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ABSTRACT

THESE MATERIALS ARE A PART OF A SERIES OF STUDIES SPONSORED BY THE CAMBRIDGE CONFERENCE ON SCHOOL MATHEMATICS WHICH REFLECTS THE IDEAS OF CCSM REGARDING THE GOALS AND OBJECTIVES FOR SCHOOL MATHEMATICS K-12. FEASIBILITY STUDIES 9-13 CONTAIN A WIDE RANGE OF TOPICS. THE FOLLOWING ARE THE TITLES AND BRIEF DESCRIPTIONS OF THESE STUDIES. NUMBER 9--"STREAMS OF IDEAS ON CHECKS, APPROXIMATIONS, AND ORDER OF MAGNITUDE CALCULATIONS." THIS PAPER SUGGESTS THAT STUDENTS SHOULD BE MADE AWARE OF MAJOR SOURCES OF ERRORS IN CALCULATIONS SO THAT THEY CAN CONCENTRATE ON LEARNING MATHEMATICAL CONCEPTS. NUMBER 10--"COMPLEX NUMBERS LEADING TO TRIGONOMETRY." THIS STUDY INTRODUCES THE READER TO TRIGONOMETRY BY THE USE OF COMPLEX NUMBERS. NUMBER 11--"THE USE OF NEGATIVE DIGITS IN ARITHMETIC." NEGATIVE INTEGERS ARE INTRODUCED AFTER THE FOUR FUNDAMENTAL OPERATIONS HAVE BEEN MASTERED. THE NOTATION 3 IS USED FOR NEGATIVE THREE. NUMBER 12--"USE OF THE SHIFT THEOREM IN DIFFERENTIAL EQUATIONS." THIS STUDY DESCRIBES THE USE OF THE SHIFT THEOREM TO SOLVE CERTAIN TYPES OF DIFFERENTIAL EQUATIONS. NUMBER 13--"TOPOLOGY IN 10TH GRADE AND AFTER." A LIST OF TOPOLOGICAL CONCEPTS WHICH ARE RELEVANT TO HIGH SCHOOL MATHEMATICS IS GIVEN IN THIS STUDY. NOT AVAILABLE IN HARDCOPY DUE TO MARGINAL LEGIBILITY OF ORIGINAL DOCUMENT. (FL)

CCSM
Feasibility Study No. 9
Friedman, Mosteller

Streams of ideas on checks, approximations, and order of magnitude calculations

These notes are intended to put together some material in the general area of order of magnitude calculations, checking, and approximations. In our initial discussions we found ourselves considerably handicapped because rules for one kind of work did not always apply to another, and this led to confusion. At the moment we have in mind, a product space of material that splits into absolute numbers and physical numbers in one direction, and into several sets on the other.

We found that in problems with physical interpretations there exist checks that are not available when we deal with purely mathematical numbers. For example, in a trigonometric problem dealing with the height of a house, common experience suggests that 1000 feet is an unreasonable height, whereas the same problem without a physical interpretation would have to rely on more formal mathematical ideas to get a check.

At the moment our breakdown reads:

		Absolute numbers	Numbers with physical interpretations
Approximate calculations	Rough		
	Careful		
Checking	Exact		
	For Blunders		

We do not know how redundant such a classification is, nor how many empty cells it may have, but it seems an improvement over discussing the whole area at once.

The conference has formulated the idea that a student should be able to do a difficult calculation in a reasonable length of time and at the end be confident that the result is correct. Part of the meaning is that the student is to be confident that he understands exactly what he is doing at the various stages and individual operations of the problem.

The other part stems from the experience of all people who have done elaborate calculations, from the time of quill pens to the use of modern calculators. That experience sums up to this: if an elaborate calculation has no check then no one is obligated to have faith in the result.

We want the student not only to have personal confidence in his ability to do the calculations, but also transmittable evidence that can engender confidence in others that the results are correct. Checks are the standard way to gain such confidence. Unfortunately checks are often as long as the original problem, and for this reason the student has to be taught to value and execute the check, and the teacher must understand that pressure for high speed does not promote reliability but does promote errors. (These remarks do not contradict the well known fact that the faster students have fewer errors.) We want the student to appreciate that a serious calculation carries with it the personal responsibility to demonstrate positively that it is correct.

Some remarks about physical equipment for arithmetical work seem appropriate in any discussion of confidence and reliability in the work.

Neatness in calculations is a positive asset, but we often confuse training for neatness with training for good calculation. Those who perform elaborate calculations, which are as difficult at their level as the problem of multiplying two 4-digit numbers, or adding up ten 4-digit numbers, are for an elementary student, have a number of physical devices to aid them to achieve order and neatness. It is strange that similar or even better aids are not available for the beginner. Elementary students are taught that they should have their columns straight but many a student never masters this. While it is well to learn that to keep your columns straight is a great asset, the question of how to manage that need not be entirely in the hands of the student. There is little reason why, in the last half of the 20th century in the richest country in the world, he should not have paper ruled both ways so that in serious calculations his columns will automatically be straight. The sizes of the boxes can be appropriate to the grade level.

Similarly, many errors stem from sloppy handwriting in calculations. While it is well to have good handwriting, first-class mathematical work can be done without it. The lore of mathematics does not seem to include the information that numbers written large make for easier reading and fewer mistakes even among sloppy writers. Indeed, the school training seems to go the other way with more and more problems to be done on smaller and smaller pieces of paper. The student is driven to tiny numbers and the poor writer is at a terrible disadvantage. In many school systems exactly the right amount of paper is given out for each arithmetic assignment, and the student is to do it on that paper oriented a certain way and on no other. The paper is of poor quality and a couple of erasures make a hole. It is no wonder that some young people with good analytical minds get weary of their mathematics during the arithmetic years.

The purpose of these remarks is to suggest that in planning an elementary mathematics curriculum there are ways to take advantage of 20th century technology so that the mathematics student's effort can be concentrated more directly on the task of learning his mathematics and a little less on the acquisition of motor skills.

The student should be put in touch with information about major sources of errors in arithmetical calculations. For example, he should know that the copying of figures from one sheet to another by hand is a major source of error in calculations. No doubt a substantial list of sources of error is available, and the information can be leaked out when the appropriate arithmetic is being taught.

Generally speaking, checks that are different from repeating the previous work are preferable to those that merely repeat. In hand calculation in which the same calculation is repeated, one is very likely to persist in the same error for a variety of reasons. For example, the same number may be misread repeatedly, or the same number trick may repeatedly be used wrongly in the same place.

Another example of error lore is that the first calculation of a given kind is often done wrong, probably because the problems of organization added to the newness of the problem itself make for extra complication. The student should be given more than one calculation of a kind so he can work up some skill in organization of problem and check. But as we have often repeated, "several" is not necessarily large.

For exact checking we have the following suggestions:

Addition:

a. Adding the other way (up instead of down)

b. In a long addition, subgrouping the addition, checking the subgroups, and then adding subtotals.

c. Casting out 9's.

Subtraction:

a. Doing the opposite operation. What will be the opposite depends on the particular method taken as standard.

b. Casting out 9's.

c. Subtract by complements; i.e. subtract the smaller number from the appropriate power of ten and add to the larger number.

Multiplication:

a. Casting out 9's. This should be done also with partial products so that it is a method not only for checking but for locating errors as well.

b. Reversing the order of the multiplication (ab instead of ba).

Division:

a. Casting out 9's.

b. Multiplication

Tricks that aid in approximate calculation (written for us rather than the student):

1. $\frac{1}{1 \pm \epsilon} \simeq 1 \mp \epsilon$

2. $(1 + \epsilon)(1 + \epsilon\ ) \simeq 1 + \epsilon + \delta$

3. $(1 + \epsilon)^{\frac{1}{2}} \simeq 1 + \frac{1}{2}\epsilon$

4. $\frac{a}{b} \simeq \frac{a + \epsilon}{b + \epsilon}$, and better accuracy is achieved if $\epsilon \simeq ax$ and $\epsilon_1 \simeq bx$.

5. $ab = \left(\frac{a + b}{2}\right)^2 - \left(\frac{a - b}{2}\right)^2 \simeq \left(\frac{a + b}{2}\right)^2$ if a is close to b.

6. Knowing the decimal equivalents of common fractions often enables one to preserve 2 or more significant digits by converting numbers to easy fraction; e. g. $167 \simeq \frac{1}{6} \times 10^3$, $17 \simeq \frac{1}{3}(50)$, $52 \simeq \frac{1}{2}(100)$.

   Thus, if weekly salary is \$150, annual salary $\frac{1}{2} \times 150 \times 100 = 7500$.

7. $(n + \frac{1}{2})^2 = n(n + 1) + \frac{1}{4}$.

8. Mixing fraction and decimal work often preserves accuracy.

9. In rounding where the first digit is small it is wise to keep an extra place or so. The reason is that the percentage accuracy goes off badly in multiplication problems when say 1.5 is rounded to 1.

10. $2^{10} \simeq 10^3$

A few principles in order of magnitude calculation:

1. Look out for the decimal, double check it. Preferably, calculate with numbers in scientific notation.

2. Look for cancellations

3. Try for compensating errors.

4. In checking for blunders by approximate checks we are counting on the blunder to make a severe error in the answer.

5. Upper and lower bounds are sometimes easy to achieve.

6. Order of magnitude calculations are a bit of an art and the student should develop ideas of his own.

In physical problems, for order of magnitude checking try to relate results with your experience. Illustrative examples:

1. Johnny is not carrying 850 pounds of butter.

2. Houses are not 1000 feet high.

3. For many liquids a pint is about a pound, so a gallon of water is not 100 pounds.

4. Wastebaskets in the home seldom contain 1000 cubic feet.

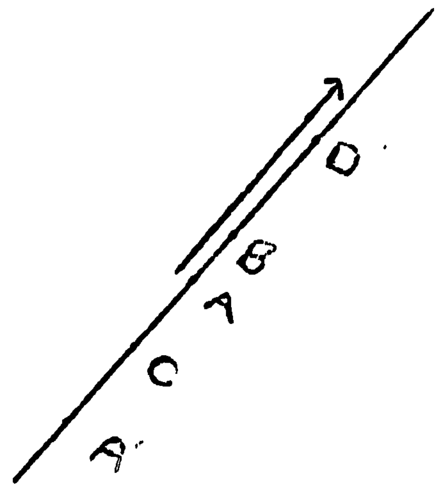
5. Comparison with other objects with known properties may help.

Friedman

Complex Numbers Leading to Trigonometry

Start with a line containing a base point 0 and vectors $\overline{OA}$, $\overline{OB}$, $\overline{OA}'$ etc. The geometrical addition of two vectors $\overline{OA}$ and $\overline{OB}$ is defined by the following procedure:

Translate the vector $\overline{OB}$ along the line until its end coincides with the tip of the vector $\overline{OA}$, then the vector from 0 to the tip of the translate of the vector $\overline{OB}$ is the sum, the vector $\overline{OD}$.



We can show that addition is commutative, i.e.

$$\overline{OA} + \overline{OB} = \overline{OB} + \overline{OA}$$

and associative, i.e.

$$(\overline{OA} + \overline{OB}) + \overline{OC} = \overline{OA} + (\overline{OB} + \overline{OC}).$$

Also, if we define the point 0 as the zero vector, then

$$\overline{OA} + 0 = \overline{OA}$$

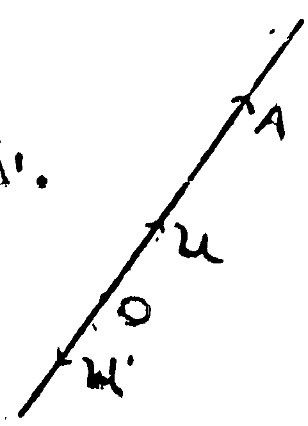
and there exists an opposite vector OA' such that

$$\overline{OA} + \overline{OA}' = 0$$

We know it is possible to map the real numbers α into the vectors as follows: Take an arbitrary vector $\overline{OU}$ as the unit vector. If $\alpha > 0$, define

$$\alpha\overline{OU} = \overline{OA}$$

where A is on the same side of 0 as U and the length of OA is α times the length of OU. If $\alpha = 0$, define $\alpha\overline{OU} = 0$. If $\alpha < 0$, define $\alpha\overline{OU} = |\alpha| \quad \overline{OU}' = \overline{OA}'$.



Notice this mapping preserves the additive structure,

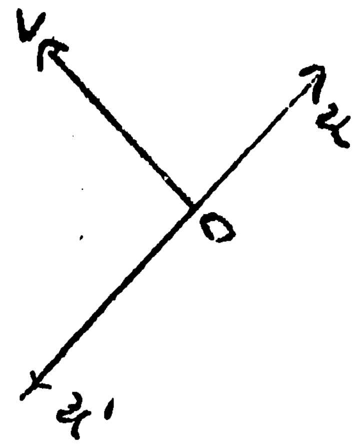
$$(\alpha + \beta)\,\overline{OU} = \alpha\,\overline{OU} + \beta\,\overline{OU},$$

also $\alpha(\beta \quad \overline{OU}) = (\alpha\beta)\,\overline{OU}.$

We may now consider the real numbers as operators which map vectors into vectors. If α is a real number, then $\alpha : \overline{OB} \longrightarrow \overline{OC} = \alpha\overline{OB}$. The

The composition of real numbers α and β corresponds to multiplication of the real numbers. The number one is the identity operator. A positive number α stretches a vector to α times its length. The number -1 rotates vector through 180° and this verifies that $(-1)(-1) = 1$. If α is negative, it both rotates and stretches. The order is immaterial because $\alpha = |\alpha| (-1) = (-1) |\alpha|$.

Let us consider the operator that rotates $\overline{OU}$ through 90° in the positive direction. Denote this operator by the symbol i so that



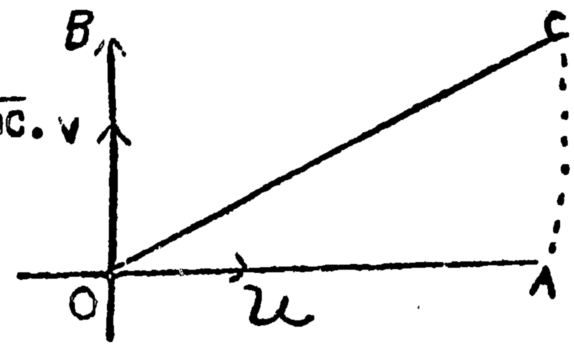
$$i\ \overline{OU} = \overline{OV}.$$

We demand also that $i(\overline{OV}) = i(i\ \overline{OU}) = \overline{OU'} = -\overline{OU}$; therefore $i^2 = -1$. Define βi as follows:

$$(\beta i)\overline{OU} = \beta(i\ \overline{OU}) = \beta\ OV.$$

Notice that βi rotates through 90° and stretches if $\beta > 0$. Obvious, that $\beta i = i\beta$ and $-i$ rotates through 270°.

We now have the operators α, βi when α and β are any real numbers. We define the operator $\alpha + \beta i$ as follows:



$$(\alpha + \beta i)\overline{OU} = \alpha\ \overline{OU} + \beta i\ \overline{OU} = \overline{OA} + \overline{OB} = \overline{OC}.$$

Obvious that $\alpha + \beta i = \beta i + \alpha$.

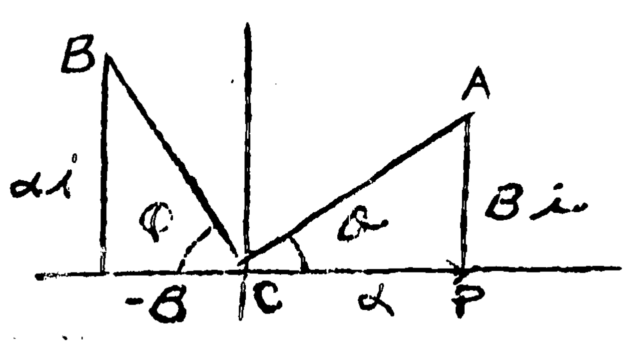
Easy to show that addition is associative. Assume that

$$\gamma(\alpha + \beta i)\ \overline{OU} = (\gamma\alpha + \gamma\beta i)\overline{OU},$$

then if γ is positive, it stretches to γ times its length but if γ is negative it rotates through 180° and stretches to $|\gamma|$ times the length.

Define

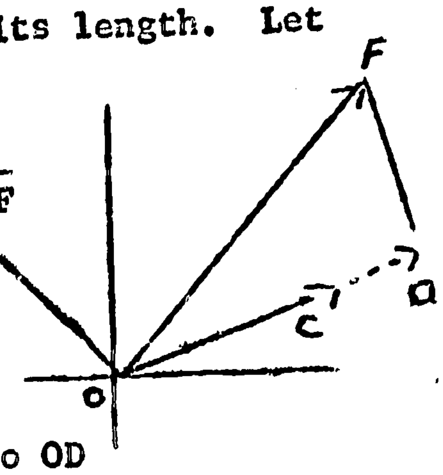
$$i(\alpha + \beta i) = \alpha i + \beta i^2 = \alpha i - \beta.$$



This definition is illustrated geometrically in the figure. Since the triangles are congruent, the angles θ and φ are complementary and the lines OA and OB are perpendicular. Thus, multiplying the vector $\overline{OA}$ by i rotates it through 90°.

The operators of the form $\alpha + \beta i$ are called complex numbers because they satisfy the usual field axioms.

It is clear that $(\alpha + i\beta)\overline{OU}$ is a vector which is obtained by rotating $\overline{OU}$ through θ and stretching it to $(\alpha^2 + \beta^2)^{\frac{1}{2}}$ times its length. Let $\overline{OC}$ be and arbitrary vector and consider the vector,



$$(\alpha + \beta i)\overline{OC} = \alpha\,\overline{OC} + i\beta\,\overline{OC} = \overline{OD} + \overline{OE} = \overline{OF}$$

The vector $\overline{OD}$ is $|\alpha|$ times the length of $\overline{OC}$ and the vector $\overline{OE}$ is $|\beta|$ times the length of $\overline{OC}$ and perpendicular to $\overline{OC}$; therefore ODF is a right angle and the ratio of the sides FD to OD is $|\beta/\alpha|$; consequently triangle OFD is similar to triangle OPA in the previous diagram. This shows that angle DOF = θ we conclude that any vector multiplied by $\alpha + \beta i$ is stretched to $(\alpha^2 + \beta^2)^{\frac{1}{2}}$ times its length and rotated through the angle θ.

Let us denote the vector of unit length which rotates through the angle θ by the symbol cis θ. It is the vector $\overline{OP}$ in the diagram. Since rotation by the angle θ followed by rotation through the angle φ is the same as rotation through the angle $\theta + \varphi$, we have

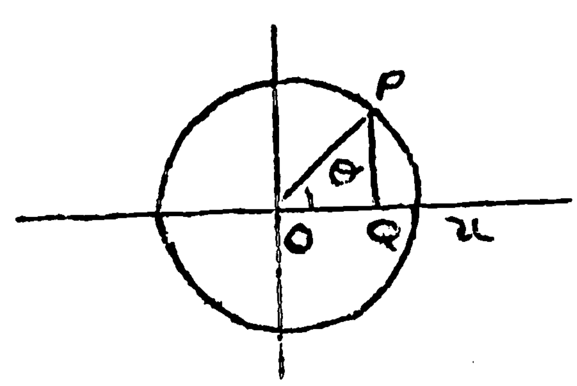
$$\text{cis}\,\theta\ \text{cis}\,\varphi = \text{cis}(\theta + \varphi).$$

It is convenient to name the real and imaginary parts of cis θ. We put

$$\text{cis}\,\theta = \cos\theta + i\sin\theta.$$

Notice $\cos\theta$, $\sin\theta$ are the x,y coordinates of the point P on the unit circle for any angle θ between 0 and 2π. Extend the definition to all values of θ. Obviously,

$$\cos^2\theta + \sin^2\theta = 1.$$



From the diagram, if $\overline{OP}$ is obtained by rotating through θ and $\overline{OP'}$ by rotating through $-\theta$, we have

$$\text{cis}(-\theta) = \cos\theta - i\sin\theta.$$

Solving, we get

$$\cos\theta = \frac{\text{cis}\,\theta + \text{cis}\,(-\theta)}{2}$$

$$\sin\theta = \frac{\text{cis}\,\theta - \text{cis}(-\theta)}{2i}$$

Also, using the addition theorem for cis θ and comparing the real and imaginary parts, we get,

$$\cos(\theta+\varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$$
$$\sin(\theta+\varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi$$ etc, etc, etc.

The Use of Negative Digits in Arithmetic

I suggest that negative digits be introduced after the four fundamental operations have been mastered. I would use a notation such as $\hat{3}$ (read "three-hat") for negative three. Using the idea of an elevator going up and down, the students would regard 3 as going up three floors and $\hat{3}$ as going down three floors. Then, it is easy to establish the rules for addition as illustrated by $3+5 = 8$, $\hat{3}+5=2$, $\hat{5}+3=\hat{2}$, $\hat{3}+\hat{5}=\hat{8}$. After these rules are understood, I would use the notation $2\hat{3}$ to mean $2\times10+\hat{3}=17$. Playing with the problem of changing numerals of the form $6\hat{7}$, $23\hat{4}$, $\hat{4}32$ to standard notation would test the student's understanding of place value. Then consider additions such as $3\hat{2} + 2\hat{1} = 5\hat{3}$ and verify that the procedure is correct by changing to standard notation. Generalize to additions with carrying such as $6\hat{4} + 3\hat{7} = 8\hat{1}$ and verify.

Try multiplication as follows:

$$\begin{array}{r} 7\hat{3} \\ \underline{32} \\ 14\hat{6} \\ \underline{21\hat{9}} \\ 22\hat{5}\hat{6} \end{array}$$

Verify the answer. Then do it the other way:

$$\begin{array}{r} 32 \\ \underline{7\hat{3}} \\ \hat{9}\hat{6} \\ \underline{224} \\ 22\hat{5}\hat{6} \end{array}$$

Finally, try the following:

$$\begin{array}{r} 3\hat{2} \\ \underline{2\hat{3}} \\ \hat{9}\,\square \\ \underline{6\hat{4}} \\ 5\hat{3}\,\square \end{array}$$

The box is left empty because we don't know what $\hat{3}\times\hat{2}$ is. By doing the problem in standard notation, we find that $\hat{3}\times\hat{2}=6$.

Similar work can be done with subtraction and division. The use of negative digits does simplify division. For example:

$$
\begin{array}{r|r}
 & 2\hat{3} \\
\hline
473 & 8245 \\
 & \underline{946} \\
 & \hat{1}\hat{2}\hat{2}5 \\
 & \underline{\hat{1}\hat{4}\hat{1}\hat{9}} \\
 & 204
\end{array}
$$

Thus, the quotient is 17 and the remainder is 204. The work is still simpler if we consistently avoid all digits larger than five. The above problem takes the following form:

$$
\begin{array}{r|r}
 & 2\hat{3} \\
\hline
5\hat{3}3 & 1\hat{2}245 \\
 & \underline{10\hat{6}6} \\
 & \hat{2}8\hat{2}5 \\
 & \underline{\hat{1}\hat{5}9\hat{9}} \\
 & 204
\end{array}
$$

Use of the Shift Theorem in Differential Equations

If $p(D)$ is a polynomial in the differential operator D over the real numbers and α is a real number, then

$p(D)e^{\alpha x} y = e^{\alpha x} p(D+\alpha)y.$

This is called the Shift theorem. To solve

$p(D)u = 0,$

put $u = e^{\alpha x} y$; then

$e^{\alpha x} p(D+\alpha)y = 0$

If $p(\alpha) = p'(\alpha) = \ldots p^{(k-1)}(\alpha) = 0$, then

$p(D+\alpha) = D^k q(D)$, where $q(D)$ is in the ring. Note that this fact would be available from the 7th grade course. In such a case the equation $q(D)D^k y = 0$ certainly has as solutions the solutions of

$D^k y = 0$

i.e. y is a polynomial of the (k-1)the degree in x. For each root of $p(t) = 0$, a similar method can be used.

The solutions thus obtained will form a complete set of solutions of the homogeneous equation. A particular solution to the non-homogeneous equation of

$p(D)u = e^{BX}$

is $u = p(B)^{-1}e^{BX}$. If $p(B) = 0$, the obvious generalization should be made. A uniqueness theorem is easily obtained by backward induction of the degree. Let $p(D)$ have degree n and let u be the solution of

$p(D)u = 0$

such that $u(0) = u'(0) = \ldots = u^{(n-1)}(0) = 0$. Suppose $p(\alpha) = 0$; put $u(x) = e^{\alpha x} y(x)$, then $y(0) = y'(0) = \ldots y^{(n-1)}(0) = 0$. We have, by the same argument as before,

$D^k q(D)y = 0$

Put $q(D)y = w$, then $w(0) = w'(0) = \ldots = w^{k-1}(0) = 0$ and since $D^k w = 0$, we conclude that $w = q(D)y = 0$,

an equation of lower degree.

Topology in 10th Grade and After

The group considering the impact of modern mathematics on curricula in grades 7 through 12 expressed the hope that some topological concepts could, in fact, be introduced. Moreover, it was felt that the notion of the continuity of a function might well become clearer if its topological nature were plainly exhibited.

Here we simply set down a list of topological concepts which are held to be relevant to high school mathematics and believed to be within the grasp of any students who could master the curriculum without them. Hopefully, these concepts cohere. Whether, in the time available, any or all of them could indeed be dealt with is not here under discussion.

It has not been thought necessary to reiterate here in detail the pedagogical principles and assumptions underlying the listed "syllabus." We recognize, of course, the decisive importance of continual exemplification through the student's own experience. For certain of the topics listed, paper, scissors, and paste are particularly valuable tools. We also emphasize the importance at this level of the student reading around and about the subject for himself; and short pamphlets may well play a very significant role in reinforcing normal instruction. Proofs may often be omitted on the first run through, and dealt with by additional reading.

There is no intention to imply, in making this list, that topology should be taught as a separate course. In most cases, the treatment of the topic listed should appear at the appropriate place in the normal mathematical development.

Table of Contents

1. Metric in Euclidean 1-space, 2-space, 3-space, n-space (recall)

   Metric yields a notion of nearness (cf. $(3-?)x(5\pm ?)= 15\pm ?$)

   Neighbourhoods, fundamental systems of neighbourhoods, especially in R^1, R^2, R^3.

   Continuity of functions from metric spaces to metric spaces; definition by means of neighbourhoods.

   Open sets; definition of continuity by means of open sets; closed sets.
   Open coverings.

2. Topological space through neighborhood axioms and open set axioms; Hausdorff space

Limits of sequences

Metrizable space; equivalent metrics

Homeomorphisms (introduced as 'more general' allowed invertible transformations); local homeomorphism; covering spaces.

Topological sum and product, universal mapping properties

Compactness; sequential compactness

Topologies on set as partially ordered system; subspace and quotient space topologies.

3. Polyhedra, simplicial complexes

Euler characteristic

Fundamental group (defined combinatorially and topologically); universal covering spaces

Jordan curve theorem for polygonal loops

Knots

4. Topological groups

Classical groups

In presenting this material the following topics should appear in examples and/or exercises:

Separability of Euclidean space

Different systems of neighbourhoods in the plane and 3-space (circles, squares, rectangles, etc.)

Equivalent metrics on circle, R^2(e.g. $|x_1-y_1|+|x_2-y_2|$) <u>as topological spaces</u>

Putting together continuous functions

Continuous image of compact is compact, elementary and familiar consequences.

Coverings of circle, torus, real projective space, fix point free transformations of S^n

Orientable and non-orientable closed surfaces.

Pictures of the real projective plane as a space with identifications; Möbius band and its relation to the projective plane; fundamental group of the projective plane.

Non-embedding problems (1-skeleton of 3-simplex in 2-space, 3 houses and 3 public utilities).

$S^n=SO(n+1)/SO(n)$, $S^1=R^1/Z$

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