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ABSTRACT

THIS IS ONE OF A SERIES OF UNITS INTENDED FOR BOTH PRESERVICE AND INSERVICE ELEMENTARY SCHOOL TEACHERS TO SATISFY A NEED FOR MATERIALS ON "NEW MATHEMATICS" PROGRAMS WHICH (1) ARE REALABLE ON A SELF BASIS OR WITH MINIMAL INSTRUCTION, (2) SHOW THE PEDAGOGICAL OBJECTIVES AND USES OF SUCH MATHEMATICAL STRUCTURAL IDEAS AS THE FIELD AXIOMS, SETS, AND LOGIC, AND (3) RELATE MATHEMATICS TO THE "REAL WORLD," ITS APPLICATIONS, AND OTHER AREAS OF THE CURRICULUM. THIS UNIT ATTEMPTS TO GENERALIZE A CONCEPT OF NUMBER FROM THE PHYSICAL WORLD. THEN BY CONSIDERING CERTAIN FINITE PHYSICAL SITUATIONS INVOLVING RELATIONS BETWEEN SETS, A WAY IS FOUND TO ABSTRACT FROM THESE SITUATIONS A NEW CONCEPT ABOUT NUMBERS. THIS LEADS ULTIMATELY TO A DEFINITION OF THE RATIONALS. OPERATIONS ON THE RATIONALS ARE DEFINED FROM GENERALIZATIONS FROM FINITE PHYSICAL WORLD SITUATIONS. THE SIX PARTS OF THIS UNIT ARE (1) ONE-TO-ONE CORRESPONDENCE AND RATIOS, (2) SUBSET AND MEASUREMENT RATIOS, (3) NEW RATIOS FROM OLD, (4) FROM RATIOS TO RATIONAL NUMBERS, (5) SOME SPECIAL RATIONAL NUMBERS, AND (6) APPLICATIONS. (RP)

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AN INTRODUCTION TO RATIONAL NUMBERS
USOE Project
J. K. Bidwell
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Unit 1. One-to-One Correspondence and Ratios

1.0 Introduction

We live and work in the physical world. In this world we see and touch many similar physical situations. One of our mental capacities is for generalization. Repeatedly seeing similar situations, we are led to generalize the common features of those situations. In fact, it is only through such generalization that we can work efficiently in a complex environment. For example, we learn through repeated experience that the light switch is close to the door on the wall. We generalize: when entering a dark room, search the wall with our hand to find the switch. In our language most descriptive adjectives are the result of generalization. We tend to symbolize our classifications with words.

In these units we will attempt to generalize a concept of number out of the physical world. Since number is an abstraction, we can only have it as a concept because we have been able to generalize. By considering certain finite physical situations involving relations between sets, we will begin to see a way to abstract from these situations a new concept about number. This will lead us to define a new set of numbers, the rational numbers, which is an infinite set of abstract objects. We will define operations on these numbers which again are generalizations from our finite physical world situations.

1.1 One-to-One Correspondences

In elementary mathematics we have the repeated experience of sets defined by having a common property; such as, the set of even numbers, the set of prime numbers, the set of all numerals having the digit "3." Some pairs of sets have as a common property that they can be matched with each other; that is, there are as many in one set as in another. We generalize this property and say,

"These sets have the same (cardinal) number." For example, these sets all have the number 3:

$$\begin{array}{ccc}
\{+ & + & +\} \\
\downarrow & \downarrow & \downarrow \\
\{0 & 0 & 0\} \\
\downarrow & \downarrow & \downarrow \\
\{* & * & *\} \\
\downarrow & \downarrow & \downarrow \\
\{1 & 2 & 3\}
\end{array}$$

Without a simple means of expressing this common property of "as many as" we would never have been able to assimilate the many physical situations facing us every day. Arithmetic as we know it today has grown out of such symbolization. However, we frequently lose sight of the original situations which gave rise to the symbols. Whenever we seek to understand a new symbolism, for example, a "new" number, we should always look for the "origin" of the symbols. We should return to the physical situations which eventually generalized into the new number. We will do just that in these units. We will begin with one-to-one correspondences and eventually develop the rational numbers.

(It is strongly suggested that you have pencil and paper at hand and that you use them to complete the diagrams of these units and to make original diagrams for the problems that follow. Through such "active" participation you will become much better acquainted with the ideas of these units.)

Suppose a person puts on a coat and buttons it up. He expects that for every button there will be a buttonhole and for every buttonhole there will be a button. He will know if he failed to button the coat properly because there will be a button and a buttonhole left over. He expects to find a one-to-one correspondence between the set of buttons and the set of buttonholes. One button for every buttonhole and one buttonhole for every button:

Many such simple one-to-one correspondences exist in the physical world: shoes for feet, tires for the wheels of a car, gloves for hands. In terms of whole numbers we know that this is how we count finite sets,

$$\begin{array}{cccccccc} \{1, & 2, & 3, & 4, & \ldots, & 19, & 20\} \\ \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow \\ \{0 & 0 & 0 & 0 & \ldots & 0 & 0\} \end{array}$$

We say "There are twenty objects in the set." In order to have a definition we say that there is a one-to-one correspondence from Set A to Set B, whenever every element in A is paired with exactly one element of B and every element in B is the mate of exactly one element of A.

Exercises

1) Which pairs of the following sets can be put in one-to-one correspondence? (Use your pencil and paper if necessary)

   A = {a, b, c} B = {1, 2, 3, 4} C = {boy, cow, horse}
   D = {+, -, x, ÷} E = {w, x, y, z} F = {z, y, x, w} G = {3, 2, 1}

2) If we have A = {1, 2, 3} and B = {6, 7, 8}, can we have more than one one-to-one correspondence from A to B?

$$\begin{array}{ccc} \{1, & 2, & 3\} \\ \downarrow & \downarrow & \downarrow \\ \{6, & 7, & 8\} \end{array} \qquad \{1, 2, 3\} \quad \{6, 7, 8\} \qquad \{1, \ , 3\} \quad \{6, \ , 8\}$$

   Complete the last two diagrams; how many different one-to-one correspondences can be made?

3) Pick out the one-to-one correspondences from the following diagrams:

   a) {1, 2, 3} {1, 2, 3, 4, 5}

   b) {a, b, c} {+, -, x}

   c) {1, 2, 3} {4, 5, 6, 7, 8}

   d) {7, 8, 9, 10} {a, b, c, d}

1.2 One-to-One Correspondences Between Partitioned Sets

Suppose Bill and Chuck share a box of apples, and that Bill receives two apples for every three apples Chuck receives. We really separated Bill's apples into groups of two each and Chuck's apples into groups of three each. Whenever we divide a set into groups in this way, we say we have partitioned the set. We really now have a new set, that is, we have Bill's apples grouped by twos:

Bill's Partitioned Set: {0 0, 0 0, 0 0, 0 0, . . ., 0 0}

↓ ↓ ↓ ↓ ↓

Chuck's Partitioned Set: {0 0 0, 0 0 0, 0 0 0, 0 0 0, . . ., 0 0 0}

Since we know that Bill received two for every three that Chuck received, we know we have a one-to-one correspondence between the partitioned sets. That is, a one-to-one correspondence between the grouped sets of apples. There are two important things we should notice here. First, we do not know exactly how many apples each boy has. Second, we do know that the boys do not have the same number of apples; that is, there is not a one-to-one correspondence between Bill's and Chuck's apples that pairs single apples. We know only that there is a correspondence from Bill's apples grouped by two's to Chuck's apples grouped by three's. We know there is a one-to-one correspondence between the partitioned sets. In order to conveniently name this correspondence so that we know the number of objects in each group of the partitioned sets, we say that their correspondence is a 2 to 3 correspondence. Since all of the correspondences we will consider are one-to-one, we will drop the phrase "one-to-one" as we did in the previous sentence.

Again, we look at another situation. Suppose we have set X partitioned into groups of four each and set Y partitioned into groups of three each. If there is one group of four in X for every group of three in Y and for every group of three in Y there is one group of four in X, then we have a

correspondence between the partitioned sets:

X = {(0 0 0 0)   (0 0 0 0)   (0 0 0 0)   . . .   (0 0 0 0)}

Y = {(0 0 0)   (0 0 0)   (0 0 0)   . . .   (0 0 0)}

We say we have a <u>4 to 3 correspondence</u> from X to Y. Notice that the arrows point to the groups of three objects each, the second number in the named correspondence.

Suppose Jim has eighteen marbles and Mary has twelve marbles. Then we have <u>no</u> correspondence between the single elements in the sets. Can we partition Jim's marbles into equal groups? We can group them by two's (mark these groups on the diagram):

Jim's Marbles: {0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0}

Can we partition Jim's marbles differently; into groups of three? Any others? Make a different partition in each diagram.

{0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0}

{0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0}

{0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0}

{0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0}

You will notice that for each diagram the number of groups times the number in each group is, of course, eighteen. We have six groups of three each in one partition. Can we partition Mary's marbles into six equal groups; yes, six groups of two each. Then Jim and Mary both have six groups of marbles:

Jim: {(0 0 0)   (0 0 0)   (0 0 0)   (0 0 0)   (0 0 0)   (0 0 0)}

Mary: {(0 0)   (0 0)   (0 0)   (0 0)   (0 0)   (0 0)}

so we have a 3 to 2 correspondence from Jim's to Mary's marbles. Draw in the arrows in the proper direction. Can we have any other correspondences? We also partitioned Jim's marbles into nine groups, 3 groups, two groups, and one group. Can we partition Mary's marbles into nine, three, two, or one equal groups? Yes,

we can have three groups of four each, two groups of six each and one group of twelve each. So we can make these correspondences:

J: { 0 0 0 0 0 0   0 0 0 0 0 0   0 0 0 0 0 0 }

M: { 0 0 0 0   0 0 0 0   0 0 0 0 }

6 to 4

J: { 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 }

M: { 0 0 0 0 0 0 0 0 0 0 0 0 }

9 to 6

J: { }

M: { }

18 to 12

(Complete the last two diagrams by grouping and drawing the arrows.)

So we see we have, in fact, four correspondences between different partitions of Jim's and Mary's marbles. We have this set of correspondences: {3 to 2, 6 to 4, 9 to 6, 18 to 12}.

Exercises

1) Alice has 12 hair ribbons and Susan has 6 hair ribbons. Name the correspondences that can be made from partitions of Alice's ribbons to partitions of Susan's ribbons. (Fill in the diagrams):

A: { 0 0 0 0 0 0   0 0 0 0 0 0 }

S: { 0 0 0   0 0 0 }

____ to ____

A: { 0 0 0 0 0 0 0 0 0 0 0 0 }

S: { 0 0 0 0 0 0 }

____ to ____

A: { }

S: { }

____ to ____

A: { }

S: { } ____ to ____

2) Suppose A is a set of 5 red cubes and B is a set of 6 blue cubes. Name all the correspondences from A to B.

3) Suppose G is a set of 24 goldfish and B is a set of 18 birds. Draw diagrams to show some correspondences from partitions of G to partitions of B. How many correspondences are there?

G: { 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 }

B: { 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 } ____ to ____.

4) Suppose we know that there is a 3 to 4 correspondence from John's blocks to Mary's blocks. What does this mean? How many blocks must John have? If John, in fact, has 12 blocks, Mary must have ____ blocks. What other correspondence could be made in this case?

5) Suppose, again, there is a 3 to 4 correspondence from A to B. Suppose A has exactly 5 less blocks than B. How many does each have? Draw a diagram.

6) Suppose Judy has red blocks and green blocks; if we know there is a 1 to 3 correspondence from red blocks to green blocks <u>and</u> that there are 24 blocks altogether, how many does she have of each color?

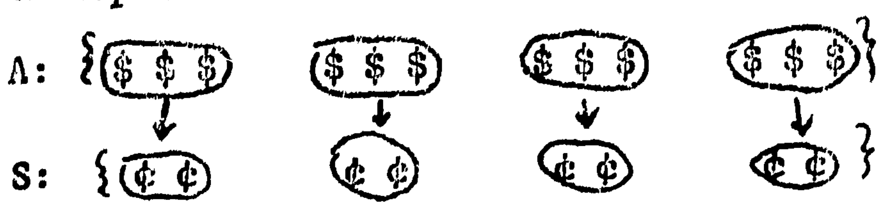
1.3 Ratio and Inverse Ratio

Given any two finite sets A and B, if we can partition A and B into the same number of equal groups, we can make a correspondence from A to B. In the previous section we saw that if A had 18 elements and B had 12 elements, we could make four correspondences between different partitions. This set of correspondences from A to B, {18 to 12, 9 to 6, 6 to 4, 3 to 2} we will define to be <u>the ratio</u> of A to B. We can name the ratio by using any of the correspondence names. So we say, the ratio of A to B is 6 to 4 or the ratio of A to B is 3 to 2.

Look back at Problem 1 of the last exercise set. We can, again, have four different correspondences. So the ratio of Alice's ribbons to Susan's ribbons is the set $\{12 \text{ to } 6,\ 6 \text{ to } 3,\ 2 \text{ to } 1,\ 4 \text{ to } 2\}$. We can say the ratio is 2 to 1, for example.

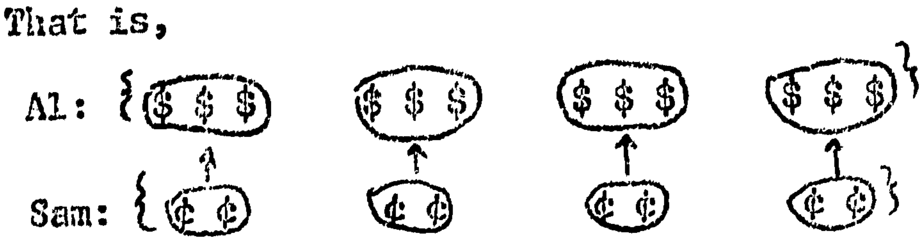
<u>Exercise</u>: Write down the ratio in Problems #2, 3, 4, 5, 6 of Exercise 1.2.

Suppose we have that Al has $12 and Sam has 8 quarters. Then we can have this correspondence:



3 to 2

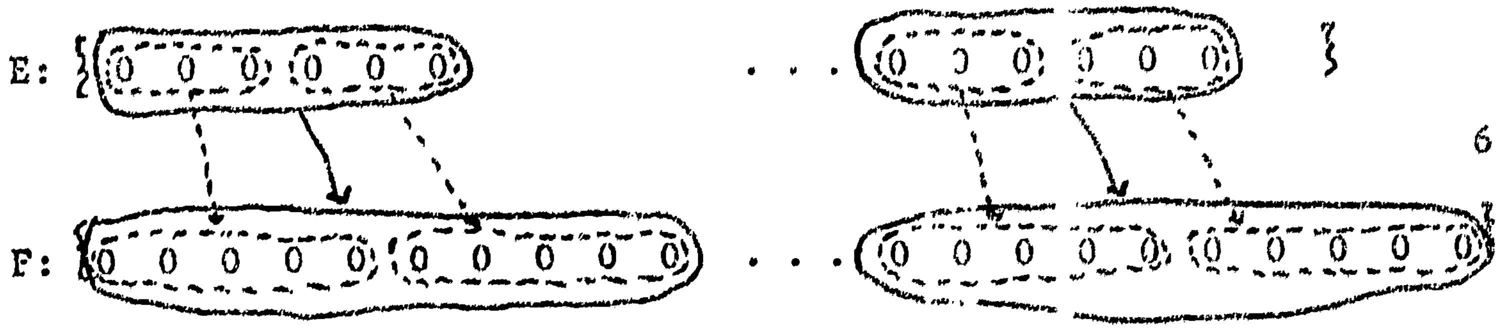
But we could just as well have the correspondence from Sam's quarters to Al's dollars. That is,



2 to 3

This correspondence we call the <u>inverse correspondence</u>. That is, the 2 to 3 correspondence is the <u>inverse</u> of the 3 to 2 correspondence. (Note: We can obtain a diagram for an inverse correspondence by reversing the arrows.) The ratio of Al's dollars to Sam's cents is $\{12 \text{ to } 8,\ 6 \text{ to } 4,\ 3 \text{ to } 2\}$. The <u>inverse ratio</u> would be, of course, $\{8 \text{ to } 12,\ 4 \text{ to } 6,\ 2 \text{ to } 3\}$. So if we know that the ratio of A to B is, say, 5 to 6, then the <u>inverse ratio</u> would be the ratio of B to A and is named 6 to 5.

It is important to notice that if the ratio of C to D is 3 to 4, then we know that there is a 3 to 4 correspondence from C to D. We do not have any other information about C and D than this. However, if we are told that the ratio of E to F is 6 to 10, then we know that there is a 6 to 10 correspondence <u>and</u> also, at least, a 3 to 5 correspondence. This is easily shown by the following diagram:



6 to 10

Exercises

1) Suppose the ratio of boys to girls in a room is 4 to 3. If 4 more girls enter the room, the ratio becomes 1 to 1. How many girls are in the room originally? How many boys originally? Draw a diagram to show the 4 to 3 correspondence. What other correspondence could be made? List the elements of the ratio of boys to girls.

2) From Exercise 1, the inverse ratio of girls to boys is ____ to ____. List the elements in the inverse ratio. Show the inverse correspondences by drawing new arrows on your old diagram of Exercise 1.

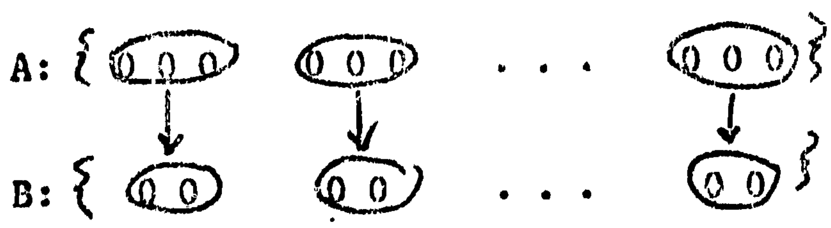
3) Suppose Kathy has some red blocks and some black blocks and the ratio of red to black is 1 to 3. Suppose she has 36 blocks altogether. How many of each color does she have? Name the possible correspondences; that is, list the elements in the ratio of red to black.

4) From Problem 3, suppose the ratio of Mike's marbles to Kathy's blocks (all of them) is 1 to 2. Draw a diagram for this! Name a correspondence from Mike's marbles to Kathy's black blocks. Name a correspondence from Kathy's red blocks to Mike's marbles.

5) Can you go back and reason through these same questions and never use numbers greater than 4? (It is possible!)

1.4 New Names for Correspondence

We have been naming the correspondence from A to B by a name such as 3 to 2 where we could draw the diagram:



3 to 2; $\frac{3}{2}$

We will now begin to use the name $\frac{3}{2}$ or 3/2 as a name for this correspondence. We call the symbol "$\frac{3}{2}$" a fraction. The number named on top of a fraction is the numerator and the number named on the bottom is the denominator. Notice that the arrow will always point to the group with the same number of elements as the denominator of the fraction. In the case above, the arrows point to groups of 2 and the fraction is $\frac{3}{2}$. The inverse correspondence above would have reversed arrows and would be named $\frac{2}{3}$ with the arrows pointing to groups of 3.

Suppose we have the ratio of C to D is 3 to 4, then the ratio may be the set $\{3/4, 6/8, 12/16\}$. The inverse ratio would be 4 to 3 and the ratio would be the set $\{4/3, 8/6, 16/12\}$. The use of the fraction notation, 3/4, to name the correspondence makes the succeeding units easier to follow. Among the correspondence names there will always be one with the smallest numerator and denominator. For example, above we have $\{3/4, 6/8, 12/16\}$. Note that 3/4 has the smallest numerator and denominator. We will call 3/4 the simple correspondence. We will call the corresponding ratio name 3 to 4 the simple ratio name. Ordinarily we will always use the simple ratio names. Notice that the simple correspondence will pair the smallest possible sets of all the correspondences.

We have seen above that given two finite sets there is at least one correspondence between them. We also agreed to call the set of all such correspondences the ratio of the two sets. Now suppose we know only, given A and B, that the ratio of A to B is X to Y. Then we know only that there is a correspondence from A to B of X/Y. Probably there are other correspondences but we cannot be sure without more information. For example, if we are told that the ratio of C to D is 3 to 5 we know only there is the correspondence 3/5 from C to D. If, however, we are told that the ratio of M to N is 12 to 18, then we know that (at least) we can make the correspondences 12/18, 6/9, 2/3, and 4/6 from M to N. (Why?)

In the units that follow we will be careful to use only those correspondences, and hence only those ratio names, which are consistent with the information given; unless special mention is made to the contrary.

Exercises

1) Henry says that a correspondence from A to B is 24/20. Name some other correspondences. What is the simple ratio name for A to B? What is the ratio from B to A? List, using fraction names, some of the known correspondences from B to A.

2) John has 36 hens. 12 of them are brown and the rest are white. Name three ratios present in this situation and illustrate with some correspondences. (Use simple correspondences).

3) Mortimer receives 50¢ for every lawn he mows. He notices that a pet shop is selling goldfish at the rate of 3 for 25¢. He would like 15 goldfish. Consider the ratios present in this situation. Draw the same conclusion as Mortimer. Make a diagram to illustrate this situation. (You are dealing with three sets: lawns, quarters, and goldfish. Study the correspondences carefully.)

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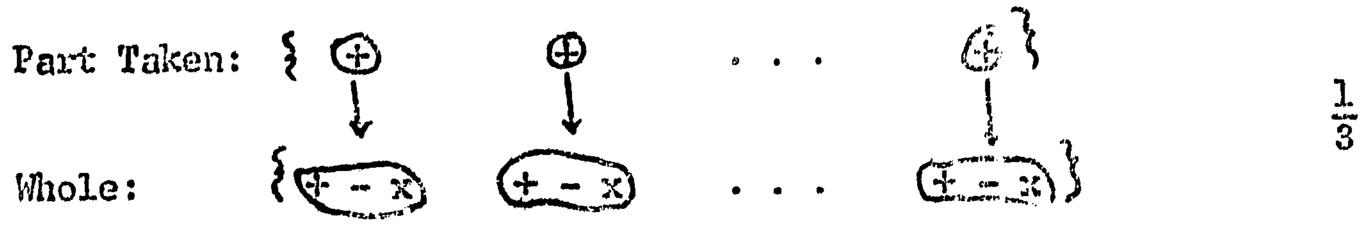
Unit 2. Subset and Measurement Ratios

2.0 Part to Whole Ratios

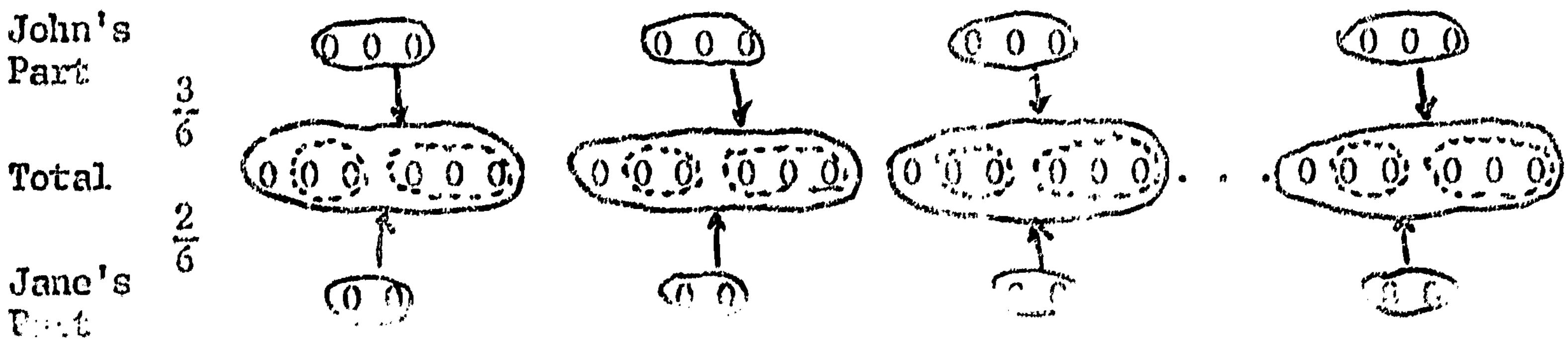
If we have a basket of apples and we remove some of them, then we can make a correspondence between the amount taken and the original total that was in the basket. Suppose for every group of three apples in the basket we take one out. Then we have taken 1 of every 3 apples and the ratio of the part taken to the total is 1 to 3 and my correspondence is 1/3. Commonly, we say we have taken one-third of the apples and we use "1/3" as a name for the amount taken. Since we understand that we always are making a correspondence of the part to the whole, we can avoid confusion in this case by saying we have taken 1/3 (read "one third") of the apples. Again if we say, "We have 3/5 of the box," we mean that a correspondence of 3/5 can be made from the part taken to the box of candy.

If we alwyas take a part of each group away, then we will always have the numerator of the correspondence name less than the denominator. That is, the number of objects in each group in the "part" will always be less than the number of objects in each group in the "whole."



Suppose we are told, "Take two-thirds of these marbles." We are to take a part of the marbles which makes a 2/3 correspondence with the whole of the marbles. So the ratio we have from part to whole is 2 to 3. We can also see that the ratio of the part taken to the part remaining is 2 to 1 and the ratio of the part remaining to the total is 1 to 3. That is, we leave 1/3 of the marbles.

Let's consider another example. Suppose Mrs. Smith has a box of stamps and she divides the stamps among Jack, Jane, and John. For every 6 stamps she gives 1 to Jack, 2 to Jane, and 3 to John. What are the various correspondences that have been made? John has 3 for every [illegible] in the box; he has three-sixths (3/6) of them or 1/2 of them. The ratio of Jack's part to the total is 1 to 6; so he has 1/6 of the stamps. We can also make correspondences between the children's stamps. A correspondence of John's to Jane's is 3/2, so the ratio of John's to Jane's is 3 to 2. Name the rest of the ratios that can be made.



Exercises

1) Discuss briefly what we mean by "three-fourths of the [illegible]." [illegible] a diagram.

2) Given a set of marbles, can we take "four-thirds" of them? Why?

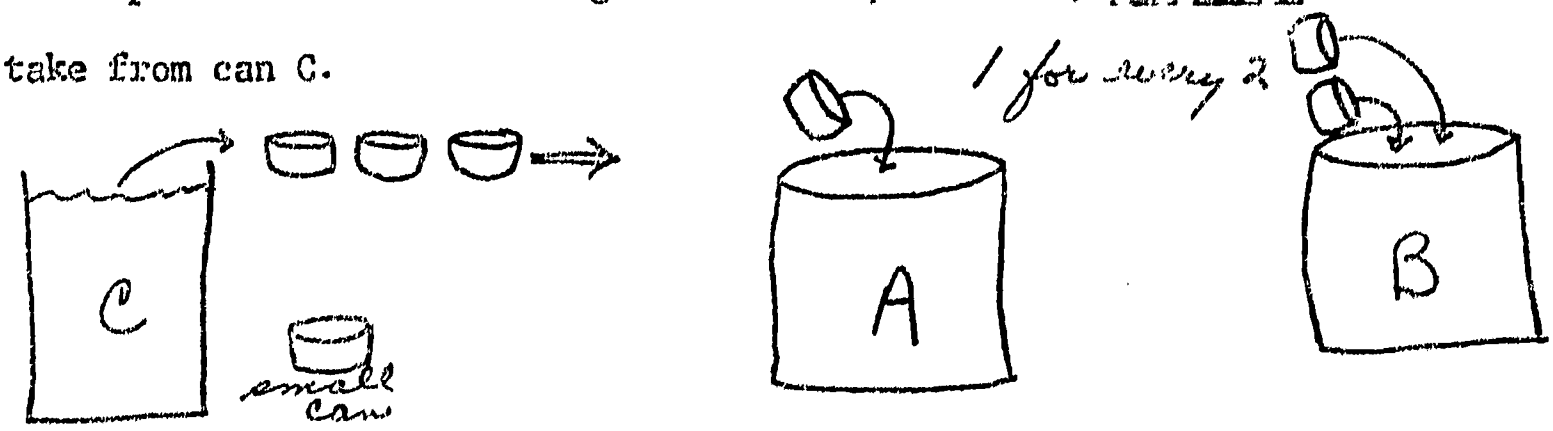
3) Mike, Bill, and Bob shared a box of candy. Bill said he got 1 for every 2 Bob got, but Mike got one-third of the box. Mike said, "I have only 9 pieces." What is the ratio of Bill's part to the box of candy? What is the ratio of Bob's part to the total? How many pieces did Bill receive? (Draw a diagram!)

4) Tim said, "John gave me one-fourth of his part and he said he got two-thirds of the money." Explain this in terms of ratios and correspondences. If the total money was $36.00, how much did John receive; how much did he give to Tim?

5) What is the difference between "Mother broke one-fourth of the dozen eggs" and "Mother broke 3 eggs"?

2.1 Measurement Ratios

So far we have dealt only with ratios between sets of distinct objects. Now we want to extend our idea of ratio to quantities which are not composed of distinct parts. For example, suppose I have a large can C of water and I want to give Sally 1/3 of it. That is, I want to take out of it a part of the water which will have the ratio of 1 to 3 to the total amount of water. Suppose I have a small can and two other large containers. I could then put one small can of water in one of the large containers (call it A), and two small cans of water I put into the other large container (call it B) for every 3 cans of water I take from can C.



If I can repeat this process until can C is empty, then I would give Sally can A in which I had been putting one small can of every three small cans. I would give her 1/3 of the water. Then can B would contain 2/3 of the water. It is important to note that without the small can with which to separate the water, I could not solve the problem! I had to use the small can as a unit of measure in order to obtain the required ratio. With the small can as a unit, I changed the problem to one of taking so many units (small cans) of water out of C to make a ratio of 1 to 3 of part to whole. A unit is an essential element in dealing with quantities of material even for ratio purposes. If the can C held exactly 24 units (small cans) of water, then if I remove 8 of them, then there would be an 8/24 correspondence from the part to the whole and the ratio of part to whole would be 1 to 3. The quantity of 8 units would be 1/3 of the water in can C.

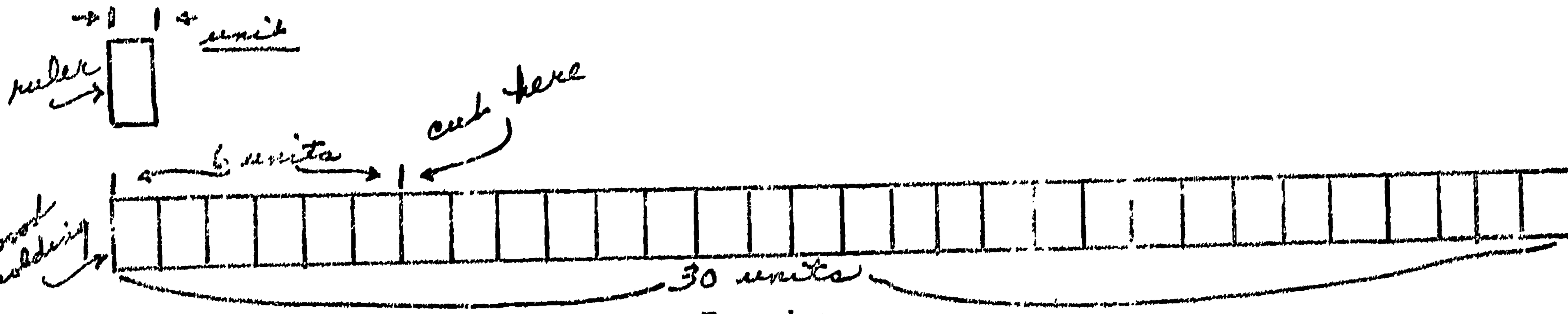
In the discussion of the last paragraph, we assumed that by using the small can as a unit, we could empty the large can with exactly 24 small cans. Now

as a matter of practicality we know that this is not likely to happen. We could possibly change the small can and obtain an exact number of such "units" in can C. But it is possible not to be able to do so. In fact, if we require the small can to have a rectangular bottom, it is not possible to get an exact number of such "units" in the round can C. We are faced here with the problem of commensurable measure and this problem has a long history. For the Greeks were also unhappy with the fact that there can be length or volumes which cannot be expressed as the ratio of two whole numbers. The solution of the problem lies in the set of irrational numbers to which we will not digress. Perhaps it is best to admit the possibility of incommensurable lengths, areas, or volumes and then to pass on in our discussion. In what follows we will assume that the measure will always be commensurable.

Suppose I have two large cans and I wish to find the ratio of the capacity of one to the capacity of the other. I can take a much smaller can as a unit and fill each can. Suppose can D is filled by exactly 15 such units of water while the other can E requires only 10 units. Then by means of the smaller can as the unit I can make a correspondence from the amount of water in the can D to the amount in E of 10/15. Hence the ratio of D to E is 10 to 15 or 2 to 3. If can D was marked 6 gallons and can E marked 4 gallons, then we already know that a standard unit has been used and a correspondence of 6/4 can be made from D to E. So again the ratio of D to E would be 3 to 2. In comparing the capacity of the two cans it is not essential to use any standard unit, such as one gallon, since the ratio of 3 to 2 would be obtained no matter what unit was used (provided it was relatively small compared with the total capacities and it came out even).

Let's look at one more example. We have several feet of wood molding and wish to cut off 1/5 of it (in one piece). We need some unit. We choose

the width of a ruler. By marking off "ruler-width units" along the molding we discover there are exactly 30 such units in the piece of wood. We then mark off 6 "ruler width" units from one end and cut there. We have a 6/30 correspondence from the cut off piece to the original whole, or the ratio of the part to the whole is 1 to 5 (or 6 to 30); i.e., we have a 1/5 correspondence from part to whole. So we have cut off 1/5 of the molding as required. A diagram will help here:

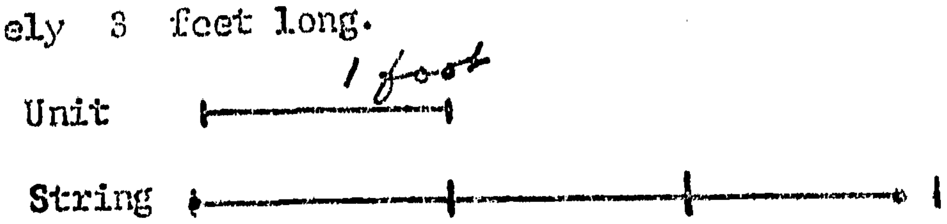


Exercise

Repeat these examples using water or sand to compare the capacities of two large containers using a small soup can as a unit. Obtain 1/7 of the length of a large table by using a "ruler width" or similar sized unit. You can invent other such exercises which do not involve measuring with any standard units of volume or length.

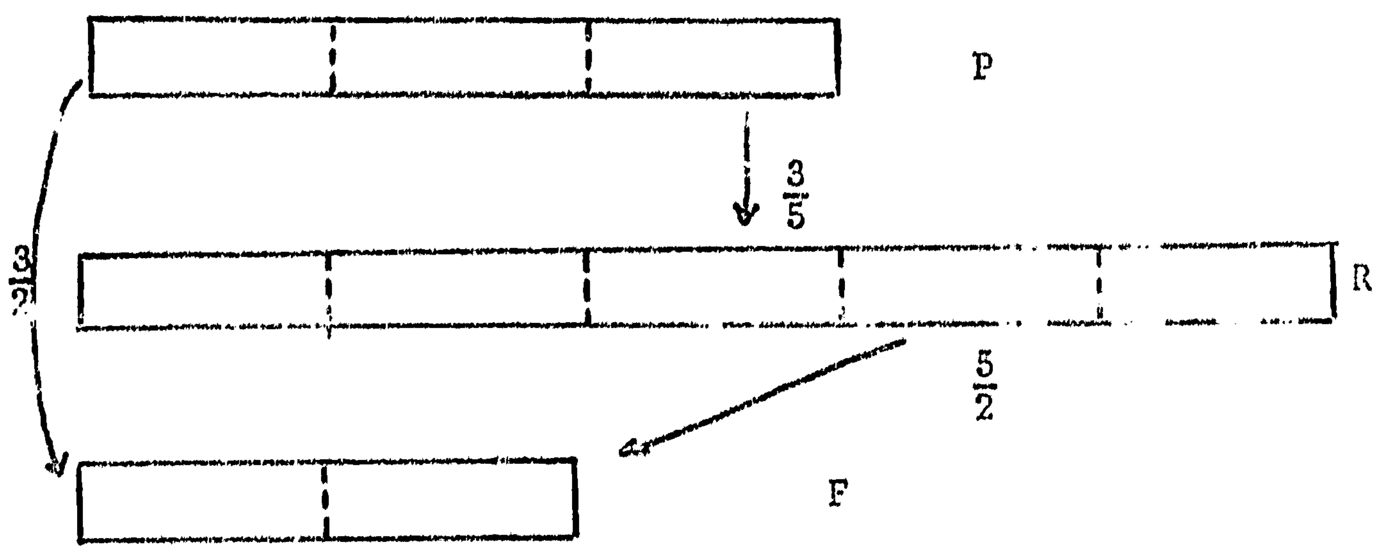
2.2 Quantity to Unit Ratio

More frequently, of course, we find the ratio of a certain quantity to a standard unit. That is, we make a correspondence from a partition of the quantity to a partition of the standard unit. Suppose we want to find the length of a piece of string in terms of string. We might be satisfied to use the foot itself as the unit to partition the string. Suppose then the string was approximately 3 feet long.



If we wish to have a more accurate comparison, we must take a smaller unit, say one inch, and use it to find a ratio between the string and one foot. We know that there are 12 inches in one foot. Suppose there are 33 inches (approximately) in the length of string. Then a 33/12 correspondence can be made from the length of string to one foot. So the ratio of the length of string to one foot is 11 to 4. Since we are using one foot as a standard unit we frequently would say that the string is 11/4 feet long (where, as usual, we mean that the denominator names the number of units of one foot which correspond to the numerator number of units in the length of string.)

Suppose we have a piece of ribbon that is 5/2 feet long; that is, the ratio of the length of ribbon to one foot is 5 to 2. (Exercise: Given a length $\overset{B}{\vdash\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!\dashv}$, how would you lay off a length A which had the ratio to B of 5 to 2?) We wish to find the measure (in feet) of 3/5 of the ribbon. Say that 3/5 of the ribbon is P and the whole ribbon is R and one foot is F. Then we can make a correspondence of 5/2 from R to F and a correspondence of 3/5 from P to R.



So for every 3 in P (via 5 in R), we have 2 in F. So we have a 3/2 correspondence from P to F or P (the part we wanted) is 3/2 feet long.

Exercises

(Please draw diagrams in these exercises.)

1) Suppose we require the weight of 2/7 of a pile of sand which weighs 14/3 pounds. (That is, find the weight of the part of the sand which has the ratio 2 to 7 to the whole pile.) Find the weight in the manner suggested above; that is, set up correspondences from the part to the whole and from the whole to one pound.)

2) The ratio of a small piece of string to one inch is 11 to 3 (that is, it is 11/3 inches long.) We know a large piece of string contains exactly 10 of the small pieces. Find the length of the large piece in inches (find the ratio of the large piece to one inch.) Can you find the ratio of the large piece to one foot if the ratio of one inch to one foot is 1 to 12?

3) A man has 2 daughters and 3 sons and he wills his [illegible] equally to all. If he has only 300 shares of stock worth $15 for every 4 shares: a) find the ratio of the sons' total of money to the daughters'; b) how many dollars does each daughter collect? c) if the father divided the estate half to sons and half to daughters, find the ratio of what one son receives to one daughter; d) how much money would one son receive?

4) A man cuts off two pieces from a piece of rope 20' long. One piece has the ratio 14 to 3 to one foot and the other has the ratio 16 to 3 to one foot. What is the ratio of the part of the rope left to the total 20' of rope? Set up correspondences for this one.

5) OPTIONAL: If a part of a group of persons in a room is described as a half of a half of a half, what is the smallest number of people who could be in the room? Translate this problem into ratios or correspondences and visualize it by drawings.

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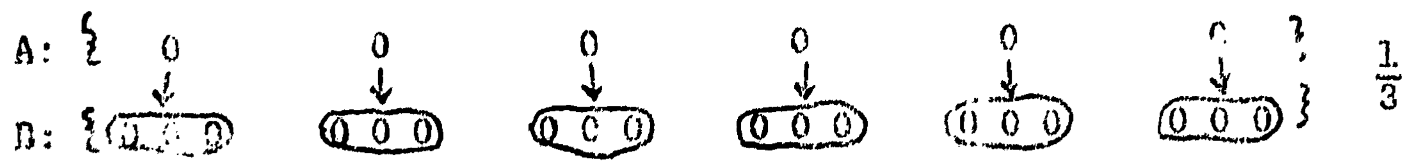
Unit 3. New Ratios from Old

3.0 Review

Let us review the discussion of Unit I. Suppose set A has 6 objects in it and set B has 18 objects in it. We can group the objects in set B by threes, forming six groups, each of these groups can be matched with the single objects of A:



That is, we can make a 1/3 correspondence from A to B. We can also make the correspondences 2/6, 3/9, 6/18 from A to B. We agreed to call the ratio of A to B the set of correspondences from these partitioned sets of A to B, in this case $\{1/3, 2/6, 3/9, 6/18\}$. We say the ratio from A to B is 1 to 3 (the simple ratio name).

Now if we have a correspondence from C to D of 2/6, we will write "$C \xrightarrow{2/6} D$", where we understand that groups of 2 each in C are paired with groups of 6 each in D. The denominator goes with the arrowhead; that is, it tells the number of objects in each group in the direction of the arrow.

Also $C \xleftarrow{5/6} D$ would mean that there is a 5/6 correspondence from D to C and C is the set grouped by sixes. We recall that there is a 6/5 correspondence from C to D, the inverse of the 5/6 correspondence.

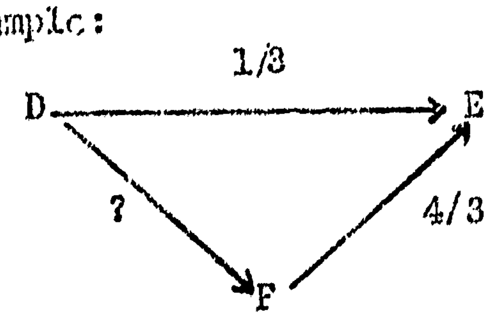
3.1 Finding a New Ratio Given Two Ratios

Suppose we have this situation:

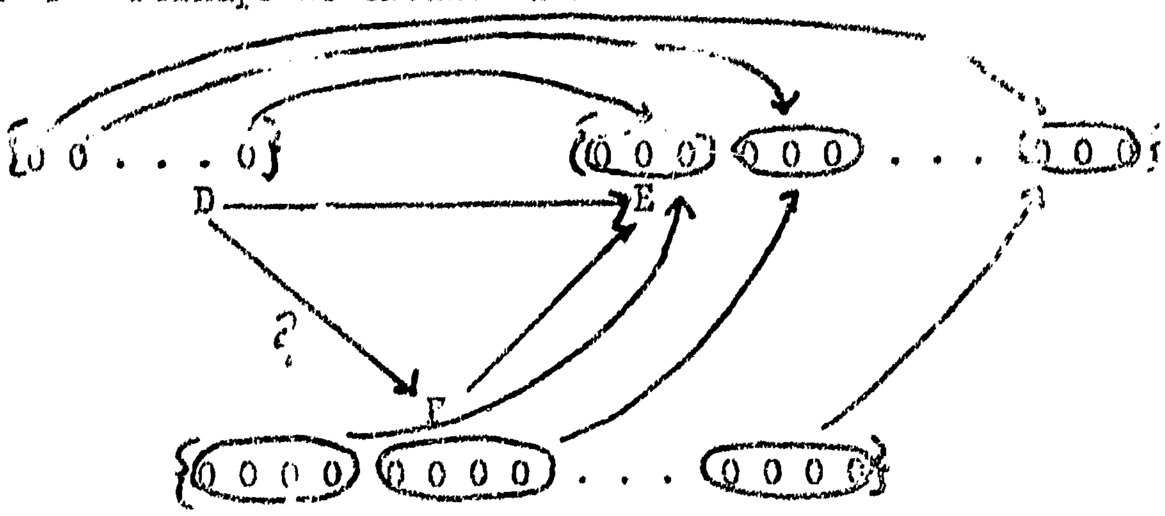
$$\begin{tikzcd} & A \arrow[dr, "4/3"] & \\ C \arrow[ur, "2/4"] \arrow[rr, "?"'] & & B \end{tikzcd}$$

We know that for every group of 2 objects in C there is a group of 4 objects in A <u>and</u> for every group of 4 objects in A there is a group of 3 objects in B. Hence, we logically can say, for every group of 2 objects in C we have a group of 3 objects in B, so we have C $\xrightarrow{2/3}$ B; i.e. the ratio of C to B is 2 to 3.

Consider this example:



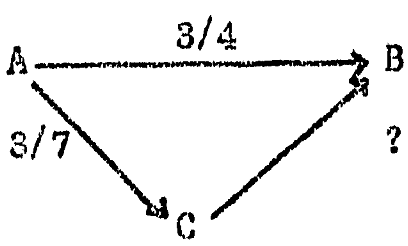
We know that for every 1 object in D there is a group of 3 objects in E <u>and</u> also for every group of 4 objects in F there is a group of 3 objects in E. Hence, we conclude that for every 1 object in D there is a group of 4 objects in F. Perhaps we should think:

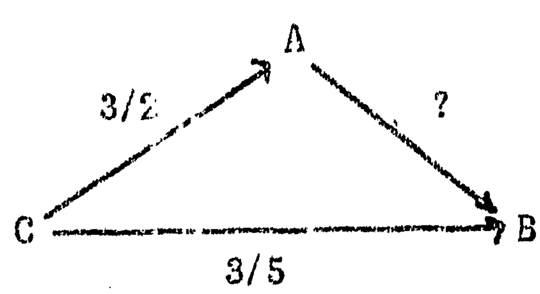


Now it is even more obvious that we have a pairing of singles in D with groups of 4 in F (via E). So we have D $\xrightarrow{1/4}$ F and the ratio of D to F is 1 to 4.

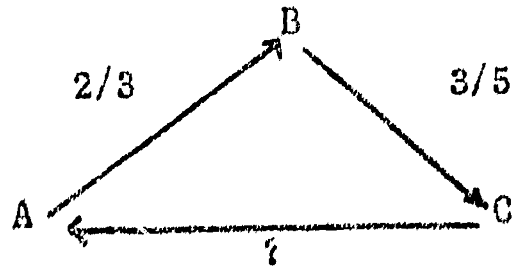
<u>Exercises</u>

Find the "?" correspondences in the following diagrams (think as in the last example). (Remember the arrowhead goes with the denominator).

1) 

2) 

3)



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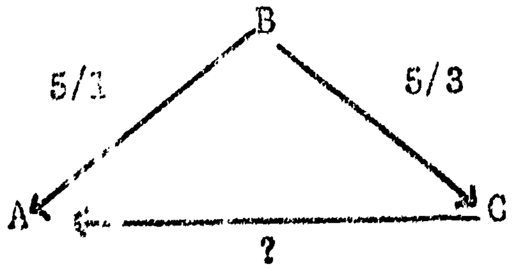

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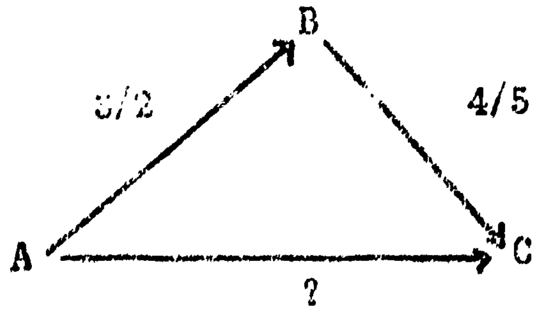

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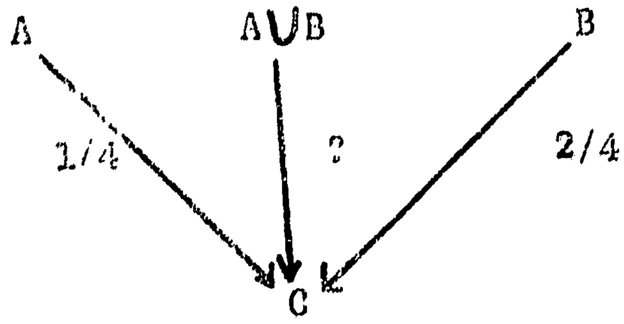
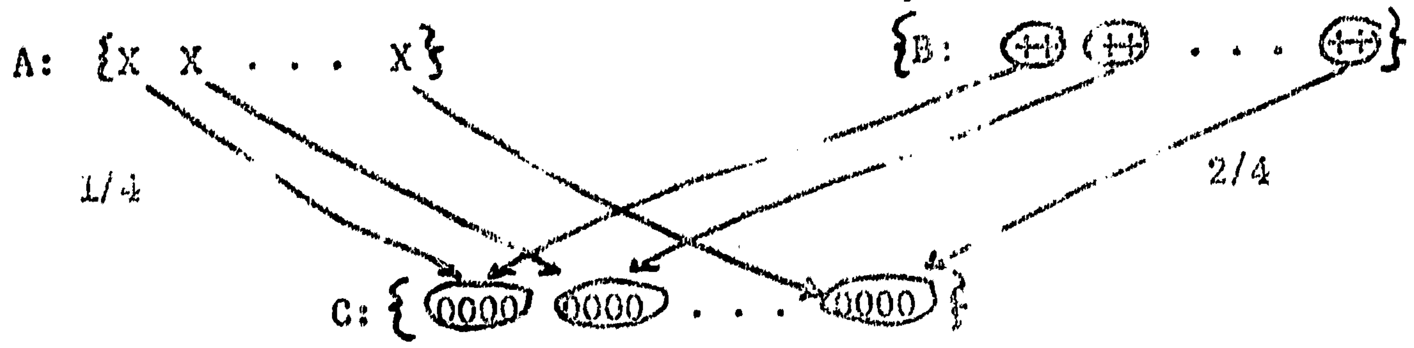
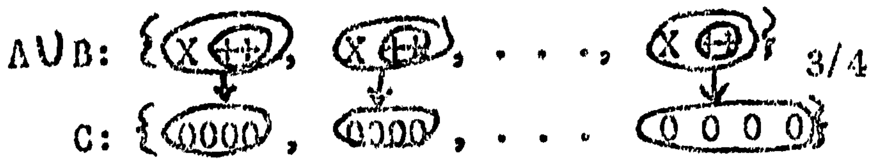


(A and B have no element in common.)

Perhaps a closer look at Problem #5 would be helpful. For every group of 3 in A there is a group of 2 in B and for every group of 4 in B there is a group of 5 in C. Now if B can be grouped by fours as well as by twos then A can be grouped by sixes as well as by threes. (Draw a diagram to see this.) So for every group of 6 in A there is a group of 4 in B and hence a group of 5 in C. So we have $A \xrightarrow{6/5} C$. Let us look at #6 also:



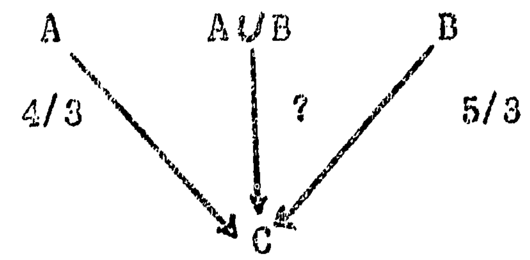
We can imagine that in A∪B we can group together the "X's" and the "(++)" according to their pairings in C so we have



Hence we have $A \cup B \xrightarrow{3/4} C$.

Exercises *

Try these problems, draw groups of objects if it is helpful. (We suppose that A, B, and C all have different objects in them.)

7)


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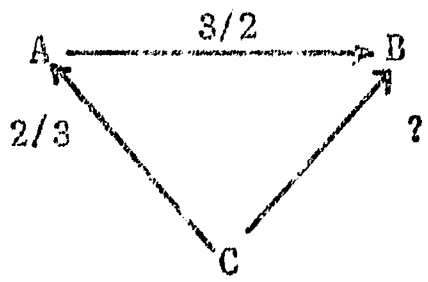

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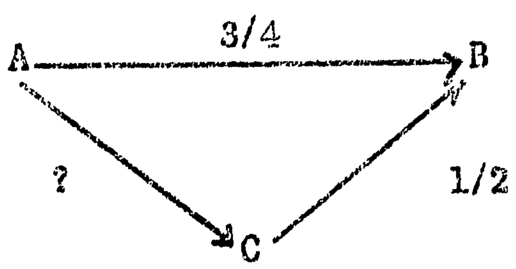

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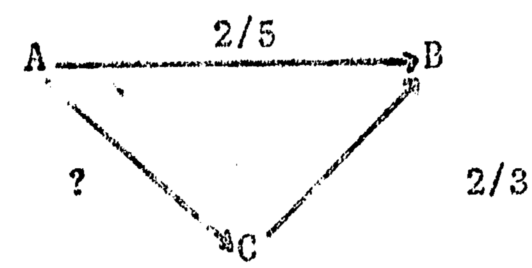

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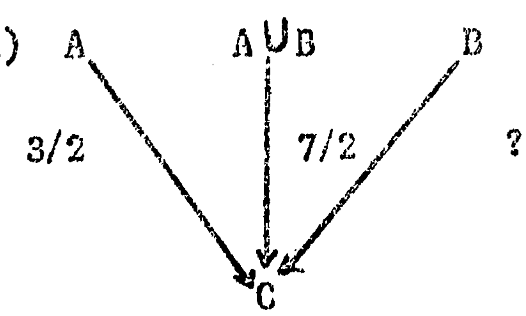

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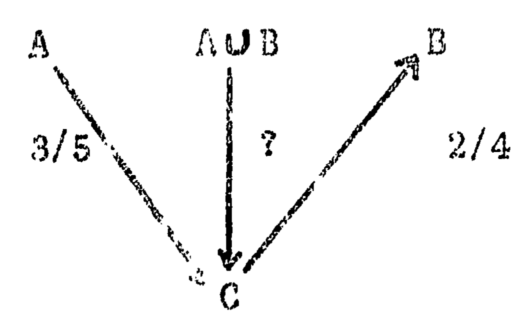

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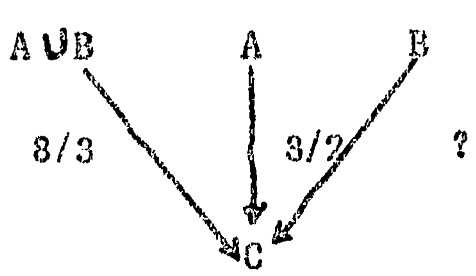

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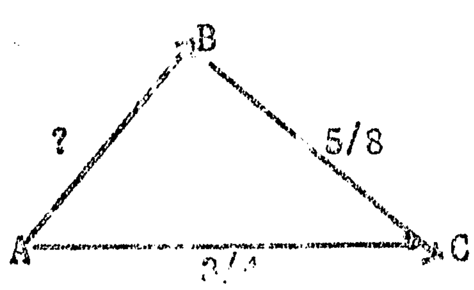


3.2 Some Physical Problems

Now we will consider some physical situations that involve the same kind of reasoning as the exercises above. Suppose in an auditorium there are 3 girls for every 5 women and also 3 girls for every 4 men. Then we have a

* Answers: 1) 7/4 2) 2/5 3) 5/2 4) 3/1 7) 9/3 8) 2/2 9) 3/2 10) 6/10 11) 4/2 12) 22/20 13) 7/6 14) 6/5

group of 3 girls for every group of 9 adults or the ratio of girls to adults is 1 to 3. If on the other hand we want to find the ratio of men to women, we clearly have one group of 4 men for every group of 5 women; so the ratio of men to women is 4 to 5. We see this by the diagram:

Men 4/5 Women

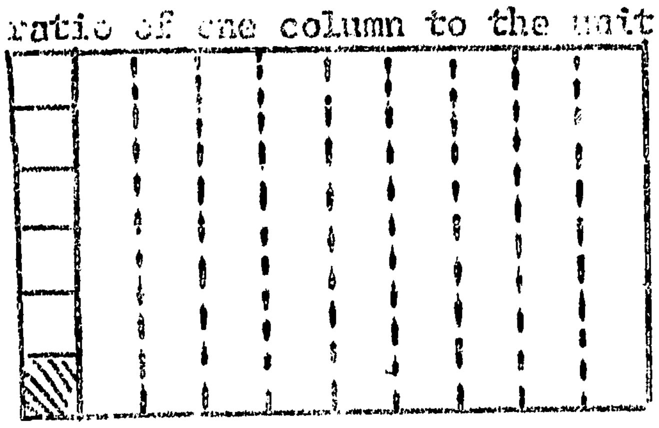
Adults

4/3 9/3 5/3

Girls

These ratios are "natural" and develop out of the correspondences given in the original ratios. Remember, whenever we can make a correspondence between groupings of elements in two finite sets, then we have a ratio established. So the building of new ratios is an outgrowth of correspondences and our rules of working with them.

We can now apply the previous ideas to find the area of a rectangle. By the area of the rectangle we will mean the ratio of the rectangle to a given square unit, where we may imagine some smaller square unit is used to determine the paired groupings. Suppose for a unit of area [illegible] square inch. Suppose we have a rectangle and wish to find its area in square inches. Then first we find how many square inches we have in a column one inch wide; suppose 6, that is, the ratio of one column to the unit is 6 to 1 We have:



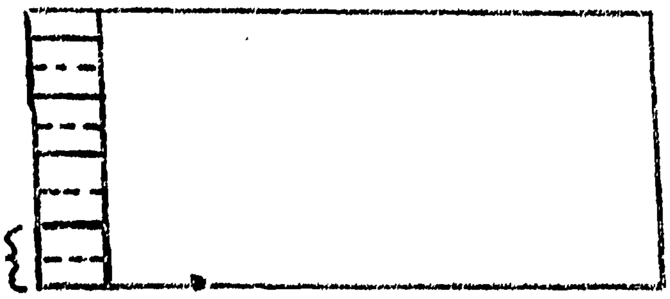
Next we consider how many columns in the rectangle. Suppose we find 10; that is, the ratio of the rectangle to one column is 10 to 1. Then we see we have:

$$\begin{tikzcd}
\text{Rectangle} \arrow[rr, "?"] \arrow[dr, "60/6 = 10/1"'] & & \text{Unit} \\
& \text{One Column} \arrow[ur, "6/1"'] &
\end{tikzcd}$$

and the solution is 60 to 1 or there are 60 square inches of area.

Now suppose we have another rectangle and wish to find its area. We find the ratio of one column to one square inch. We have this diagram:

one square inch

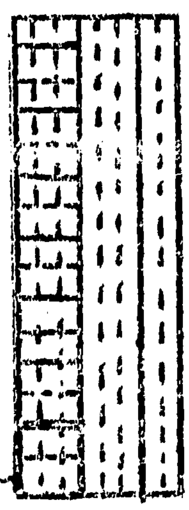


That is, we have the ratio of 9 to 2. Suppose the ratio of rectangle to one column is 10 to 1. Then we have

$$\begin{tikzcd}
\text{Rectangle} \arrow[rr, "90/2"] \arrow[dr, "90/9 = 10/1"'] & & \text{Unit} \\
& \text{One Column} \arrow[ur, "9/2"'] &
\end{tikzcd}$$

so the ratio of rectangle to one square inch is 90 to 2 or 45 to 1 so we say there are 45 square inches of area in this rectangle.

Now consider the more typical problem. Consider this rectangle:



one square inch

We see the ratio of one column to one square inch is 15 to 2 and the ratio of the rectangle to one column is 8 to 3. So that we have

$$\begin{tikzcd}
\text{Rectangle} \arrow[rr, "60/3 = 20/1"] \arrow[dr, "60/8 = 15/2"'] & & \text{Unit} \\
& \text{One Column} \arrow[ur, "8/3"'] &
\end{tikzcd}$$

so that the ratio of the rectangle to one square inch is 20 to 1, or there are 20 square inches of area.

In general, if the ratio of the rectangle to one column is ℓ to 1, and the ratio of one column to one square unit is w to 1, then we have

$$\begin{tikzcd}
\text{Rectangle} \arrow[rr, "\ell w/1"] \arrow[dr, "\ell w/w = \ell/1"'] & & \text{Unit} \\
& \text{One Column} \arrow[ur, "w/1"'] &
\end{tikzcd}$$

so the ratio of the rectangle to one square unit is ℓw to 1, or the area is ℓw square units. We can return to this problem later in the applications unit.

Exercises

1) If the ratio of A to B is 2 to 3 and the ratio of B to C is 5 to 5, find the ratios of A to C and $A \cup C$ to B. Set up diagrams as in the earlier problems.

2) If A is a subset of B and the ratio of A to B is 2 to 5, suppose C is the rest of B that is not A. Find the ratio of C to A and $A \cup C$ to B. Draw diagram.

3) A grandfather owned 7/8 of the stock in a company. He has 3 sons and one son had 2 sons. Suppose all share equally. How much of the stock did one grandson inherit? Set up ratios and diagrams (if necessary).

4) Take a small square piece of paper and estimate the area of a rectangular table top by the method suggested above. Find the ratio of one column of squares to one square and the ratio of the table top to one column of squares. Then find the ratio of the table top to the square. (If you take an ordinary sheet of paper, you can make an 8-1/2" x 8-1/2" square, which has the approximate ratio to one foot of 4 to 9. Hence you could estimate the number of square feet in the table top.)

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Unit 4. From Ratios to Rational Numbers

4.0 Rational Numbers

After years of counting sets, man developed the arithmetic of whole numbers to make the counting and combining of large sets much easier to do. Long practice of counting 2 groups of 3 things each for a total of 6 led to the shorthand $2 \times 3 = 6$. Similar "shorthand" work led to the arithmetic we now consider so functional in everyday life that we rarely think "behind" any of the simple algorithms we use to compute. But it is very important that children be led through a development of the algorithms of whole numbers that shows their efficiency in a meaningful way. Just as we develop the computational work in whole numbers out of counting, so we will develop a computational scheme for our ratio computations which we developed in the previous unit. In order to do this, we first need to have a set of numbers and operations on these numbers.

What should be our new numbers? We want something to stand in place of a ratio. A ratio was the set of one-to-one correspondences that could be made between two finite sets. Since our numbers must be general and apply to many different situations, to be valuable, we must generalize our idea of ratio.

We saw that the ratio of A to B might be the set of correspondences: $\{1/2, 2/4, 3/6, 6/12\}$. We were limited by the actual number of elements in A and B. If we doubled the sizes of A and B, forming the sets A' and B', we would increase the number of possible correspondences. In fact, we would then have $\{1/2, 2/4, 3/6, 6/12, 4/8, 12/24\}$. If we tripled the sizes of A and B, we would increase again the number of correspondences.

We will consider this situation more carefully. We will also attempt to deal with it more abstractly. Remember we called any symbol of the form a/b,

where b is not zero, a fraction. Notice that a ratio is named by a set of fractions and that a correspondence is named by a fraction. Now we will say that two fractions are equivalent if they name two correspondences that can be made between the same pair of sets. In our case above we can say 1/2 is equivalent to 6/12 (we write 1/2 = 6/12) because they name two correspondences between A and B; similarly 1/2 is equivalent to 12/24 (1/2 = 12/24), because they name two correspondences between A' and B'. What about 1/2 and, say, 16/32? We sense that they should be equivalent, that is, we feel that 1/2 = 16/32. We must find two sets with these correspondences. Let C have 16 elements and D have 32 elements. Then we have both a 1/2 correspondence and a 16/32 correspondence. So, indeed, we have two sets, C and D, and two correspondences between them named by 1/2 and 16/32, so 1/2 = 16/32.

So our set of correspondences {1/2, 2/4, 3/6, 6/12} can be enlarged; if we write down the set of fractions equivalent to 1/2. We have, in this enlarged set {1/2, 2/4, 3/6, 4/8, 5/10, 6/12, . . ., 100/200, . . . 500/1000, . . .}. This "enlarged" ratio we will call a rational number. We will name it by using any fraction in it, say 1/2 or 5/10. We will usually use the fraction for the simple correspondence, in this case 1/2. So we have the rational number 1/2:

1/2 = {1/2, 2/4, 3/6, . . ., 50/100, . . .}.

We can now refer to the rational number 1/2 in the same manner we referred to the ratio 1 to 2 earlier. Remember that rational number is an extended idea of ratio. So that for any ratio situation we have a specific rational number.

Suppose we want to talk about the rational number, 5/7, then we are thinking of the fraction names for all of the correspondences that can be made between any two sets that have the correspondence 5/7, such as 10/14, 15/21, 50/70, etc. So 5/7 = {5/7, 10/14, 15/21, . . ., 50/70, . . .}.

4.1 Multiplication of Rational Numbers

Now our problems we had in Unit 3 can be translated into rational number problems. For example,

$$\begin{tikzcd} & B \arrow[dr, "4/5"] & \\ A \arrow[ur, "3/4"] \arrow[rr, "3/5"'] & & C \end{tikzcd}$$

was a situation where we obtained the ratio 3 to 5 from the ratios 3 to 4 and 4 to 5. Now we see that this translates into obtaining the rational number 3/5 from the rational numbers 3/4 and 4/5. We worked the previous ratio problem by considering our correspondences; we want to work the rational number problem by using the fractions which name the correspondences. The situation above gives us a rational number from two rational numbers; that is, we have an operation on rational numbers. We will call it multiplication. In general,

$$\begin{tikzcd} & B \arrow[dr, "b/c"] & \\ A \arrow[ur, "a/b"] \arrow[rr, "a/c"'] & & C \end{tikzcd}$$

We see that a/b and b/c gives a/c. So we have that the multiplication of a/b and b/c (write a/b x b/c) is a/c (write a/b x b/c = a/c). So our first example becomes

$$3/4 \times 4/5 = 3/5.$$

We see with little difficulty that 3/2 x 2/7 = 3/7 and 3/11 x 11/4 = 3/4. (Just think of the associated correspondences and draw a diagram.)

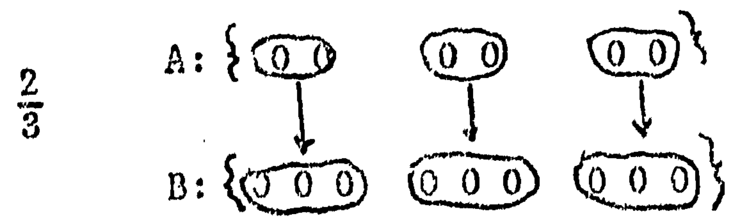
Suppose we have rational numbers 3/4 and 1/2 and wish to multiply them. What is the answer? We look to this ratio problem for the answer:

$$\begin{tikzcd} & B \arrow[dr, "1/2"] & \\ A \arrow[ur, "3/4"] \arrow[rr] & & C \end{tikzcd}$$

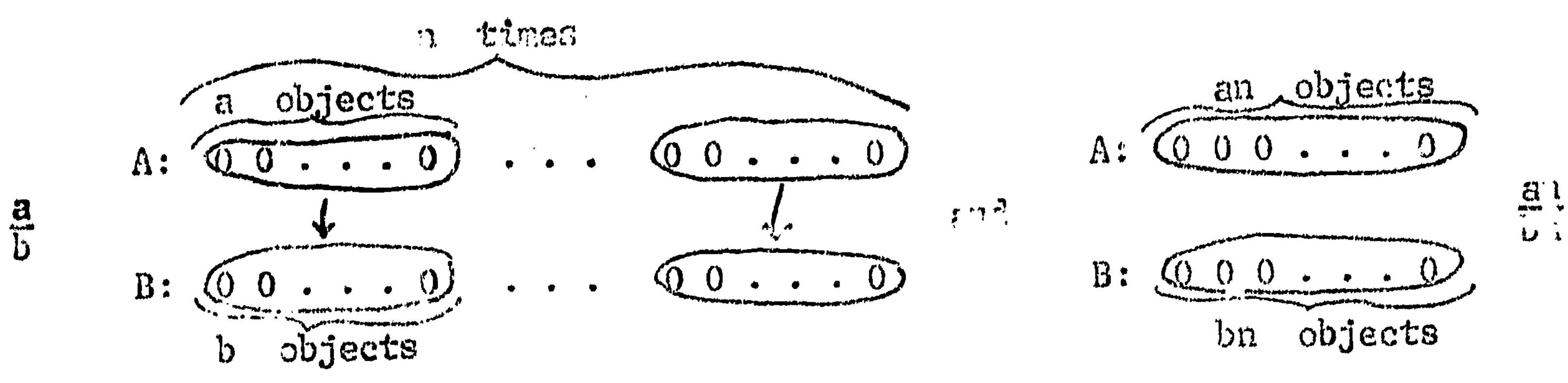
To solve this problem, we had to consider another correspondence from B to C, namely, 4/8. Then from 3/4 and 4/8 we obtained 3/8 as the new correspondence from A to C and so 3 to 8 was the new ratio.

We follow the same algorithm in rational numbers. Thus 3/4 x 1/2 becomes 3/4 x 4/8 or 3/8 by writing a different name for the rational number 1/2. Since any fraction in a rational number can be used to name it, this is perfectly all right. We change fractions, as necessary, to obtain the form a/b x b/c and we know the answer is a/c.

Now we would like a neat computational rule for multiplying any two rational numbers a/b and c/d. We go back to our correspondences and notice one more fact. If given the correspondence 2/3 we can get the name of an equivalent correspondence (an equivalent fraction) by multiplying the numerator and denominator by the same number. That is, we have $\frac{2 \times 3}{3 \times 3}$ is an equivalent fraction to 2/3. This is so because $\frac{2 \times 3}{3 \times 3} = \frac{6}{9}$ and if we have a set A of 6 objects and a set B of 9 objects, we have clearly a 6/9 correspondence and by grouping we have:



So we have 2/3 and $\frac{2 \times 3}{3 \times 3}$ equivalent fractions. Given a/b we see that $\frac{a \times n}{b \times n}$ is an equivalent fraction by considering



We have both correspondences a/b and $\frac{an}{bn}$ from A to B so we have equivalent fractions a/b and $\frac{an}{bn}$.

Now our general problem of $a/b \times c/d$ can be answered by replacing a/b and c/d by the proper equivalent fractions, namely $a/b = \frac{a \times c}{b \times c}$ and $c/d = \frac{b \times c}{b \times d}$ so

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times c} \times \frac{b \times c}{b \times d} = \frac{a \times c}{b \times d}$$

That is, in brief, the multiplication of $a/b \times c/d$ is the rational number $\frac{a \times c}{b \times d}$.

So if given $3/4 \times 1/2$ we see that we can write

$$\frac{3}{4} \times \frac{1}{2} = \frac{3 \times 1}{4 \times 1} \times \frac{4 \times 1}{4 \times 2} = \frac{3 \times 1}{4 \times 2} = \frac{3}{8} \quad \text{or} \quad \frac{3}{4} \times \frac{1}{2} = \frac{3 \times 1}{4 \times 2} = \frac{3}{8}$$

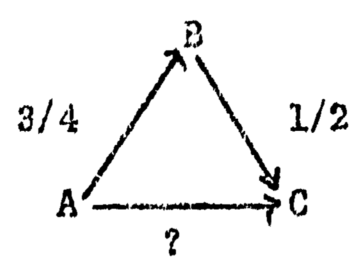
and if given $1/8 \times 3/4$ we can write

$$\frac{1}{8} \times \frac{3}{4} = \frac{1 \times 3}{8 \times 3} \times \frac{8 \times 3}{8 \times 4} = \frac{1 \times 3}{8 \times 4} = \frac{3}{32} \text{ or, in brief,}$$

$$\frac{1}{8} \times \frac{3}{4} = \frac{1 \times 3}{8 \times 4} = \frac{3}{32}.$$

You will see that this is exactly what was done to obtain the answer to some of the ratio problems in Unit 3.

Now the problem



can be quickly solved, $3/4 \times 1/2 = 3/8$. So we have an <u>efficient computation</u> for our previous problem situations, if we use our new operation on rational numbers. (Reread Section 4.0.)

<u>Exercises</u>

1) Multiply: $1/2 \times 3/1$, $2/3 \times 7/2$, $1/3 \times 3/4$, $2/3 \times 1/5$; use the rule $a/b \times b/c = a/c$.

2) Go back to Unit 3 and pick out problems like $A \longrightarrow B \longrightarrow C$ and solve by multiplying.

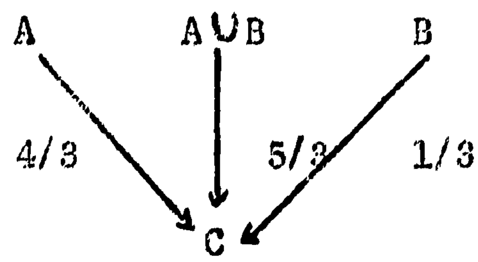
3) Solve the area problems in Unit 3 by multiplying.

4) If there are 3 girls for every 4 men and 4 men for every 7 boys, find the ratio of girls to boys by multiplying.

5) Solve $2/3 \times \square = 7/8$ by going back to a ratio diagram.

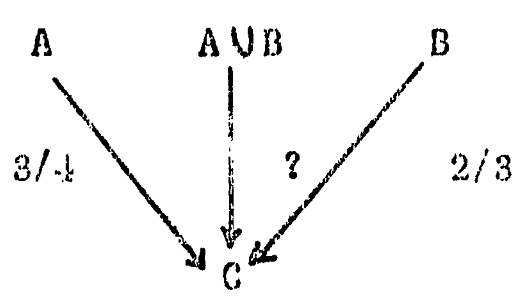
4.2 Addition of Rational Numbers

Now we will look at another operation on rational numbers that comes from our ratio work. Consider the situation

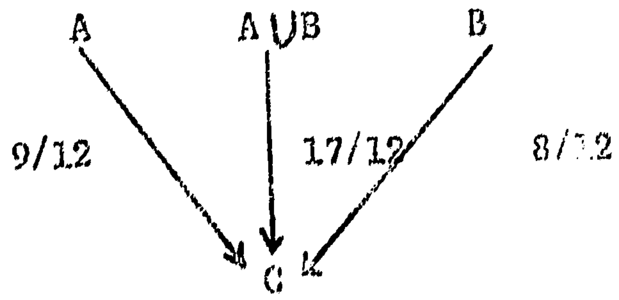


where we consider the union of the sets A and B. Remember, we obtained a new ratio (5 to 3) from the given ratios 4 to 3 and 1 to 3. This problem suggests that the rational number 5/3 can be obtained from 4/3 and 1/3. We see that multiplication would give $4/3 \times 1/3 = 4/9$, so apparently we have another operation for rational numbers. The use of union of sets suggests we call this operation addition, and we will use the sign "+". We write $4/3 + 1/3 = 5/3$. Our past work with ratio problems suggests the rule $b/a + c/a = \frac{b+c}{a}$. So we adopt this rule and say that the addition of the rational numbers b/a and c/a is the rational number $\frac{b+c}{a}$. We call $\frac{b+c}{a}$ the sum of b/a and c/a. We can say that $2/5 + 6/5 = 8/5$ and $1/4 + 7/4 = 8/4 = 2/1$.

Now what about $3/4 + 2/3$? We want a fraction for their sum. Consider the associated situation using correspondences:



What did we do in our ratio problems? We used new correspondences that we knew could be made, namely $3/4 = 9/12$ and $2/3 = 8/12$. Then our problem was solved



and the ratio of C to $A \cup B$ was 17 to 12. So we follow the same pattern and replace our fractions by equivalent fractions so as to have the form $b/a + c/a$.

$$3/4 + 2/3 = 9/12 + 8/12 = 17/12$$

We can do this because any fraction in the rational number can name it.

Another example is

$$2/5 + 1/4 = 8/20 + 5/20 = 13/20.$$

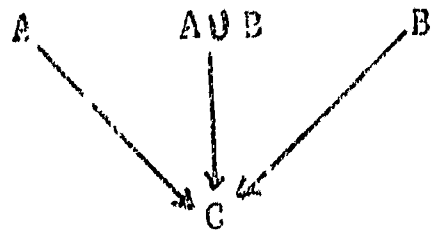
This suggests a brief computational form again:

$$a/b + c/d = \frac{a \times d}{b \times d} + \frac{b \times c}{b \times d} = \frac{a \times d + b \times c}{b \times d}$$

Exercises

1) Add: $1/2 + 3/2$, $4/6 + 1/6$, $7/8 + 1/2$, $1/7 + 1/8$, by the rules given above. Draw ratio diagrams for the 2nd and 3rd problem to review the connection!

2) Go back to Unit 3 and work the problems of form by addition.



3) If there are 3 girls for every 4 men and 7 boys for every 4 men, find the ratio of children to men by addition of the rational numbers involved.

4) Solve $2/3 + \square = 7/4$ by going back to a ratio diagram.

We have obtained two operations on our new numbers by working from our ratio problem of Unit 3 and applying our extended ratio ideas (rational numbers) to them. Remember, the set of rational numbers and their brief computational

forms were developed in order to make easier the work involved in solving the kinds of problems we found in Unit 3. The purpose was to become more efficient, exactly as the whole number algorithms are more efficient than counting.

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Unit 5. Some Special Rational Numbers

5.0 Introduction

We have seen that rational numbers are a natural extension of our ideas on ratio. Once we have established a set of numbers and operations on those numbers, we next want to investigate to see if any of the numbers have "special" properties. You will recall that, in the whole number system, 0 and 1 play important roles as identities, that is, for any whole number w, $w + 0 = w$ and $w \cdot 1 = w$. We would like to find some similarities in our set of rational numbers. We would like this new set of numbers to have some mathematical properties that would make it a number system.

5.1 Identities and Inverses

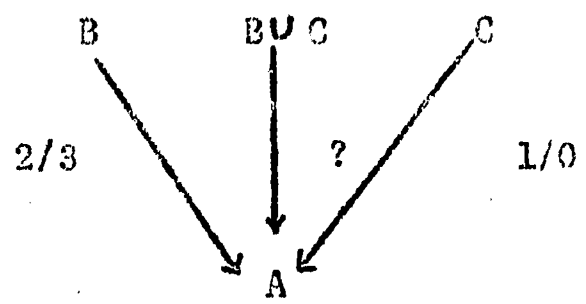
Given any rational number a/b, we want to find a rational number which when multiplied with a/b gives a/b. We have $\frac{a}{b} \times \frac{x}{y} = \frac{ax}{by}$, and we would like $ax = a + by = b$ to be true. If $x = 1$ and $y = 1$ we have what we wish; so $\frac{a}{b} \times \frac{1}{1} = \frac{a \cdot 1}{b \cdot 1} = \frac{a}{b}$, so $\frac{1}{1}$ acts as an identity for rational number multiplication. Remember that $\frac{1}{1} = \{1/1, 2/2, 3/3, 4/4, 5/5, \ldots\}$ where 5/5, for example, is a fraction for a correspondence. We could also show that there is no other rational number except 1/1 which has this property.

We would also like to find a rational number that is an additive identity. We want some $\frac{x}{b}$ so that for every $\frac{a}{b}$, $\frac{a}{b} + \frac{x}{b} = \frac{a}{b}$. Now by our addition rule $\frac{a}{b} + \frac{x}{b} = \frac{a + x}{b}$, so $a + x = a$ is required. Now in whole numbers, $x = 0$ is a solution. So we are lead to consider $\frac{0}{b}$ as an identity element. Our first question is: Is $\frac{0}{b}$ a rational number?

What is the meaning of 0/b in a correspondence? It says we have a one-to-one correspondence between two sets A and B where groups of 0 elements

each in A are matched with groups of b elements each in B. Since we understand all the groups to be of equal size, this requires A to be the empty set. Although there would be no physical example for this situation, we can see that it is an extension of our earlier ideas of correspondence. Since we would like our system to have a "zero" element in it, and since the use of the fraction $\frac{0}{b}$ leads to no contradiction, we accept $0/1 = \{0/1, 0/2, 0/3, . . . 0/b, . . .\}$ as a rational number.

However, we have just introduced a problem also. Before we saw that any a/b correspondence had an inverse correspondence of b/a. Is the inverse of 0/1, namely 1/0, a correspondence, and is $1/0 = \{1/0, 2/0, 3/0, . . . , b/0, . . .\}$ a rational number? First of all, 1/0 is as good a correspondence as 0/1 since we merely can consider reversing the arrows in any 0/1 diagram. So we consider 1/0 as a rational number candidate. But we have problems immediately. Suppose we wish to find 2/3 + 1/0. Returning to our correspondences which motivated our definition of addition, we find that in no way can we have a solution for this situation,



since 1/0 requires A to be empty and 2/3 requires it to be non-empty. So 1/0 gives us trouble. But even if we abandon our correspondences and just follow the addition definition we have

$$2/3 + 1/0 = \frac{2 \cdot 0 + 3 \cdot 1}{3 \cdot 0} = 3/0 = 1/0 \qquad \text{and also}$$

$$1/2 + 1/0 = \frac{1 \cdot 0 + 2 \cdot 1}{2 \cdot 0} = 2/0 = 1/0.$$

So we are led to believe 2/3 = 1/2, which is not so. So we must abandon 1/0

as a rational number. We can show, in fact, that $0/1$ is the only rational number with no inverse in the sense of inverse correspondences.

We now have all of the rational numbers we can form out of the whole number system. Any set of equivalent fractions of form a/b where a and b are whole numbers and $b \neq 0$ is a rational number. We also can now say that $0/1$ is the additive identity for rational numbers; that is, $a/b + 0/1 = a/b$ for any a/b.

We can also observe that our inverse ratios and hence inverse rational numbers have an interesting property that whole numbers do not have. Given a whole number, say 5, we know that $5 \cdot 1 = 5$ and also that no whole number multiplied by 5 will give 1. But given a rational number a/b, $a \neq 0$, we have $a/b \times 1/1 = a/b$ and also we have a <u>unique inverse</u> of a/b, namely b/a, and $a/b \times b/a = ab/ab = 1/1$. We have a way of multiplying to return to 1/1. We call b/a the <u>multiplicative</u> inverse of a/b. Knowing such a rational number allows us to solve equations we cannot similarly solve in whole numbers. We cannot solve $3x = 7$ in whole numbers, but a similar rational number problem, $3/5 \times X = 7/4$ can be solved, for $(5/3 \times 3/5) \times X = 5/3 \times 7/4$ or $1/1 \times X = 35/12$ or $X = 35/12$. So we see that in a real sense we have <u>more</u> that we can do in the set of rational numbers.

5.2 <u>Rational Numbers and Whole Numbers</u>

Now we can return to our ideas of counting and look at it from a different point of view. If we say we have 6 apples, or 5 quarts of water, or 3 pints of strawberries, we are, in each case, comparing our amount to some understood unit. When we say we have 6 apples, we mean we are using 1 apple as a basic unit. So we have a disguised ratio situation, namely, the ratio of our amount to one apple is $6/1$. Similarly 5 quarts of water can be viewed: the ratio of this amount of water to one quart is 5/1 [illegible] pints of strawberries becomes the ratio of this amount of strawberries to one pint, namely, $3/1$.

Thus in this way any whole number may be "extended" and viewed as a ratio. Hence a whole number a can be associated with the rational number $a/1$. This association would be stronger if our operations agree also. We see that $3 + 2 = 5$ is associated with $5/1$ and $3/1 + 2/1 = 5/1$. Also $3 \cdot 2 = 6$ is associated with $6/1$ and that $3/1 \times 2/1 = 6/1$. So we see that any whole number problem in addition or multiplication can be translated into a rational number problem and still obtain the same results. It is just this fact that allows us to be "careless" in mixing whole number and rational number arithmetic.

Consider, for example, the "computation", $3 \times 1/2$; since we have 2 different number systems represented our ordinary rules don't really apply. Suppose we forget this for the moment. We can translate "$3 \times 1/2$" into a whole number situation, or "repeated additions"; i.e., $3 \times 1/2 = 1/2 + 1/2 + 1/2 = 3/2$. Or we can translate "$3 \times 1/2$" into a rational number problem, $3 \times 1/2 = 3/1 \times 1/2 = 3/2$. We obtain the same answer in either case. Here is a second example: "$3 + 1/3$." As a whole number sum this has no meaning, but as a rational number sum: $3 + 1/3 = 3/1 + 1/3 = 10/3$. A third more complex situation is $3 \times (2 + 1/4)$. As a rational number this is, $3/1 \times (2/1 + 1/4) = 3/1 \times (9/4) = 27/4$. As whole number ideas we have: $3 \times (2 + 1/4) = 2 + 1/4 + 2 + 1/4 + 2 + 1/4 = 6 + 3/4$; as in the second example we can have $6 + 3/4 = 24/4 + 3/4 = 27/4$. So they are the same.

In fact, generally speaking, one can do the above "intermixing" of whole number and rational number arithmetic operations (because of the way they were defined) and be led to no "wrong" answers. However, such action would be done wisely <u>only</u> as an added feature and should not be given as the standard form for the operation.

In the physical world the preceding "blending" of whole numbers and rational numbers is sometimes disastrous in attempting to solve real world problems. For verbal problem work it is essential to maintain the distinction to clarify

problem situations. The situation of the whole number 3 multiplied with the rational number 2/3 never occurs in the real world according to our physical understandings about correspondences which served as models for our rational numbers. Some problems may be thought of as whole number problems as well as rational number problems, but not as a mixture of the two. In the unit to follow on applications only rational number situations will be considered, but we will see where many problems could be equally solved by whole number methods.

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Unit 6. Applications

6.0 Introduction

In the preceding units we have developed the concepts of ratio and correspondence in physical situations and have [illegible] defined the set of rational numbers and operations on this new set of numbers. In this unit we wish to bring to bear all of our preceding work on the solution of various types of practical applications. Some of these problem types occur in regular junior high school texts. Some will simply illustrate how the correspondence notion can clarify regular rational number problem types.

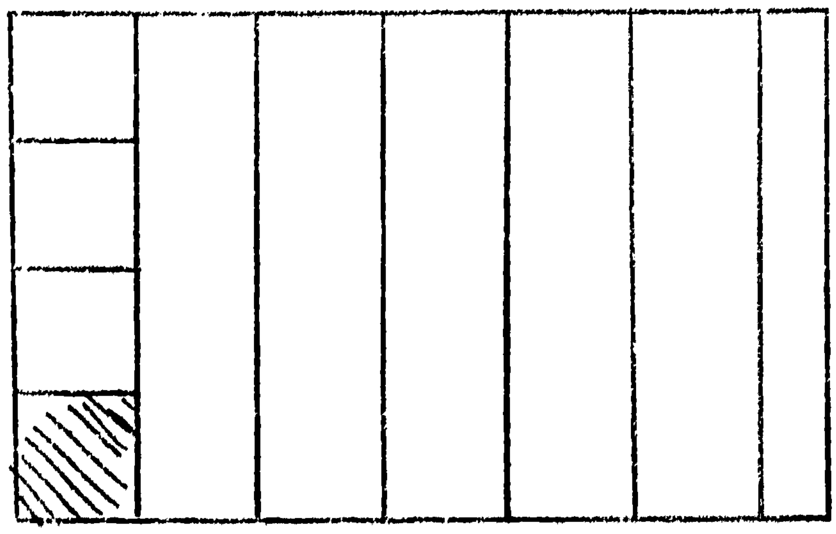
6.1 Area of Rectangles

First, we have some unfinished business to consider. In Unit 2, we introduced and worked with areas of rectangles by means of correspondences. We can now finish this area problem. Recall, given a rectangle and a standard unit of area measure, that we computed the area by forming two correspondences: total rectangle to one column and one column to unit area. We may have had, for example,

$$\begin{tikzcd} \text{Total Rectangle} \arrow[rr, "14/3"] \arrow[dr, "14/4 = 7/2"'] & & \text{One Unit} \\ & \text{One Column} \arrow[ur, "4/3"'] & \end{tikzcd}$$

so we say that the area of the rectangle is 14/3 square units. According to our definition of multiplication of rational numbers we can say that $7/2 \times 4/3 = 14/3$ is the rational number corresponding to the area of the rectangle. Or, more loosely, that 14/3 is the <u>number</u> of square units of area.

Consider this diagram:



The square unit of area, say <u>one square inch</u>, has an edge of one linear unit, in this case <u>one inch</u>. Now we observe that in one column there are 4 square units and also that, of course, the height of the column is 4 linear units. Similarly, the ratio of the total rectangle to one column (which essentially is a measure of "how many" columns) is 13 to 2 and that the ratio of the length of the rectangle to one linear unit is also 13 to 2. Whereas before we considered two ratios 4 to 1, followed by 13 to 2, we can now consider the two rational numbers $13/2$ and $4/1$ and can think of them as measuring the length of the rectangle and the height (or width) of the rectangle. The rational number operation used is multiplication; so the area is $13/2 \times 4/1 = 26/1$ square units. So the product of the rational number of the length of the rectangle (l) and the rational number of the height (or width) (w), namely $l \times w$, is the rational number of the area of the rectangle, where we understand that square units must be used. Hence, the shortcut <u>formula</u> for area: $A = l \times w$.

Let us do another example of area, retracing our steps. Suppose we are told that the rectangle has a length of $7/3$ inches and a height of $15/2$ inches. We can infer from this that we have an area of $7/3 \times 15/2$ square inches or $105/6$ square inches. Retracing our previous argument we further infer that the ratio of the total rectangle to one column was 7 to 3 and that the ratio of one column to one square inch was 15 to 2. So, by diagram,

$$\begin{tikzcd}
\text{Total Rectangle} \arrow[rr, "105/6"] \arrow[dr, "105/45\,=\,7/3"'] & & \text{One Square Inch} \\
& \text{One Column} \arrow[ur, "15/2\,=\,45/6"'] &
\end{tikzcd}$$

and the ratio of Rectangle to One Square Inch was 105 to 6.

Again we see that rational number multiplication gives us an easy way to compute the area of a rectangle. It is also important to remember that if we consider only the "product of length and width" argument to obtain a number, then we lose the essential nature of the problem. This is the problem with all computational short-cuts; they always give easy answers, but often the sense of the problem is lost. Thus it is important to keep the idea of the ratio of Total Area to one square unit of area in the foreground as the meaning of "area." We will see more examples of the distinction between meaning (or understanding) of the situation and the computation of the answer.

6.2 Proportion and a Rule

It is common to include with ratio the concept of a proportion. Since many texts use the two terms frequently, we should be clear on their meaning in the context of these units. A problem will introduce the idea. If John receives 3 of every 5 apples and there are 25 apples, how many does John receive? A "standard" solution is to say, "Set up the proportion, $3/5 = \boxed{n}/25$, and solve." The answer is, of course, 15 apples. How should we view the idea of proportion in the light of our previous discussions? What is 3/5? Isn't it a correspondence from John's apples to the total number of apples? What would $\boxed{n}/25$ be then? That's right, another correspondence from John's apples to total apples. So this proportion apparently asserts: "find $\boxed{n}$ so that $\boxed{n}/25$ and 3/5 are two correspondences for the same ratio." (Or translating into rational number language, we can also say, "$\boxed{n}/25$ and 3/5 are fractions

for the same rational number.) Hence, to say two fractions are in proportion is to say that they are equivalent, or to say two correspondences are in proportion means that they are in the same ratio.

Now how would we "solve" $3/5 = \boxed{n}/25$? Here we can "see" that the answer is 15. To be more explicit, we can write an equivalent fraction to 3/5 by multiplying the numerator and denominator by the same number so $\frac{3 \times m}{5 \times m}$ is any equivalent fraction to 3/5. If $m = 5$, then we have $\frac{3 \times 5}{5 \times 5} = 3/5 = \boxed{n}/ 25$ or $15/25 = \boxed{n}/25$. Now, from our knowledge of correspondences we can assert that $\boxed{n}$ must be 15. So we apparently have a rule that requires changing the denominators (or numerators) so they are the same number. Then the numerators (or denominators) must name the same number also. So if $a/b = c/d$, then $ad/bd = bc/bd$ and we can say $ad = bc$. Also if $ad = bc$, then $ad/bd = bc/bd$ or $a/b = c/d$. So we have the rule: $a/b = c/d$ means the same as $ad = bc$. Or, in words, "a/b and c/d are in proportion" means the same as "ad = bc." This rule allows us to check whether any stated proportion is true. For example, is $3/4 = 9/16$ true? Does $3 \times 16 = 4 \times 9$? No! So the proportion is false. Suppose we have $3/4 = 36/48$. Is this true? Is $3 \times 48 = 4 \times 36$ true? Yes! So it is a true proportion. We can make frequent use of these ideas in the problems dealing with proportions.

6.3 Sample Ratio Problems From Textbooks

It would be much too lengthy a project to attempt to cover most of the problems found in current texts. Instead, we will consider a selection of three of the most common types and discuss them. We will first consider each type by solving a typical member of the type.

I) "Find the ratio of 2 pounds to 6 ounces." What is called for here is clear. A diagram will make it even clearer:

$$\text{"2 pounds"} \xrightarrow[32/16 = 2/1]{} \text{1 pound} \xrightarrow[16/1]{} \text{1 ounce} \xrightarrow[1/6]{} \text{"6 ounces"}$$

and our past work says that we have

$$\text{"2 pounds"} \xrightarrow[32/1]{} \text{1 ounce} \xrightarrow[1/6]{} \text{"6 ounces"}$$

So the ratio of 2 pounds to 6 ounces is 32 to 6 or <u>16 to 3</u>. A common approach to this problem is to "change" 2 pounds to 32 ounces (to make "the units the same"). This amounts to forming a ratio in itself. However, note that this type diagrams easily and involves little of difficulty after that. We may apply rational numbers now by multiplying in a similar fashion the associated rational number, (2/1 x 16/1) x 1/6, which gives the rational number 16/3.

Consider another sample of the same type: Find the ratio of 36 yards to 48 feet. We diagram:

$$\text{"36 yards"} \xrightarrow[36/1]{} \text{one yard} \xrightarrow[3/1]{} \text{one foot} \xrightarrow[1/48]{} \text{"48 feet"}$$

which yields

$$\text{"36 yards"} \xrightarrow[36/1]{} \text{one yard} \xrightarrow[3/48 = 1/16]{} \text{"48 feet"}$$

so the ratio of 36 yards to 48 feet is 36 to 16 or 9 to 4.

Exercises

(Please draw diagrams for each of these problems.)

1) Find the ratio of 3 inches to 4 feet.
2) Find the ratio of 6 gallons to 4 pints.
3) Find the ratio of 2 rods to 11 inches. (1 rod = 16-1/2 feet)
4) Find the ratio of 3 quarts to 5 pints.
5) Find the ratio of the capacity of a pail to 3 pints if the ratio of the pail to 2 quarts is 5 to 2.
6) Find the ratio of capacity of pail A to pail B if the ratio of 3 gallons to pail A is 3 to 7 and the ratio of pail B to 1 quart is 9 to 2.

II) "80% of the class were present, that is, 16 were present. How many people are in the class?" This is a typical percentage problem. It translates into the same problem type above. 80% of the class present is a disguised ratio: that is, the ratio of those present to the total class was 80 to 100 or 4 to 5. We have $4/5 = 16/n$; and since $\frac{4 \times 4}{5 \times 4} = \frac{16}{20} = \frac{16}{n}$ we see that 20 were in the class. Normally for any percent problem, changing the percent form of a ratio to our standard form obviates the problem. This is, x% names the ratio x to 100.

Another example: "70% of the tomato plants grew; Sue had set out 30 plants; how many grew?" The ratio of plants that grew to total plants is 70 to 100 or 7 to 10, or 21 to 30 so 21 plants grew out of 30.

Here is an example of the third type of percent problem: "John made 8 baskets out of 32 baskets tried in a basketball game. What was his percent of baskets made?" We have the ratio given 8 to 32 or 1 to 4 and we seek the percent, that is, the ratio name with 100 for a second number. If 1 to 4 is the ratio, another name would be 1×25 to 4×25 or 25 to 100 or 25%. So John made 25% of the baskets. The percent name for a ratio is a poor one from our point of view since it suggests that correspondences of the form x to 100 <u>can</u> be made; where in reality, as in the last problem, John only attempted 32 baskets. We can say that <u>if</u> we can group by 100's, <u>then</u> the corresponding groups will have size x.

Exercises

1) John sold 75% of his papers; he started with 60 papers; how many did he sell?
2) Mike lost 60% of his marbles; he had only 20 left; how many did he have before he lost any?
3) Mary made 3 out of every 15 cookies shaped like bells; what percent of the cookies were bell-shaped?
4) Henry found that the interest on his savings account was $130 for the year. He knew the bank paid 4% interest. How much was his principal at the beginning of the year?

6.4 Problems Using Rational Numbers

A common problem type is one which involves a simple multiplication or division of rational numbers. Such a problem could be: The Jones family spends 3/10 of its income for food; if their annual income was \$5200, how much did they spend on food for the year? We would normally say: "Well multiply!" Thus the student may produce the answer in this way: "3/10 x 5200/1 = 15600/10 = 1560/1. \$1560." But he may very well fail to understand what is involved. Suppose we set up a diagram to illustrate the situation:

$$\begin{tikzcd} \text{Part} \arrow[rr, "?"] \arrow[dr, "3/10"'] & & \$ \\ & \text{Whole} \arrow[ur, "5200/1"'] & \end{tikzcd}$$

We wish to discover the ratio from the part of the income to dollars. We discover the ratio is 1560 to 1 so we say they spent \$1560 for food. Consider the much harder problem for students: "The Smith family spends 3/10 of its income for food; the last year's food costs were \$1275 ; what was the total income? Again by diagram:

$$\begin{tikzcd} \text{Whole} \arrow[rr] & & \$ \\ & \text{Part} \arrow[ul, "3/10"] \arrow[ur, "1275/1"'] & \end{tikzcd}$$

Now we see that, by our past work, that we can change the diagram by considering the inverse ratio of whole to part:

$$\begin{tikzcd} \text{Whole} \arrow[rr] \arrow[dr, "10/3"'] & & \$ \\ & \text{Part} \arrow[ur, "1275/1"'] & \end{tikzcd}$$

and hence our answer is 1275/1 x 10/3 = 12750/3 = 4250/1. So the income was \$4250.

By drawing the diagrams and thinking in terms of ratios between sets, the student can eliminate the guesswork involved. Hence he can introduce an element of concreteness into his work. A similar example would be: "Sue cut off 3/4 of the length of ribbon; she measured the part cut off and found it was 25-1/2 feet long; how long was the total ribbon?" We understand that we are to find the ratio of the length of the ribbon to one foot. By a diagram we have: (Remember, 25-1/2 feet indicates a ratio of 51 to 2):

$$\begin{tikzcd} \text{Whole} \arrow[rr, "?"] & & \text{one foot} \\ & \text{Part} \arrow[ul, "3/4"] \arrow[ur, "51/2"'] & \end{tikzcd}$$

So we have by taking the inverse of the 3/4 correspondence:

$$\begin{tikzcd} \text{Whole} \arrow[rr, "?"] \arrow[dr, "4/3"'] & & \text{one foot} \\ & \text{Part} \arrow[ur, "51/2"'] & \end{tikzcd}$$

and so by multiplication we have $4/3 \times 51/2 = 204/6 = 34/1$. So the whole ribbon was 34 feet long; that is, the ratio of the length of ribbon to one foot was 34 to 1.

<u>Exercises</u>

(In each case, work by diagrams rather than by other methods.)

1) Jim spent 3/4 of his money for the baseball game. He had only \$6.40 when he started from home. How much did he have left?

2) Marilyn had cut off 6-1/4 feet of ribbon from the roll. The whole roll was supposed to have 25 feet in it. What percent had she cut off?

3) Tom paid \$2.50 for 1/3 of the roll of wire. If the roll contains 180 feet of wire, how much did one foot of wire cost?

4) Mike budgeted 2/5 of his paper route money for buying a bicycle. He earns \$3.50 every week. How many weeks will it take to buy a \$42.00 bicycle?