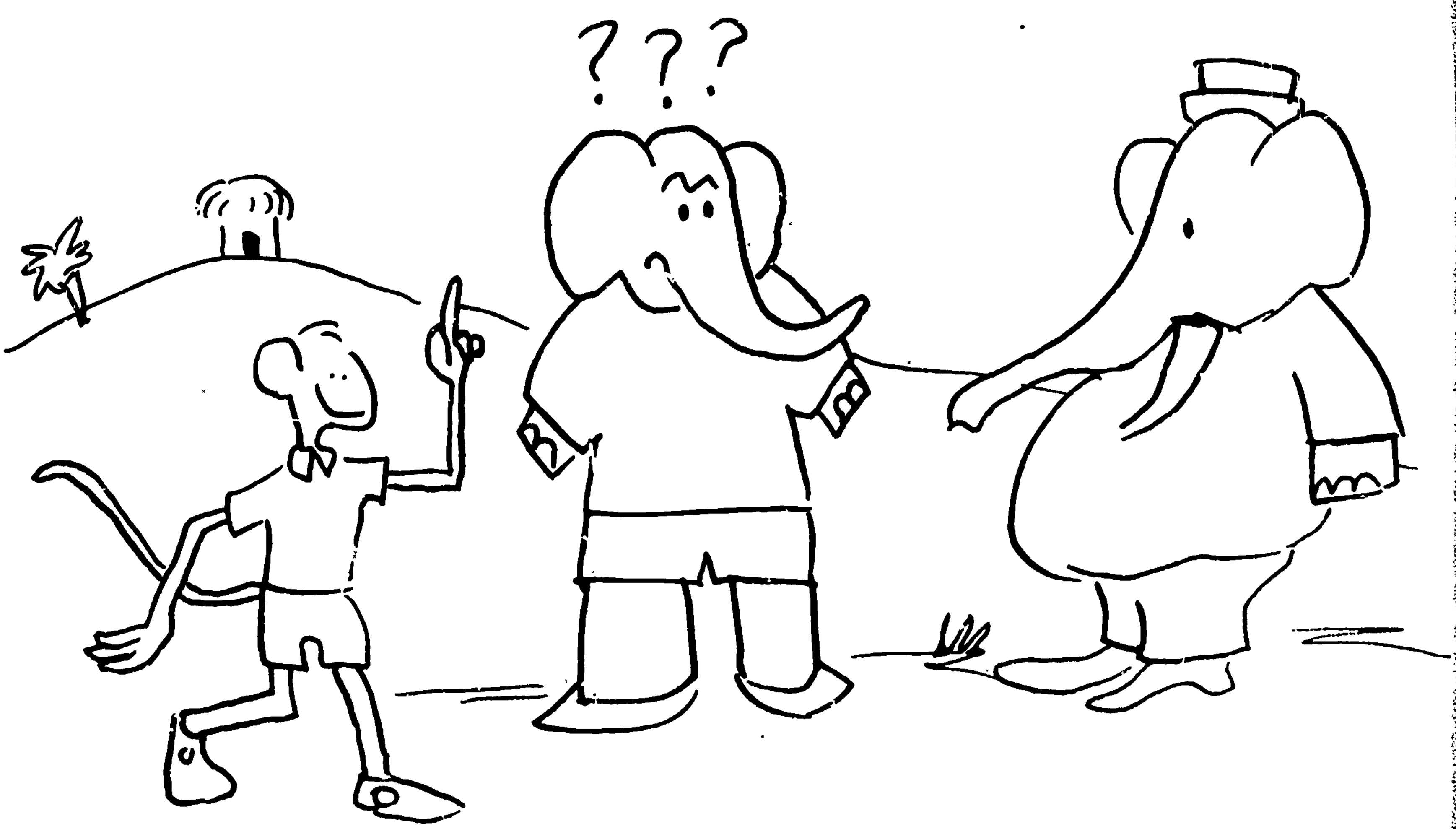
Harvey thought. And thought. But he could not find the number. Finally Sidney said, "2 is prime and 2 is even."

EMIRP gave him another peanut.

Harvey was embarrassed. Then it was Sidney's turn to appear stupid.

"All odd numbers are prime," he said without thinking, hoping to get still another peanut.

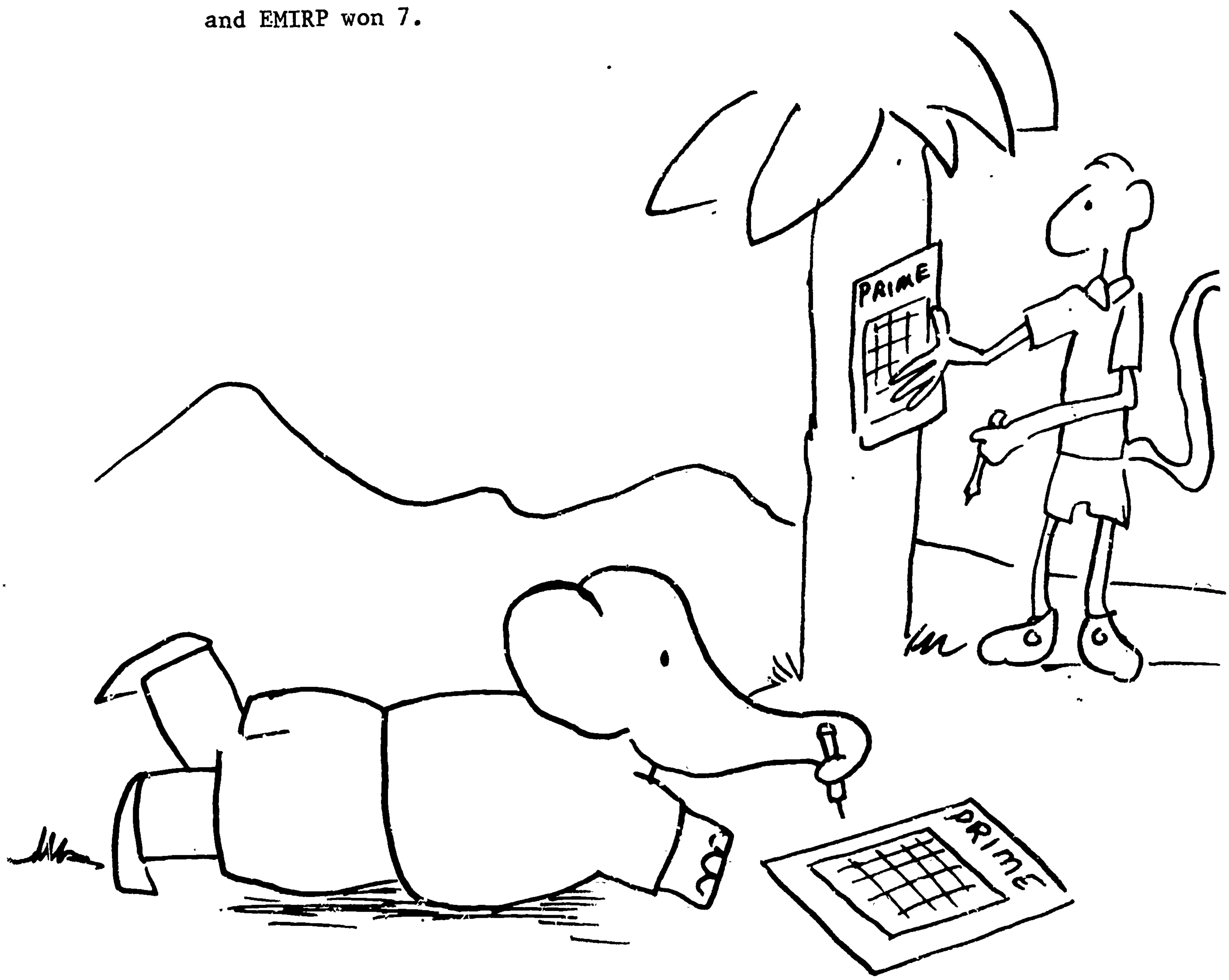
"Not so," smiled Harvey. "15 = 3 x 5."



It was Sidney's turn to be embarrassed. "There must be a lot of prime numbers," said Sidney, trying to change the subject. "I wonder how many there are?" Do you know?

"That's an excellent question," said EMIRP. He then showed the boys how to find all of the prime numbers less than 100. They discovered 25 prime numbers less than 100.

"I have invented a game. It is called PRIME," said the teacher. He explained how to play, and for the next three hours they played PRIME. "This is more fun than a barrel of monkeys," said Harvey. They played 31 games. Harvey won 11, Sidney won 13, and EMIRP won 7.



"Now," said the wise old elephant, "we must get on to the main topic for today."

"What is that?" asked Sidney.

"The fact that every counting number greater than 1 is divisible by at least one prime number."

"Holy hay!" said Harvey.

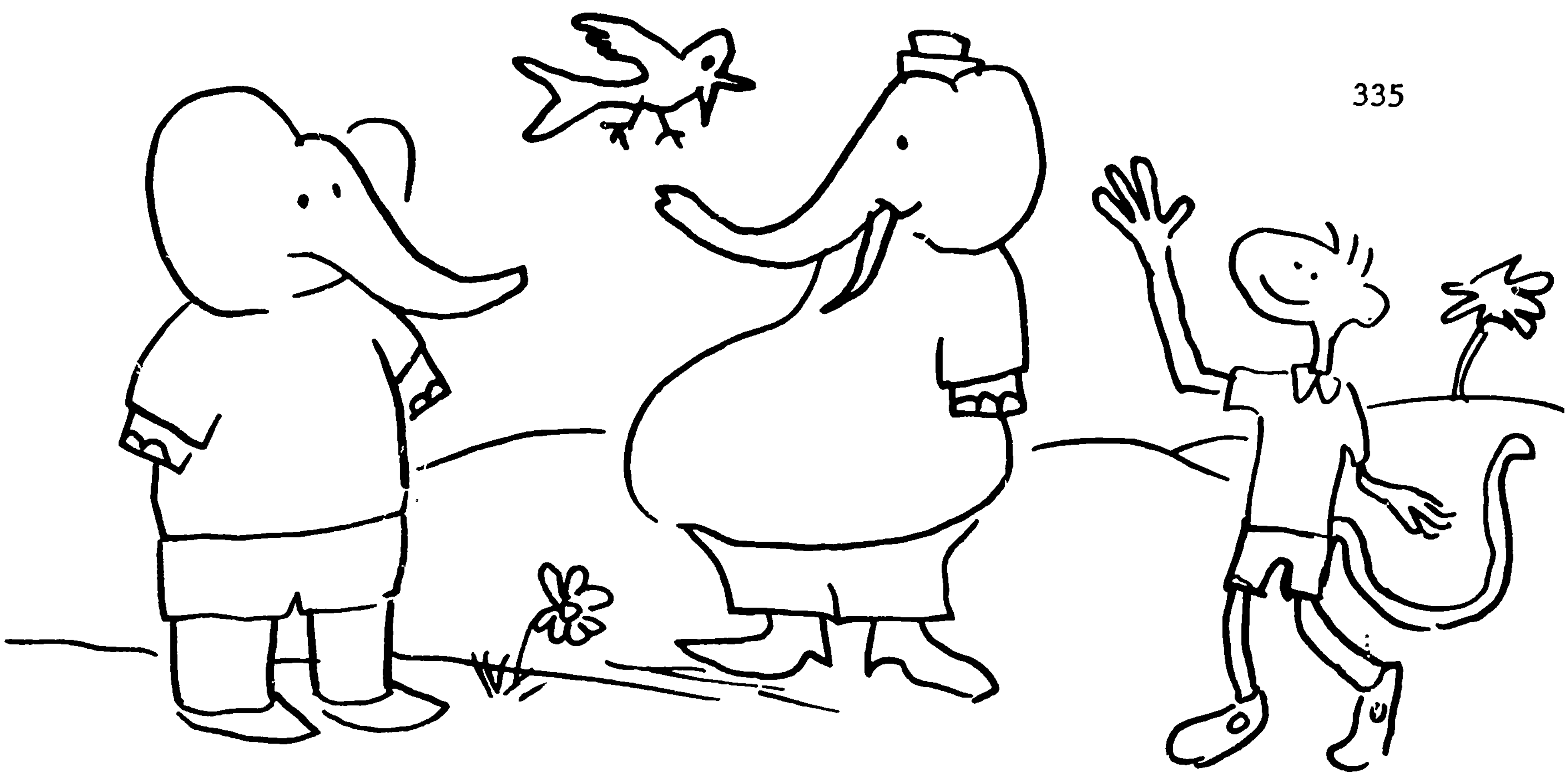
"Buckets of bananas!" exclaimed Sidney. "Is that really true?"

"Well, try it! Can you think of a prime number which divides 20?" asked EMIRP.

"Easy peasy," said Harvey, "2 and 5."

"Good. For each of the following numbers, give at least one prime which divides it."

1.	15	11.	48
2.	16	12.	96
3.	21	13.	33
4.	7	14.	41
5.	23	15.	45
6.	17	16.	51
7.	28	17.	37
8.	13	18.	82
9.	25	19.	99
10.	24	20.	75



Just then the bird returned. "Sidney. Have you forgotten. Today is Bananaday. All the monkeys are waiting for you and Harvey to figure out how many bananas they get.

"Blue bananas!" said Sidney. "We were having so much fun with EMIRP that I forgot."

And off they ran.

Than night Sidney ate 37 bananas, Harvey ate 41 bales of hay, and Mr. EMIRP ate 79 bales of hay. He would have eaten 80, but 80 is not prime.

LESSON ELEVEN

SUMMARY

The purpose of this final lesson is to review the unit and to summarize what has been done. The lesson includes a short discussion of mathematical proof.

BEHAVIORAL OBJECTIVES:

1. The student can write the proofs of all six theorems.
2. Given a list of statements, the student can name a reason for each statement.

MATERIALS:

1. Final two episodes of the story
2. Three sheets with incorrect proofs
3. Three practice sheets for writing proofs
4. Practice sheets on prerequisites
5. Practice sheets on reasons

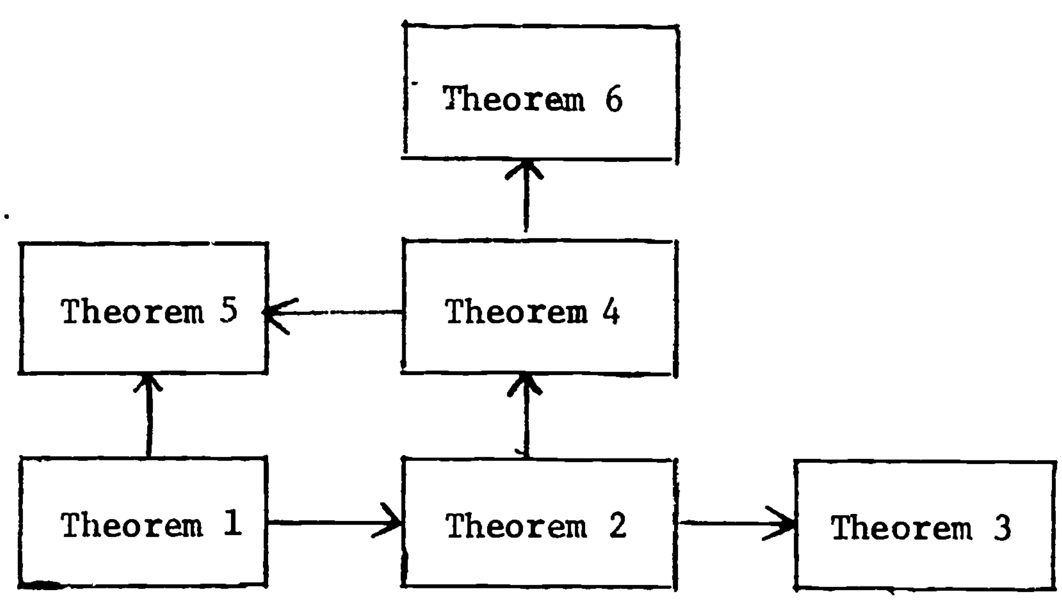
PROCEDURE: Use Theorem 1 to explain the axiomatic nature of mathematics. This unit is the first contact with formal proof that the students have had, hence it is not expected that they have a complete or mature understanding of proof. What is intended there is to expose the students to the basic ideas involved in an axiomatic system and to use the theorems in the unit to illustrate it.

Begin with an explanation of definitions. Then explain that in order to prove some results, we must accept a few statements without proof. These are usually properties which are very obvious. For example, the substitution principle, the distributive law, and Axiom 1 were all accepted as being obviously true. In order to prove other results, we may use the following:

1. definitions
2. laws, axioms, or principles which are accepted to be true without proof
3. previously proven theorems

Point out that in proving Theorem 1 we used a definition, the distributive law, and the substitution principle. Do the same for each of the six theorems.

Then draw the following diagram on the board:



Explain that we actually proved Theorem 2 in order to prove Theorem 4, and that we proved Theorem 4 in order to prove Theorem 6.

After this discussion, distribute sheets #1, #2 and #3. Then the practice sheets, and finally the sheet with prerequisites. Circulate about the room and correct any errors. The stories may be used for homework.

Following this lesson, the posttest will be administered.

JOURNAL ON SUMMARY

Wednesday, May 7, 1969

After the Lesson 10 Mastery Test was administered, the teacher discussed the nature of mathematical proof and reviewed the unit. The diagram of the proofs appeared to be helpful to the students. The story on Theorem 6 was given for homework.

Thursday, May 8, 1969

The teacher passed out all of the drill and practice sheets and gave individual help to those who needed it. The divisibility items were forgotten by three students. The story on Lake Mills was given for homework.

ANALYSIS: The exercises seemed to be effective.

Friday, May 9, 1969

The posttest was administered to both the control and experimental group in the same manner as the pretest. After the students had finished the test, the investigator interviewed each student individually in an effort to determine how well they understood the proofs. Each student was asked what plan he would use to prove each of the theorems. The investigator's subjective observations were as follows:

1. Nine of the ten students appeared to understand the proofs of the first three theorems. One student could not explain why certain steps were taken, nor why.

2. Seven of the ten students appeared to understand the proofs of Theorems 4 and 5. Three students were unable to explain why certain steps were taken in the proofs.

3. Eight of the ten students appeared to understand the proof of Theorem 6.

NAME THE REASON FOR EACH OF THE FOLLOWING STATEMENTS:

Statements	Reasons
1. IF $X = 6$, then $A + X = A + 6$	1.
2. $AB + AX = A(B + X)$	2.
3. If an assumption leads to a false statement, then the assumption is false.	3.
4. Every whole number greater than 1 is divisible by some prime number.	4.
5. IF $N \mid A$, then there is some whole number K such that $A = NK$	5.
6. $M \mid (C - D)$ and $M \mid D$, then $M \mid C$	6.
7. $(2 \times 3 \times 7) + 5$ is a whole number	7.
8. $AB + AC$ is divisible by A	8.
9. 3 will not divide $(2 \times 3 \times 5 \times 7) + 2$	9.
10. IF $X = YZ$, THEN $Y \mid X$	10.

SHEET #1

IF $N|A$ AND $N|B$, THEN $N|(A + B)$.

STATEMENT	REASON
1. $A = NP$ $B = NQ$	1. AXIOM 1
2. $(A + B) = NP + NQ$	2. SUBSTITUTION
3. $(A + B) = N(A + Q)$	3. DEFINITION OF DIVIDES
4. $\therefore N|(A + B)$	4. DISTRIBUTIVE LAW

IF $N\nmid A$ AND $N|B$, THEN $N\nmid(A + B)$.

STATEMENT	REASON
1. $N|(A + B)$	1. GIVEN
2. $N|B$	2. THEOREM 2
3. $N|A$	3. THEOREM 1
4. $N\nmid A$	4. ASSUME
5. $\therefore N\nmid(A + B)$	5. LAW OF CONSTRICTION

SHEET #2

GIVEN ANY SET OF PRIME NUMBERS $\{2, 3, 5, \cdots, P\}$,

THERE IS ALWAYS ANOTHER PRIME NUMBER.

STATEMENT	REASON
1. $(2 \times 3 \times 5 \times \cdots \times P) + 1$ IS A WHOLE NUMBER	1. FACT
2. NONE OF THE NUMBERS IN THE SET WILL DIVIDE $(2 \times 3 \times 5 \times \cdots \times P) + 1$	2. THEOREM 5
3. $\therefore$ THERE MUST BE ANOTHER PRIME NUMBER	3. LAW OF CONTRADICTION

IF $N \mid A$, $N \mid B$, AND $N \mid C$, THEN $N \mid (A + B + C)$.

STATEMENT	REASON
1. $A = NC$ $B = ND$ $C = NE$	1. DEFINITION OF PRIME
2. $(A + B + C) = NC + ND + NE$	2. SUBSTITUTION
3. $(A + B + C) = N(A + B + C)$	3. AXIOM 1
4. $\therefore N \mid (A + B + C)$	4. DEFINITION OF DIVIDES

SHEET #3

IF $N \nmid A$ AND $N|B$, THEN $N \nmid (A - B)$.

STRAWBERRY	REASON
1. $N\|(A - B)$	1. ASSUME
2. $N\|B$	2. GIVEN
3. $N\|A$	3. THEOREM 1
4. $N \nmid A$	4. GIVEN
5. $\therefore N \nmid (A - B)$	5. LAW OF CONTRADICTION

IF $N|A$ AND $N|B$, THEN $N|(A - B)$.

STATEMENT	RAISIN
1. $N = AP$ $N = BQ$	1. DEFINITION OF DIVIDES
2. $(A - B) = NP - NQ$	2. SUBSTITUTION
3. $(A - B) = N(A - B)$	3. DISTRIBUTIVE LAW
4. $\therefore N\|(A + B)$	4. DEFINITION OF DIVIDES

PRACTICE SHEET

If $N|A$ and $N|B$, then $N|(A - B)$.

STATEMENT	REASON

Given any set of prime numbers $\{2, 3, \cdots, P\}$, there is another prime.

STATEMENT	REASON

PRACTICE SHEET

If $N \mid A$ and $N \mid B$, then $N \mid (A + B)$.

STATEMENT	REASON

If $N \nmid A$ and $N \mid B$, then $N \nmid (A - B)$.

STATEMENT	REASON

PRACTICE SHEET

If $N|A$, $N|B$, and $N|C$, then $N|(A + B + C)$.

STATEMENT	REASON

If $N \nmid A$ and $N|B$, then $N \nmid (A + B)$.

STATEMENT	REASON

REVIEW OF PREREQUISITES

Apply the distributive law to the following expressions:

1. (9 x 8) + (9 x 53) =
2. AQ + AR =
3. 6t + 6K =
4. AS + AT =
5. MB + MN =
6. 89B + 89V =
7. If an assumption leads to a contradiction, what can you conclude?
8. Circle the prime numbers in the following list:

   22 33 23 57 78 13 17 99

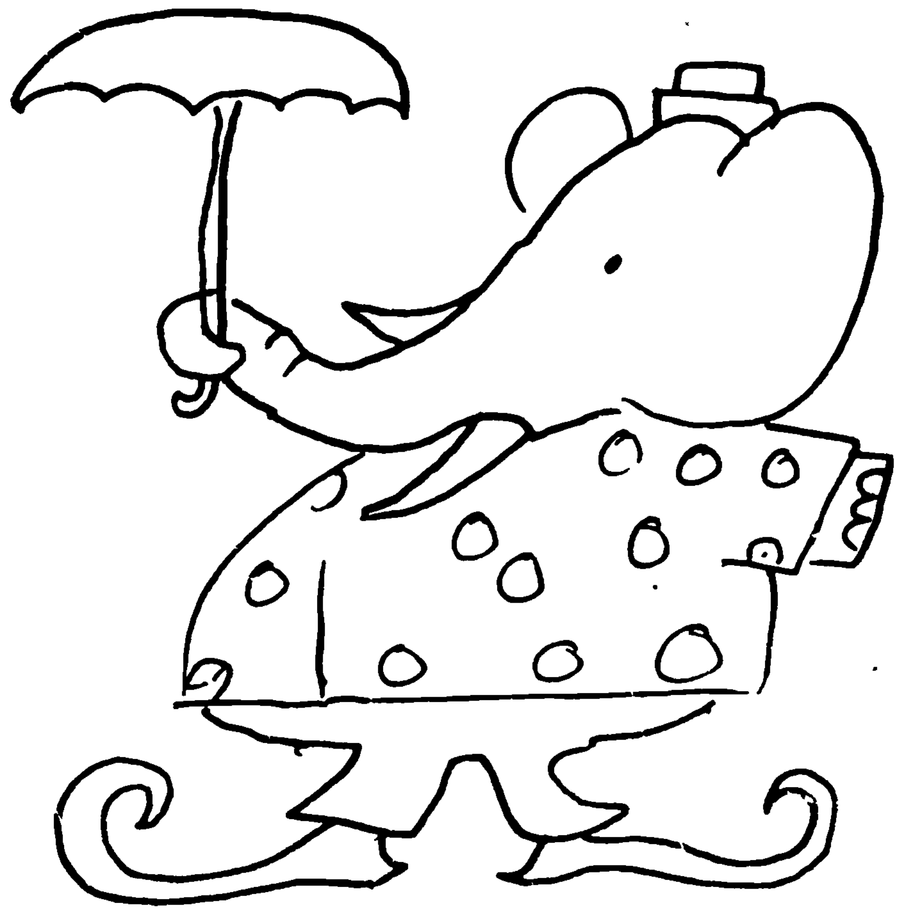
9. 4 x 2 = 8 which is even, 8 x 4 = 32 which is even, and 6 x 4 = 24 which is even. This proves that if you multiply an even number by an even number the product is always another even number.
   TRUE FALSE

Give one division fact for each of the following:

10. 45 = 5 x 9
11. W = JK
12. R = FG
13. (A + B) = rt
14. WE = 6
15. (M - X) = 3K
16. If M = 6, the M + 11 =
17. If P = 45, then P - 39 =
18. If M = 7, then $1 + 2 + 3 + \cdots + M =$
19. If T = 3, $T^2 + 70 =$
20. $1 \times 2 \times 3 \times \cdots \times 6 =$

Write an equation to illustrate each of the following:

21. $6|24$
22. $T|R$
23. $3|(A + B)$
24. $D|(V - P)$
25. Sidney knows that T = 6. He assumes that k + p = 17 and this leads him to conclude that $T \neq 6$. What can he conclude?



EMIRP was all excited. He had just received a package from his Aunt Hattie. "Wow!" said EMIRP as he opened it. Inside the package was an umbrella, a pair of slippers, and a polka dot coat. He had no sooner put them on when he heard Harvey calling, "Mr. EMIRP. EMIRP."

EMIRP stepped outside. He saw Sidney and Harvey running up the path to his house. "A terrible thing has happened. King Herman was on an airplane headed for Biami Beach. The plane has been sky-jacked and flown to HUBA," said Harvey.

"That's O.K. They always let the passengers go."

"But that's just it," replied Sidney. "They have arrested Herman as a spy and will not let him go."

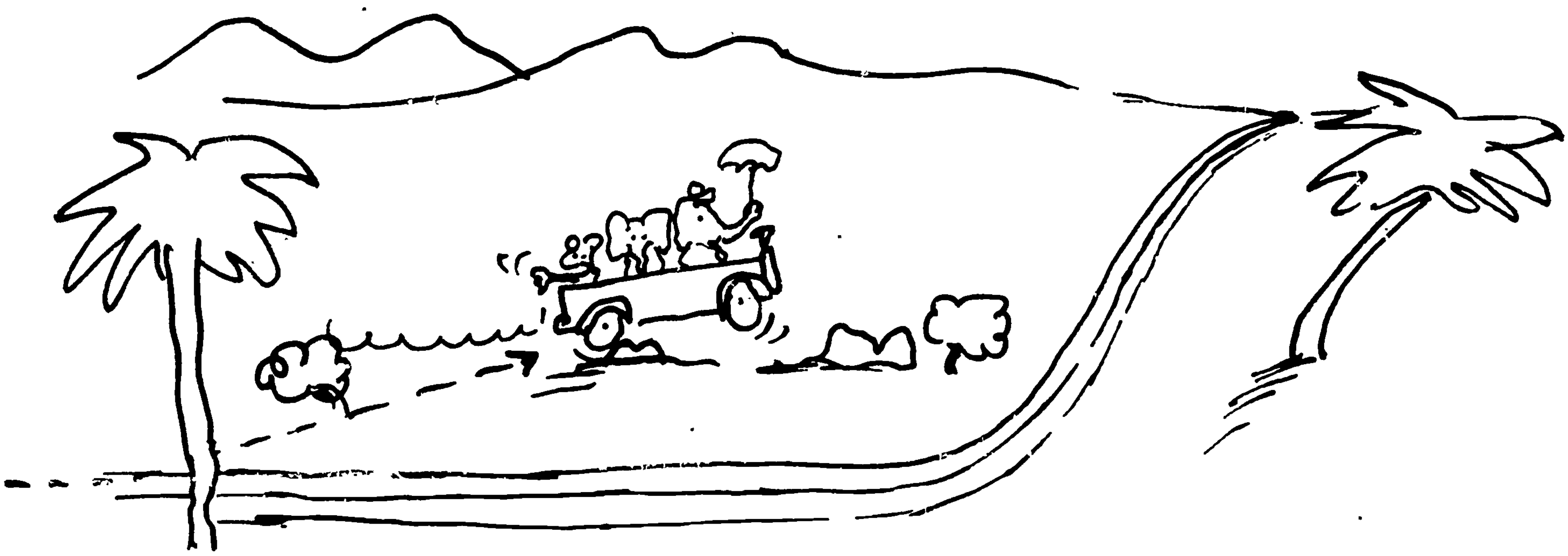
"Let's go!" said EMIRP, and they climbed into his 1933 Monkeymobile.

"Now," said EMIRP as they sped along," do you remember our last theorem?"

"Yes," said Harvey. If $N \nmid A$ and $N \mid B$, then $N \nmid (A + B)$."

"Oh yeah," added Sidney. "For example, $5 \nmid 11$ and $5 \mid 55$, so $5 \nmid 66$."

"Very good," remarked Emirp as the car



swerved off the road. BUMPETY-BUMP! With his superb driving skill EMIRP guided the car back onto the road.

"How many primes are there?" asked EMIRP.

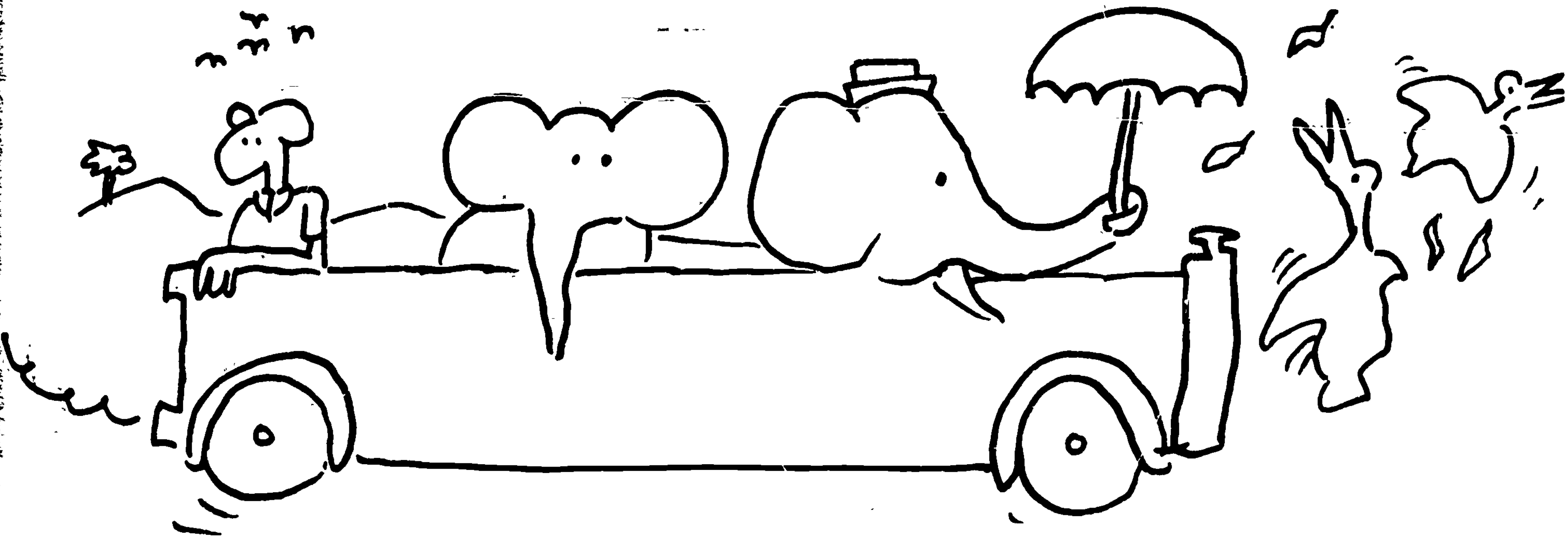
"I think there are just two," said Sidney. "2 and 3." He had forgotten the lesson on prime numbers.

"Consider the number $(2 \times 3) + 1$. $2 \mid (2 \times 3)$ but $2 \nmid 1$. And $3 \mid (2 \times 3)$ but $3 \nmid 1$," said EMIRP.

Then Harvey said, "By our last theorem, that means that neither 2 or 3 will divide (2 x 3) + 1."

"Correct," said EMIRP. A flock of geese was in the road. Emirp honked the horn and blasted straight ahead. The geese flew. Feathers flew. Eggs flew. "Hey!" said Sidney. "There are 3 eggs in the car."

"Wunnerful, wunnerful," said EMIRP.



When they got to the airport, they quickly got into a plane. "We're off to HUBA," said EMIRP as the plane left the ground.

"I've changed my mind," said Sidney. "I think the set of prime numbers is 2, 3, 5 ."

"Consider the number (2 x 3 x 5) + 1.

$2 \mid (2 \times 3 \times 5)$ but $2 \nmid 1$

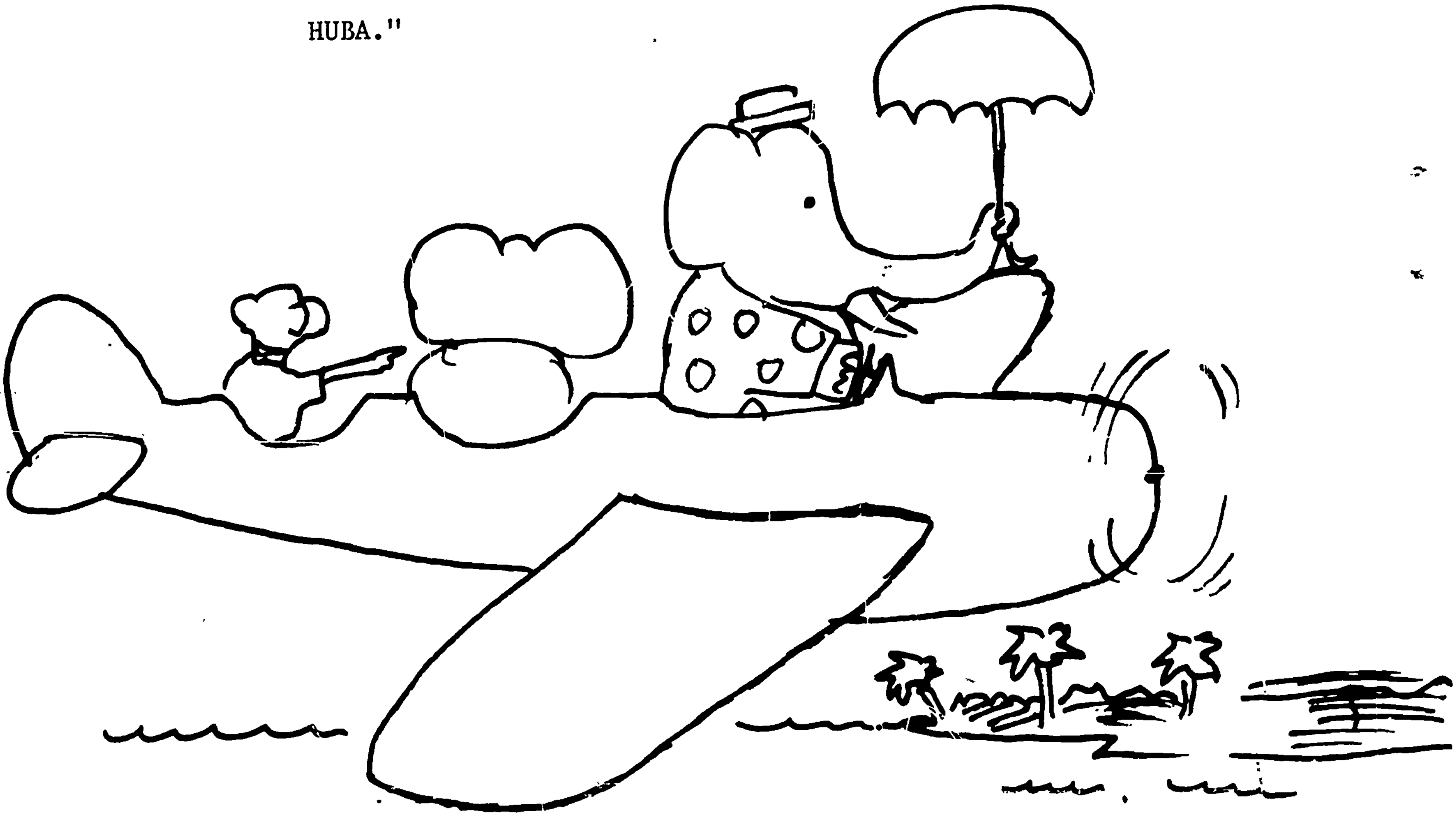
$3 \mid (2 \times 3 \times 5)$ but $3 \nmid 1$

$5 \mid (2 \times 3 \times 5)$ but $5 \nmid 1$.

So by our theorem, 2, 3, and 5 will not divide the sum (2 x 3 x 5) + 1. But some prime number <u>must</u> divide it.

So there is a prime number other than 2, 3, and 5."

"Look!" shouted Harvey. "There's the island of HUBA."



EMIRP landed the plane. A lion appeared.

"What do you want?" he asked.

"We are good and we are brave and we have come to rescue Herman ," replied EMIRP.

"I am not supposed to let anyone go to the prison. But if you will give me that gorgeous umbrella, I will let you go."

"O. K." said EMIRP.

The lion held the umbrella in his tail and growled, "Now I am the grandest lion in HUBA."

"This way to the prison," said Sidney.

Harvey said, "I think there are an infinite number of primes. Suppose you thought you had all the primes, say,

2, 3, 5, . . ., P , where P was the largest prime. Consider $N = (2 \times 3 \times 5 \times \ldots \times P) + 1$. As before $2 | (2 \times 3 \times \ldots \times P)$ but $2 \nmid 1$ so $2 \nmid N$. In the same way, 3, 5, . . ., P will not divide N. But some prime number must divide N. This means there is another prime besides 2, 3, 5, . . ., and P."

"Wunnerful, wunnerful," said EMIRP.

When they got to the prison, another lion appeared. "What do you want?" he asked.

"We are good and we are brave and we have come to rescue **Herman** ."

"Well, I'm not supposed to let anyone in this building, but if you will give me that polka dot coat, I will."

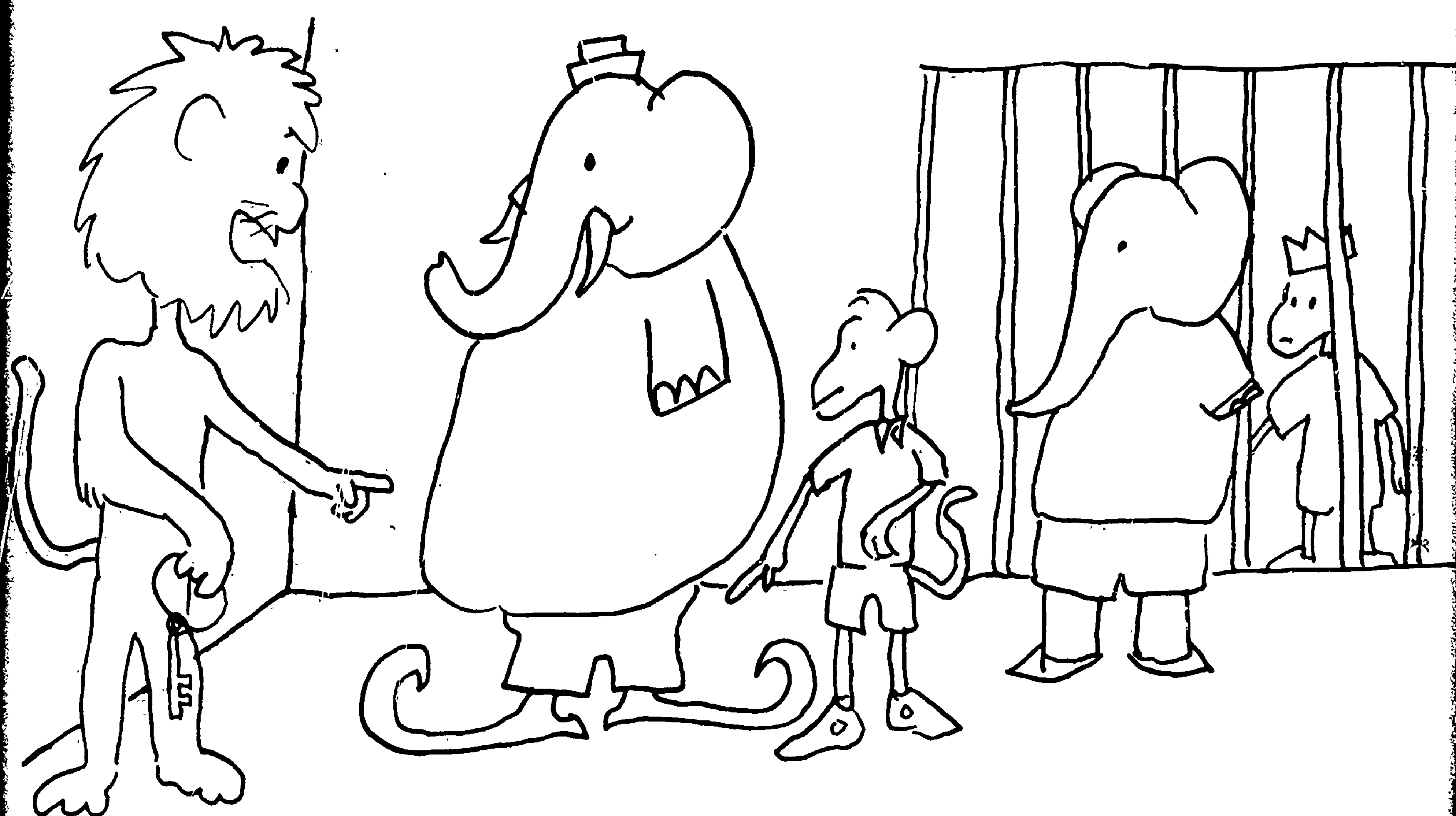
"It's a deal," said EMIRP. The lion put the coat on and growled, "Now I am the grandest lion in HUBA."

"Whew! Golly! Gosh!" said Sidney. "Who would have guessed that there is an infinite number of primes?"

"I would have," said EMIRP as they soon found Herman.

"Get me out of here," said the King. Just then a third lion appeared. "What do you want?" he asked.

"We are good and we are . . . ." The lion interrupted him. "I know, I know," said the lion impatiently.

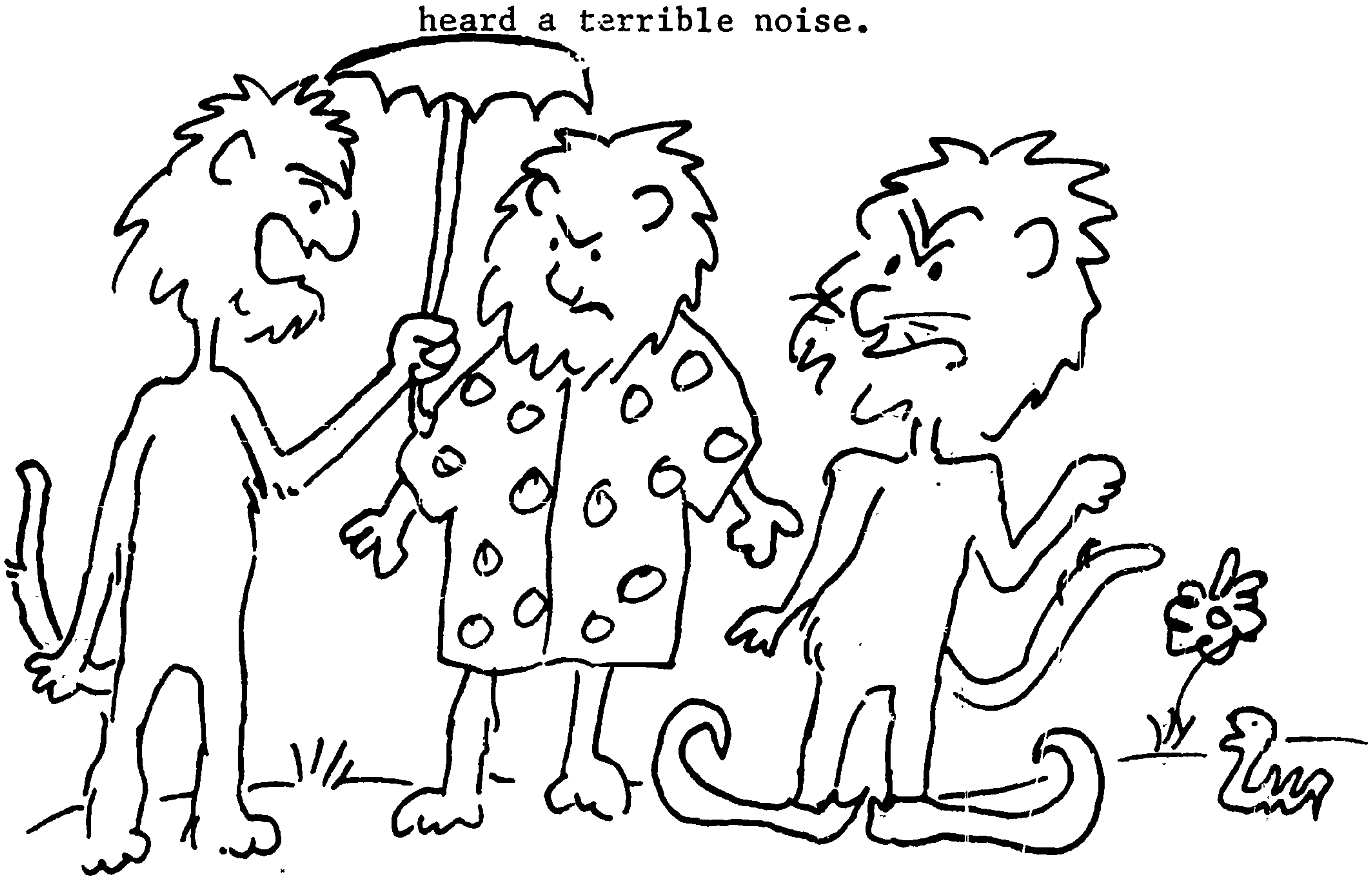


"If you will give me those beautiful slippers, I will give you the key to the cell," pleaded the lion.

"It's a bargain," said EMIRP.

As the lion walked away he roared, "Now I am the grandest lion in HUBA."

They quickly let King Herman out of jail. Just as they were about to climb into the airplane, they heard a terrible noise.

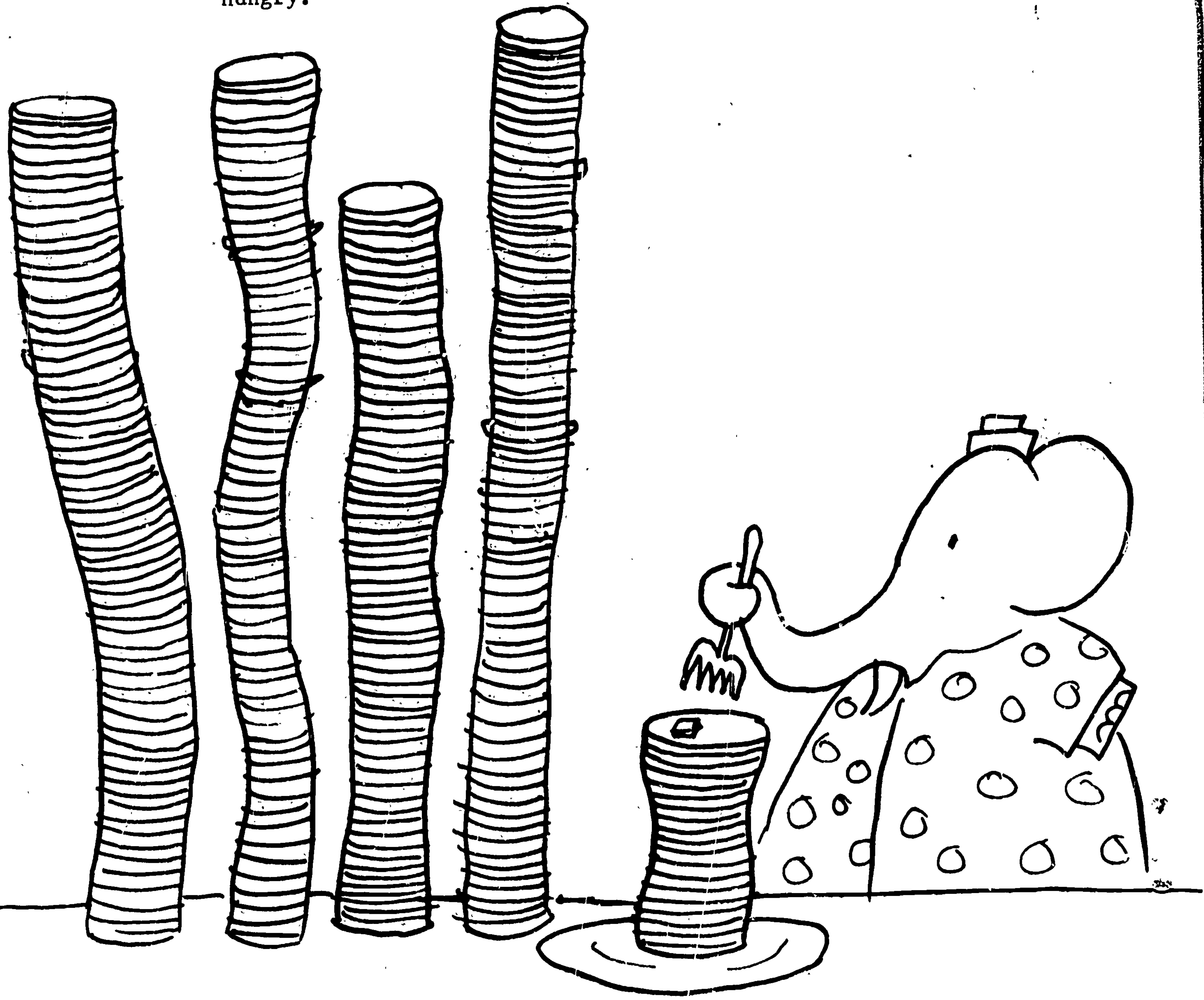


The three lion had met and were arguing over who was the grandest lion in HUBA.

"Quick," said EMIRP, "Get the jars out of the airplane!"

The lions put the umbrella, the coat and the slippers down. They grabbed each others' tails and ran around a tree. Well, you probably know the rest of the story. EMIRP got his clothes back, they all flew safely back to Monkeyville, and they had pancakes for supper.

Sidney ate 31, King Herman ate 37, Harvey ate 101, and Mr. EMIRP ate 9973 pancakes because he was so hungry.



Mr. Emirp, Harvey, and Sidney were about to begin another math lesson when the mailmonkey came down the road. "Letter for Emirp," he said as he handed Emirp a letter.

> Dear Mr. Emirp,
>
> I am teaching a unit on proof to ten sixth-grade students at Lake Mills, Wisconsin. They are having some difficulty learning to prove theorems.
>
> You are the greatest teacher in the world. Could you come to Lake Mills and help me teach these students how to prove theorems?
>
> Mrs. Gornowicz

It was decided that all three of our friends would travel across the ocean to Wisconsin to help Mrs. Gornowicz. The next morning they set sail for America. Half way across the Atlantic Ocean a huge sea monster stopped the boat.

"Who dares to pass through my kingdom?" asked the monster.

"I am Emirp the Great, and this is Harvey and Sidney. We are going to Lake Mills to help Mrs. Gornowicz teach."

"Not THE Mrs. Gornowicz?" asked the monster.

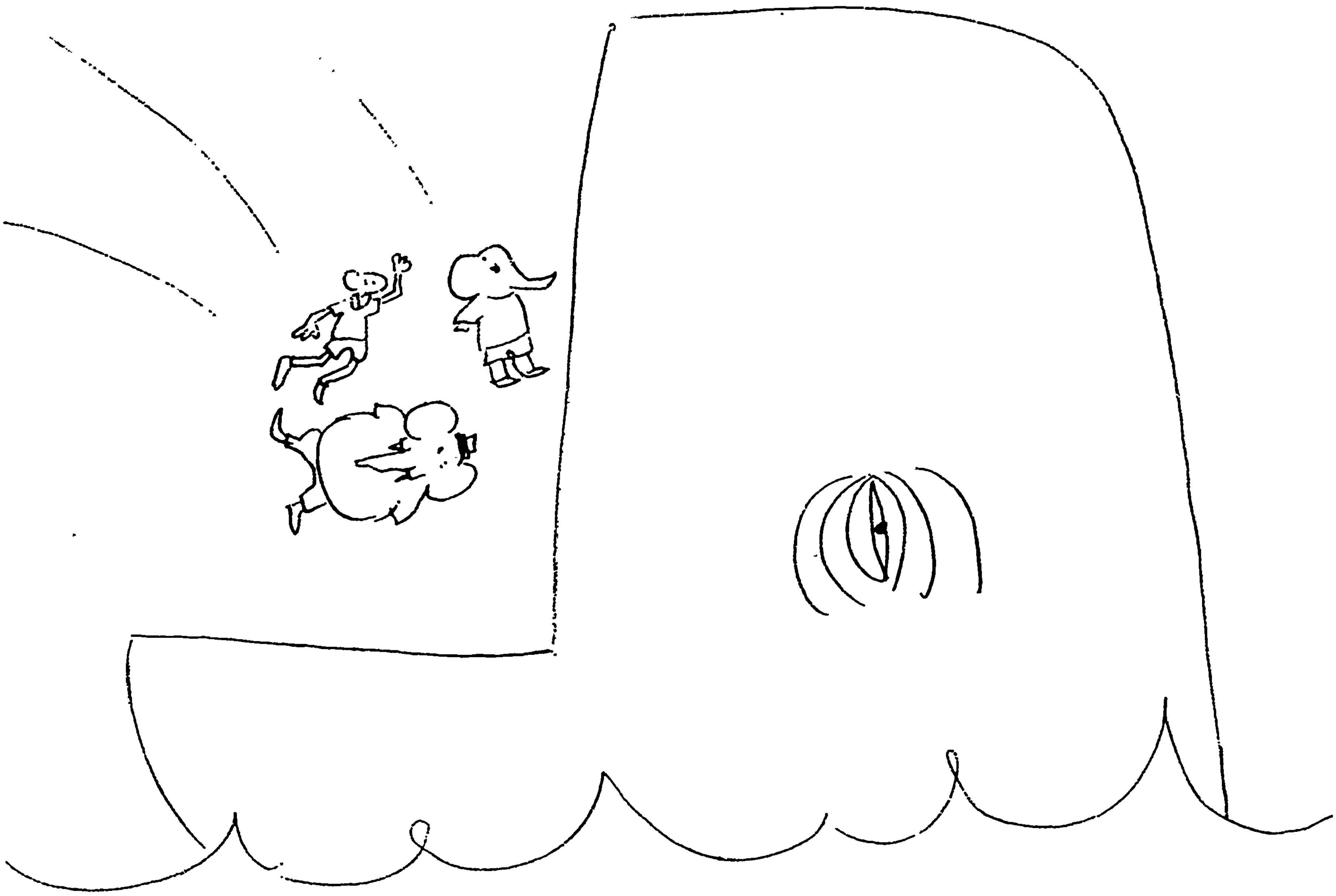
"Yes," replied Emirp.

"Well, there is a theorem that I have been trying to prove for the last 200 years. I just can't do it. If you can prove this theorem, I will let you pass. Otherwise I shall eat you for dinner."

If $N \mid A$ and $N \mid B$, then $N \mid (A-B)$.

STATEMENT	REASON
1.	1.
2.	2.
3.	3.
4.	4.

"That's easy," said Emirp. He showed the monster how to prove it, and our heroes continued on their way. Soon a terrible storm arose. Huge waves bashed the boat. Suddenly a wave hurled the boat through the air and our three friends fell into the ocean. Just then a gigantic whale swallowed our friends.



"Gosh, it's dark," said Sidney. "Where are we?"

"I can tell by all this blubber that we are in the belly of a whale," said Emirp.

"Shucks," said Harvey, "we'll never get to Lake Mills now."

"Do not despair," said Emirp calmly. "I'll think of something."

Sidney began to cry. "We've let Mrs. Gornowicz down."

"Not yet," said Emirp. "If I remember correctly there should be a jet liner flying to New York, and it should be directly over our heads in about two minutes. Let's find the whale's water spout and make him sneeze. He will blow us up into the air. The plane should be flying at 24,000 feet. If we get sneezed that high, maybe we can land on the wings of the jet and get a free ride to New York.

"But how will we make him sneeze?" asked Harvey.

"I have a small can of sneezing powder in my hip pocket," answered Emirp.

Emirp waited until the precise moment, then spread the sneezing powder in the whale's mouth. He sneezed and choked and sent a spray of water into the air. Our three adventurers flew exactly 24,002 feet into the air, and they landed on the wings of the jet. It landed safely in New York.

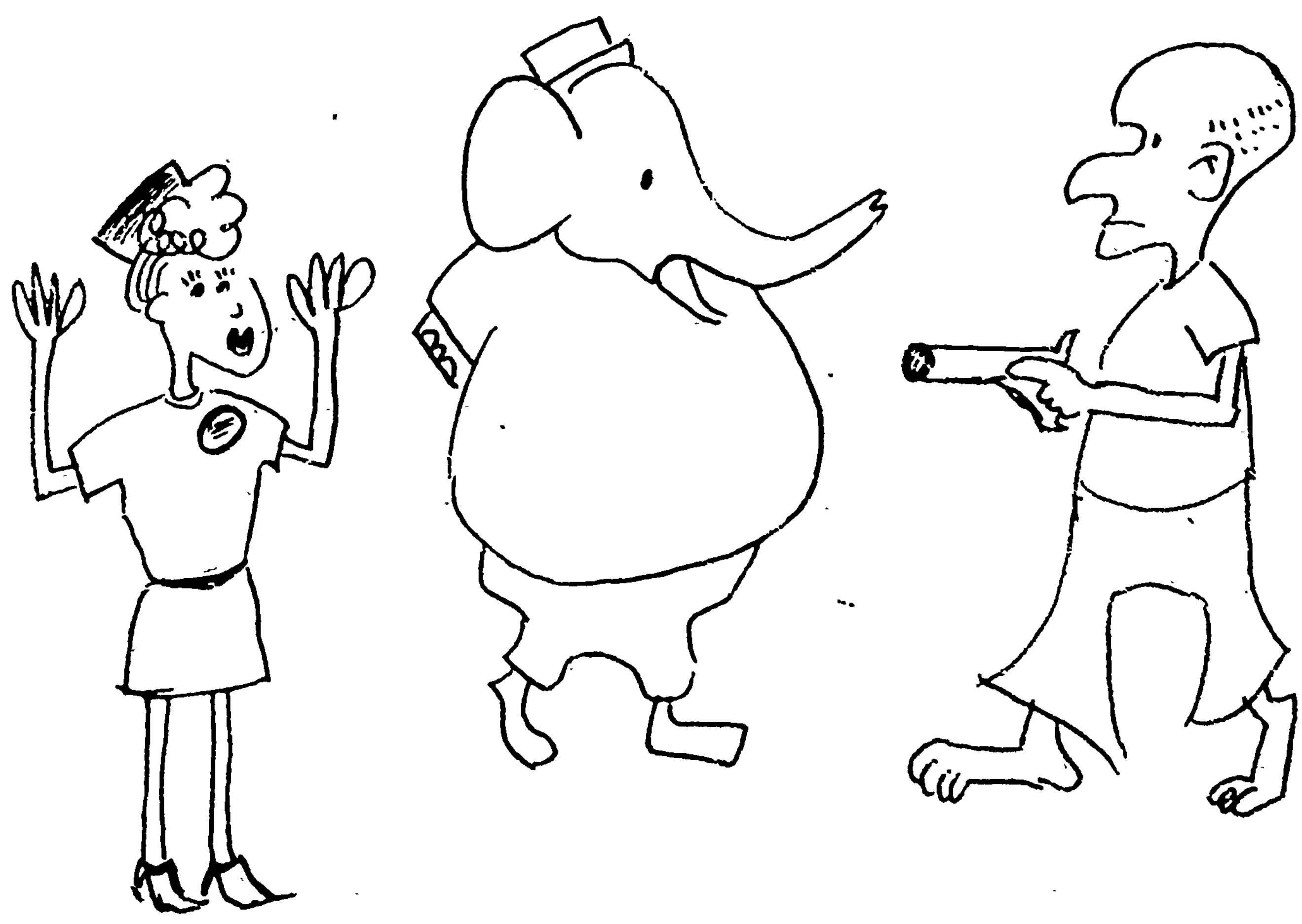
"Emirp strikes again," said Sidney.

They quickly caught a plane for Chicago. Up in the air a man pointed a gun at a stewardess and shouted, "Tell the pilot to fly to Cuba."

"Oh, you can't do that," said Emirp.

"Why not?" asked the sky-jacker.

"I am Emirp the Great and we are going to Lake Mills to help Mrs. Gornowicz."



"Not THE Mrs. Gornowicz?" said the man.

"Yes," said Emirp.

"Well, if you can prove this theorem I will let this plane go to Chicago."

If $N \nmid A$ and $N \mid B$, then $N \nmid (A + B)$.

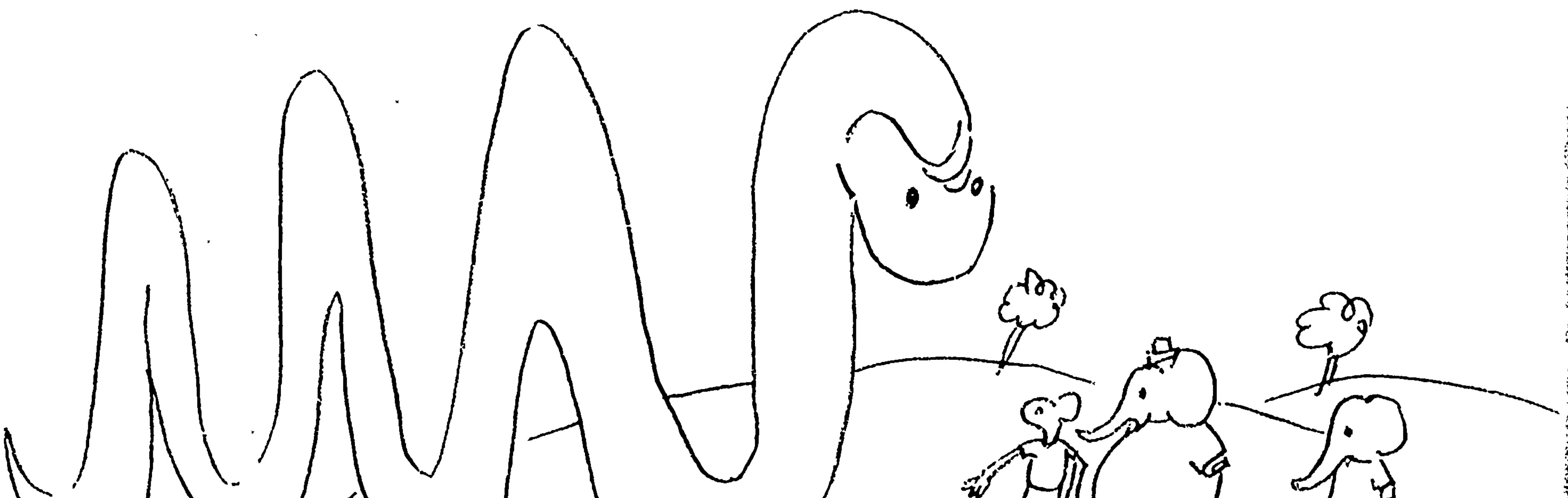
STATEMENT	REASON
1.	1.
2.	2.
3.	3.
4.	4.
5.	5.

Sidney proved the theorem, and the plane landed safely in Chicago. From there they took a bus to Lake Mills.

"Look!" shouted Sidney. "There's the school."

But before they could go any further, the Rock Lake Monster stopped them.

"Yum, yum," he said. "Elephant and monkey meat. You'll make a fine supper. I'm tired of eating fish and children."



"You can't eat us. We have to help Mrs. Gornowicz," said Emirp.

"Not THE Mrs. Gornowicz," said the monster.

"Yes," said Emirp.

"Well, in that case, if you can prove this theorem I will spare you."

> Given any set of primes {2, 3, 5, ..., P}, there is always another prime.

STATEMENT	REASON
1.	1.
2.	2.
3.	3.

Sidney proved the theorem, and our friends ran into the school room.

"Mr. Emirp!" shouted Mrs. Gornowicz.

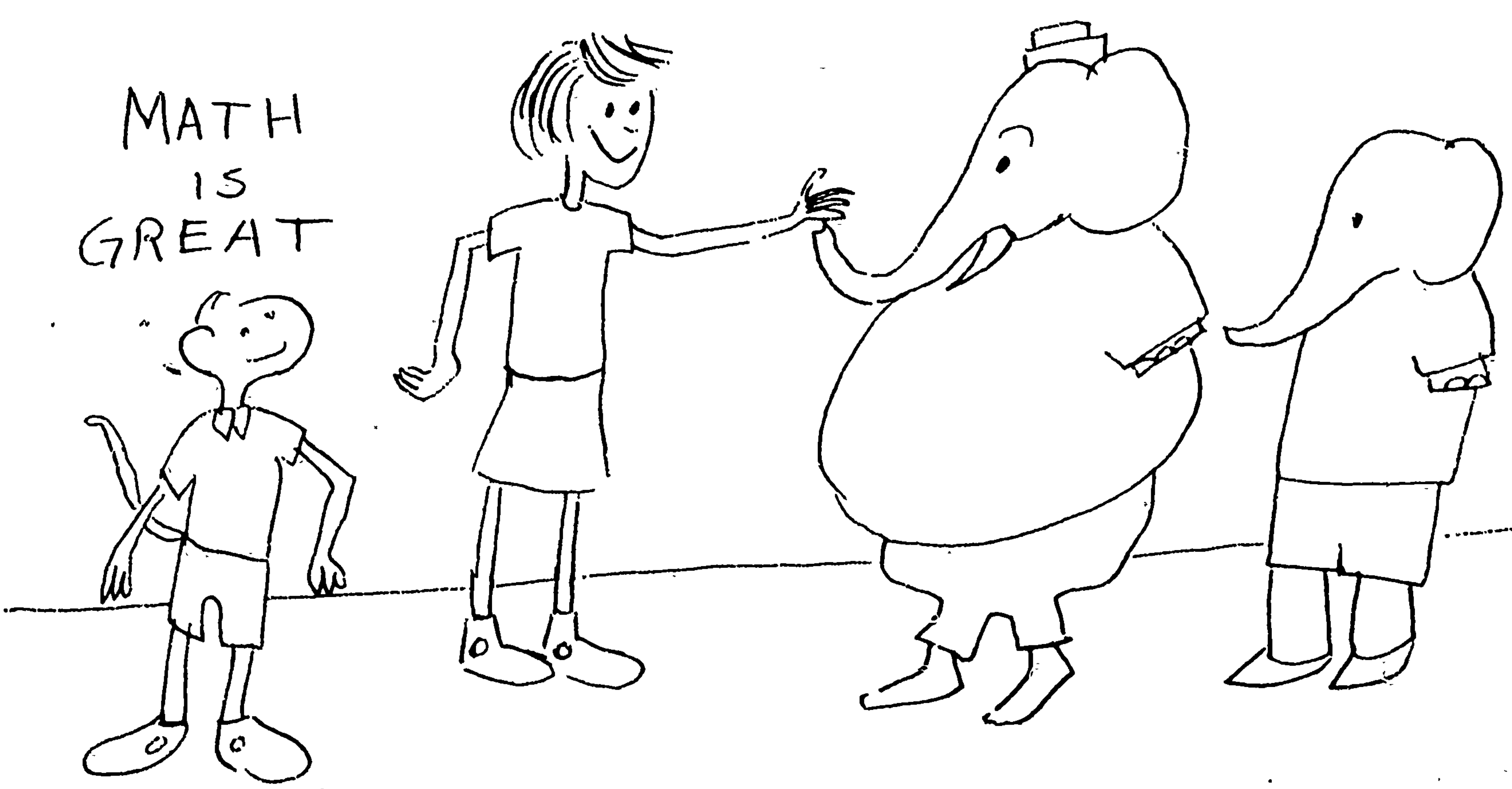
"Mrs. Gornowicz!" shouted Emirp.

"I have come to help you."

"Oh dear," said Mrs. Gornowicz. "This is the last day of class and the bell will ring any minute."

Ring -g-g.

"Elephant feathers!" said Emirp. "This calls for a Diet Rite Cola."



Technical Report No. 111

A FORMATIVE DEVELOPMENT OF A UNIT ON PROOF FOR USE IN THE ELEMENTARY SCHOOL PART III

Report from the Project on Analysis of Mathematics Instruction

By Irvin L. King

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Wisconsin Research and Development
Center for Cognitive Learning
The University of Wisconsin

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APPENDIX D

RESULTS OF LESSON MASTERY TESTS

The pages of this Appendix contain tables which summarize the results of the mastery tests used in the experiment. "Item criterion" is the ratio of the number of items correct to the number of total items. In Table A, for example, the item criterion of 9/10 has 10 students reaching that criterion. This means that 10 students answered 9 of 10 items correctly. The same table indicate that 9 students answered 10 of 10 items correctly. An asterisk (*) indicates where criterion was not reached.

I. DISTRIBUTIVE LAW MASTERY TEST

TABLE A

Identifying instances of the distributive law

Item Criterion	No. Reaching Criterion
10/10	8
9/10	10

TABLE B

Applying the distributive law

Item Criterion	No. Reaching Criterion
10/10	9
9/10	10

II. DIVISIBILITY MASTERY TEST

TABLE C

Stating divisibility facts

Item Criterion	No. Reaching Criterion
10/10	4
9/10	6
8/10	7*
6/10	8
3/10	10

*(80/80) criterion not met.

TABLE D

Writing equations

Item Criterion	No. Reaching Criterion
10/10	8
9/10	9
8/10	10

III. SUBSTITUTION MASTERY TEST

TABLE E

Substitution

Item Criterion	No. Reaching Criterion
10/10	10.

IV. MASTERY TEST FOR FIRST THREE PREREQUISITES

TABLE F

Identifying instances of distributive law

Item Criterion	No. Reaching Criterion
5/5	7
4/5	10

TABLE G

Applying distributive law

Item Criterion	No. Reaching Criterion
5/5	9
0/5	10

TABLE H

Substitution

Item Criterion	No. Reaching Criterion
5/5	8
4/5	10

TABLE I

Stating divisibility facts

Item Criterion	No. Reaching Criterion
5/5	10

TABLE J

Writing equations

Item Criterion	No. Reaching Criterion
5/5	8
1/5	9
0/5	10

V. THEOREM 1 MASTERY TEST

All ten students wrote correct proofs.

TABLE K

Applying Theorem 1

Item Criterion	No. Reaching Criterion
2/2	10

TABLE L

Giving numerical examples

Item Criterion	No. Reaching Criterion
2/2	10

VI. THEOREM 2 MASTERY TEST

All ten students wrote correct proofs

TABLE M

Applying Theorem 2

Item Criterion	No. Reaching Criterion
2/2	10

TABLE N

Giving numerical examples

Item Criterion	No. Reaching Criterion
2/2	10

VII. MASTERY TEST ON FIRST THREE THEOREMS

All ten students wrote correct proofs for all 3 theorems

VIII. LAW OF CONTRADICTION MASTERY TEST

TABLE O

Forming opposites

Item Criterion	No. Reaching Criterion
5/5	10

TABLE P

Law of Contradiction

Item Criterion	No. Reaching Criterion
3/3	6*
2/3	10

* Criterion not met

TABLE Q

Law of the Excluded Middle

Item Criterion	No. Reaching Criterion
2/2	10

IX. THEOREM 4 MASTERY TEST

All ten students wrote correct proofs

TABLE R

Giving numerical examples

Item Criterion	No. Reaching Criterion
2/2	10

TABLE S

Applying Theorem 4

Item Criterion	No. Reaching Criterion
2/2	10

X. MASTERY TEST ON BOTH THEOREM 4 AND THEOREM 5

Nine students wrote correct proofs for Theorem 4

All ten students wrote correct proofs for Theorem 5

XI. MASTERY TEST ON THEOREM 6

All ten students wrote correct proofs

All ten students wrote statements for Axiom 1

TABLE T

Interpreting use of 3 dots

Item Criterion	No. Reaching Criterion
5/5	9
4/5	10

APPENDIX E

PRETEST-POSTTEST

____
Name

PRETEST-POSTTEST

Apply the distributive law to the following expressions:

1. $(7 \times 8) + (7 \times 1) =$

2. $AB + AF =$

3. $5T + 5K =$

4. $CX + CY =$

5. $NP + NQ =$

6. For each of the following numbers, give one prime number which divides it:

14 15 21 23 49 45 100

7. When you add or multiply two whole numbers, the result is always another whole number. TRUE FALSE

8. If $T = 8$, then $1 + 2 + 3 + \ldots + T =$

9. If $A = 6$ and $X = 2$, then $A + X + 3 =$

10. If $P = 7$, then $(2 \times 3 \times P) + 4 =$

11. $2 + 4 + 6 + \ldots + 12 =$

Write an equation to illustrate each of the following facts:

12. $4 \mid 24$

13. $A \mid B$

Give one division fact for each of the following equations:

14. $70 = 2 \times 35$

15. $A = BC$

16. $X = TM$

17. If an assumption leads to a contradiction, what can you conclude?

18. Circle the prime numbers in the following list:

2 9 11 27 42 41 49 35 33 81

Write a proof for each of the following theorems (problems 19-24):

19. If $N \mid A$ and $N \mid B$, then $N \mid (A + B)$.

STATEMENT	REASON

20. If $N \nmid A$ and $N \mid B$, then $N \nmid (A - B)$.

STATEMENT	REASON

21. If $N \nmid A$ and $N \mid B$, then $N \nmid (A + B)$.

STATEMENT	REASON

22. Given any set of prime numbers $\{2, 3, 5, \ldots, P\}$, there is always another prime number.

STATEMENT	REASON

23. If $N \mid A$ and $N \mid B$, then $N \mid (A - B)$.

STATEMENT	REASON

24. If $N \mid A$ and $N \mid B$ and $N \mid C$, then $N \mid (A + B + C)$.

STATEMENT	REASON

25. 2 + 2 = 4, 6 + 8 = 14, 24 + 6 = 30, 4 + 4 = 8. These examples prove that the sum of two even whole numbers is always an even whole number. TRUE FALSE

APPENDIX F

SUMMARY OF DATA ON STUDENTS IN THE EXPERIMENT

TABLE U

Summary of Data on the Experimental and Control Groups

	I.Q.		STEP READING		STEP MATHEMATICS		MATH GRADES	
Matched pair of students #	E	C	E	C	E	C	E	C
1	135	131	96	90	94	99	A	A
2	127	132	82	82	94	94	A	A
3	125	130	90	72	88	88	B+	B+
4	119	127	77	68	81	84	B+	A-
5	117	117	49	62	84	84	B+	B+
6	113	122	82	77	94	99	A-	A-
7	112	118	65	77	84	84	B	B
8	112	124	55	55	67	76	B	B
9	109	111	86	77	74	88	B-	B
10	107	101	52	25	79	52	C+	C+
Mean	117.6	121.3	73.4	68.5	83.9	86.2	B+	B+
Standard Deviation	8.93	9.84	16.0262	17.293	8.596	13.482		

Henmon-Nelson I.Q. test was administered in the Spring of 1969.
STEP test (from 4A) was administered after the study was completed.
Math grades are for the first semester of the 1968-1969 school year.
"E" stands for Experimental Group.
"C" stands for Control Group.

APPENDIX G

CERTIFICATE

384/385

THE WISCONSIN RESEARCH and DEVELOPMENT CENTER FOR COGNITIVE LEARNING

RESEARCH
LEARNING

May
1969

Lake Mills
Middle School

This is to certify that has completed an experimental unit on proof and has mastered the concepts therein.

teacher

principal

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REFERENCES

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