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ABSTRACT

This unit describes an experience in informal geometry that is based on work with construction paper and milk cartons. The description is mostly of work actually carried out by children in the elementary grades involving such mathematical concepts as congruence, symmetry, the idea of a geometric transformation, and some basic notions of elementary group theory. The purposes of the unit are (1) to give students experience in visualizing two and three dimensional objects, and (2) to give students opportunity to learn to raise questions, pose problems, and learn to solve them. (RP)

A Teacher's Guide for a Unit in Informal Geometry

BOXES SQUARES and OTHER THINGS

MARION I. WALTER

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

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PREFACE

This unit describes an experience in informal geometry that is based chiefly on work with construction paper and milk cartons.

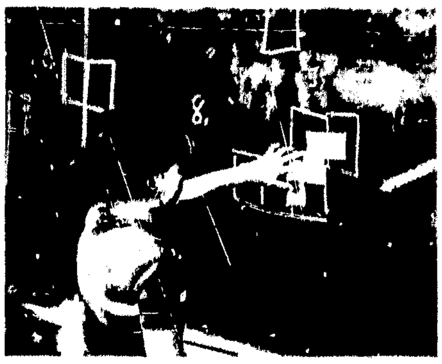
The description here is mostly of work actually carried out by children. Consider it a sample; after reading through it or using it once you will think of many other ideas. So will the children. The description of the work given in this booklet is definitely not the only way to go about it, nor are the many suggested problems related to the work more than a small sample of possible ones. The work can also lead in directions other than the ones suggested here.

No special mathematical training on the part of the teacher is assumed. The mathematics underlying the unit is explained separately for those who are interested.

All or parts of the unit can be explored by children in large classes and in classes where they work in small groups. Some children will want to investigate their own problems alone. You may want to use all these types of arrangements in any one class, for the unit lends itself to a variety of teaching and learning styles.

Most important of all—this unit was developed with the aid of students and teachers who enjoyed it. It is to be enjoyed, experienced, and explored—not memorized, dreaded, and disliked.

Checking a pattern to see if it fits. Here the drawing is much larger than the pattern—it doesn't seem to bother the student. ▶



◀ While trying to cut a particular pattern these students decided to come close to the flat pattern.

ACKNOWLEDGEMENTS

The children who worked with this unit are the ones who in a real sense created it. I owe them a special debt.

Part of the work in the classrooms and some parts of the first versions of this guide were supported by the Cambridge Conference on School Mathematics.

Since 1964 many teachers and several of my students have used this unit. I wish to thank all of them for their comments and useful suggestions. It is not possible for me to thank all of them by name, but among those I wish to thank are Mrs. Teresa Barton, Miss Carol Christie, Mrs. Marilyn Flanigan, Mrs. Grace Galton, Miss Nina Gould, Mrs. Mary Hatch, Miss Jane Hoyt, Mr. Philip Hunt, Mr. Norton Levy, Miss Jan McCoy, Mr. Derry Ridgway, and Mrs. Rosly Walter.

I also wish to thank members of the Cambridge Conference for their encouragement.

The reviewer of the manuscript made many helpful suggestions as far as details, content, and organization were concerned. Although he or she has remained anonymous, I am most grateful for the help I received.

I wish to express my thanks also to various patient typists—especially Mrs. Mary Sullivan and Miss Jenny Olmsted.

Checking to see if pattern folds into box.

This student is trying to copy "her" pattern on the board.

CONTENTS

INTRODUCTION

Before reading the teacher's guide it is a good idea for the reader actually to do the few things suggested below.

Close your eyes. (You'll be able to "see" better!)
Visualize a box with sides that are squares.
How many sides does it have? (Were your eyes still closed?)
If the box has no top, *how many sides does it have?*
Imagine it flattened out.
How does your box look flattened out?
Open your eyes!
Make a drawing of how the box you visualized looked flattened out.

Will the five squares in figure 1 fold into a box if no additional cutting is allowed? What about the pattern shown in figure 2?

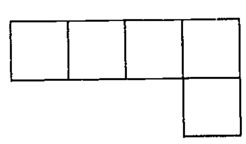


Figure 1

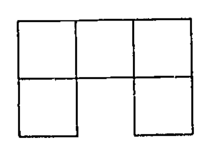
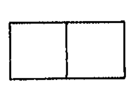
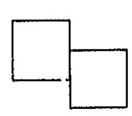


Figure 2

Can you find other ways to arrange five squares, regardless of whether they fold into boxes or not? (Whole sides must be touching, like this:

 This:  is not allowed.)

Two ways of arranging five squares are already drawn in figures 1 and 2. Draw some of the patterns you can think of. Which of your patterns fold into boxes?

For Whom Is the Unit Intended?

This unit is intended for use with children in the elementary grades. The guide is written for teachers. It describes work primarily for grades four to seven. For lower or higher grades suitable amendments can be made easily. Suggestions for work in the first grade are given. The guide has been used by teachers with third-, fourth-, fifth-, sixth-, eighth-, and tenth-grade children. The unit has also been used with teachers in training. Small parts of the unit have been done by first graders. Of course for young children the pace will be slower; the amount of material that can be discussed will be reduced, and the questions raised by the children will be different. Teachers of all grades may wish to adapt the unit in many ways; in any case, different children will become interested and absorbed by different and often new problems.

Description of the Unit

The purposes of the unit are several. One is to give students experience in visualizing two- and three-dimensional objects. Often this involves deciding how a figure would look after it has been moved or folded in a certain way. Another purpose is to give students opportunity to learn to raise questions, pose problems, and learn to solve them. Most of the time students can use materials such as paper or cardboard to obtain answers or to check them. The work will introduce students (and teachers) in a natural way to important mathematical concepts: the idea of a geometric transformation, the concept of symmetry, and some basic notions of elementary group theory. For example, the children will make acquaintance with such notions as congruence, rotation, reflection, symmetry, and commutativity, not by "learning" the definition of the word but by direct experience. They may at first make up their own words; for example, the identity element was called the "do nothing." Teachers who are not familiar with these concepts can work with the material and get acquainted with them. Some teachers and graduate students in mathematics have gained deeper insight into these concepts while using this unit even though they felt they understood them perfectly before beginning. See the section titled "Mathematics Involved in this Unit" for further explanation.

In other words, the aim is *not* to present a systematic and abstract development of a mathematical topic nor is the aim to make students adept at memorizing definitions or proving theorems.

Another aspect of the unit that is different from the usual geometry taught in the early grades is that its emphasis is *not* on verbal activity. This material will call for much use of the students' (and teachers') visual faculty; thus the nonverbal children (who are often poor achievers) have a real chance to participate and often to shine in performance. So often in geometry nearly all emphasis is put on learning to reason correctly and none on learning to visualize. The latter should not be neglected. The ability to visualize is not only important in mathematics but in many other branches of knowledge and daily activity. Chemistry, architecture, medicine, plumbing, dressmaking, tile laying, and engineering are only a few fields in which the ability to visualize is necessary.

This unit has a broader outlook than some geometry units. For example, many high school graduates associate the word "congruent" with triangles only; in this unit congruences are used for a much larger class of figures, and notions of algebra are combined with notions of geometry.

There are many other starting places for the type of work described in this unit. If it is successful in any way, teachers will think of many other physical situations that could open up whole new units in a similar spirit. This one came about, broadly speaking, through thinking of boxes and squares; thinking of cylinders, balls, hexagons, etc., clearly opens up other possibilities. Many additional problems are suggested; teachers and students themselves will find plenty of exciting new ones. The bibliography should also be useful in extending the work.

Development of the Unit

In the summer of 1964, while I was a member of the Cambridge Summer Conference, meeting at the Morse School, my aim was to create some mathematically significant visual problems for children.

The germ for the idea came from a game called "Hexed." It consists of a rectangular box in which are placed some flat plastic shapes; they fit in exactly. The problem is to try to put the pieces back into the box after they have been taken out.

It is a most frustrating game, for although there are many ways of fitting the pieces back into the box, it is extremely difficult to find even one way of doing it. While struggling over this problem, I noticed that some pieces, if made of paper, could be folded to form a box without a top, while some could not. This gave me an idea for a first lesson. From then on, the lessons were planned on a day-to-day basis, and what was done on any day depended heavily on the ideas generated by the students during the previous session, on their reactions, and on their questions. During the summer, a class of sixth graders and a class of first graders took part in the work. Subsequently I wrote several versions of a Teacher's Guide based on a variety of classes that used this material and on "feedback" received from the teachers.

Materials Needed

1. Construction paper that has been ruled with two-inch squares on both sides, as for graph paper, taking care that the rulings on both sides coincide. This can be done by hand or by mimeographing. Twenty two-inch squares can be drawn on standard small sheets of construction paper.

Several such sheets are needed for each student and several for the teacher.

2. Several sheets of one-half-inch-square grid paper.

3. "Graph" paper with equilateral triangles ruled on it. This is called Isometric 4 × 4 to the inch and is available from the Keuffel and Esser Company.

4. Several boxes of various shapes. Use boxes with a one-piece lid or with no top. Avoid boxes with four flaps as a "lid." Several boxes with square sides and no top for each student. Milk cartons are most useful. Cut them off so the height is equal to the width.

5. A felt board and felt squares (about 2 in. × 2 in.) for use by younger children.

6. Some small cubes (¼ in., ½ in., or 1 in.)

Trying to fit 4 different patterns a student cut from 4 x 5 piece back into a rectangle.

Trying to cut a particular pattern from a milk carton.

THE UNIT—A GUIDE

In the work below, sentences in italics indicate that the teacher is asking the question or giving the instruction. The teacher can work with the whole class or with a small group. Obviously every teacher is going to approach each lesson in his own way. What is written here are suggestions and records of what actually has taken place in a classroom. The amount of time spent on each part will depend on the age of your students, their interest and past experience. Do not try to hurry through it; feel free to investigate questions brought up by students. You may want to put some of the problems on cards so that children can work on their own, singly or in small groups.

Obtaining the Different Patterns

TEACHER:

Close your eyes! Visualize a box.

How big is the box you visualized? (Some students may give actual dimensions; others may say, "As big as a desk."

How many sides does it have?

Visualize a box without a top.

How many sides does it have?

Visualize a box with sides that are squares.

Visualize a box with square sides and no top.

Find examples of boxes in the room and decide whether they have sides that are squares. You may need to clarify the meaning of square, corner, edge, side, top.

TEACHER: *Close your eyes. What does a box with square sides and no top look like flattened out? Can you draw it?*

Probably many students will draw figure 3.

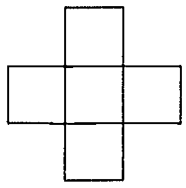


Figure 3

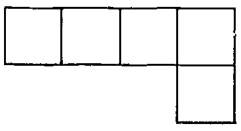
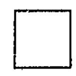
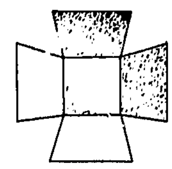
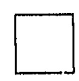


Figure 4

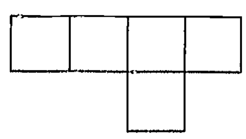
When this question was asked of a group of college students, all except one drew figure 3. Only one drew figure 4, which is also correct and is but one of eight possibilities.

Of course some will think of just squashing the box and drawing that. Good for them! But you might want to add, "Flatten the box by cutting along the edges." The discussion of how the box looks flattened out may lead to students' flattening a box by cutting along edges to see if their picture is correct. We will come back to this later. Some students may draw pictures such as this  and claim all the sides are piled on top of each other. Or they may draw this: 

Ask a student to draw his pattern on the board. (Pick one who has drawn figure 3.) Then ask if anyone has a different pattern. If someone draws



discuss it and the fact that all five squares are on top of each other. Get as many pictures on the board as students can contribute. Some may have drawn



TEACHER: *Are there other ways of arranging five squares?*

Someone may suggest figure 5. Some students may want to take this pattern and try to fold it into a box.

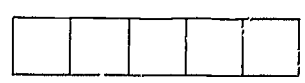


Figure 5

TEACHER: *Does this pattern fold into a box?*

No, but it is a way of arranging five squares with whole sides touching. (One student wanted to take this piece home to try. He came back the next day convinced it would have to be cut in order to make a box.)

TEACHER: *Can anyone find another way of arranging five squares, regardless of whether they fold into a box or not?*

Someone may suggest figure 6 or figure 7. Discuss the fact that they

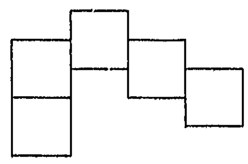


Figure 6

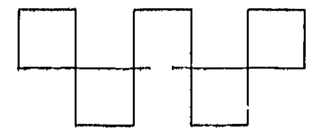


Figure 7

have no chance of folding the pattern in figure 6 into a box, since the whole edges are not touching, and figure 7 would fall apart if one tried to cut it out. Make the rule: "Use five squares and have whole edges touching." That is, figure 8 is allowed, but figure 9 is not.

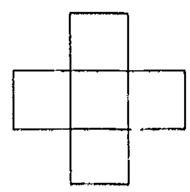


Figure 8

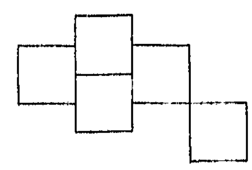


Figure 9

Draw on the chalkboard a few of the patterns obtained so far. Then ask the students to draw as many *different* patterns as possible using five squares. Unless it already has come up, do not say anything yet about what "different" means. Give the students half-inch-squared paper to draw on. (Even without squared paper, students tend to keep the sides horizontal and vertical without being told to do so.) One sample paper, obtained in an actual class, is shown in figure 10.

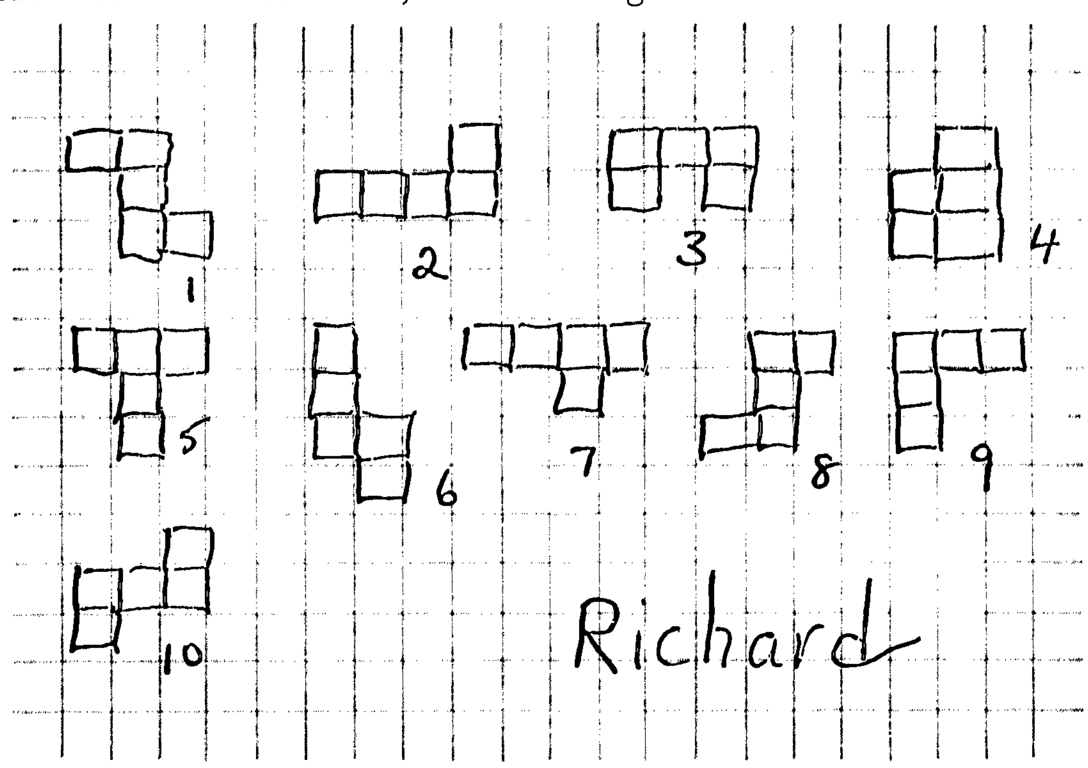


Figure 10

Ask those who finish early (they probably won't find *all* patterns) to mark those patterns that they think will fold into a box. They may need to cut out each pattern from construction paper to check whether it folds into a box. If any students have difficulty in finding new patterns, give them five cutout squares to move around so they can find new patterns.

When you think that they have drawn as many different patterns as they can find, or as many as they can find without losing interest, collect the papers. Then ask a student to draw one pattern on the board. (Ask students to draw the squares with sides about two inches long so they will match the paper cutouts.) They can then take turns drawing the new patterns on the board.

Sooner or later someone will probably draw figure 12 when figure 11

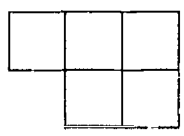


Figure 11

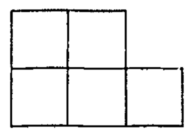
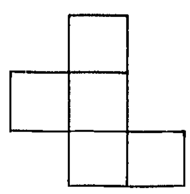


Figure 12

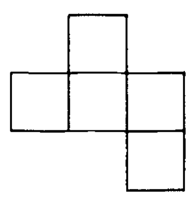
has already been drawn, or vice versa.

Someone will probably call out, "They are the same." The two patterns are the same in the sense that one paper pattern can be made to match both. You may want to have a student show that one can match both the pattern in figure 11 and the pattern in figure 12 with the same paper cutout. (Use paper patterns cut from construction paper. You can hand the student the correct pattern so that he does not get a chance to see all the other patterns that you have while he looks for the one he needs. Of course he can cut out his own pattern.)

Eventually the students should have all twelve patterns drawn on the board. At first several of them are likely to be repeated. For example, this pattern



may be drawn when this one



is already on the board.

Some students will immediately recognize these patterns to be the same; others will need to check by using paper cutouts. Notice that the paper cutout must be flipped over after matching one pattern if it is to fit the other. Some duplicates may not be recognized until later. The twelve different patterns of five squares are shown in figure 13. Those that fold into boxes are marked with a *B*.

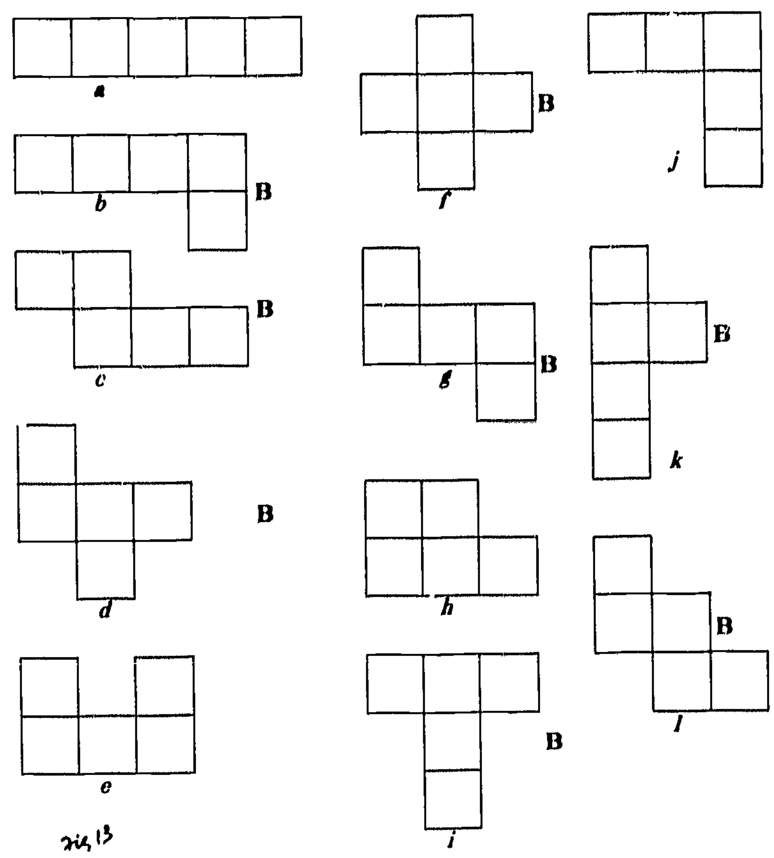


Figure 13

TEACHER: *Which of the patterns fold into a box without a top? Which square becomes the bottom of the box?*

The work up to now might take two, three, four, or more lessons. It depends on your students and on what questions come up.

Establish the rule that a pattern counts only once, regardless of which position it is in; that is (*a*), (*b*), and (*c*) of figure 14 are the same pat-

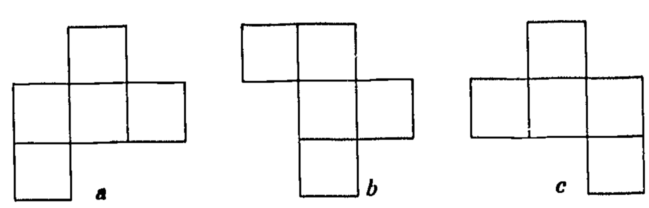
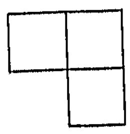
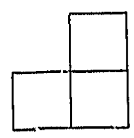
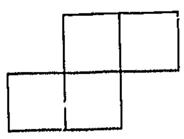
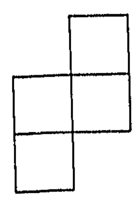


Figure 14

tern because one can find a paper pattern that will fit onto each of them exactly. Then you may want to ask the students again (perhaps on the next day) to try to draw all the patterns on the half-inch-squared paper. You may want to return the sheets on which they had made their first attempt at listing all patterns. The students themselves can compare their first and later attempts and notice which patterns they left out the first time.

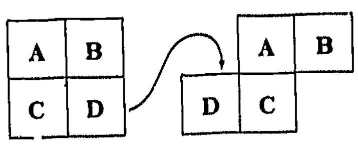
In a first grade one would not, of course, begin by asking the children to visualize a box flattened out. One might begin by discussing squares and looking for and finding things in the room that were "square." Then children can be asked to make designs using, say, three squares. How many different designs can they make? Or one might begin by looking at boxes and counting the number of sides.

For example, in one first-grade class the children were each given three paper squares and asked to make different patterns—with whole edges touching. When they had laid out a pattern using their separate squares, they were given patterns of that shape. When some students suggested  as being a different pattern than  , other students showed how one could turn or flip the one pattern so that it fitted on the other. The same type of work was carried out with four squares. They learned quickly to recognize when a pattern was duplicated (in the sense described previously), each time actually moving the paper pattern to show that

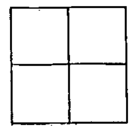
 and  ,

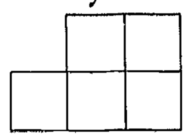
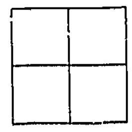
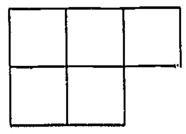
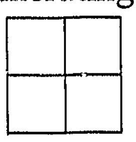
for example, could be covered by the same pattern.

A game was played on a felt board: Move one square to make a "different" pattern. For example:

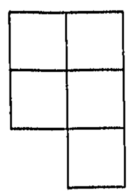
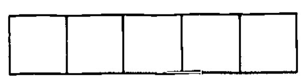


The first graders were able to do this so accurately and quickly that it soon turned out to be too easy a game.

They tried to make boxes, starting with three- and four-square patterns, and on experimenting found that  , for instance, was not useful for making part of a box. Some children combined various four-square patterns or asked for extra squares.

The five-square patterns were obtained at the felt board—the students coming to it in turn to make a new pattern. More difficulty was encountered here in avoiding duplication. This pattern  was duplicated particularly often. After putting  on the felt board, the children realized that they had no alternative for the fifth square if they did not want to duplicate  , but only one child thought of removing the fourth square. Several children then realized why there was no way out at that point, and some verbalized by enumerating all the possibilities for the position of the fifth square once  has been put on the felt board.

They built boxes without tops out of some of the shapes and found that

 and  ,

among others, were not box-makers.

The Game of Listing Patterns

Even after one has found all twelve patterns once, it is not always easy to find them again. For this reason, and to gain more practice in recognizing the shapes in different positions, play the following game.

Divide the class into two teams. Each team plays at its own chalkboard if there are two in the room, or at its own half of the board if there is only one. Each player has a turn and must draw a pattern on the board. Let the students line up at the board, or decide beforehand on the order in which they have turns. The team that gets all twelve patterns (or thinks it has) sits down. (Try to arrange it so that the two teams cannot peek at each other's patterns while they are playing!) Then each team looks at the other's drawings and tries to check for duplicates. A challenger can show that two patterns are duplicates by taking a paper pattern and showing that it fits on both. Cross out one pattern in each pair of duplicates (leaving the other on the board). Of course, one pattern may be duplicated several times. In this way each team can count how many "different" patterns the other team found. Require some speed and allow no erasing and no hinting from another teammate!

Children like to play this game on several days, and it often causes much excitement. They soon realize that the last man on the team has a much tougher time than the first man. It is interesting to note the order in which the patterns are drawn. Some students like to start with the easy ones such as the cross; others like to start with the "hardest" one they can think of! Although students "know" that there are only twelve different patterns, they often do not hesitate to draw many "more" and do not realize that they have duplicates.

To sum up, the first phase consists of giving the students practice in finding and drawing the different five-square patterns and in recognizing duplicates. Some students will need the concrete material (that is, the five separate squares) to find the different patterns. Paper cutouts should be used some of the time to see if two or more patterns are the same. Besides providing a check, this will give the students an opportunity to become familiar with carrying out the motions that will be discussed later. Some students may at first have difficulty in turning the paper pattern. For example, turning (*a*) of figure 15 so that it will fit (*b*) and (*c*) is

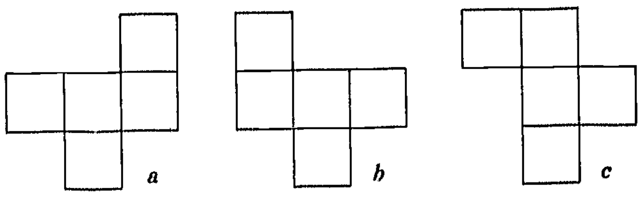


Figure 15

not easy for all students. But the practice of turning and flipping the pattern over to make it fit is valuable for later work.

The children need to check some patterns more often than others. For example, a paper cutout is needed much more often to check whether

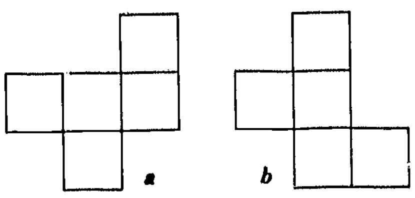


Figure 16

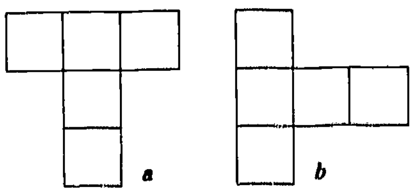


Figure 17

(*a*) and (*b*) of figure 16 match than to check whether (*a*) and (*b*) of figure 17 match.

Some Further Constructions and Destructions

Making Boxes and Flattening Boxes

While this work is proceeding, some of the students may want to construct boxes out of the patterns that fold into boxes.

At some time when you have all twelve patterns on the board, ask students to mark those that fold into boxes (there are eight). Give each pattern a letter name, as shown in figure 18. Ask the students which pat-

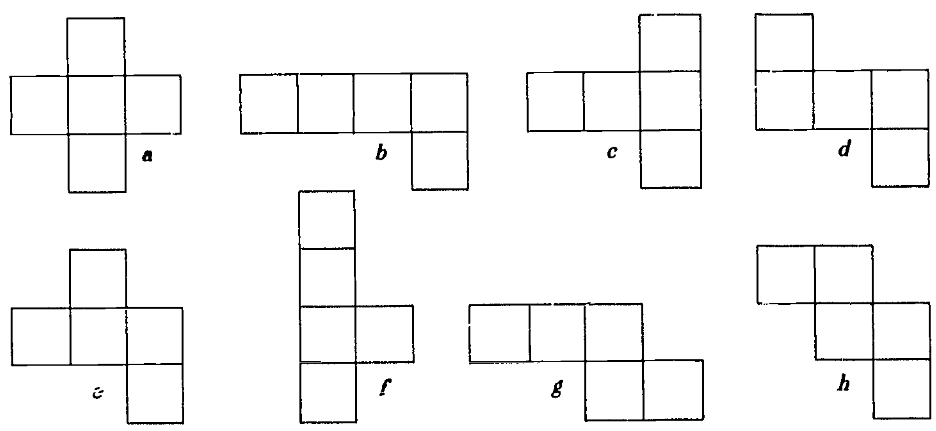


Figure 18

tern they think will be easy to cut from a box and which will be difficult. They will probably all agree that (*a*) is easy. Ask each student to choose one pattern. Give each a cut milk container that is in the shape of a box with square sides and no top. Ask him to put the letter of the pattern he chose on the bottom of the box and then to cut the box so as to obtain that pattern. Some students will succeed; some will obtain a pattern, but not the one they chose; and some will find they have cut too many edges and the pattern is about to fall apart! Let them "do" as many as they want—perhaps some at home.

One class of fifth graders were asked if they could give advice on how to cut the milk carton to obtain the chosen pattern without making a mistake. Among suggestions given were these:

"Think of the pattern in your head and fold it."

"Find which blocks (squares) are attached to which and how."

"Do not cut unless you have planned it."

"See where the bottom of the carton would be on your pattern."

"Look at the bottom of the milk carton. Then look at the pattern.

If that pattern were folded into a box, which square would be the bottom? Use that as a guide to cut it out."

"Cut the pattern on paper. Fold it into a box and look where it's cut or where it's connected. That way you get it 1, 2, 3, . . .!"

"I couldn't find anything."

A more challenging way to do the cutting is to permit the person to look at the pattern only before but not *while* cutting the milk carton. It is even more difficult if one is asked not to cut any edge until one has chosen *all* the edges that need cutting to obtain a particular pattern.

Cutting the Construction Paper to Obtain Patterns

If you have supplied the students with a few pieces of construction paper ruled with twenty two-inch squares as shown in figure 19, the question of economical cutting often comes up.

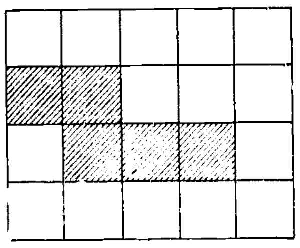


Figure 19

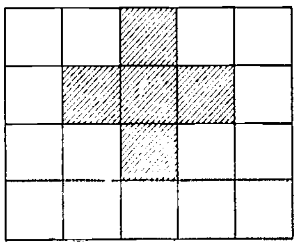


Figure 20

Some students, having cut out the pattern in figure 19, find they then cannot cut out the cross pattern in figure 20. Some students will cut out a pattern—say the cross—from the middle of the sheet and find, for example, that they cannot cut out a second cross. They soon realize that it pays to plan. Students will notice that since there are only twenty squares, four is the maximum number of five-square patterns that can be cut out of one sheet. This suggests a problem.

Teacher: *Can you cut four different patterns from the construction paper?*

Figure 21 shows one solution. Let your students find their own. Other rectangles, 3 x 5 for example, can be investigated.

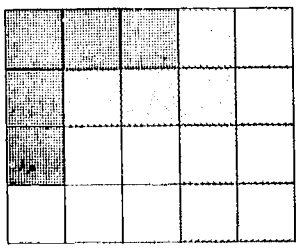


Figure 21

Then the question can be raised whether *any* four patterns can be cut out. Some students may suggest finding those patterns that can be cut out four times from the same construction paper. These are practical problems, for they enable one to get the maximum number of patterns out of the construction paper; other problems are suggested in the section on tessellations.

Using the Patterns to Do Some Arithmetic

In order to tie in the work with number concepts, first graders can do the work described below. Give the children a three-square and a five-square pattern. Ask them to find as many different patterns as they can that combine the two. (A few are shown in figure 22.) You may wish to have students work individually and then have one student show a solution on the felt board. It would be helpful to use felt squares of two different colors on the felt board—one to represent the three-square pattern and one to represent the five-square pattern. Again, the students can check to see if they have found a new pattern by using paper cutouts.

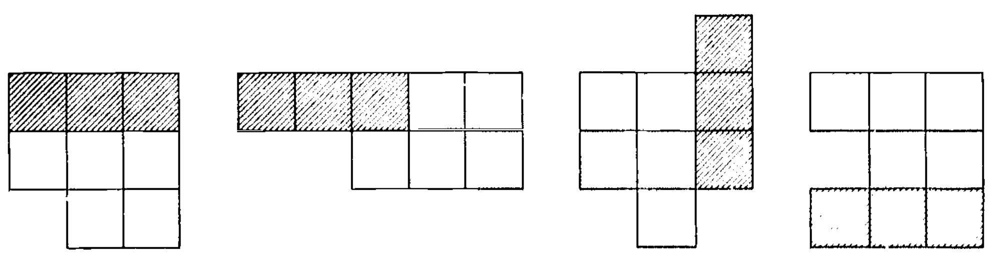


Figure 22

Of course, you can turn the problem around and ask, "How can you make the patterns in figure 23 from a three-square and a five-square pattern?"

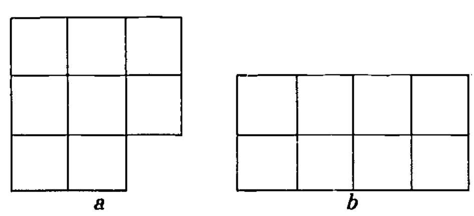


Figure 23

The children may discover that they can make figure 23 (b) in two "essentially different" ways. (See figure 24.)

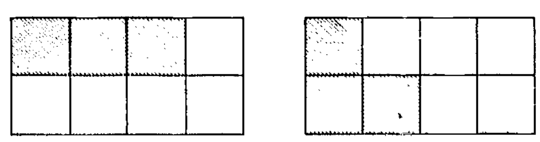
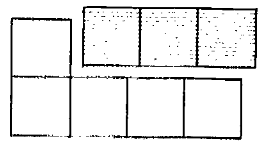


Figure 24

At first it might be easier if you let them count the patterns in figure 25 as different. Then let them show that one paper cutout made with a red three-square pattern and a blue five-square pattern taped together to form an eight-square pattern  will match all the patterns in figure 25 in shape and color.

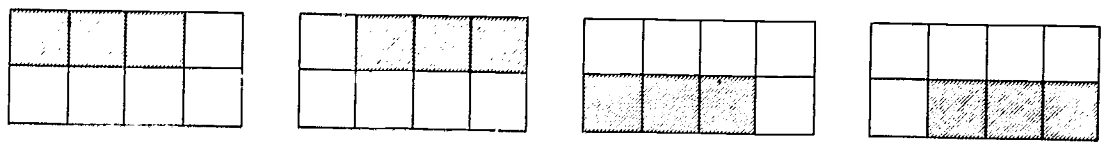


Figure 25

They can make the pattern shown in figure 23 (*a*) in four different ways. (See figure 26.)

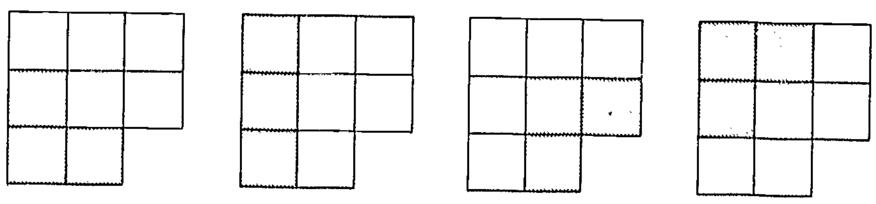
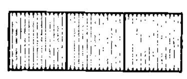
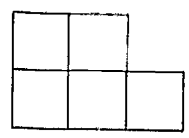


Figure 26

Notice again that we will consider the arrangements shown in figure 27 as "the same," because the same paper cutout (made of a three-square pattern  and a five-square pattern  taped together) matches all of them in shape and color.

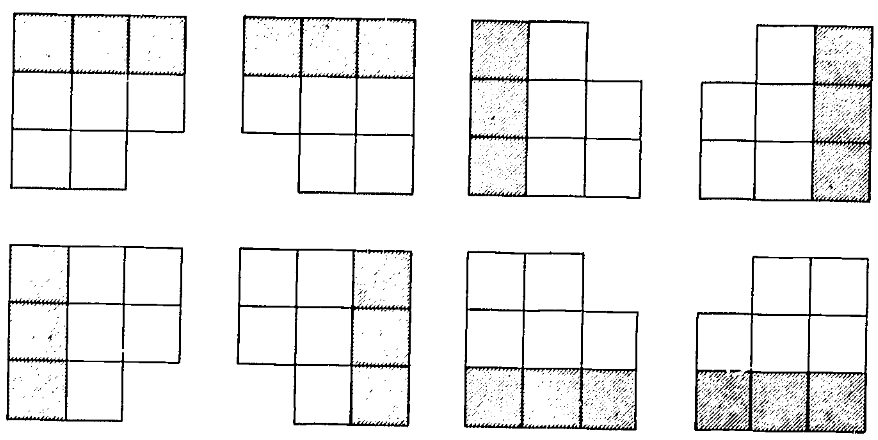


Figure 27

Doing problems of this type gives children experience in recognizing patterns of eight (or any number that you choose), as well as an opportunity to see that $3 + 5 = 8$ in many different pattern forms.

Exploring the Patterns

The children may feel that some patterns are "nicer" than others. They can explore special properties of them. For example, which patterns can you fold in half so that the two parts match exactly? Which patterns can you cut into two parts so that the two parts match exactly? On which patterns can you place a mirror and still see the whole pattern? Which patterns can you see by drawing only one half of a pattern and using a mirror?

The children can use Mirror Cards as a side activity [VI, 1].* Mirror Cards consist of sets of cards that have patterns on them, together with a mirror. There is also a Teacher's Guide [IV, 4]. The basic problem is to match, by use of a mirror, the pattern on one card with the pattern on another. For example, can you, by using a mirror on the card shown in figure 28(*a*) match the pattern shown on the card in figure 28(*b*)?

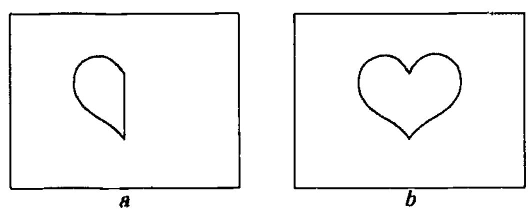


Figure 28

The sets vary in difficulty, ranging from some that first graders can do, to some that adults find challenging.

"Dangerous" and "Safe" Patterns

After the game of listing the patterns has been played several times, and after the students have tried to list the patterns themselves, they may realize that some patterns are duplicated more often than others. Some are "dangerous" in the sense that they are drawn several times; others are more "safe" in that almost no one repeats them.

Teacher: *Which patterns are "dangerous" and which are "safe" in playing the game; that is, which patterns did you repeat most often? Which least often?*

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* Numerals in brackets indicate references listed in the bibliography.

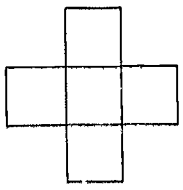


Figure 29

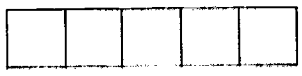


Figure 30

The students will probably agree that figures 29 and 30 are among the safe ones—that is, they are not duplicated often; and figures 31 and 32

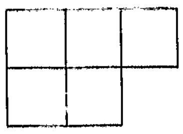


Figure 31

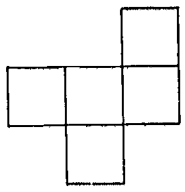
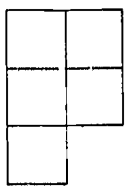
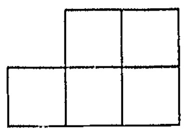


Figure 32

are among the most dangerous ones. That is, figure 31 is repeated as



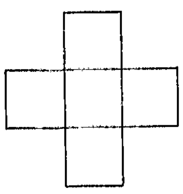
and

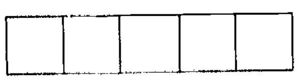


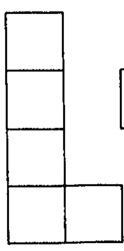
for example.

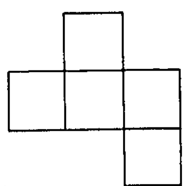
Let the students decide which patterns fall into the dangerous class, which into the safe class, and which fall in between. They may want to make a table, part of which might look like the following:

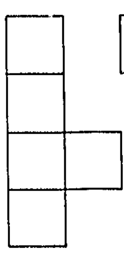
Safe	*Dangerous*

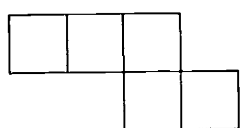




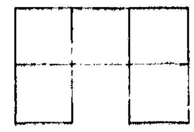




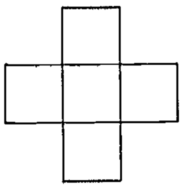




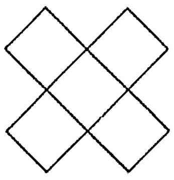
There will probably be some pieces on which there is disagreement.

For example, some students may feel that this pattern 

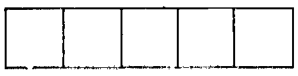
is safe, and others will think that it is dangerous. Then raise the question (if they don't) of why some pieces are hardly ever drawn twice while others often are repeated.

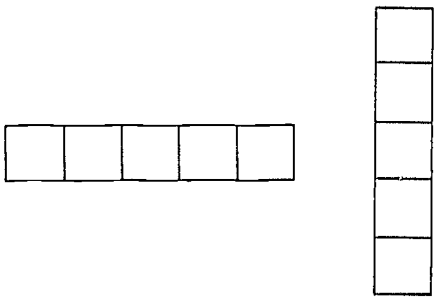
Someone may suggest that the "cross pattern"  is safe

because no matter which way you turn it, it looks the same. Someone else may say that this is not true, because one can turn it a little and it

will look different. 

Establish the fact that we are keeping the edges horizontal and vertical. A discussion of the meaning of horizontal and vertical may be necessary. Someone then may say that the "straight pattern"

 is safe because there are only two ways of drawing it if we keep the lines horizontal and vertical:

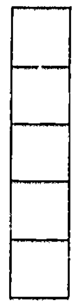


This can then lead to the following discussion.

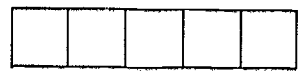
Teacher: *How many different positions does each piece have?*

You may want to start this work at the board and let students continue individually at their seats, or whatever you may feel is best.

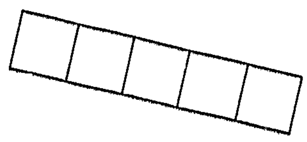
They have just found that the "straight pattern" has two positions, namely,



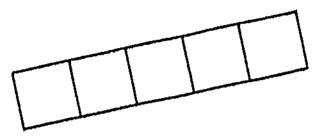
and



Again, someone may say there are more positions, because one can also draw

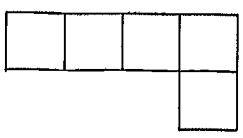


and

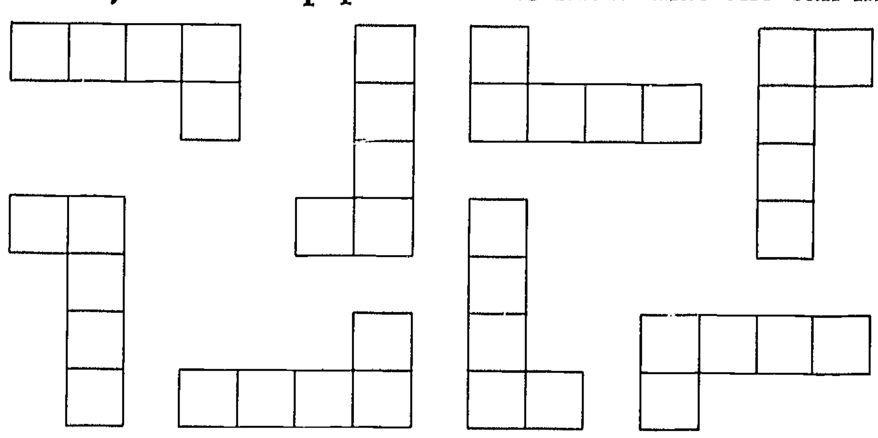


This could lead to a discussion of how many positions there would be if we did not have the rule that sides should remain horizontal and vertical. The students will probably realize that in that case there would be as many different positions as they pleased. This notion of "as many as we please" brings in the idea of infinitely many.

Each student should find out how many positions the pieces have, or at least enough of them so that a summary can be made on the board. It is interesting to observe how the students go about this task. Some will, in a very haphazard way, try to "think" of ways of drawing the pattern and very carefully draw the different positions on the one-half-inch ruled paper, perhaps finding only five out of eight possible positions of



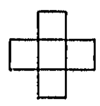
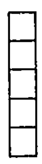
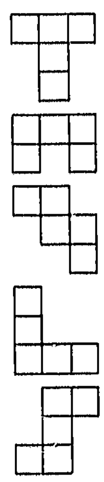
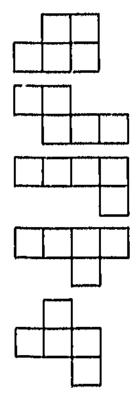
Others will be very sophisticated and say that there must be eight positions. They will use a paper cutout to show that one can make four



quarter-turns, then flip the pattern (that is, turn it over) and make four more quarter-turns, as shown at bottom of page 20. Some students may need to use a paper cutout to find the different positions. Eventually the class should be able to make table 1.

Table 1

Number of positions for each pattern

1	2	3	4	5	6	7	8	9	10
		—		—	—	—		—	—

Some students can probably tell from table 1 that the patterns they thought were fairly safe (i.e., were not often duplicated in different positions) have only a few positions—while those that are "dangerous" have the most positions. In other words, the patterns that were thought of as "dangerous" are so because there are eight possible ways of drawing them. It is for many of the "dangerous" ones that students need paper cutouts in order to find their number of different positions.

Which Motions Keep Patterns in the Same Position?

Teacher: *Why is it that this pattern (figure 33) has eight different positions, this one (figure 34) only two different positions, and this one (figure 35) only four?*

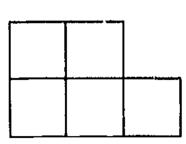


Figure 33

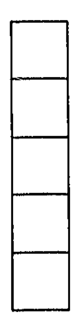


Figure 34

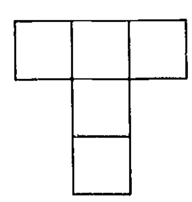


Figure 35

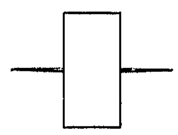
Students will probably realize that some movements that give new positions for figure 33 do not give a new one for figure 34 or 35.

TEACHER: *What motions can one make so that this pattern (figure 35) will still fit in the same location?*

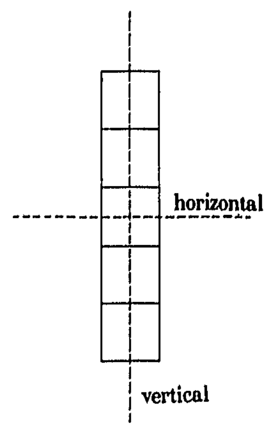
Look at figure 35. Some students may realize that if you "turn it over," it looks the same. Some may realize that both halves of the figure look the same, and they may feel that that has something to do with it.

TEACHER: *What motions can you make so that this pattern (figure 34) will still fit in the same position?*

The words used will depend on your students. "Half turn," and "do nothing," both keep the pattern in figure 34 in the same location. Flipping it over does too. One can flip about the horizontal and the vertical axes. One might call these horizontal and vertical flips. Students may have difficulty in learning which is a horizontal and which is a vertical flip. One way to help them is to draw a horizontal axis on the board and show that they are turning the piece *about* that axis. One can also stick toothpicks between two layers of the pattern as shown

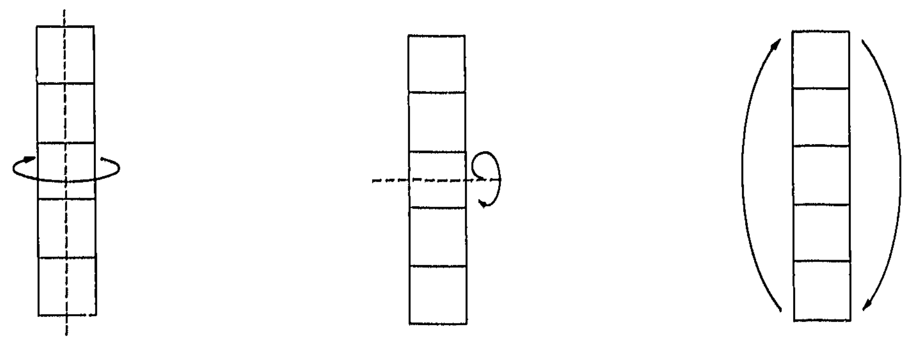


and turn it by keeping the hands fixed and rolling the toothpicks between the fingers. The same holds for a vertical flip.



Students should have practice in carrying out these operations with a paper pattern. (The reason for including a "do nothing" as a motion is made clear in the section "The Mathematics Involved in this Unit.")

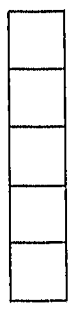
Some children may suggest a "whole turn." A "whole turn" is equivalent to a "do nothing" as will be clear once we mark the figure. Establish direction of turn (say, clockwise) for the half turn, and as shown below for the horizontal and vertical flip. (They can investigate later that the direction of motion in *this* case does not matter.)



Vertical Flip **Horizontal Flip** **½ Turn**

We can see that one can do a half turn, a vertical flip, a horizontal flip, and a "do nothing." After carrying out each of these, the pattern still fits in the original location. After the students are familiar with the motions that move the pattern (including the "do nothing" movement) so that it fits its original location, they are probably ready for the next game.

The Game "What Motions Were Made?"



Consider once again the pattern in figure 34: We will call this pattern the "straight pattern." How can one move the pattern so that after the motion has been made the pattern will still fit into its original location? Recall that the motions are a horizontal flip, a vertical flip, a half turn, and a "do nothing." Can the students tell which of these motions has been made if their eyes were closed at the time it was made?

For example, let the students see the paper pattern as you hold it on the board. Ask them to close their eyes. While their eyes are closed, make one of the motions, say, a vertical flip. When they open their eyes, can they tell with certainty (of course they can guess or peek!) which motion you made? What could they do so they could tell with certainty what motion was made? They will probably suggest marking the figure. What they suggest will not necessarily be adequate or economical. Try all their suggestions. An example of what has happened in the past may clarify this idea. One class suggested marking it on one side as in figure 36.

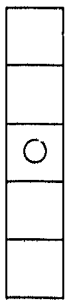


Figure 36

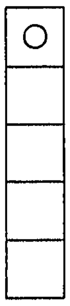


Figure 37

Half the class were asked to keep their eyes closed while the game was played once. A half turn was made. When they opened their eyes the figure appeared the same as it had before. They realized that they could not tell whether a half turn or a "do nothing" had been made. It was then suggested that the figure be marked as in figure 37. The game was played again. Again, half the class kept their eyes closed. Figure 38 shows the move.

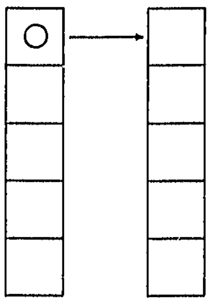


Figure 38

Now they realized they had to mark the back of the figure too. They could only tell that the figure had been flipped—they could not tell whether it had been given a vertical or a horizontal flip. It was suggested that the back be marked with another little circle directly behind the one in front.

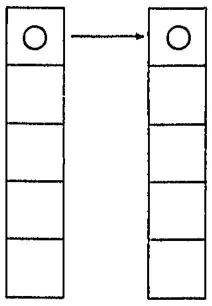


Figure 39

Making the move shown in figure 39 aroused amusement and the realization that the marking was still not adequate because they could not tell whether a "do nothing" or a vertical flip had been made. It was suggested that a little square be used on the second side. Thus, finally,

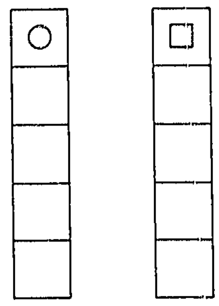
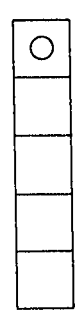


Figure 40

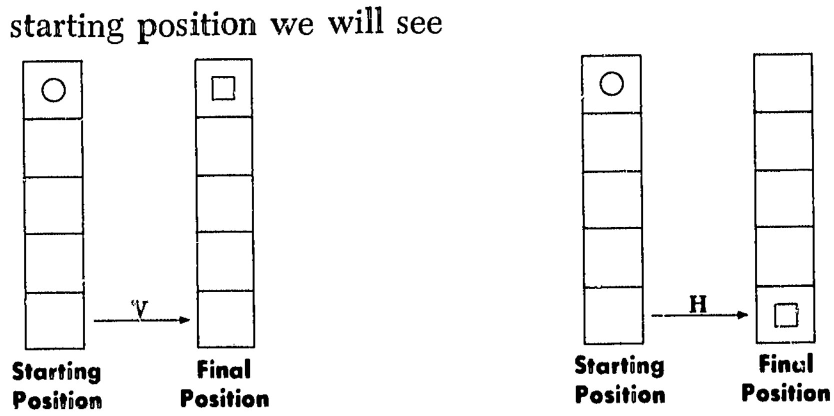
the piece was marked with the square directly behind the circle mark, as in figure 40.

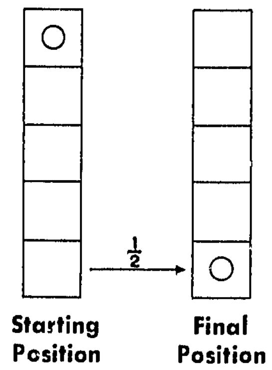
After several suggestions have been tried, you will probably end up with adequate, economical markings. If the students suggest numbering the squares 1, 2, 3, 4, . . . , 10, play the game and then ask if fewer markings would do. It is easier to work with the little square marked behind the circle than with the marks on opposite sides and on opposite ends. They may suggest marking the figure with a "1" in the middle of one side and a "2" behind it on the other. Of course this will work provided you do not draw a "1" like this: I .

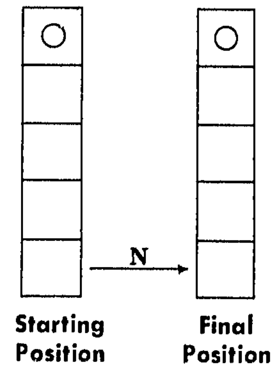
Abbreviations for the motions will be useful. For example, vertical flip (V), horizontal flip (H), half turn $(\frac{1}{2})$ and do nothing (N) have been suggested by students. Students should have their own paper patterns and practice the motions so that they are familiar with them. Before playing the game, be sure to show the starting position of the piece. Thus if we consider



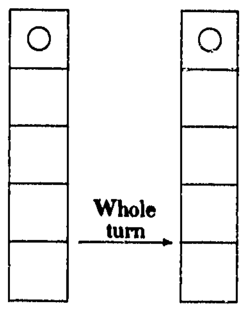
to be the starting position we will see







Notice that a "whole turn" would give the same final position as a "do nothing." That is why we will consider only the more economical "do nothing."



Now the students can play the game several times (as long as they are interested). Each time half the class could keep their eyes closed, and the other half watch while the motion is made. They will probably be very quick at seeing, for example, that the motion illustrated in figure 41 is a horizontal flip.

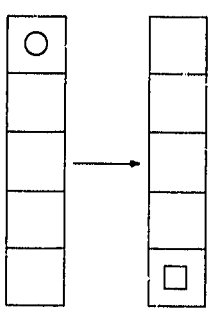


Figure 41

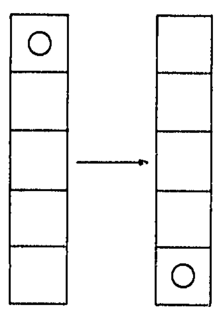


Figure 42

The students will soon realize that if only *one* motion is made, they can tell for certain which motion it was even when their eyes were closed. For example, in figure 42, the result must have come from a half-turn if one motion was used.

After a few games, students may begin to suggest the result of more than one motion. They may realize that *H H H H H H V* is equivalent to V; that is, they will realize that *H H H H H H V* gives the same result as V. They may find the rule that an even number of half turns, or horizontal flips, or vertical flips are equivalent to not moving the piece.

Can they picture the result of *H* followed by *V*? We can denote *H* followed by *V* as H^*V, and *V* followed by *H* as V^*H. (Notice that in this case the H^*V and V^*H give the same result. We can say $H^*V = V^*H$. For these motions "followed by" is commutative. This is discussed in the section, "The Mathematics Involved in this Unit."

When the students spontaneously begin to give two or more motions (one carried out after the other) that give the desired result, play the game using *two* motions. For example, they may suggest that the motion illustrated in figure 42 was done by H^*V instead of by ½. When the students are ready for this, announce that you are going to make two motions while their eyes are closed.

Again, show the pattern in starting position and ask half the students to close their eyes. Perhaps do the motions shown in figure 43.

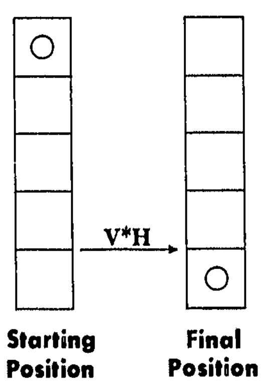
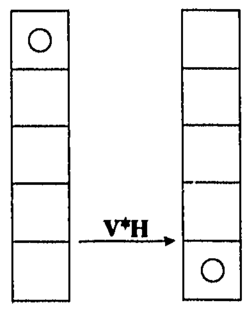


Figure 43

They may not realize, at first, that it is a matter of luck now whether they pick the two that you actually carried out. Let them find out that there are four possible ways of getting to this new position by using *two* motions. For example, the change shown in figure 43 can be done in one move by using a ½ turn; but by using two motions it can be done by H^*V, V^*H, $N^*½$, $½^*N$. (Recall that we have agreed to call a "do nothing" a motion although it is one of doing nothing.) There is no way of their knowing which you actually did. List these four possibilities next to the diagram on the board as shown in figure 44.



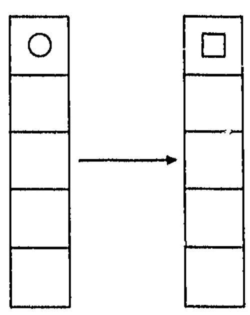
Starting Position   Final Position

Possible "Correct" Answers

One Motion	Two Motions
½	V^*H
	H^*V
	$½^*N$
	$N^*½$

Figure 44

Another example: The change shown in figure 45 can be done in two motions by N^*V, V^*N, $H^*½$, $½^*H$. In one motion it could be done by V.



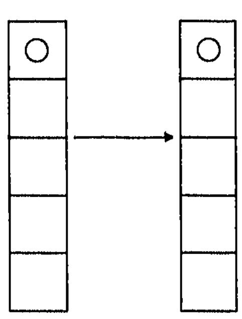
Starting Position   Final Position

Possible "Correct" Answers

One Motion	Two Motions
V	N^*V
	V^*N
	$½^*H$
	$H^*½$

Figure 45

The change in figure 46 could be made in two motions by $½^*½$, H^*H, V^*V, N^*N; in one motion by N.



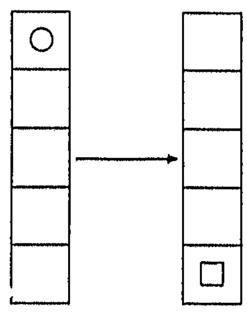
Starting Position   Final Position

Possible "Correct" Answers

One Motion	Two Motions
N	$½^*½$
	H^*H
	V^*V
	N^*N

Figure 46

Figure 47 could be made in two motions by $N*H$, $H*N$, $V*\frac{1}{2}$, $\frac{1}{2}*V$; in one motion by H.



Possible "Correct" Answers

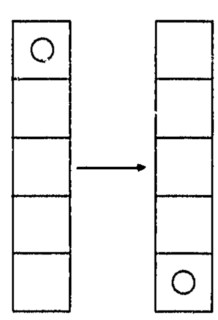
One Motion	Two Motions
H	$N*H$
	$H*N$
	$V*\frac{1}{2}$
	$\frac{1}{2}*V$

Starting Position **Final Position**

Figure 47

This game of guessing the motions will probably cause quite a bit of excitement. Don't be surprised if the students want to play the game of making three motions. We have found that young children are so able at these visualizing problems that the only way to keep up with them (be sure that they are correct when they produce two or more possible motions) is to say, "Well, let's check," and actually let them carry out the motions they suggest with a marked paper pattern to see if they are right. This not only saves us from memorizing any results, but it is also valuable for the students who don't quite agree or who are not as fast.

That is, if a student suggests that

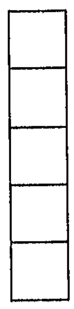


was carried out in two motions by $H*\frac{1}{2}$, the student could actually take the pattern and carry out $H*\frac{1}{2}$ to see whether this is correct. Of course, you could tell that it is incorrect without doing the actual motions because $H*\frac{1}{2}$ would turn the paper pattern over.

Making a Table

After the game has been played a sufficient number of times and the students have found out which two-motion moves are equivalent to (i.e., give the same result as) a one-motion move, try to summarize the information in a table.

You may wish to review the two-motion moves that leave the straight pattern in the same position.



N: $H*H$, $V*V$, $\frac{1}{2}*\frac{1}{2}$, $N*N$

$\frac{1}{2}$: $H*V$, $V*H$, $\frac{1}{2}*N$, $N*\frac{1}{2}$

H: $N*H$, $H*N$, $V*\frac{1}{2}$, $\frac{1}{2}*V$

V: $V*N$, $N*V$, $H*\frac{1}{2}$, $\frac{1}{2}*H$

A good way to introduce the idea of a table is to talk about the addition tables and the multiplication tables.

Addition Table

+	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

Multiplication Table

×	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16

Then ask the students how they could fill in a motion table (table 2).

Table 2

		Second Motion			
	*	N	½	H	V
First Motion	N				
	½				
	H				
	V				

They may want to use a paper cutout to find out what one-motion move gives the same result as $H*V$. They may be able to visualize it without one, or they may remember it. Each student could be asked to fill in table 3 on his own.

Table 3

		Second Motion			
	*	N	½	H	V
First Motion	N	N	½	H	V
	½	½	N	V	H
	H	H	V	N	½
	V	V	H	½	N

After they have filled in the table, make sure they understand how to read it—both ways—i.e., not only how to fill it in, but also how to answer: "If I want a final result that is the same as I would get with the motion H, and I start with V, what must I follow V by?" and "What other two-motion moves give the same result as H?"

What do the students notice about the table? The number of things they notice may be large—some things students have noticed are the following:

The first row is like the first column.
N occurs down the diagonal.
Each row contains H, V, N, and ½.

Two parts of the table match (the shaded and the unshaded parts indicated below).

*	N	½	H	V
N				
½				
H				
V				

You may want to ask them if they could erase part of the table and be able to put it back without recalculating. Someone may suggest erasing the part shown as shaded.

They will notice that $H*V$ gives the same result as $V*H$; $\frac{1}{2}*H$ the same result as $H*\frac{1}{2}$. In other words the order in which the two motions are made on the straight pattern does not matter. In *this* case we say "followed by" is commutative. Do they think that "followed by" is always commutative? (We will come to a case in a minute where it is not. But do not tell them this.)

Examining Another Pattern

Next, ask students to examine the motions that they can make with the cross pattern (fig. 48) and yet have it fit back in the same location.

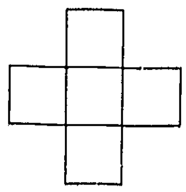
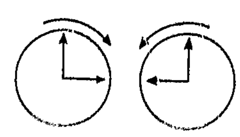
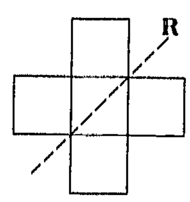


Figure 48

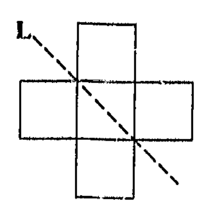
This is a bit harder because not only do the N, ½, H, and V leave it in the same position, but there are others as well. The students will probably notice ¼ and ¾ turns first. Deciding on one direction of turning is important, since a ¼ turn clockwise, for example, does not give the same result as ¼ turn counterclockwise. Compare this to the difference between ¼ past 12 and ¼ to 12. Establish one of the directions as the one



your class will consider; clockwise is usual. Notice also that a ¼ clockwise turn gives the same result as a ¾ counterclockwise turn. Hence we need not also consider a ¾ counterclockwise turn. For the straight pattern we considered only one ½ turn because a ½ turn in either direction gives the same result. Then there are two more flips (about the axes drawn below) which make the piece fit back in the same location.



Right-Slant Flip Line



Left-Slant Flip Line

The students can make up names for these flip lines. We have sometimes called them left- and right-slant flip lines. In other words, the cross pattern has four special flip lines: *H*, *V*, *L*, and *R*. It is not easy to flip the pattern about the *R* or *L* line. One way of doing it correctly is to draw the *L* line on the board, then hold the pattern as shown in figure 49 at the two corners and, keeping the hands in those positions, turn the pat-

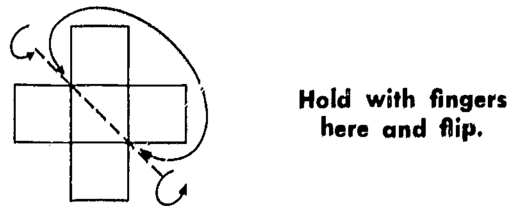
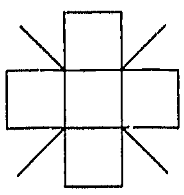


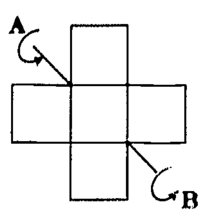
Figure 49

tern. It is easy to slip up and combine the *L* flip with a turn, for example. Students need to practice carrying out this motion, and you may have to show them how to do it.

It might help students who have difficulty in carrying out right- and left-slant flips if you place four toothpicks between two layers of a pattern as shown.



To carry out a left-slant flip hold the sticks at *A* and *B* and, keeping the hands still, turn the stick with the fingers until the pattern has been flipped.



You may wish to further consider "flip lines." The "flip lines" are lines along which you can fold the paper pattern and obtain two halves that match. Also, if a mirror is placed along these lines, the whole pattern can be seen. Students may want to examine these patterns with a mirror, especially if they have used the Mirror Cards. Some of the worksheets in the Mirror Card Teacher's Guide could be used by the students as additional work with this unit.

You might also want to discuss what properties a paper pattern has if one can make a ¼, ½, or ¾ turn and still have it fit back on the same location.

Draw the patterns in figure 50. For each one draw a straight line through P. What do you notice? Do it again drawing a different line. What do you notice? Note that each pattern in figure 50 has a point P such

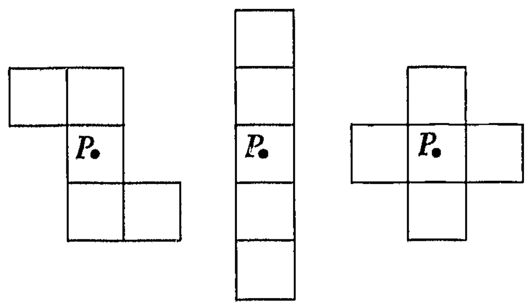


Figure 50

that if you draw any straight line through it and cut along that line, the pattern will fall into two matching parts. (That is, you can pick up the two parts and fit them on top of each other by turning them around. Note that you don't need to flip them.)

For those patterns that also allow $\frac{1}{4}$ and $\frac{3}{4}$ turns (the cross pattern, for example) you can cut *any two* perpendicular lines through P, and it will divide the paper pattern into four matching parts (you can fit them on top of each other by turning). If a $\frac{1}{4}$ or $\frac{3}{4}$ turn does not leave the pattern in the same location, then you cannot draw *any* two perpendicular lines through P to obtain 4 matching parts.

After the eight motions (N, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, H, V, R, L) that leave the pattern in the same position have been established, the game of what motion was made can again be played.

First, the pattern must be marked. Ask for suggestions for marking it. Carry out each one until an economical adequate marking is obtained. If the students suggest marking a circle in the middle square of the pattern, play the game (eyes closed) as shown in figure 51. They will

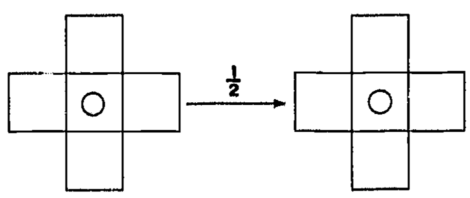
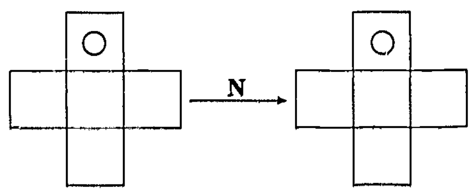
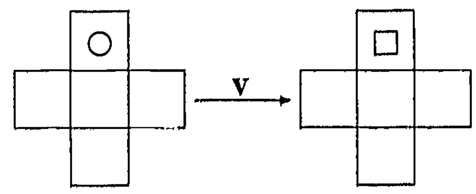


Figure 51

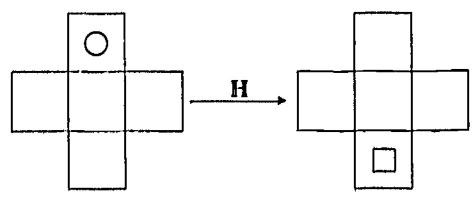
realize that the marking is not adequate. Work with them until an adequate one is obtained. If they suggest numbering the squares 1 to 10, ask if fewer marks are possible. Then continue. A marking system that is easy to work with is one that has a circle in one square (not the middle one) and a different mark on the other side directly behind it. Of course you can use any system that the children suggest and that "works." It is useful to practice the one-motion moves until the students are really good at carrying them out with a paper pattern.



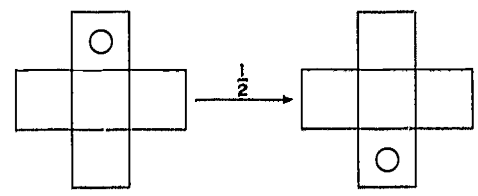
Starting Position **Final Position**



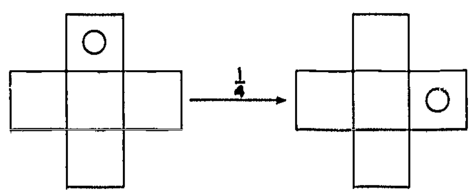
Starting Position **Final Position**



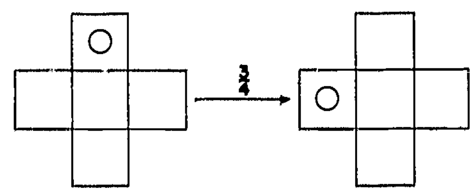
Starting Position **Final Position**



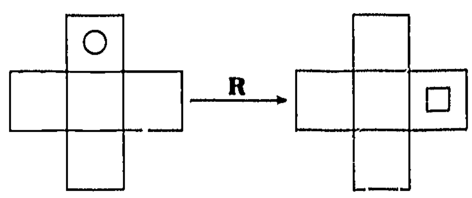
Starting Position **Final Position**



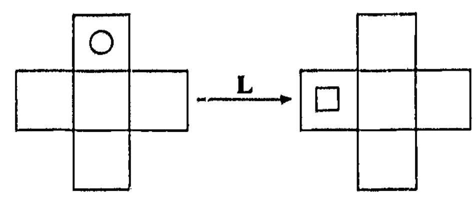
Starting Position **Final Position**



Starting Position **Final Position**

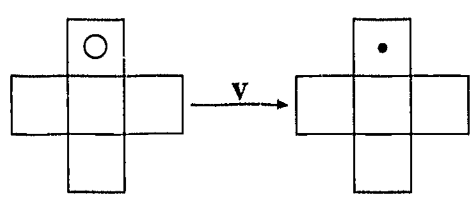


Starting Position **Final Position**



Starting Position **Final Position**

Now show the starting position, have half the students close their eyes, and make a motion. Let the students say what it was. For example, carry out the motion illustrated in figure 52. After one motion becomes

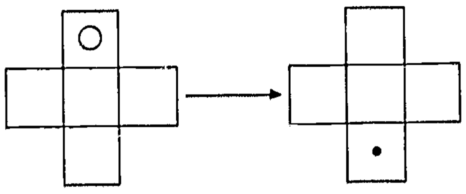


Starting Position **Final Position**

Figure 52

too easy to guess, make two motions. They will find again that if a two-motion move was made they cannot be sure which two-motion move it was. Furthermore, they will now find eight possibilities. For example, if you make ¼**H*, they may guess *N***R*, *V**¼, *H**¾, *R***N*, ¾**V*, *L**½, ½**L*, for two-motion moves, but they will be sure that it was *R* if *one* motion was made.

Play often when and while they are still interested. Now many students may need a paper cutout to convince themselves that the two motions suggested are correct. Again, each answer can be checked by a student at the board who can use the paper cutout. Another example is given in figure 53. What two-motion move might have been made to arrive at the final position shown?

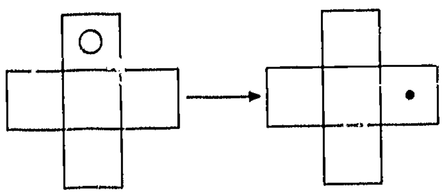


Starting Position **Final Position**

Figure 53

The students may guess *N***H*, *V**½, ¼**L*, ¾**R*, *H***N*, *L**¾, *R**¼, ½**V* for two-motion moves.

To what one motion are the two equivalent? (*H*) You may want to write down *H* (one motion); *N***H*, *V**½, ¼**L*, ¾**R*, *H***N*, L*¾, *R**¼, ½*V (two-motion moves). Another example is shown in figure 54.



Two Motions	One Motion
¼* *H*	*R*
½* *L*	
*N** *R*	
*R** *N*	
¾*V	
*L** ½	
V* ¼	
*H** ¾	

Figure 54

Eventually, you can again have the students complete a table—a compact way of storing the information. Although you can list the motions in any order at the head of the table, it is worthwhile to list the first four in the same order as in table 3 because the students will then be able to notice that the corner of the table for the cross pattern looks exactly like the whole table for the straight pattern.

Table 4

Second Motion

Followed by * (First Motion)	N	½	H	V	¼	¾	R	L
N								
½					¾			
H	H							
V		H				L		
¼								
¾					N			
R								
L								

Second Motion

Followed by * (First Motion)	N	½	H	V	¼	¾	R	L
N	N	½	H	V	¼	¾	R	L
½	½	N	V	H	¾	¼	L	R
H	H	V	N	½	L	R	¾	¼
V	V	H	½	N	R	L	¼	¾
¼	¼	¾	R	L	½	N	V	H
¾	¾	¼	L	R	N	½	H	V
R	R	L	¼	¾	H	V	N	½
L	L	R	¾	¼	V	H	½	N

What do students notice about table 4 now? If we look at the shaded and unshaded part of the table shown below, the two parts do not match as they did for table 3.

The motion $\frac{1}{4} * H$ does not give the same result as $H * \frac{1}{4}$. We say that not all the pairs of motions commute. The students may notice again that every motion occurs once in each row and once in each column.

Second Motion

First Motion

Followed by *	N	½	H	V	¼	¾	R	L
N	N	½	H	V	¼	¾	R	L
½	½	N	V	H	¾	¼	L	R
H	H	V	N	½	L	R	¾	¼
V	V	H	½	N	R	L	¼	¾
¼	¼	¾	R	L	½	N	V	H
¾	¾	¼	L	R	N	½	H	V
R	R	L	¼	¾	H	V	N	½
L	L	R	¾	¼	V	H	½	N

Can they answer questions like the following: If I start with the motion V, and if I want a final result that is the same as I would get with the motion ¼, what motion must I follow V by? Is there always an answer to such a question? If I start with any motion, say V, can I always follow it by something so the net result is N? In other words, can I always undo a motion by following it with a second motion? If I carry out two motions, could I have obtained the same result in one motion instead?

What about more than two motions? For example: Can you find one motion that is equivalent to $H * \frac{1}{4} * V * L$? (That is a hard one!) $H * \frac{1}{2} * H$ might be easier. Some students might be able to do this by visualizing in their heads—others will need to have a paper cutout. After they have had plenty of exercise in visualizing, they will notice that they can also read the answers from the table (even more than two motions can be read off from the table by repeated application), and thus the table is really a laborsaving device. However, the students should have considerable practice either in moving the pattern or in visualizing it in their heads before looking at the table. Another question may arise: What is the meaning of $H * V * \frac{1}{2}$, say? Is one first to do H, then V, and then ½? Yes, we will agree to carry out the motions from left to right.

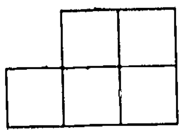
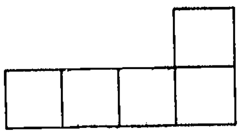
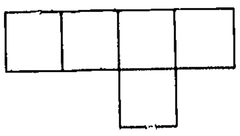
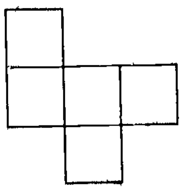
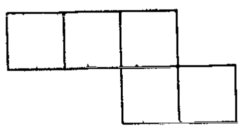
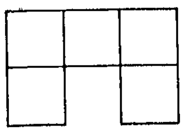
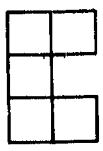
Now suppose we want to use the table to find out the result of $L * V * \frac{1}{2}$. The table tells us which one motion is equivalent to any pair of motions. Does $L * V * \frac{1}{2}$ mean $(L * V) * \frac{1}{2}$ or $L * (V * \frac{1}{2})$? If we check, $L * V = \frac{1}{4}$. So $(L * V) * \frac{1}{2} = \frac{1}{4} * \frac{1}{2} = \frac{3}{4}$. On the other hand,

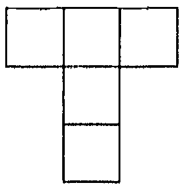
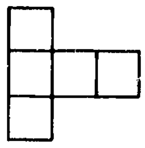
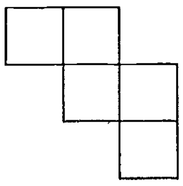
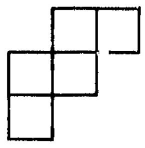
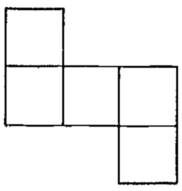
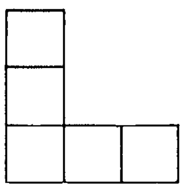
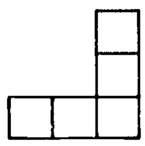
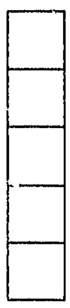
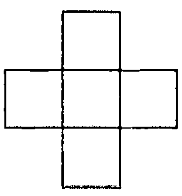
$L^*(V^*\frac{1}{2}) = L^*(H)$ since from the table $V^*\frac{1}{2} = H$. Therefore, $L^*(V^*\frac{1}{2}) = L^*H = \frac{3}{4}$.

So we see that in this example $(L^*V)^*\frac{1}{2} = L^*(V^*\frac{1}{2})$. Older students may want to check this for several examples.

The students may want to examine motions that leave the other shapes in the same location. The different motions are listed in table 5.

Table 5

Pattern	*Single Motions That Give Back Original Position*	*Number of Different Positions Possible, Keeping the Sides Horizontal and Vertical*
	N	8
	N	8
	N	8
	N	8
	N	8
	V, N; or H, N, if we consider 	4

Pattern	*Single Motions That Give Back Original Position*	*Number of Different Positions Possible, Keeping the Sides Horizontal and Vertical*
	H, N; or V, N, if we consider 	4
	R, N; or L, N, if we consider 	4
	½, N	4
	R, N; or L, N, if we consider 	4
	H, V, ½, N	2
	¼, ½, ¾, N, H, V, R, L	1

RELATED PROBLEMS AND EXTENSION OF THE WORK

The work can be extended in many ways. There are many questions that the students can investigate in class or at home. They will probably think of many of these questions themselves and suggest many more that are not included here. The more they pose their own problems the better. If they don't think of any at first, perhaps with a little guiding they will come up with some. When problems are their own, students will be more motivated to solve them.

You will find it useful to read through all the problems before giving any of them to the children. They are not necessarily in order of difficulty, but for many of the questions you will find it useful to have done the previous ones. For younger children you will have to reword some of the problems, and you may want to introduce the problems by example.

More related problems are suggested in the various Teacher's Guides and articles mentioned in this section and listed in the bibliography. Some of the problems are suggestions that the teacher can use for making up others.

Making Squares and Rectangles

1. *a.* Choose various objects in the classroom and check to see if they are square. Do you need to use a marked ruler for this? Could you do it if you only had a piece of string? You must check that all sides have the same length. What else must you check? Find squares and rectangles in your street or house. Draw squares by tracing around square objects.

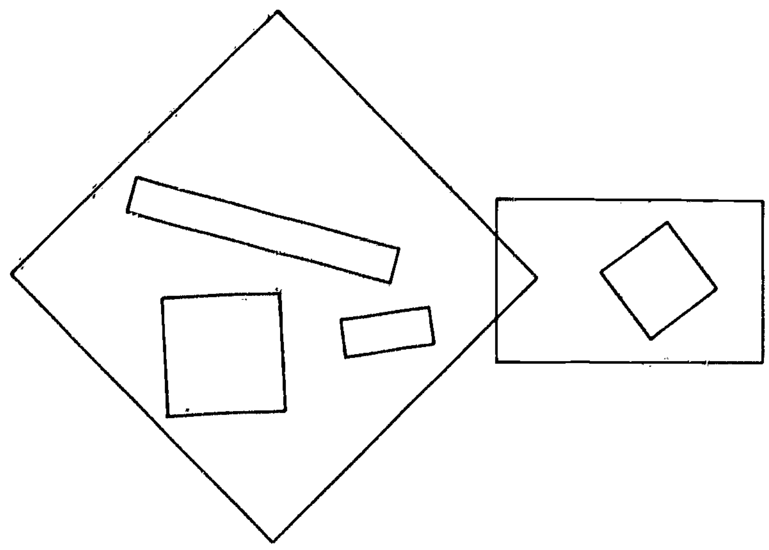
   b. Now try to draw a square without tracing around a square object. This is called freehand drawing.

2. Construct a square, using a ruler and the corner of an index card.

3. Construct a square, using a ruler and a compass.

4. Take a piece of paper. Wax paper would be best, but any paper will do. Can you fold it so that the creases show a square? Can you do it even if the piece of paper has no corners to begin with?

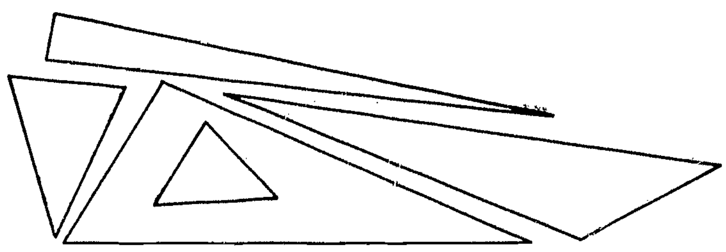
5. Find a quick way of drawing many squares.
6. Repeat some of the problems above for a *rectangle* that is not square.
7. In the figure below which shapes are drawings of squares? Of rectangles that are not squares? Of rectangles?



8. Build three-dimensional shapes out of sticks [IV. 1].

Making Triangles

1. *a.* Find objects that have shapes that are triangles.
   b. Are the sides of the triangle of the same length?
2. Check to see whether two or three sides of any of these triangles have the same length. How did you check? Can you find another way?

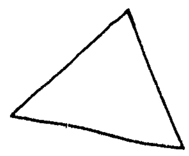


A triangle is called equilateral if all its sides have the same length, "Equi" means equal, and "lateral" means side.

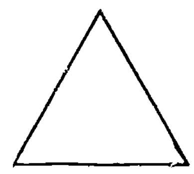
3. *a.* Draw three points. Join them by lines. Did you draw a triangle? Does that always work?

b. Do it again and try to get an equilateral triangle. Do it several times. Check to see if you drew "almost" an equilateral triangle. Do it again by just drawing three line segments—without first drawing three dots. When you draw triangles by either of these "guessing" ways, the triangle often comes out close to an equilateral one. This is called a freehand drawing.

c. Can you do a more accurate job by using ruler and compass?



Freehand



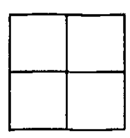
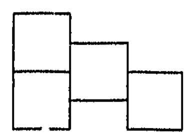
Ruler and Compass Construction

Arrangements of Patterns

A. *In two dimensions*

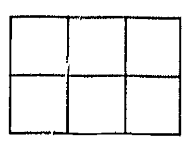
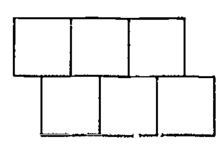
Some of the questions can be done by drawing or by using paper cutouts, or by using Pattern Blocks.

1. How many different arrangements of four squares are there? Keep whole sides touching. For example,

 is allowed, but  is not.

Two patterns are considered the same if the same paper cutout can be made to match each of them. Draw them all.

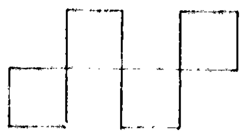
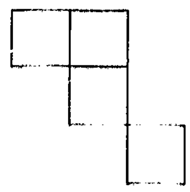
2. *a.* How many different arrangements of six squares are there? Keep whole sides touching. For example,

 is allowed, but  is not.

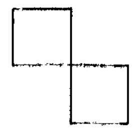
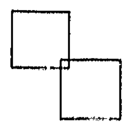
b. Which of your six-square patterns fold into boxes? Make models out of construction paper so that you can check your answers.

3. How many different arrangements of four squares are there if you

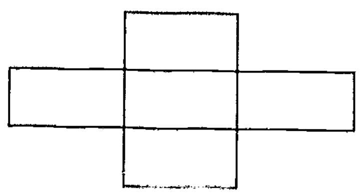
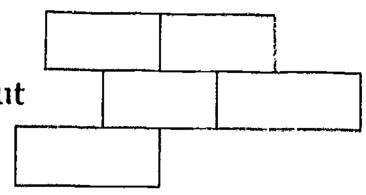
make the rule that squares can touch only at the corners? For example,

 is allowed, but  is not.

Two patterns are considered the same if the same paper cutout can be made to match both. Of course, you will have to pretend that you can glue two squares together at corners only.

Imagine:  Actual: 

4. *a.* How many different arrangements are there of five rectangles if the rectangles are not squares? For example, the length of each rectangle might be twice its width. Keep whole sides touching. For example,

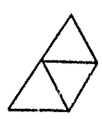
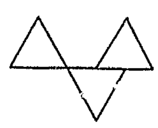
 is allowed, but  is not.

b. Does the number of patterns change if the shape of the rectangle changes?

c. Which patterns fold into boxes without tops?

5. Make up other problems involving rectangles and squares.

6. *a.* An equilateral triangle is one whose sides are all of the same length. How many different arrangements of three equilateral triangles can you draw? (Use Isometric "graph" paper, described on p. 4 .) Keep whole sides touching. For example,

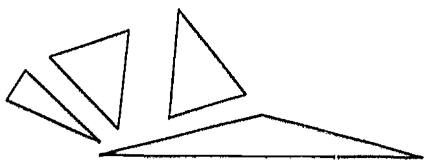
 is allowed, but  is not.

Two patterns are considered the same if the same paper cutout can be made to match each of them.

b. Repeat with four equilateral triangles. Make models of these patterns out of construction paper and see what you can build.

c. Repeat with five equilateral triangles and make models.

7. An isosceles triangle is one that has two equal sides. Examples of isosceles triangles are shown in the figure below. Make up some problems with isosceles triangles.

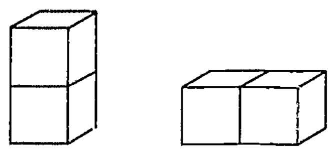


8. Use the Pattern Blocks. These are a set of colorful wooden blocks. The shapes are squares, triangles, hexagons, diamonds (rhombi), and trapezoids. Many of the problems suggested here can be done by using the blocks instead of drawing the pictures. In addition, countless other patterns can be made. The Pattern Block Guide gives many suggestions [IV. 4].

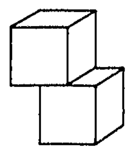
B. *In three dimensions*

For these questions you will need to make or buy some small cubes of the same size which can be glued together.

1. *a.* Two cubes make only one pattern if one keeps whole faces of cubes touching. Below, both arrangements are considered the same because if we turn one pattern around it will be in the same position as the other. By turning them, we can see that the two pieces are identical.

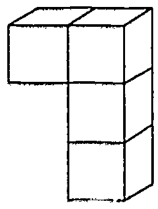


Notice that we are keeping whole faces touching. This arrangement is *not* allowed:

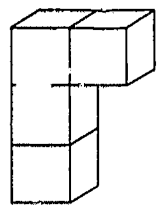


b. How many different arrangements of three cubes can you make?

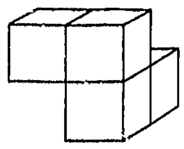
c. How many different arrangements can you make with four cubes? Keep whole faces touching. Notice that



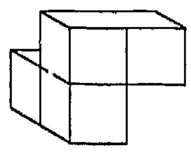
and



are the same pattern because you can turn one around and make it fit in the same location as the other (provided you move the other out of the way). You can think of it another way. If one were a thin hollow shell, the other would fit inside it. Notice that

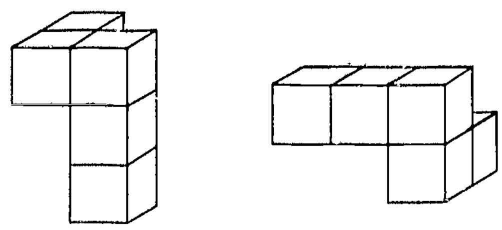


and



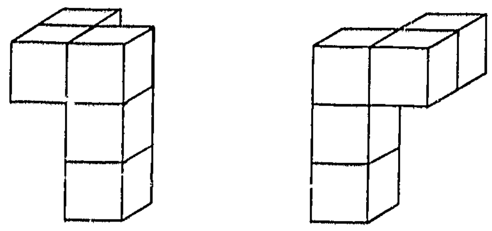
cannot be made to fit in the same location. They "look" as if they match but they do not. However, there is something special about them. One is the mirror image of the other. Look at one of them in the mirror and you see the other. These have the same property as a pair of shoes. The two shoes can never occupy the same space (at different times of course) but one shoe is the mirror image of the other.

d. How many different arrangements can you make with five cubes? Keep whole faces touching. How many pairs are mirror images of each other? One pair of images is drawn here.



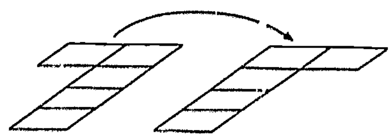
In these positions it is not obvious that one is the mirror image of the other.

However, if we draw them like this, it is



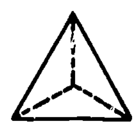
more obvious. Can you build these models and place the mirror between them so that you can see the second by looking at the mirror image of the first?

e. How many five-cube patterns are there if we consider mirror images to be the same arrangement? In the plane, when we used squares we could actually place two mirror images in the same location without using a mirror. We merely had to pick the piece up and turn it over. We took it out of the plane and



turned it around. In three dimensions we can't take a pattern "out of" three dimensions to turn it around.

f. There are many puzzles connected with certain sets of these arrangements. Piet Hein, who is Danish, invented the Soma Cube [I. 9].

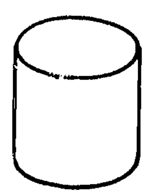
2. If you find models of small regular tetrahedrons,  you may want to make up problems of your own.

3. See the game of TAP, page 53.

Making Other Three-Dimensional Patterns

1. What three-dimensional shapes can you make of construction paper? Can you make—

a. a long box?
b. a cylinder without any ends (that is, a shape like a tube)?



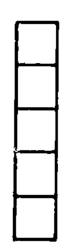
c. a house with a roof?
d. a pyramid?
e. a ball?
f. a doughnut?
g. other shapes?

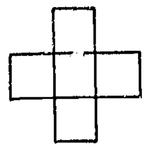
2. What three-dimensional shapes can you make of modeling clay? Can you make—
   a. a cube?
   b. a cone?
   c. a pretzel?
   d. a pyramid?
   e. a house?
   f. a tent?
   g. what else?
3. What is the difference between the three-dimensional figures made of paper and those made of clay?
4. What three-dimensional figures can you make with wire? What can you make with straws? Can you make—
   a. a cube?
   b. a cone?
   c. a pretzel?
   d. a pyramid?
   e. a house?
   f. a tent?
   g. others?
5. Wh t three-dimensional figures can you make out of other materials or other objects? You may want to use scrap materials often thrown away. For example—plastic lids, egg cartons, cardboard tubes, and containers of all shapes.

Using a Mirror with the Patterns

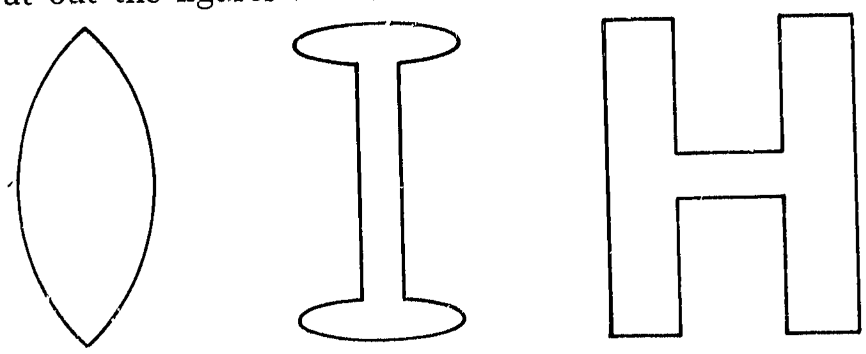
1. Investigate patterns with a mirror [VI. 1, 2].
2. Play with the Mirror Cards [IV. 4].
3. Draw designs and investigate each with a mirror.
4. Draw a triangle and investigate with two mirrors. How many triangles can you see? Repeat with other shapes.
5. Use the Pattern Blocks and one or two mirrors [IV. 4].

Turning and Flipping a Pattern

1. In the unit, you investigated the straight pattern  and found

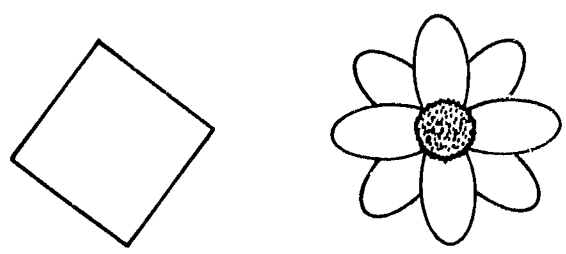
it has four motions that leave it in the same location, and the cross pattern  has eight. You marked the pieces to help you. Mark each of the other five-square pieces and find out how many motions leave each in the same location.

a. Cut out the figures below and check to see if each of them has



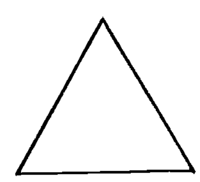
four motions that leave it in the same location. Make up two more patterns with this property.

b. Examine the two figures below to find out if each has the same motions as the cross pattern which leave it in the same location.



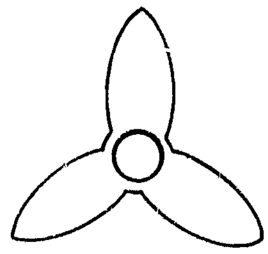
Can you draw other figures that have these eight motions that leave them in the same location?

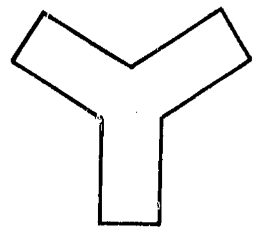
3. Cut out an equilateral triangle. Draw an outline of it on a sheet of paper and place the triangle on it. Mark the triangle so that you could tell what motion was made even if your eyes were closed.



What turns can you make that leave the triangle in the same location? A ½ turn will not leave it in the same location. What flips can you make that leave the triangle in the same location? Did you find six motions altogether? Make a table showing which one motion gives the same result as any two motions.

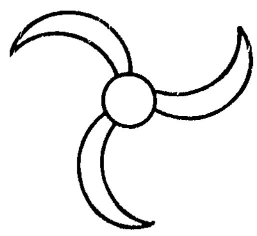
4. Examine these two figures:





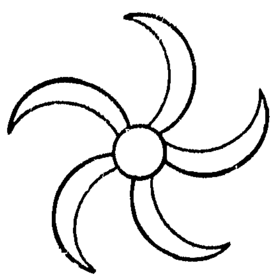
Does each have the same motions as the triangle that leave it in the same location? Can you draw other figures such that each has the same six motions that leave it in the same location?

5. Examine this figure.

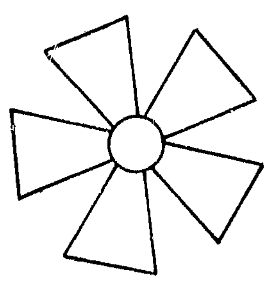


What turns leave it in the same location? What flips? This figure has only three motions that leave it in the same location—they are all turns.

6. What are the turns that leave this figure in the same location? Do any flips leave it in the same location?



7. What turns leave this figure in the same location? What flips?

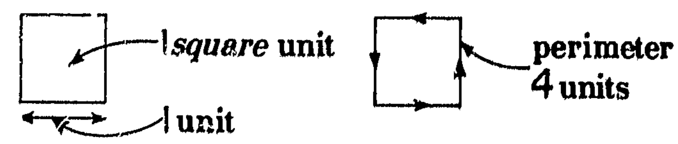


Did you find ten motions altogether?

8. Make up other problems.

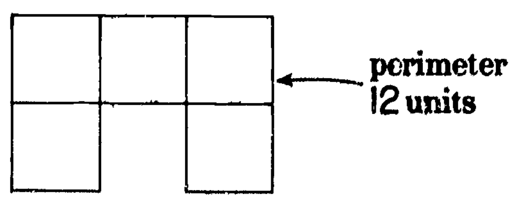
Perimeter, Area, and Volume Problems

1. Draw a square.



If the length of the side of your square is 1 inch, then the area of the square is 1 square inch. The perimeter (how far it is to trace all the way around) would be 4 inches.
If the length of the side of your square is 1 "my own" unit, then the area of the square is 1 "my own" square unit and the perimeter is 4 "my own units."

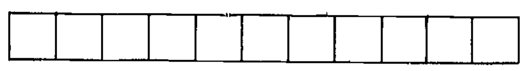
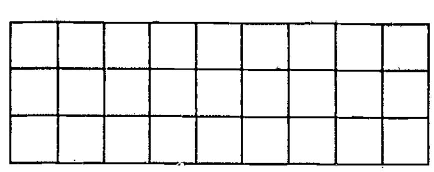
2. *a.* Draw all the five-square patterns. Each has an area of 5 square units. What is the perimeter of each pattern? Be careful! They don't *all* have a perimeter of 12.



   b. Repeat with all the six-square patterns.

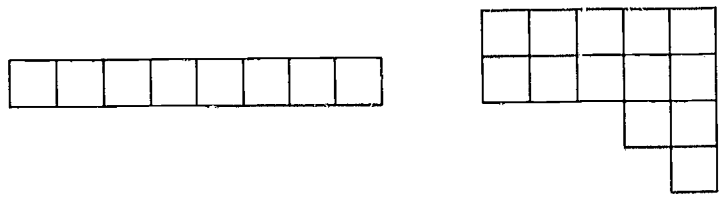
3. *a.* Take 24 squares all of the same size. Call the length of each side 1 unit; then the area of each square is 1 square unit. Make as many different rectangles out of these 24 squares as you can. Each rectangle has an area of 24 square units. What is the perimeter of each?
   b. Repeat with 36 squares.
   c. Repeat with 37 squares.
   d. Repeat with 60 squares.

4. *a.* Take some graph paper. Call the length of each side of a square 1 unit. How many rectangles can you draw with a perimeter of 24 units? Here are two:

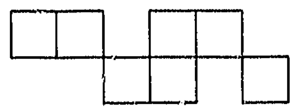



   Draw others. How many squares does each contain? What is the area of each? Which has the largest area? The smallest?
   b. Repeat with a perimeter of 36 units.

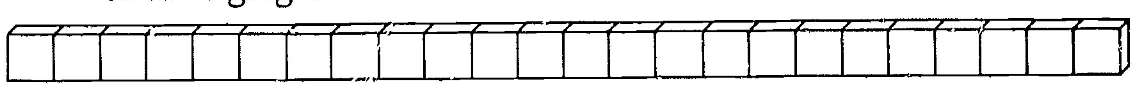
5. Make shapes that have a perimeter of 18. Keep whole sides touching. These shapes are allowed:



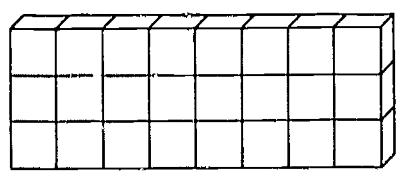
but this is not:



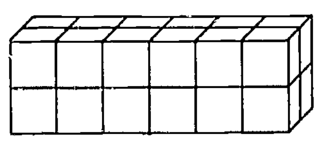
6. *a.* Imagine that you have 24 milk cartons with the tops cut off so that all sides are squares. (You can use any cubes instead.) One way of arranging them is this:



This is a $24 \times 1 \times 1$ array.
Other ways to arrange them would be an $8 \times 3 \times 1$ array:



and a $6 \times 2 \times 2$ array:



How else can you stack 24 cut milk cartons? You need not draw every arrangement.

b. Repeat with 48 cartons.

c. Repeat with 240 cartons.

d. Repeat with 315 cartons.

e. Make up problems of your own.

f. For each of the arrangements you made, imagine that you want to cover the stack with gummed paper. How many square units of paper would you need for each?

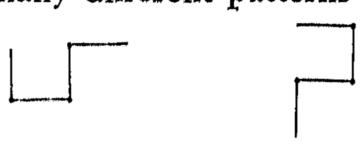
7. If the edge of each *cut* milk carton is 2 inches, then the volume of each is 8 cubic inches.
   a. How much room is needed for 150 milk cartons?
   b. For 600 milk cartons?
   c. About how many will fit into a cubic yard?
   d. If each cut carton weighs 1 ounce, how many pounds will 600 cartons weigh?
   e. How much space will 30 of the milk cartons (*not* cut) take up?
8. Use the Geoblocks [IV. 4]. These consist of a variety of wooden shapes including cubes of various sizes. The Teacher's Guide suggests a number of problems.

The Game of "Turn a Pattern" (TAP)

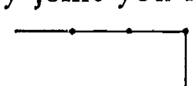
This game gives children additional practice in visualizing patterns in two and three dimensions. The game can be played in two dimensions by drawing line segments on paper, and in three dimensions by using sticks and joints. Instead of drawing line segments, pieces of magnetic strips on a metal board might be used. Any number of players can play.

1. *In two dimensions*

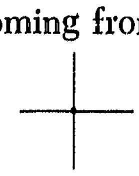
Rules. Use four sticks (line segments) of the same length. Make a right-angled turn at every joint. Not more than two sticks can come from a single joint. How many different patterns can you make? Watch out! These two figures



are the same pattern, because you can move one pattern so it coincides with the other. One person makes a pattern, then the next person tries to make a different pattern. Score 1 for a new pattern, 0 for one that has already been made, and (if two or more persons play) 1 for noticing a duplicate.

Variation *A.* Rules as above, except that at every joint you may go straight ahead or make a right-angled turn. That is,  is allowed.

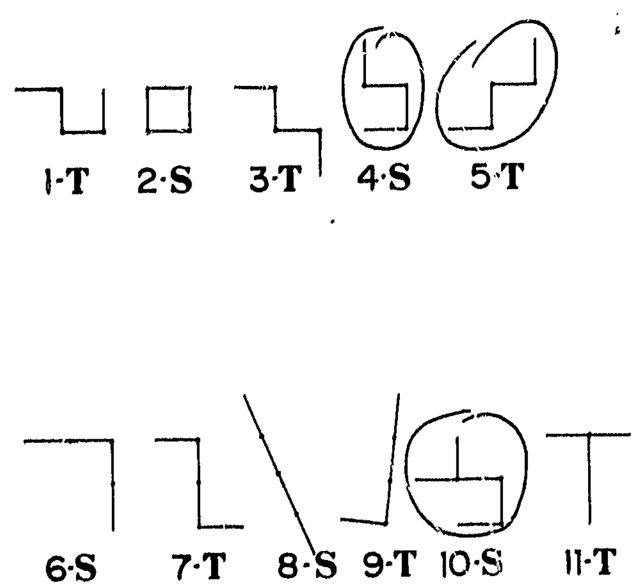
Variation *B.* Rules as above, except that there are no restrictions on the number of sticks or lines coming from a single joint.

That is,  is allowed.

This is how a two-player game might look under the rules with both variations A and B; that is, with four sticks, right-angled turns or no turns at every joint, and any number of sticks coming from a single joint. Note that number 4 scored zero because it duplicated 1; number 5 duplicated 3; and number 10 used five sticks instead of 4.

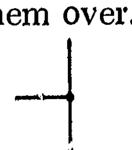
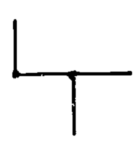
Score

	Tom	Sue
1	1	
2		1
3	1	
4	1	0
5	0	1
6		1
7	1	
8		1
9	1	
10		0
11	1	
Total	6	4



No one could think of another pattern. Are there any more? Can you find more? Which of the patterns would not be allowed if only two sticks could come from one joint?

Variation C. Same rules as above except that now the definition of duplicates is different. To make patterns match, you may rotate them but may not flip them over. That is,

 and 

are different patterns since you must flip one over (as well as rotate it) to make it coincide with the other, and that is not allowed. To make this notion clearer, imagine the lines black on one side, red on the other. Count patterns the same only if they can be made to coincide with motions that would not change the exposed color of the pieces.

Variation *D.* Same rules as above except that five sticks may be used.

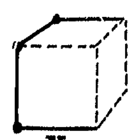
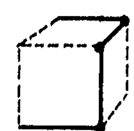
Variation *E.* Instead of taking turns drawing a whole pattern, take turns drawing one line at a time. The aim is to cause other players to repeat a pattern and enable yourself to draw a new pattern. The person who completes a pattern has either completed a new one or has repeated one that had already been drawn. Score 1 for completing a new one; 0 for completing one already drawn.

All the variations can be played by a player alone, by two, or by several. The challenge is to find all possible patterns under the given set of rules with one or several of the variations agreed upon. The players may make up other variations.

2. *In three dimensions using sticks and joints*

The game is basically the same but, of course, more difficult.

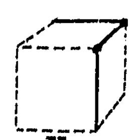
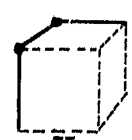
Rules. Use four sticks of the same length. Make a right-angled turn at every joint. Not more than two sticks may come from each joint. How many different patterns can you get? Here it is more complicated to state what is meant by "patterns being different." Certainly, if patterns can actually be made to coincide, we can say they are the same. Thus

 and 

x y

are the same pattern.

Now,

 and 

x y

cannot be made to coincide, but one is the mirror image of the other. We can have the following cases:

1. *X* is the mirror image of *Y*, but they cannot be made to coincide.
2. *X* is the mirror image of *Y*, and they can be made to coincide.
3. *X* and *Y* are not mirror images, but they can be made to coincide.
4. *X* and *Y* are not mirror images, and they cannot be made to coincide.

Certainly we would agree to count the patterns the same in cases 2 and 3 and not the same in case 4. We will also consider the patterns in case 1 to be "the same." That is, two patterns are considered the same when they can be made to coincide, *or* are mirror images of each other.

One can, of course, make the same type of variations as in the two-dimensional case; that is, one can change the number of sticks, the rule about allowed angles, and the rule about the allowed number of sticks from each joint.

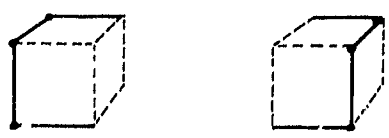
Students, after much practice, may want to investigate the following problems:

For the paper version—For some patterns, it makes no difference whether turning over is allowed. These patterns

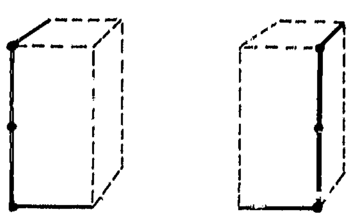
can be made to coincide by rotating or turning them over; for these,

flipping the pattern is required. Investigate the nature of the patterns of each type.

For the three-dimensional version—These patterns



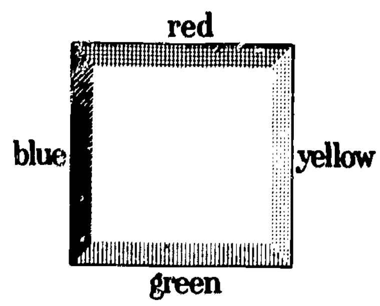
can be made to coincide, and one is a mirror image of the other. These



are mirror images of each other, but cannot be made to coincide.

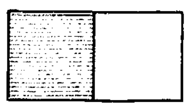
Coloring Problems

1. How many different ways can you color the four edges of a square with four colors? Here is one way. The front and back are colored the same; imagine that the color "goes through" the paper. Two "colorings" are the same if the two squares can be made to coincide so that the same colors are on top of each other.

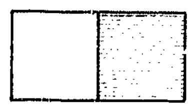


2. How many different ways can you color the three sides of a triangle?

3. Imagine that you have two squares and two colors. The interior of each square is to be colored with one color. There are two ways of coloring them:



or



Now imagine you have three squares and three colors. How many ways can you color the three squares in a row? Do it again with four colors and four squares.

4. *a.* Think of a six-sided box all of whose sides are square—a cube. Make many models out of anything you want. Get a friend to help you. Suppose you have *two* colors. How many different ways can you color a cube? Each face has only one color! You will have to decide whether to call mirror images "the same" or "different."

   b. Repeat with three, four, five, or six colors. [I. 7]

5. Make up other coloring problems with other shapes.

6. Look at many dice. What can you say about the way they are numbered?

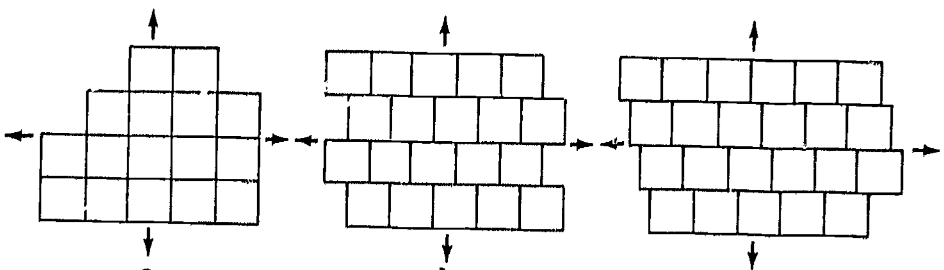
Tessellations

The problems below can be done by drawing the shapes, or using paper cutouts or wooden blocks. The Pattern Blocks [IV. 4*a*] are particularly useful for these problems. The Pattern Blocks Teacher's Guide suggests many more problems, and I strongly recommend it. Several of the references throughout the Bibliography are useful, for they describe work done by children.

To tessellate means "to form into or adorn with mosaic; to lay with checkered work" (Webster's Dictionary).

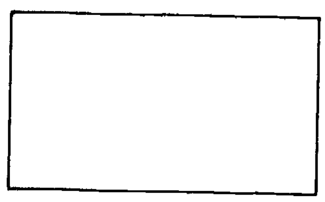
Mosaic means "a surface decoration made by inlaying in patterns small pieces of colored glass, stone, or other materials." (Webster)

Squares of the same size can be used to cover a "floor," which we will imagine goes on and on as far as we wish. For example, here are a few ways of covering or tessellating such a "floor."



a b c

1. *a.* Tessellate a "floor" with arrangements of squares other than the ones shown. Which of the patterns (*a*), (*b*), (*c*), or yours would



fit exactly into a rectangular floor as shown? Why will some of the patterns fit into a rectangle while others will not?

b. Could you cover an oval-shaped floor with squares?

c. Color your arrangements of squares in an interesting way.

2. *a.* Take many equilateral triangles of the same size. Can you place them so that they cover a large space without gaps?

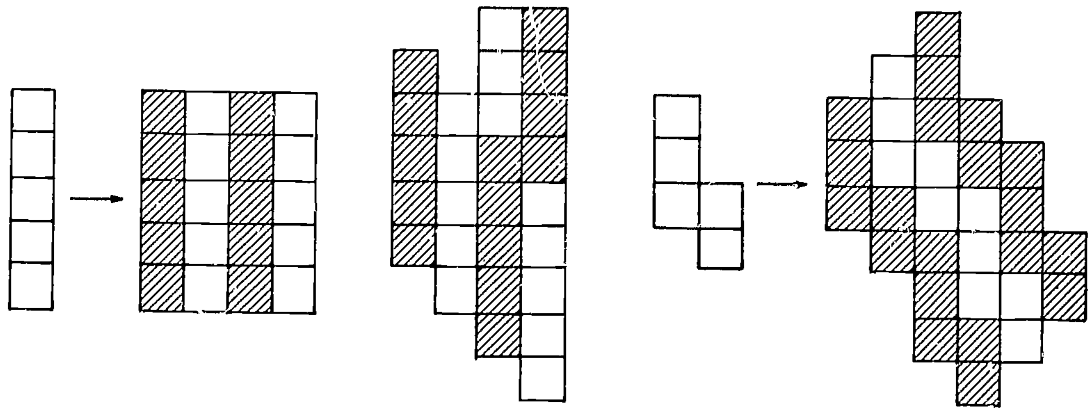
b. Can you do it in more than one way?

c. Color your pattern in an interesting way.

d. Can you cover a rectangular floor with equilateral triangles so there are no gaps?

e. Can you cover a triangular floor with equilateral triangles so there are no gaps?

3. With which of the five-square patterns can you cover a "floor" that goes on and on? A few of the patterns are shown here. Will any of the other patterns cover without gaps a floor that goes on and on?

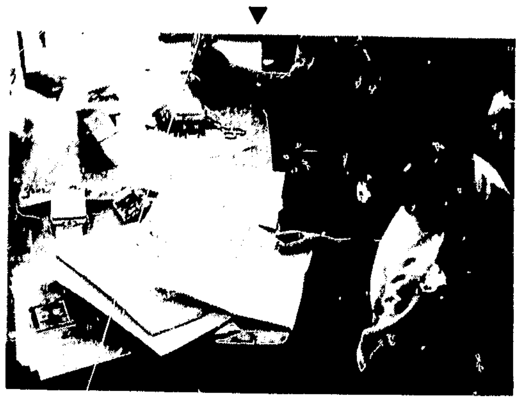


4. Make some other shapes such as parallelograms and see whether you can cover a plane with them.

Related Materials

1. Soma cubes [I. 7]
2. Pattern blocks [IV. 4]
3. Mirror cards [IV. 4]
4. Geo sticks [IV. 1]
5. Geoboards [IV. 2]
6. Films [V]

Using a letter code to see if 4 different patterns can be cut from one 4 x 5 sheet.



THE MATHEMATICS INVOLVED IN THIS UNIT

Someone watching this unit being taught may not immediately recognize that it deals with mathematics—at least not with the usual mathematics taught. The fact that many children do not associate this work with arithmetic, which some of them already dislike, may be an advantage. There are, however, significant mathematical ideas involved in the unit, and the children become acquainted with many of them.

Congruence

Figure 55 shows examples of pairs of congruent figures. In each

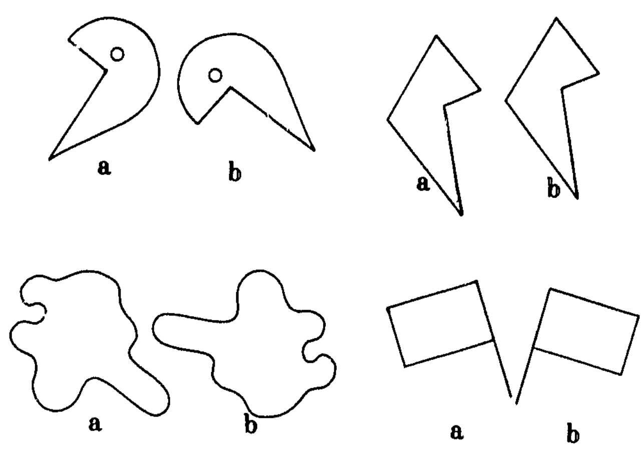


Figure 55

case figures (a) and (b) could be made to coincide by moving one or both of them, and therefore they are said to be *congruent.* Of course, more than two figures can be congruent.

Much of the work in plane geometry deals with congruent triangles. This unit gives students experience with a variety of congruent figures. Students actually make two congruent figures coincide by moving one to the position of the other; and they find out which motions are needed to do it.

By restricting figures to certain positions (such as vertical and horizontal sides for the squares), the motions that have to be made are the more easily recognized ones. To give an example, most children will see that a quarter (clockwise) turn is needed to turn the pattern in figure 56 from position (*a*) to position (*b*).

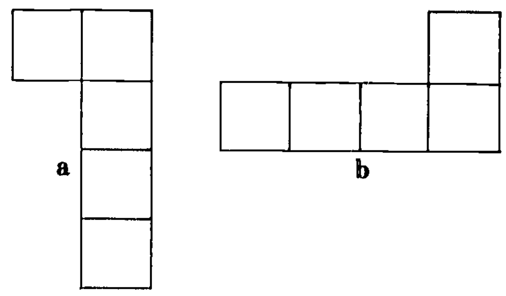


Figure 56

It is not nearly as easy to name the motion that will move the pattern in figure 57 from position (*a*) to position (*b*).

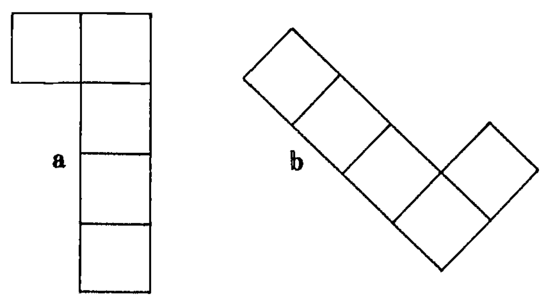


Figure 57

While playing the game of TAP, the students deal also with congruence of three-dimensional figures (see page 53).

Symmetry

This work also deals with the concept of symmetry. Symmetry, which can be noticed all around us, is an exciting topic. Figures 58–61 exhibit symmetry. Figures 62–64 do not.

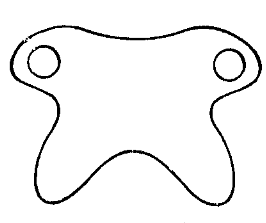


Figure 58

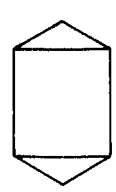


Figure 59

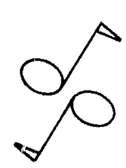


Figure 60

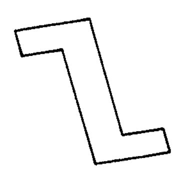


Figure 61

Figure 62

Figure 63

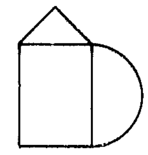
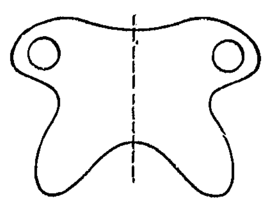
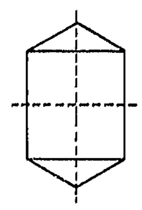


Figure 64

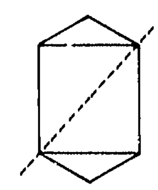
Figure 58 is said to be *symmetric with respect to a line.* Figure 65 has that line drawn in.



One line of symmetry
Figure 65



Two lines of symmetry
Figure 66



Not a line of symmetry
Figure 67

If you "flip" or turn the whole of figure 65 (out of the plane) about the line, then the figure looks as if it had not been moved. In fact, unless you had marked it in some way (as we did the pieces used in the unit) you could not tell by looking at it that it had been moved. The line is called a line of symmetry. If you fold the piece of paper along this line, the two parts of the figure fall on top of each other. If you place a mirror along this line (facing either way), you will see in the mirror a pattern that seems to be exactly the pattern on the paper behind the mirror.

The pattern in figure 59 has two lines of symmetry. Look at figure 66 where the two lines of symmetry have been drawn in. As for figure 66, if you "flipped" or turned the whole figure about either line of symmetry, the figure would look as if it had not been moved. If you fold the piece of paper along either line of symmetry of the figure, the two parts of the figure fall on top of each other. Place the mirror on a line of symmetry. Do you see in the mirror a pattern that exactly matches the pattern on the paper behind the mirror? Look at figure 67 and check to see that the line drawn in there does not have this property.

The patterns in figures 60 and 61 also exhibit symmetry. These patterns are *symmetric with respect to a point.*

In figures 68 and 69 the point is drawn in. If we rotate the original

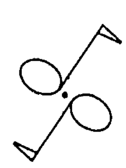


Figure 68

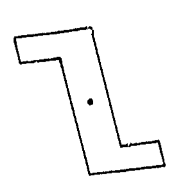


Figure 69

pattern by a half turn about the point shown, then the pattern looks as if it had not been moved; it ends up in the same location. Note that the pattern is kept in the same plane in these examples; for a flip it had to be taken out of the plane. Cut out patterns that match those in figure 70 and then, on a new sheet, trace around each, obtaining its outline. Place

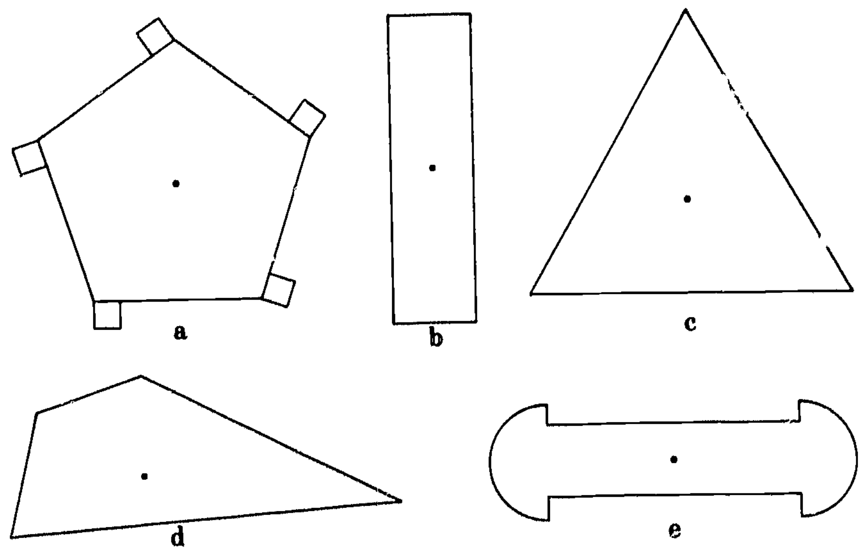


Figure 70

a thumbtack at the point indicated on each cutout pattern after you have placed it to fit its outline. Give each pattern a half turn. We will agree to use clockwise turns throughout. Now which pattern still fits in its outline? Those that do, (*b*) and (*e*), are symmetric with respect to a point—the point where the thumbtack was. Check to see that the thumbtack could not have been placed at any other point.

Look again at those patterns of figure 70 that did not fit in the same location after a half turn. These are redrawn in figure 71. Notice that

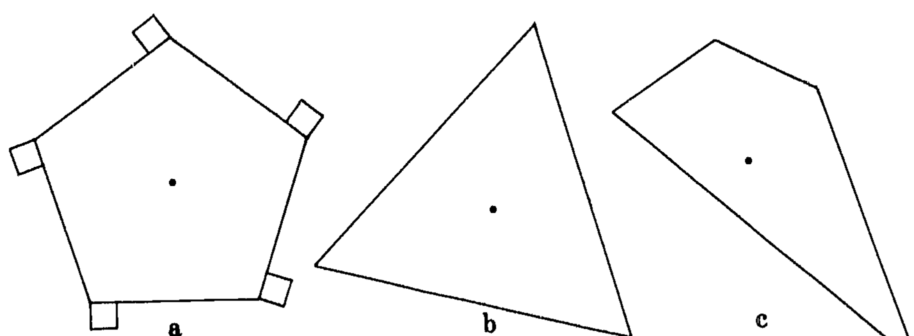


Figure 71

pattern (*a*) of figure 71 fits back into the same location after a ⅕, ⅖, 3/5, 4/5, 5/5, (a "do nothing") turn. Check this by putting the thumbtack at the point indicated and giving the pattern a 1/5, 2/5, etc., turn. This pattern has *five-fold rotational symmetry.* Pattern (b) fits back into the same location after a 1/3, 2/3, 3/3 (a "do nothing") turn. This has *three-fold rotational symmetry.* Pattern (c) has no symmetry properties. One can also say that the patterns that are symmetric with respect to a point—those of figures 68, 69, 70(b) and 70(e), for example, have two-fold rotational symmetry.

Look again at figure 71(a). Are there any other motions besides a 1/5, 2/5, 3/5, 4/5, 5/5 (or "do nothing") turn that leave the pattern in the same location? There are only those five. Look now at Figure 71*b*. We found that a 1/3, 2/3, and 3/3 (a "do nothing") turn placed the pattern back in the same location. Are there other motions that place it back in the same outline? Yes! There are three "flips" because there are three lines of symmetry. These are shown in Figure 72. If our eyes are closed while

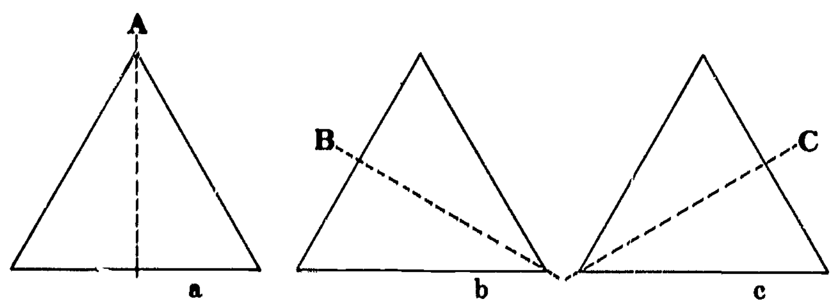


Figure 72

one of the motions is made which places the triangle back into the same location, we will not be able to detect it. In the game "what motion did I make?" the pattern had to be given distinguishing marks so that one could tell which of the permissible motions might have been made. Let us place the triangle with one side horizontal and mark the front and back of one corner F and B respectively (see fig. 73). Of course the pattern can be marked in many other ways.

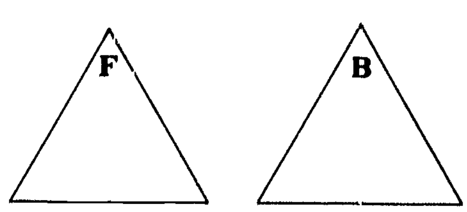


Figure 73

Let us call the flips about the axes as shown in figure 74 a vertical flip (V), a left-slant flip (L), and a right-slant flip (R). Again the children may suggest many other names. Practice these motions with a paper cutout.

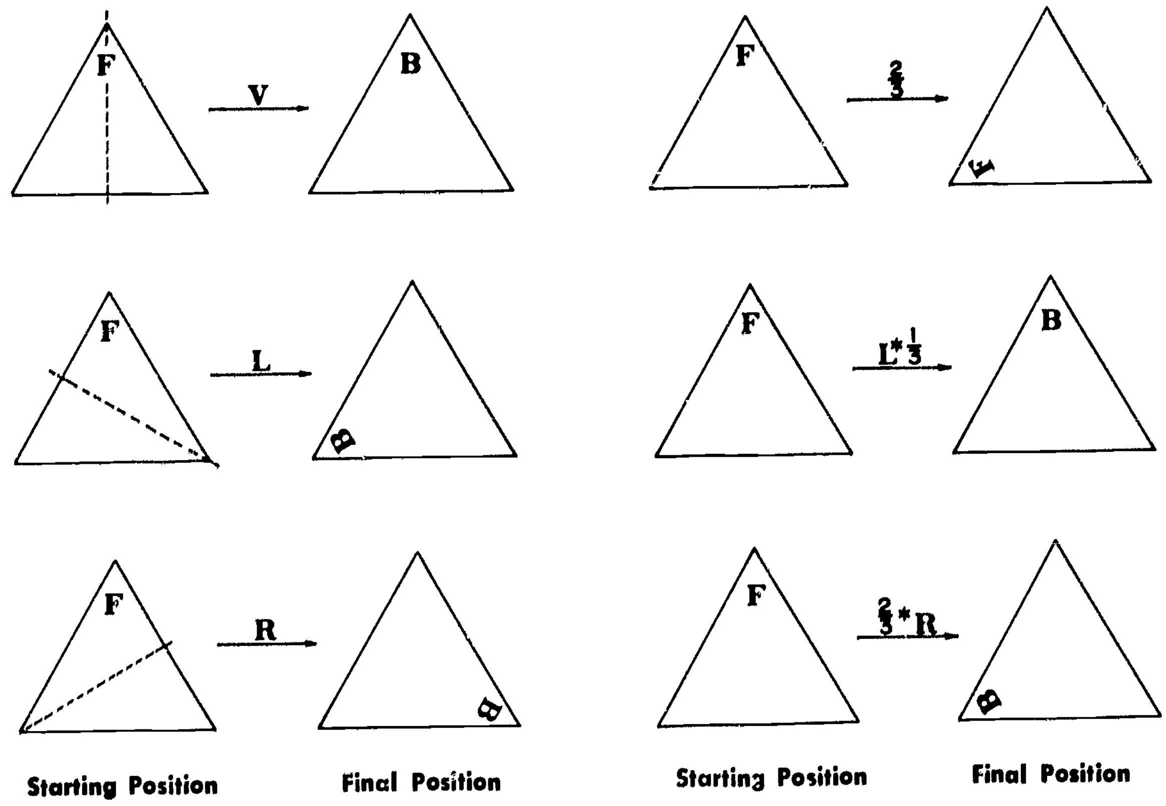


Figure 74

You can then complete the following table.

*	N	⅓	⅔	L	R	V
N			⅔			
⅓						
⅔					L	
L		V				
R						
V						

You can also check that the four group properties hold. Note that the inverse of the motion ⅓ is the motion ⅔ because $⅓*⅔ = ⅔*⅓ = N$.

There are other patterns that have six motions that leave them in the same location. For example, see figure 75. Each pattern in figure 76

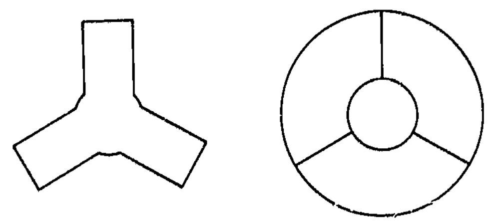


Figure 75

has six motions that leave it in the same position, but the motions are all "turns"; no flips leave these patterns in the same location. Permissible motions here are $\frac{1}{6}$, $\frac{2}{6}$, $\frac{3}{6}$, $\frac{4}{6}$, $\frac{5}{6}$, and $\frac{6}{6}$ ("do nothing") turns.

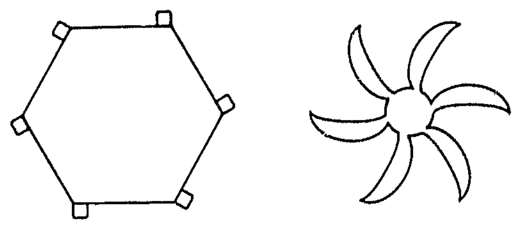


Figure 76

The pattern in figure 70*a* had five such permissible motions. Draw another pattern that has five such motions putting it back into the same location. The different motions that put a pattern back into the same location provide a way of describing its symmetry.

Geometric Transformations

There are many ways of transforming (that is, of changing or moving) a pattern. Figure 77 shows pictures of (*a*) how a pattern looked

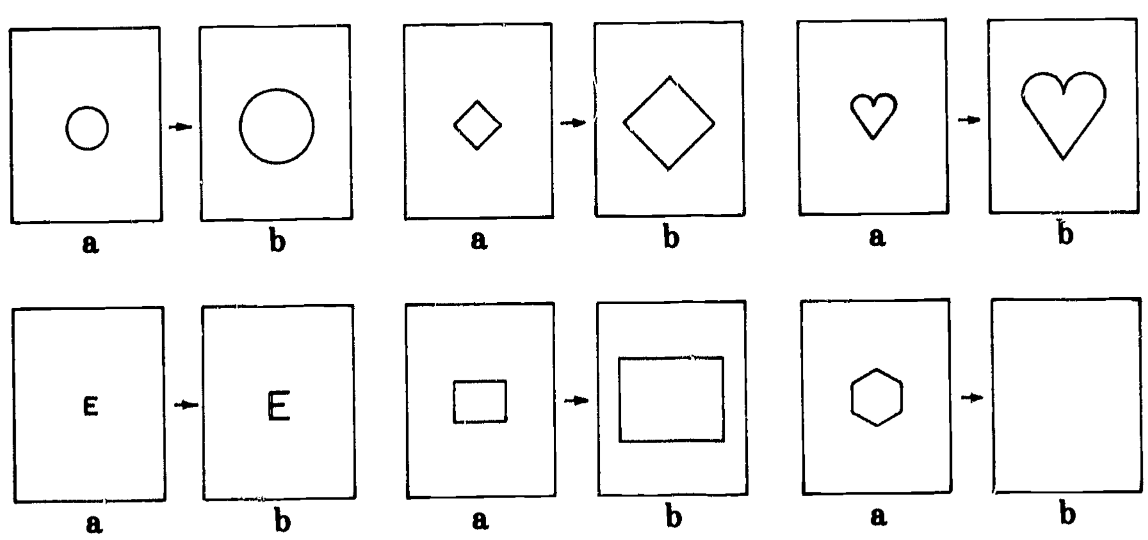


Figure 77

before being transformed, and (b) how it appeared, on the *same* page, after a transformation.

Can you complete the empty page, assuming that for each of the patterns in figure 77 the "rule of change" or transformation is the same? What is the "rule of change" or transformation for these examples? What has been done to each pattern? Each pattern has been *enlarged*; each drawing (b) is an enlargement of the drawing (a). In this case, each dimension was doubled. For example, the radius of the big circle is twice the radius of the small circle. The height of the large heart is twice the height of the small heart. Notice that the *shape* of each pattern before and after enlargement is the same. For example, the circle remains a circle, the square remains a square, and the rectangle is transformed into a rectangle with the same proportions (the length of each rectangle is twice its width). The *size* of the pattern does *not* remain the same.

An easy way to obtain enlargements is to make a shape out of wire or cardboard and to hold it under a lamp parallel to a table or screen as shown in figure 78. Investigate what happens if the lamp, or screen, or

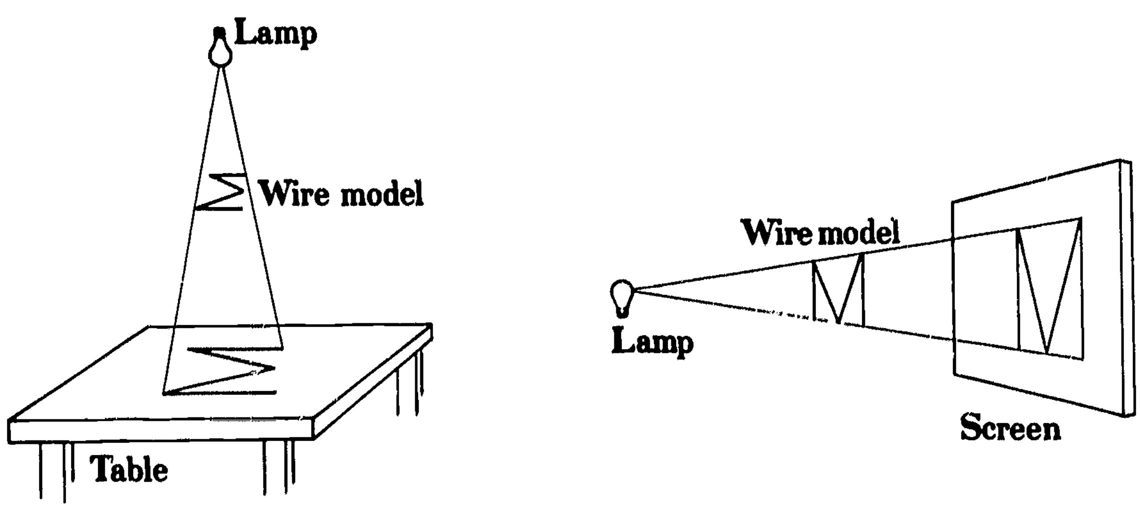


Figure 78

pattern is moved. What happens to the shape of the shadow if the pattern is not held parallel to the table or screen?

Another way to obtain enlargements is to use different graph paper or grids or a geoboard to make a pattern. (A geoboard is a board with a square array of nails on it. Rubber bands are used to make various shapes.) For example, in figure 79 a pattern on a geoboard is shown in (a). If the pattern is copied on a geoboard with nails spaced twice as far apart, the pattern will be enlarged, as shown in (b). [IV. 2]

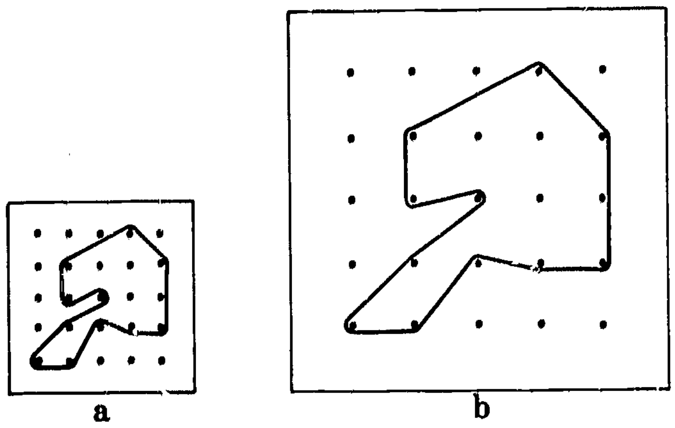


Figure 79

Figure 80 shows how to obtain enlargements by using graph paper.

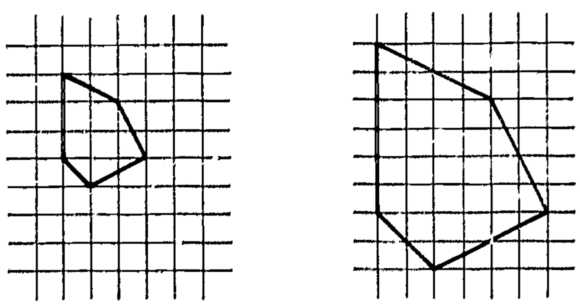


Figure 80

A different way to change a pattern is shown in the pairs of patterns in figure 81. Again, imagine that the rectangle in each pair represents the same page before and after the change or transformation was made.

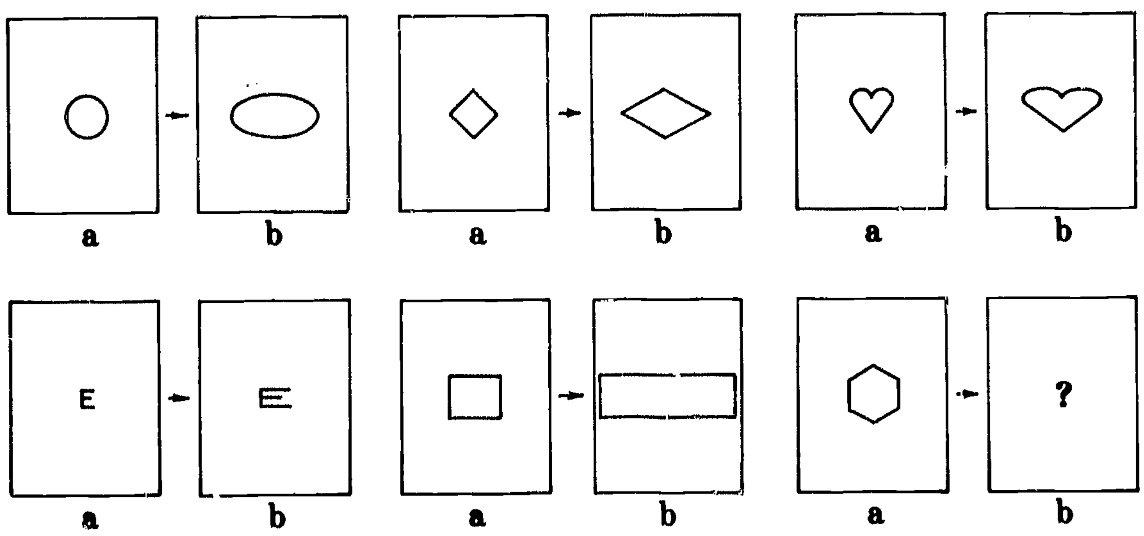
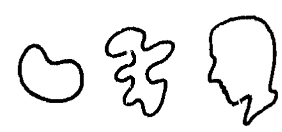


Figure 81

What has been done to each pattern (a) in figure 81? Can you complete the empty page? What is the rule of change or transformation? Each pattern has been *stretched* along the horizontal direction.

Notice that the shape of each pattern has been changed. The circle has become what looks like, and is in fact, an ellipse. The "E" can still be recognized as an "E," but its proportions have been changed. The rectangle, although still a rectangle, has now a base that is four times as long as its height. Although neither size nor shape remain the same if the pattern is stretched in this way, there are some properties that do remain the same (or invariant). For example, straight lines remain straight, and closed figures remain closed. A simple closed figure is one that can be traced without lifting the pencil from the paper and in which the figure is completed when the pencil returns to the starting point without crossing its own path.

Examples of closed fiigures are 

while ones that are not closed are 

One way of drawing stretched figures is to draw the original pattern on graph paper and then to "copy" it on graph paper that has a different horizontal scale. Figure 82 shows how this can be done.

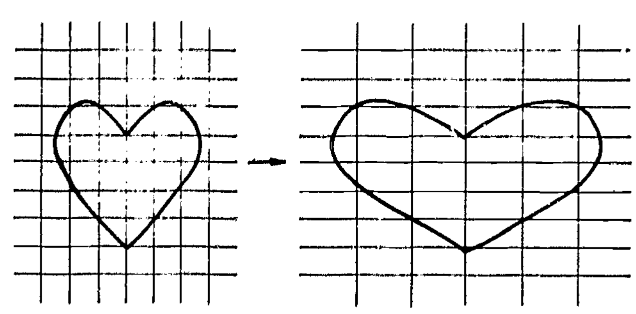
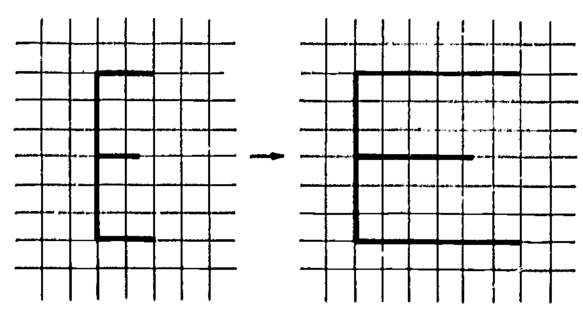


Figure 82

Figure 83 shows how an enlargement can be made if the horizontal *and* the vertical scale are changed in the same manner.

Figure 83



Another set of transformations is shown in figure 84. Again, the rectangular outlines in each pair denote the edge of the same page before and after the change.

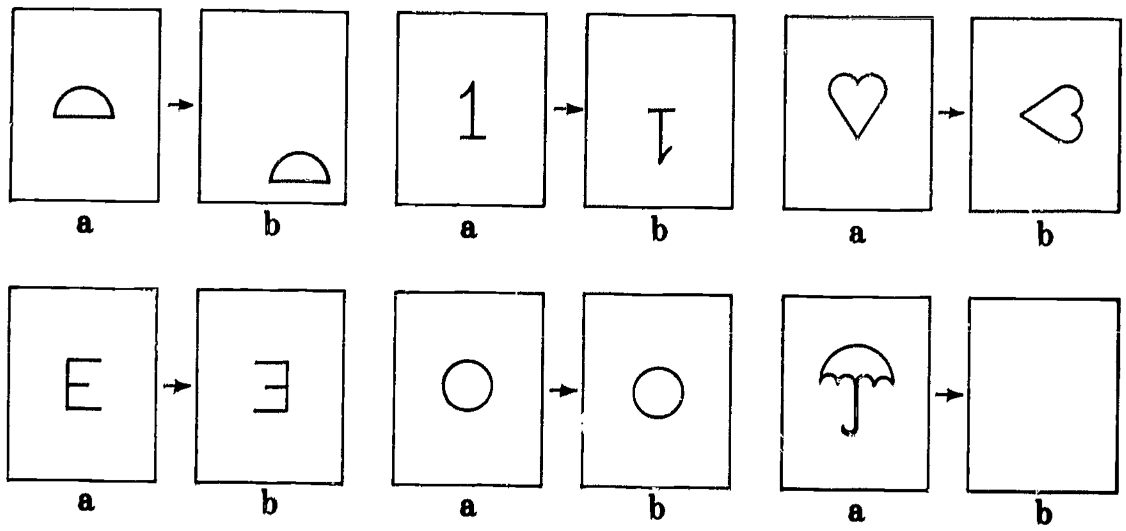


Figure 84

What has happened to each pattern (a) of figure 84? Can you complete the last drawing? In how many ways? Each pattern has been moved. The case where the pattern is moved back to the same location it began in is also considered a motion. What remains unchanged? The size and shape of each pattern after it has been moved is the same as it was before.

These types of transformations are called *rigid motions*—the pattern does not become deformed as it did in enlarging or stretching. If one pattern can be transformed onto the other by a rigid motion, they are called congruent (see page 60).

How can one carry out such a transformation? Which patterns in figure 84 might have been moved by rotating the original? By flipping or reflecting the original? By sliding (but not rotating or flipping) the original pattern? Such a sliding motion is usually called a translation. It is an interesting fact that every rigid motion can be achieved by combining a translation, a rotation (perhaps N) and, if need be, a reflection. Just as any color could be made from red, blue, and yellow, so any given pattern in one location on the page can be moved to any other given pattern by one of the motions mentioned above. There are many other types of transformations. Figure 85 shows more drastic ones.

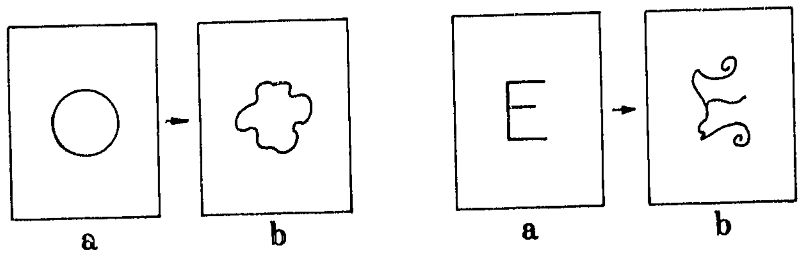


Figure 85

What do all these transformations have to do with geometry, or with this unit? Plane geometry, which students study in school, is only one type of geometry. There are others. In plane geometry, one studies those properties of a shape which remain unchanged when the shape is moved by rigid motions. Thus, in plane geometry one is concerned about those properties that deal with the shape and size of patterns. For example, there are theorems that deal with triangles and circles, such as "The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides," and "The angle inscribed in a semicircle is a right angle."

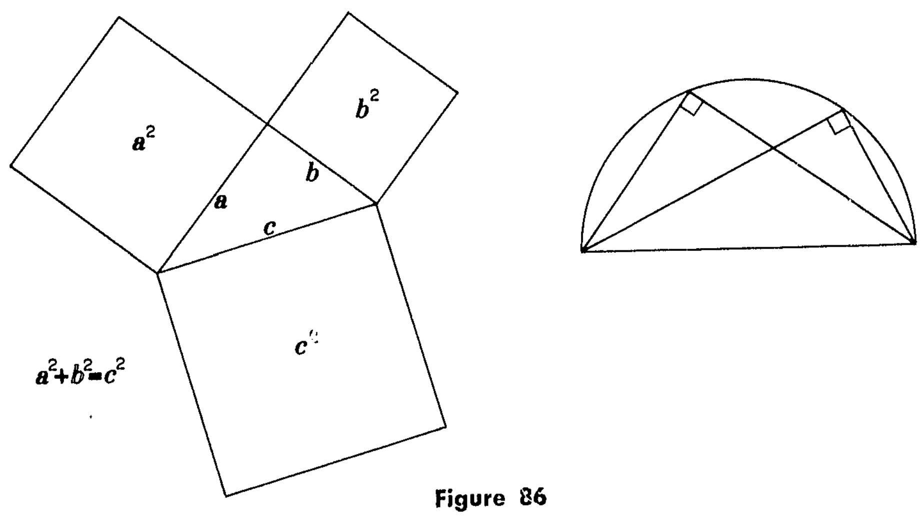


Figure 86

One is not concerned with *where* on the page a pattern is drawn or "which way round" the pattern is because that does *not* remain unchanged under a rigid motion.

In projective geometry one studies the properties of a figure that remain unchanged by "making shadows"; the pattern does *not* have to be held parallel to the screen. This shadowing is called projection. Thus in projective geometry one does not consider "circles" or "right angles," because these do not remain unchanged under projection. One studies those properties of figures that do remain unchanged under projection; for example, straight lines remain straight lines.

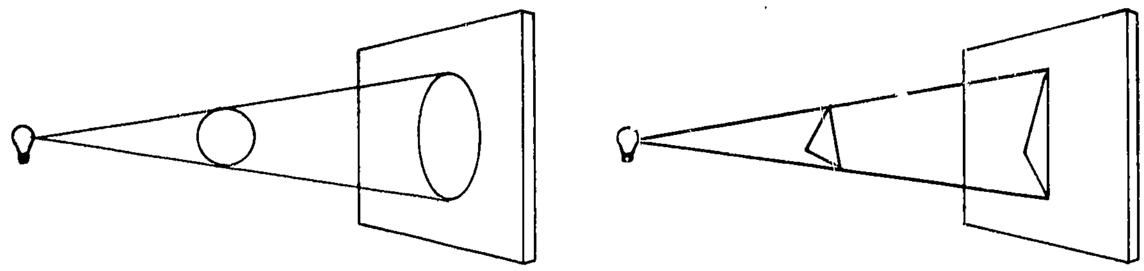


Figure 87

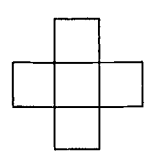
This unit gives students experience with one type of geometric transformation—the rigid motion—and with rotations and reflections in particular. Some of the exercises deal with other transformations.

Groups

In this unit we have seen examples of an abstract concept called a group. In order to examine this concept, consider the following sets:

1. *The sets of the integers.* This can be indicated by writing (. . . , −4, −3, −2, −1, 0, 1, 2, 3, 4, . . .). One cannot write down all of them because there are infinitely many. The three dots indicate "and so on."

2. *The rational numbers.* (Any number that can be written as the quotient of two integers.) Examples of rational numbers are $\frac{41}{37}$, $\frac{9}{4}$, $\frac{-11}{2}$, $\frac{-3}{1}$, $\frac{4}{5}$. There are infinitely many of them.

3. *The motions that leave*  *in the same location.* Among them are the half turn, $\frac{1}{2}$, and vertical flip, V. There are eight of them, and so we can write all of them down: $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, N, V, R, H, L.

4. *The set of integers from 0 to 9.* This is the set (0, 1, 2, 3, 4, 5, 6, 7, 8, 9).

Together with each of the sets listed above, we will consider one operation. For set (1) we will consider the operation addition. You know how to add two integers. For example, $14 + 191 = 205$. Here "+" is shorthand for "add." For set (2) consider the operation multiplication. "×" is shorthand for "multiply." For set (3) consider the operation "followed by." We found that $H{*}V$ gives the same ***final result*** as $\frac{1}{2}$, and we wrote $H^*V = \frac{1}{2}$. The symbol "*" is shorthand for "followed by." For set (4) we will consider the operation addition.

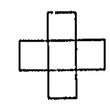
Do these examples—that is, the sets of elements together with the specified operation—have anything in common? There are several properties that they have in common.

Property I

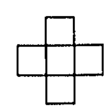
Is it true that if any two of the elements are combined (using the elements and operation under consideration) the final result is the same as one of the given elements?

Example 1: For instance, $14 + (-5) = 9$, and 9 is one of the elements in the set, since 9 is an integer. Clearly this type of statement is true for *any* two integers we choose, if we add them.

Example 2: For instance, $\frac{4}{3} \times \frac{11}{7} = \frac{44}{21}$, and $\frac{44}{21}$ is one of the elements in the set, since $\frac{44}{21}$ is a rational number. Again, this type of statement can be made for *any* two rational numbers if we multiply them.

Example 3: For instance, $H^*V = \frac{1}{2}$, and $\frac{1}{2}$ is one of the elements in this set, since $\frac{1}{2}$ leaves  in the same location. A similar statement can be made for *any* two such motions if we "follow one by the other." If N, the "do nothing" motion were not included, then this statement would not be true. For example, $\frac{1}{4}*\frac{3}{4}$ could not be replaced by a single motion.

Example 4: If we consider the set of integers from 0 to 9, namely (0, 1, 2, 3, 4, 5, 6, 7, 8, 9), then the sum of any two integers is not *always* one of the integers in this set. For example, although $3 + 5 = 8$, and 8 is in the set, $6 + 7 = 13$, and 13 is not one of the integers in the set $(0, 1, 2, \ldots, 9)$.

Any set of elements, together with an operation, is said to be *closed* with respect to (or "under") that operation if the final result of combining any two of the elements is the same as one of the elements of the set. If it is not so, even for one pair of elements, then the set is not closed under the operation. So we can say that the integers are closed under the operation addition, the rational numbers are closed under the operation multiplication, and the motions that leave  in the same location are closed under the operation "followed by." The set of integers from 0 to 9 are *not* closed under addition since, for example, $2 + 9 = 11$, and 11 is not in the set of integers 0 to 9. Because of the failure of closure we will now no longer consider example (4) for discussion.

Are the integers closed under the operation multiplication? Division? Are the rational numbers closed under the operation addition? Are the positive multiples of 5, (5, 10, 15, 20, 25 . . .), closed under addition? Under multiplication?

Property II

Example 1:

$$(14 + 2) + 5 = 16 + 5 = 21.$$
$$14 + (2 + 5) = 14 + 7 = 21.$$
$$\text{Thus} \quad (14 + 2) + 5 = 14 + (2 + 5).$$

A similar statement can be made for *any* three integers we choose, whether positive or negative or zero. We say that addition of integers $(\ldots, -4, -3, -2, -1, 0, 1, 2, 3, \ldots)$ is associative.

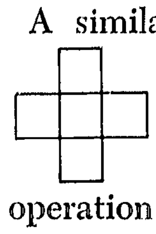
Example 2:

$$\left(\frac{4}{3} \times \frac{2}{5}\right) \times \frac{1}{7} = \frac{8}{15} \times \frac{1}{7} = \frac{8}{105}.$$
$$\frac{4}{3} \times \left(\frac{2}{5} \times \frac{1}{7}\right) = \frac{4}{3} \times \frac{2}{35} = \frac{8}{105}.$$

Thus $\left(\frac{4}{3} \times \frac{2}{5}\right) \times \frac{1}{7}$ gives the same final result as $\frac{4}{3} \times \left(\frac{2}{5} \times \frac{1}{7}\right)$. The associative property holds for the multiplication of rational numbers.

Example 3:

$$(H^* V)^* \frac{1}{2} = \frac{1}{2}{}^* \frac{1}{2} = N.$$
$$H^*\left(V^* \frac{1}{2}\right) = H^* H = N.$$

Thus $(H^*V)^*\frac{1}{2}$ gives the same final result as $H^*\left(V^*\frac{1}{2}\right)$, and we write $(H^*V)^*\frac{1}{2} = H^*\left(V^*\frac{1}{2}\right)$. A similar statement can be made for any three motions that leave  in the same position. The associative property holds for the operation followed by for these motions.

Property III

Example 1: Consider the following problems:

$$14 + ? = 14. \qquad ? + 14 = 14.$$
$$-13 + ? = -13. \qquad ? + -13 = 13.$$

We know that the integer 0 is the answer to all these problems. For any integer that we choose from (. . . , −4, −3, −2, −1, 0, 1, 2, 3, . . .), it is true that if zero is added to it, the result is the integer we chose. The same can be said if that integer is added to zero, instead of zero being added to that integer. Zero is called the identity or neutral element for addition.

Example 2: Consider the problems below:

$$\frac{4}{3} \times ? = \frac{4}{3}. \qquad ? \times \frac{4}{3} = \frac{4}{3}.$$
$$\frac{7}{113} \times ? = \frac{7}{113}. \qquad ? \times \frac{7}{113} = \frac{7}{113}.$$

The solution to all these problems is the rational number $\frac{1}{1}$ (which of course is equal to 1). That is, 1 is the element such that if any rational number is multiplied by it (or it is multiplied by any rational number), the result is that rational number. We say that 1 is the identity element for the set of rational numbers under multiplication.

Example 3: Consider the following problems:

$$II * ? = II. \qquad ? * II = II.$$
$$\frac{1}{4} * ? = \frac{1}{4}. \qquad ? \times \frac{1}{4} = \frac{1}{4}.$$

The "do-nothing motion" N has a special property; if it follows or is followed by any one of the motions we choose, the final result is the same as the motion we chose. N is called the identity element in this case. Thus each of our sets has an identity element for the operation considered. Does the set (5, 10, 15, 20, 25, . . .) have an identity element under addition? Under multiplication?

Property IV

Consider the following problems:

Example 1:

$$4 + ? = 0. \qquad ? + 4 = 0.$$
$$-7 + ? = 0. \qquad ? + -7 = 0.$$
$$1836 + ? = 0. \qquad ? + 1836 = 0.$$

Notice that 0 is the identity element for example 1 (the integers under addition). If we choose *any* integer of the set, can we find another integer to add to it (or to which it can be added) so that the final result is 0? Here are three examples that illustrate that this can always be done.

$$4 + (-4) = 0. \qquad (-4) + 4 = 0.$$
$$-7 + 7 = 0. \qquad 7 + (-7) = 0.$$
$$1836 + (-1836) = 0. \qquad (-1836) + 1836 = 0.$$

We call -4 the inverse of 4, and 7 the inverse of -7. Every integer has an inverse under the operation addition.

Example 2:

$$\frac{3}{4} \times ? = 1. \qquad ? \times \frac{3}{4} = 1.$$
$$\frac{11}{19} \times ? = 1. \qquad ? \times \frac{11}{19} = 1.$$

Note that 1 is the identity of the set of rational numbers together with the operation multiplication.

$$\frac{3}{4} \times \frac{4}{3} = 1. \qquad \frac{4}{3} \times \frac{3}{4} = 1.$$
$$\frac{11}{19} \times \frac{19}{11} = 1. \qquad \frac{19}{11} \times \frac{11}{19} = 1.$$

We call $\frac{4}{3}$ the inverse of $\frac{3}{4}$, and $\frac{3}{4}$ the inverse of $\frac{4}{3}$. Is it true that for every rational number we can find another rational number to multiply it by (or which it can multiply) so that the final result or product is 1? Consider $0 \times ? = 1$. There is no rational number by which 0 can be multiplied to get a result of 1. So we will *not* be able to say that *every* element of the set of rational numbers has an inverse. Zero is the only culprit; it has no inverse. However, if we exclude 0 and consider only the positive and negative rational numbers, then every element has an inverse with respect to the operation multiplication. Excluding the 0 from the set of rational numbers does not destroy any of the properties previously discussed; for example, the rationals without 0 are closed under multiplication.

Example 3: Here the identity element is N. Can you find the solution to the following problems?

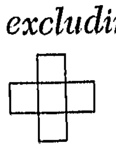
$$H * ? = N. \qquad ? * H = N.$$
$$\frac{1}{4} * ? = N. \qquad ? * \frac{1}{4} = N.$$

For any motion in our set we can find another to follow it by (or to precede it by) so that the final result is the same as doing an N. Notice that

$$H^{*}H = N.$$
$$\frac{1}{4}{}^{*}\frac{3}{4} = N.$$
$$\frac{3}{4}{}^{*}\frac{1}{4} = N.$$

Thus, the motion $\frac{3}{4}$ is called the inverse of the motion $\frac{1}{4}$. Each element of this set of motions (together with the operation "followed by") has an inverse. Thus the inverse of $\frac{1}{4}$ is $\frac{3}{4}$ because $\frac{1}{4}{}^{*}\frac{3}{4} = N$, and $\frac{3}{4}{}^{*}\frac{1}{4} = N$.

We can amend our original list of sets as follows:

	Elements	*Operation*
Example 1:	The integers	addition
Example 2:	The rational numbers *excluding 0*	multiplication
Example 3:	The motions leaving  in the same location	"followed by"

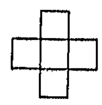
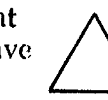
We see that each set together with the specified operation has the closure property, the associative property, an identity element, and an inverse element for each element in the set. Any set of elements together with an operation that satisfies these four properties is called a group.

The integers under addition, the rationals under addition, and the rationals *without zero* under multiplication form groups. The real numbers form a group under addition, and if zero is omitted from the set they form a group under multiplication. Many sets of motions in geometry form groups. This unit gives the children an easy introduction to groups on an informal level. They do not need to learn any definitions yet, but they can gain some firsthand experience with them.

Notice that in example 1, $4 + 3 = 3 + 4$, and similar statements can be made for *any* two elements in that set. The operation here (addition) is *commutative*. Consider the set of elements together with the specified operation in example 2. For example,

$$\frac{4}{11} \times \frac{3}{5} = \frac{12}{55} \text{ and } \frac{3}{5} \times \frac{4}{11} = \frac{12}{55}.$$

Similar statements can be made for *any* pair of rational numbers permitted in this set. The operation (multiplication) is commutative. Example 4 is not a commutative group. Although $H^{*}V = \frac{1}{2}$ and $V^{*}H = \frac{1}{2}$, and

Set	*Sample elements of set*	*Operation*	*Closure Property*	*Associative Property*	*Identity Element*	*Inverse Element*	*Group*
1. Integers	$-4, 193$	Addition	✓	✓	✓	✓	Yes
2. Rational numbers	$\frac{4}{3}, 0, \frac{-3}{11}$	Multiplication	✓	✓	✓	✗	No
3. Rational numbers without 0	$\frac{4}{3}, \frac{-3}{11}$	Multiplication	✓	✓	✓	✓	Yes
4. Motions that leave  in same location	$H, \frac{1}{2}$	Followed by	✓	✓	✓	✓	Yes
5. Positive multiples of 5	5, 10, 15, 20, 630, 1755	Addition	✓	✓	✗	Inverse cannot exist without identity	No
6. 1 and multiples of 5	1, 5, 10, 15, 630, 820	Multiplication	✓	✓	✓	✗	No
7. Integers	$0, -4,$ 193, 143	Multiplication	✓	✓	✓	✗	No
8. Rational numbers	$\frac{4}{3}, 0, \frac{-3}{11}$	Addition	✓	✓	✓	✓	Yes
9. Motions that leave  in same location	$R, \frac{1}{3}$	Followed by	✓	✓	✓	✓	Yes

hence $H*V = V*H$, not *all* pairs of elements commute. For example, $\frac{1}{4}*V$ does not give the same final result as $V*\frac{1}{4}$. That is, $V*\frac{1}{4} \neq \frac{1}{4}*V$. For a group to be a commutative group it must be true that for *every* pair of elements of the set considered, the result is the same if their order is interchanged; if even one pair of elements does not commute, the group is not commutative. There are sets of elements that together with an operation do not form groups but in which the operation has the important commutative property. There are many other examples of groups and other concepts.

CONCLUSION

Students in grades 1, 3, 4, 5, 6, 8, and 10 who have worked with all or a part of this material seemed able and eager to work with two- and three-dimensional patterns. Judging by the classes that I observed and by the reports from teachers and observers, almost all the students actively participated. This was not a unit in which they were fed facts that they had to learn and give back on examinations. Rather, almost all new information was obtained by the students themselves, with genuine excitement over discoveries and accomplishments. The teachers reported that the students were unusually engrossed in the work. Some of the poorer students participated more and performed better than in regular mathematics classes.

The students were able, most of the time, to settle arguments, decide particular questions, and test hypotheses on their own. In several classes, students spontaneously came to the board in small groups to settle arguments by use of paper models and chalk; they asked for no help from the teacher. Rarely did I hear a student ask a teacher, "Am I right?" The students simply checked and found out by themselves.

I feel that some of the students gained confidence in their own ability to think, predict, and decide. Other benefits might also have been derived from this unit. When one student suddenly realized, after having used models to decide whether various patterns would fold into a box, that "you can do it in your head," he may have gained valuable insight about the power of abstract thinking. The student who, after cutting up a box to obtain a particular flat pattern said, "You really have to think very carefully first—don't rush into it," had had a lesson far more valuable than one in which he might merely have been told to "think first."

Physical objects were used for the following problems:

1. Find all five-square patterns. (Individual squares were used.)
2. Which patterns fold into boxes?
3. Do two given patterns match?
4. How many positions does a given pattern have?
5. What motions can be made so that a pattern still fits back in the original position?

6. What motion was made with a marked pattern?
7. How does one cut a box to obtain a particular pattern?
8. How many ways are there of arranging a given number of sticks, according to certain rules, in two and three dimensions?
9. How many ways are there of arranging four and five cubes?

Of course, the extent to which they were used varied not only from grade level to grade level, from student to student, but also from problem to problem. Similar comments can be made about work stemming from the problems and exercises given in the section entitled "Related Problems." Clearly, first graders, who have done the part of the unit dealing with finding patterns, noticing duplicates, matching patterns, and making a variety of three-dimensional objects, spent much more time with the concrete material on this part of the work then did the sixth graders.

I would like to stress again that not only does the section entitled "Related Problems and Extension of Work" deal with other work in informal geometry which uses concrete material, but the bibliography will, I hope, be used to continue this type of approach.

I hope that this informal geometry unit fulfills some of the suggestions made in the *Goals for School Mathematics* [I. 2]. For grades 3 to 6, the report recommended that

> in the later grades of elementary school, relatively little pure geometry would be introduced, but more experience with the topics from K–2 would be built up.

And what was stated for grades K–2 under geometry was, in part, the following:

> Some of the aims of this study are to develop planar and spatial intuition of the pupil, to afford a source of visualization for arithmetic and algebra, and to serve as a model for that branch of natural science which investigates physical space by mathematical methods. The geometric portion of the curriculum seems to be the most difficult to design. Therefore the geometry discussed here for grades K, 1, and 2 represents a far more tentative grouping than was the case for the work in real numbers described earlier.
>
> The earliest grades should include topics and experiences like these:
>
> 1) Identifying and naming various geometric configurations.
>
> 2) Visualization, such as cutting out cardboard to construct 3-dimensional figures, where the child is shown the 3-dimensional figure and asked to find his own way to cut the 2-dimensional paper or cardboard. . . .
>
> 3) Symmetry and other transformations leaving geometrical figures invariant. The fact that a line or circle can be slid into itself. The symmetries of squares and rectangles, circles, ellipses, etc., and solid figures like spheres, cubes, tetrahedra, etc. This study could be facilitated with mirrors, paper folding, etc.
>
> 4) Possibly the explicit recognition of the group property in the preceding. . . .

BIBLIOGRAPHY

I. Books for Background Reading

1. Allendoerfer, C. B., and Oakley, C. O. *Principles of Mathematics.* New York: McGraw-Hill Book Co.
   Chapter 3 deals with groups.

2. Cambridge Conference on School Mathematics. *Goals for School Mathematics.* Boston: Published for Educational Services, Inc., by Houghton Mifflin Co., 1963.

3. Cundy, H. M., and Rollett, A. P. *Mathematical Models.* Oxford: Clarendon Press, 1961.

4. Denbow, and Goedicke, V. *Foundations of Mathematics.* New York: Harper & Bros., 1959.
   Section 2 of chapter 5 discusses a beautiful and simple to understand example of a group that deals with moving furniture.

5. Dienes, Z. P., and Golding, E. W. *Exploration of Space and Practical Measurement.* New York: Herder & Herder, 1966.

6. Gardner, K. L. *Discovering Modern Algebra.* New York: Oxford University Press, 1966.

7. Gardner, Martin. *Martin Gardner's New Mathematical Diversions from "Scientific American."* New York: Simon & Schuster, 1966.
   Among chapters you will want to look at are—
   Chapter V, "Paper Cutting." Includes a problem dealing with folding certain shapes of paper into cubes.
   Chapter XIII, "Polyominoes and Fault Free Rectangles." Poses problems dealing with five-square patterns (pentominoes) and four-square patterns tetrominoes).
   Chapter XVI, "The 24 Color Squares and the 30 Color Cubes." Discusses and shows pictures of 30 ways of coloring a cube. Also poses other problems.

8. ———. *The Scientific American Book of Mathematical Puzzles and Diversions.* New York: Simon & Schuster, 1959.
   Chapter 13, "Polyominoes," gives a definition and origin of the word "polyominoes." Also introduces "monomino," "domino," "tromino," "tetromino," "pentomino," "hexomino" (one, two, three, four, five, six squares). Shows the 12 pentominoes and the 35 hexominoes. Indicates that no formula has been found giving the number of different *n*-ominoes in terms of *n*.

9. ———. *The Second Scientific American Book of Mathematical Puzzles and Diversions.* New York: Simon & Schuster, 1961.

Among chapters that you will want to look at are—

Chapter 1, "The Five Platonic Solids." Includes a nice way of making a regular tetrahedron out of an envelope.

Chapter 6, "The Soma Cube." Discusses and shows pictures of all the "irregular" shapes that can be made out of three or four cubes. The cube that can be made from them is called a "soma" cube. There are pictures of many other shapes that can be made from these pieces.

10. Golomb, Solomon W. *Polyominoes.* New York: Charles Scribner's Sons, 1965.

Golomb "invented" polyominoes. This book deals with many aspects of polyominoes and introduces the reader to a host of new and fascinating problems.

11. Holloway, G. E. T. *An Introduction to the Child's Conception of Geometry.* London: Routledge and Kegan Paul, 1967.

12. ———. *An Introduction to the Child's Conception of Space.* London: Routledge and Kegan Paul, 1967.

These two books are published in conjunction with the National Froebel Foundation and deal with "further aspects of Piaget's work."

13. Isaacs, Nathan. *The Growth of Understanding in the Young Child.* London: Educational Supply Association, 1961.

A brief introduction to Piaget's work.

14. Kutozov, B. V. *Geometry.* Translated by Louis I. Gordon and Edward S. Shater. Studies in Mathematics, vol. 4. Chicago: Published for School Mathematics Study Group by University of Chicago Press, 1960.

15. National Council of Teachers of Mathematics. *Enrichment Mathematics for the Grades.* Twenty-seventh Yearbook. Washington, D.C.: The Council, 1963.

The chapters you will especially want to see are—

Chapter 4, "Modular Arithmetic," by Francis J. Mueller.

Chapter 9, "Geometry in the Grades," by Irvin H. Brune. Discusses why informal geometry should be taught in the elementary grades. Among other examples, there is a brief discussion on tessellations.

Chapter 10, "Topology," by Donovan A. Johnson. This article describes some of the mathematics involved when transformations such as those shown on page 70 of the unit are made.

Chapter 23, "Geometry and Transformations," by Daniel E. Sensiba. Background material on transformations.

16. ———. *More Topics in Mathematics for Elementary School Teachers.* Thirtieth Yearbook. Washington, D.C.: The Council, 1969.

See especially Booklet No. 18, "Symmetry, Congruence, and Similarity," and pertinent parts of Booklet No. 14, "Informal Geometry."

17. Prenowitz, Walter, and Swain, Henry. *Congruence and Motion Geometry.* Thinking with Mathematics Series. Boston: D. C. Heath & Co., 1966.

Mainly for high school teachers.

18. School Mathematics Project. *Book 1 and Teacher's Guide*, 1965; *Book 2 and Teacher's Guide*, 1965; *Book 3 and Teacher's Guide*, 1967; *Book T and Teacher's Guide*, 1964; *Book T–4 and Teacher's Guide*, 1965. Cambridge University Press.

These books are for children aged 11 and up. The books, together with the Teacher's Guides, could serve as valuable background reading for teachers. They concern themselves throughout with geometric topics and stress transformations and symmetry. Many of the chapters in the series deal with topics directly related to this unit. In particular see—

Book 1: Chapter 8, "Polygons and Polyhedra"; Chapter 10, "Area—Tessellations"; Chapter 13, "Symmetry."

Book 2: Chapter 3, "Similarity and Enlargement," Chapter 5, "Reflection and Rotation."

Book 3: Chapter 2, "Isometrics"; Chapter 12, "Shearing."

Book T: Chapter 4, "Simple Transformations"; Chapter 7, "Shearing"; Chapter 8, "Enlargement"; Chapter 15, "Geometry in Three Dimensions."

Book T–4: Chapter 2, "Transformations Combined."

19. Steinhaus, H. *Mathematical Snapshots.* 3d rev. American ed. New York: Oxford University Press, 1969.

You will want to dip into many of the chapters.

20. Stover, Donald W. *Mosaics.* Mathematics Enrichment Series. Boston: Houghton Mifflin Co., 1968.

Gives a discussion of all the different mosaic patterns in the plane. Requires some background of plane geometry. For junior high school students this work would be useful for introducing some plane geometry.

21. Weyl, Hermann. *Symmetry.* Princeton, N.J.: Princeton University Press, 1952.

Although much of the book requires a strong mathematics background, it is worth looking at this classic just for the pictures.

22. Yaglom, I. M. *Geometric Transformations.* Translated from the Russian by Allen Shields. New York: Random House, 1962.

Background reading on geometric transformations. Presupposes knowledge of high school plane geometry. This is a monograph project of the School Mathematics Study Group.

II. Books Giving Further Ideas for Work in the Classroom

1. Association of Teachers of Mathematics. *Notes on Mathematics in Primary Schools.* New York: Cambridge University Press.

This book is a "must" from cover to cover. For this reason I hesitate to single out any chapter. However, in connection with this unit you will want to look particularly at Chapter 2, "Dot Patterns and Patterns on a Pegboard," Chapter 6, "Three Dimensions and Representing Shapes," Chapter 7, "Tessellations, Symmetry, Holefitting, Similarity, Enlargement," and Chapter 11, "Open Situations." There is a most useful bibliography on pp. 332–38. The paperback costs $4.95 and is available from the Cambridge University Press, 32 East 57th Street, New York, N.Y. 10022.

2. ———. *Paper Folding.* Mathematics Teaching Pamphlet No. 8. This is based on an article by R. M. Frye in *Mathematics Teaching,* no. 14, 1963.

The eight-page pamphlet shows how to fold many shapes, including a square and a series of squares.

3. ———. *Pegboard Games.* Mathematics Teaching Pamphlet no. 13, 1967.

The last pegboard game (on p. 22) deals with making squares. But don't skip all the other good ideas for games that are in the rest of the book.

4. Elliott, H. A.; Maclean, James R.; and Jordan, Janet M. *Geometry in the Classroom. New Concepts and Methods.* Toronto: Holt, Rinehart & Winston of Canada, 1968.

5. Mold, Josephine. *Solid Models.* Cambridge: The University Press, 1967.

Gives careful instructions on how to make models of the five regular solids (cube, tetrahedron, octahedron, dodecahedron, icosahedron) as well as various other ones.

6. Nuffield Mathematics Project. Introductory Guide: *I Do, and I Understand.* Teacher's Guides: *Pictorial Representation, 1; Beginnings, 1; Mathematics Begins, 1; Shape and Size, 2; Computation and Structure, 2; Shape and Size, 3; Computation and Structure,* 3. New York: John Wiley & Sons.

For this unit the three books, *Beginnings, 1, Shape and Size, 2,* and *Shape and Size, 3,* are most pertinent.

7. Page, David A. *Ways to Find How Many.*

This is a workbook for children which uses cubes and squares to present problems in estimating. It is available from the University of Illinois Arithmetic Project, Education Development Center, 55 Chapel St., Newton, Mass. 02160.

8. Pearcy, J. F. F., and Lewis, K. *Experiments in Mathematics,* Stage 1, 2, 3. Boston: Houghton Mifflin.

These booklets contain experiments which children can explore on their own. Most deal with some aspects of geometry, but the variety is great and many of the topics included relate directly to this unit.

9. Schools Council for the Curriculum and Examinations. *Mathematics in Primary Schools.* 2d ed. London: Her Majesty's Stationery Office, 1966.

This is Curriculum Bulletin No. 1 and is available for \$2.00 from British Information Services or SEE, Inc., 3 Bridge St., Newton, Mass. 02160. The whole book is a "must." See especially Chapter 5, "Children, Shapes and Space," pp. 50–68.

10. Wenninger, Magnus J. *Polyhedron Models for the Classroom.* Washington, D.C.: National Council of Teachers of Mathematics, 1966.

Gives careful instructions on how to make the five regular (platonic) solids, the thirteen Archimedean solids, as well as many others.

III. Books Related to Art

1. Escher, M. C. *The Graphic Work of M. C. Escher.* Translated from the Dutch by John E. Brigham. New York: Meredith Press, 1967.
   Black and white reproductions of many of Escher's works, including many tessellations or "tiling" designs.

2. Macgillavry, Caroline H. *Symmetry Aspects of M. C. Escher's Periodic Drawings.* Utrecht: Published for the International Union of Crystallography by A. Oosthoek's Uitgeveersmaatschappij, NV, 1965.
   Even without any mathematical background one can enjoy Escher's remarkable drawings—many of them in color. There are 41 plates of intriguing plane filling or "tiling" designs on tessellations.

3. Rowland, Kurt. *Learning to See,* Books 1, 2, 3 (4 and 5 to come). Toronto: Ginn & Co., 1968.
   Notes for teachers are available for each book.

4. ———. *Pattern and Shape; The Development of Shape; The Shapes We Need; The Shapes of Towns.* Looking and Seeing, nos. 1, 2, 3, 4. Toronto: Ginn & Co., 1966.
   Notes for teachers are available for each book. These books are available from Ginn & Co., 35 Mobile Drive, Toronto 16, Canada. At the time of this printing, they were not available in the United States.

IV. Concrete Materials and Guides Related to This Unit

1. For a comprehensive list of concrete materials see this article:
   Davidson, Patricia S. "An Annotated Bibliography of Suggested Manipulative Devices." *The Arithmetic Teacher,* 15 (1968):509–24.

2. Geoboards
   Available from Sigma Enterprises, Box 15485, Denver, Colo. 80215.

3. Guides
   a. Bradford, John, and Bartram, Harlan. *The Geosquare Teacher's Manual.* Available from Sigma Enterprises, Box 15485, Denver, Colo. 80215.
   b. Cohen, Donald. *Inquiring in Mathematics via the Geo-board.* Available from Walker Teaching Programs and Teaching Aids, 720 Fifth Ave., New York, N.Y. 10019.

4. Materials from Elementary Science Study
   a. Pattern Blocks and Teacher's Guide for Pattern Blocks
   b. Mirror Cards and Teacher's Guide for Mirror Cards
   c. Tangrams and Teacher's Guide for Tangrams
   d. Geo Blocks and Teacher's Guide for Geo Blocks
   e. Teacher's Guide for Light and Shadows

   For information write to Webster Division, McGraw-Hill Book Co., Manchester Rd., Manchester, Mo. 63011.

V. Films

1. *Dance Squared.* National Film Board of Canada. 16mm, color, sound. Available from International Film Bureau, 332 South Michigan Ave., Chicago, Ill. 60624.
2. *Mathematics Peep Show.* Ray and Charles Eames. 11 minutes, color. No rental charge. Available from Herbert Miller, Inc., Zeeland, Mich. Six to eight weeks advance booking time is needed. This film contains a delightful two-minute sequence on symmetry.
3. *Notes on a Triangle.* National Film Board of Canada. 16mm, color, sound. Available from International Film Bureau, 332 South Michigan Ave., Chicago, Ill. 60624.
4. *Symmetry.* Polytechnic Institute of Brooklyn. 1967, 16mm, color, sound. Available from Contemporary Films, 267 West 25th St., New York, N.Y.

VI. Periodicals

The Arithmetic Teacher

Published monthly, eight times a year, October through May. The National Council of Teachers of Mathematics, 1201 Sixteenth St., N.W., Washington, D.C. 20036. Every teacher should have access to this journal. Each issue has rich ideas. The following issues are among those particularly pertinent for this unit because they contain many articles dealing with geometry in the elementary grades: Volume 14, February 1967 and October 1967; Volume 15, December 1968. The following articles by the author are related to this unit:

1. Walter, Marion. "An Example of Informal Geometry: Mirror Cards." 13 (1966):448–52.
2. ———. "Some Mathematical Ideas Involved in the Mirror Cards." 14 (1967):115–25.
3. ———. "A Second Example of Informal Geometry: Milk Cartons." 16 (1969):368–70.

The Mathematics Teacher

Published monthly, eight times a year, October through May. The National Council of Teachers of Mathematics, 1201 Sixteenth St., N.W., Washington, D.C. 20036. "Devoted to the interests of teachers of mathematics in the junior high schools, senior high schools, junior colleges, and teacher-education colleges." In connection with this unit see especially—

4. Ranucci, Ernest R. "A Tiny Treasury of Tessellations." 61 (1968):114–17.
5. Sanders, Walter J., and Denis, Richard J. "Congruence Geometry for Junior High School." 61 (1968):354–69.

   This article discusses congruence as well as the three basic motions of translations, rotations, and reflections and shows how they "provide a wealth of opportunities to explore topics in geometry."

Mathematics Teaching

The Bulletin of the Association of Teachers of Mathematics. Published quarterly by the Association of Teachers of Mathematics, Vine Street Chambers, Nelson, Lancashire, England. Consists of articles of interest to both the high school and elementary school teacher as well as to mathematics educators. Among the articles most related to this unit are—

6. Berryman, J. P. "An Investigation." No. 39 (1967), pp. 37–38.
   A short interesting discussion involving six squares.
7. O'Brien, Thomas. "Some Problems Involving Cubes." No. 45 (1968), pp. 29–31.
   Discusses various problems in which students are asked to predict what is on the faces of the cube after the cube has been turned.
8. Parker, John; Hadley, Judith; and Cooke, Charles. "Tessellations." No. 40 (1967), pp. 17–20.
   A discussion of which octominoes (eight squares) tessellate (cover) the plane.
9. Underwood, Val. "Polyominoes." No. 41 (1967), pp. 54–57.
   Anne and Fay (two students) discuss polyominoes.
10. Walter, Marion. "Polyominoes, Milk Cartons and Groups." No. 43 (1968), pp. 12–19.
    A very brief version of this unit written for high school teachers.

Mathematics Teachers' Forum (formerly known as the *Nuffield Mathematics Project Bulletin*)

Published six times per year by Fanfare Publishing House, 174 Chingford Mount Rd., London E4, England. All these bulletins give suggestions useful for class use with children aged 5–13 years. In connection with this unit, see especially—

11. Caine, P. A. "Permutations and Group Structure." Bulletin no. 14 (1967):4–5.
    A discussion of how eleven-to-thirteen-year-old children used the permutation of three colored cubes to arrive at a group table.
12. Dunn, James A. "Patterns with Pentagons." Bulletin no. 18 (1968):1–2.
13. Fletcher, T. J. "The Effelation." Bulletin no. 12 (1967):4–5.
    Discusses a tessellation using the letter *F*.
14. Gow, Margaret M. "Polyominoes." Bulletin no. 7 (1966):12–15.
    Describes work done by below-average-ability children aged nine and ten years. The work involves problems using five and then six squares.
15. Petrie, J. W. H. "More Patterns and Tessellations." Bulletin No. 20 (1968): 4–9.
16. Robinson, Nancy. "Tessellations." Bulletin no. 6 (1965):10–15.
    The author describes her work with nine- and eleven-year-olds.
17. Shaw, H. "Tessellations," "Solid Tessellations," and "The Regular Polyhedra." Bulletin no. 20 (1968):4–9.
18. "Symmetry." Bulletin no. 5 (1965):3–10.
    This article describes how nine-year-olds worked with symmetry of strip patterns.

19. Wheeler, David. "Unfinished Business: Note Number 6 on Tessellations." Bulletin no. 10 (1966):9–11.

20. Wyvill, R. A. "Polos and Polyominoes." Bulletin no. 17 (1968):10–11.

Primary Mathematics (formerly *Teaching Arithmetic*)

Published three times a year. Pergamon Press, 44–01 21st St., Long Island City, N.Y. 11101. This journal is published in England and should be available to every elementary school teacher. In connection with this unit, see especially this article:

21. Berryman, J. P. "Corners." vol. 6, no. 3, pp. 9–14.

An introduction—assuming minimal knowledge of geometry—to making many polyhedra. The article includes two worksheets that have been used by children.

Scientific American

Published monthly by Scientific American, 415 Madison Ave., New York 17, N. Y. See "Mathematical Games" by Martin Gardner. The mathematical games sections have many intriguing problems (and solutions). Many deal with problems related to this unit.

Scripta Mathematica

Published by Yeshiva University, 186 St. and Amsterdam Ave., New York, N. Y. 10033.

22. Boyd, Rutherford. "Mathematical Ideas in Design." June 1948.

A discussion and pictures of fascinating designs "which can be carried out by the use of the most elementary mathematical figures and operations."

23. Weyl, Hermann. "Symmetry." June 1948.