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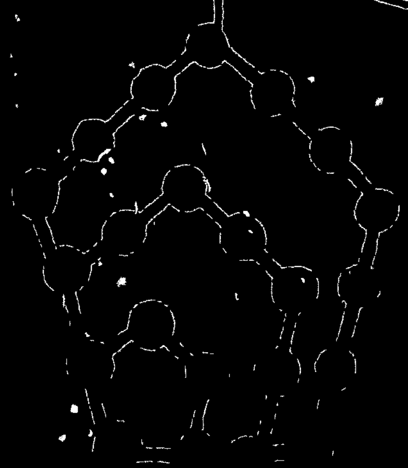
ABSTRACT

The development of number concepts from prehistoric time to the present day are presented. Section 1 presents the historical development, logical development, and the infinitude of numbers. Section 2 focuses on non-positional and positional numeration systems. Section 3 compares historical and modern techniques and devices for computation. Section 4 considers the numbers of today which include the real numbers, complex numbers, and hypercomplex numbers. (RP)

THE NUMBER STORY

六千五百三十二

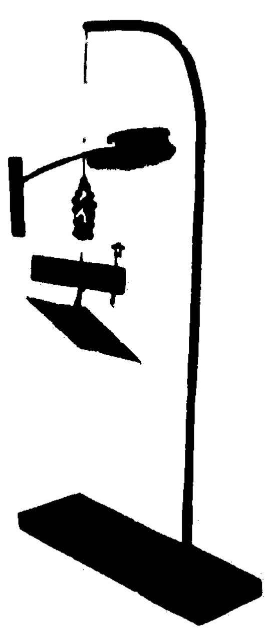
$$(a, b) = (c, d) \iff ad = bc.$$
$$(a, b) + (c, d) = (ad + bc, bd)$$
$$(a, b)(c, d) = (ac, bd).$$



$$\sqrt{c} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{9 + \cdots}}}}}}}}$$

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Mobility in numbers . . .



is the theme of this cover design. The design value of number symbols throughout history led us to make this mobile, which is built of wood, wire, copper tubing, acetate and photostats. We suggest that a cooperative project between the art and mathematics departments to create designs using mathematical symbols might stimulate interest in both subjects. Try it also using geometric forms.

KENNETH FRYE
Chief of NEA Art Section

THE NUMBER STORY

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INTRODUCTION

During the sixth century B. C., the world's earliest scientific society was holding meetings in Greece. Pythagoras, a pioneer in the teaching of mathematics, had founded a brotherhood of young men and women in which secrecy, magic and religion were conglomerated with number inquiries. These inquiries laid a profound groundwork for the future development of mathematics.

Pythagoras discovered that the pitch of a note produced by a plucked string is closely related to its length; mathematics may, therefore, be used for describing musical intervals. With this insight he conjectured that mathematical language is qualified to help unravel nature's secrets, and thus mathematical physics was born.

The startling discovery that revealed an interdependence between number and physical phenomena intrigued the master. He thought he had found the nucleus of man's most cherished dream—an understanding of the universe in which he lives. "Number is a universal measure of everything," he reiterated, "number rules the universe," "the essence of all things is number."

What did he mean by "number"? How have more than two millennia in the evolution of mathematics affected his notions? Was Pythagoras right in his philosophy that number is the key for an understanding of our world?

For more than 2000 years generations of mathematicians have endeavored to understand numbers fully. The concept of number has been extended and mathematical reasoning has become more rigorous and generalized. Has modern man, on the basis of these developments, lost Pythagoras' viewpoint?

1. THE GROWTH OF NUMBER NOTIONS

Historical Development

Somewhere in prehistoric time, some human being—maybe several at the same time—became inquisitive. It must have been a Pekingman who, looking perhaps at a group of animals, queried: *How many?* This question aroused the number notion, but whoever first exhibited this mental curiosity must forever remain anonymous.

What is it that enables man to conceive of number, to develop number systems? Is counting instinctive? Can all animals count?

NATURAL NUMBERS (TALLY NUMBERS)

Animal experiments have been reported indicating a non-linguistic ability for counting. This appears to be a capacity for comparing groups of elements side by side as well as for remembering unnamed numbers. These two facets are unified in man's faculty of number perception. Beyond this animal manner of thinking of numbers to which no names are assigned, man is the only animal possessing the faculty for language and the use of symbols.

In establishing simple number relationships, numbers are not always used. The young child who sets the table for the family dinner does not necessarily need to know that there are five in the group. She just puts "one for daddy, one for mummy, one for Johnny . . . " and thus employs a most important mathematical principle. It is the principle of matching the elements of two sets and establishing a one-to-one correspondence, mathematically speaking. The elements in the one set are the dishes; in the other are the members of the family.

This matching process leads to more than a comparison between two sets. If a model collection is known, say the fingers on one hand, then establishing a one-to-one correspondence with the members of this set means that there must be five elements in the given collection. Birds'

wings, clover leaves, legs of four-legged animals are other examples of such model collections. The origins of number words are often traced to these beginnings.

The use of the principle of correspondence leads to the concept of cardinal number. The artifice of counting requires linguistic ability. Without it, man would not have been able to arrive at insights into natural phenomena. Ordinal numbers were necessary to translate events into the language of mathematics.

An artificial collection must be created, a sequence in ordered succession. Counting, then, consists of matching the numbers of the given collection with the ordered numbers of this model. The last term of the model used in the matching process is called the ordinal number of the collection. This number at the same time fixes the plurality of the collection, its cardinal number. Two important principles penetrate our number system. They are the correspondence principle which leads to cardinal numbers and the principle of succession, the basis of ordinal numbers.

To obtain the abstract set of the model is very difficult—the mathematician would write it as $\{1, 2, 3, \cdots\}$. Fingers are used invariably to overcome the obstacles. Even after our prehistoric ancestor had already established some names for counting, he would have thought of "three bears" or "three trees" but not of "three." The step from understanding threesomes of that kind to understanding "three" is an enormous advance, and marks the initial important climb in mathematics. The very essence of mathematics is its abstractness, and with this first abstractive encounter all humans in their maturing process step into the world of mathematics.

Barbarous tribes have invariably disclosed some number concept even if a very limited one. Some primitive tribes have been found who conceive of one, two, and sometimes three things, but beyond that it would be "many."

Yet civilizations thousands of years before Christ already had a well-developed idea of number. The Egyptians, Sumerians, Babylonians, Indians, and later the Greeks and Romans understood large numbers and had various ways of recording them and operating with them. About 3500 B. C., the Egyptians recorded 120,000 prisoners, 400,000 captive oxen and 1,422,000 captive goats. The Greek Archimedes, a mental giant who anticipated his age by about two thousand years, invented a nomenclature by which he could express numbers up to $10^{8\times10^{6}}$. How early civilizations wrote their numbers and computed with them will be told later.

Counting numbers are called natural numbers or tally numbers, probably because they occurred to man almost naturally. In some

mathematical treatises it seems expedient to include zero among the natural numbers and to conceive of them as the set $\{0, 1, 2, \cdots\}$. In our discussion, we will mean the ordinary counting numbers, the collection $\{1, 2, 3, \cdots\}$.

FRACTIONS

Two activities made necessary the invention of "new" numbers. One was measuring; the other, sharing. In measurement if one takes, for example, a certain unit of length and then compares a given length with it, left-over parts usually occur. These could be taken care of by deciding on sub-multiples of the units and naming them separately. The Romans relied largely on this method. The alternative was to invent numbers that constitute parts of the natural numbers, namely fractions.

Problems of sharing also invariably led to these numbers. After all, man had acquired possessions by this time and wanted to assure each of his children a just share. These partitives or partitioners were not even considered numbers, but only broken parts, and they posed real difficulties in writing as well as in computation.

A considerable portion of the early Egyptian arithmetic consisted in explanations of fractions and operations with them. Fractions were limited to unit fractions—that is, those of the type $\frac{1}{a}$, where a is a natural number. The only exception was $\frac{2}{3}$. Elaborate tables were developed to express fractional quantities in terms of unit fractions. The reader may gain some appreciation of the mental struggle involved by an attempt to represent $\frac{11}{19}$ as a sum of unit fractions. He might obtain $\frac{1}{2} + \frac{1}{19} + \frac{1}{38}$ or $\frac{1}{2} + \frac{1}{13} + \frac{1}{494}$.

The Babylonians also studied fractions intensively. They, however, used sexagesimal fractions—that is, fractions of the type $\frac{a}{60^n}$, where a and n are natural numbers. Our system of minutes and seconds is a heritage from the Babylonians.

The Romans adopted the custom of expressing the common fractions $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{6}$ in terms of a new unit. They primarily used twelve and multiples of twelve as their subdivisions, and they talked about "unciae" as one-twelfth of the unit of weight. The terms *ounces* and *inches* are derived from this word.

The Egyptian and the Babylonian systems were used by the Greeks, who also employed common fractions. The modern treatment and the terminology of common fractions were given by Robert Recorde (1510–1558). To him, incidentally, we also owe the equality sign. He said, when introducing this symbol in his *Whetstone of Witte*, "I will sette as I doe often in woorke vse, a pair of paralleles, or Gemowe (i.e., twin) lines of one lengthe, thus =, because noe 2. thynges can be moare equalle."

The system of sexagesimal fractions (Babylonian) and duodecimal ones (Roman) prepared the way for decimal fractions. Christoff Rudolff in the first part of the 16th century wrote in the usual notation of decimal fractions, putting a comma for the decimal point. The complete systematic treatment of decimal fractions, however, had to wait for Simon Stevin (1548–1620). His discovery will be better understood after our discussion of the Hindu-Arabic decimal system in Chapter 2.

IRRATIONAL NUMBERS

Pythagoras' strong interest in number has been mentioned. Number to him meant the collection of natural numbers and fractions; these suffice for the ordinary needs of daily life, for counting individual items and measuring various quantities. Around 500 B. C., the master decided to work on a simple but important problem in class. He wanted to find the length of the diagonal of a square whose side is one unit. No difficulty could be posed since the relationship that may have been first generally proved by him, the "theorem of Pythagoras," could be used. Alas, the relationship led to the fact that the diagonal d would have to equal $\sqrt{2}$ in our present symbols. This convinced him that there was *no* number (number meaning positive integers and fractions) such as $\sqrt{2}$.

By indirect reasoning, we may assume the $\sqrt{2}$ to be a number, say $\sqrt{2} = \frac{a}{b}$, where a and b are positive integers. For convenience consider these as relatively prime (having no common factor), since a fraction may always be reduced to lowest terms. Squaring both sides of the equation and clearing fractions gives $2b^2 = a^2$. This means that a^2 must be even, and consequently a itself must be even, say $a = 2a_1$ with a_1 an integer. Substituting and reducing leads to $b^2 = 2a_1^2$. But this forces b to be even also which contradicts the original assumption that a and b have no common factor. Therefore, there is *no* number (of the type known) which gives the measure of the diagonal of such a square.

Pythagoras' grief was overwhelming because number, this omnipotent concept, had failed him. Number did not rule the universe after all. Legend will have it that a devoted student Hippasus, also in despair, forgot the school rule of secrecy within the group. Drowning was his punishment.

While this incident dealt a fatal blow to the Pythagoreans, the difficulty was resolved masterfully about 370 B. C., by Eudoxus (408 B. C.–355 B. C.). He cleared up the "scandalous situation" by a rather involved definition of these new quantities, admitting them into the realm of number. We now write these numbers as $\sqrt{a}, \sqrt[3]{a}, \cdots \sqrt[k]{a} \cdots$ where a may be any natural number, and $a \neq b^k$ with b as another natural number.

ZERO AND NEGATIVE NUMBERS

In the early A. D.'s the Hindus, joyously playing with numbers, contributed many developments in the evolution of the number concept. They were not handicapped by the restrictive standards of Greek thought. They were imaginative formalists with a flair for flowery language, and they worked on mathematical problems simply for pleasure. A notable contribution they made with respect to symbolizing numbers will be discussed in the second chapter.

Astronomical studies in Babylonia had already suggested the usefulness of zero and negative numbers. The Greeks, mainly Diophantus, met equations at least one of whose roots was a negative number. But such solutions were meaningless, and were rejected. About 1150 Bhāskara admitted negative integers for x and y in an equation of the type $ax + by = c$, where a, b, and c are integers. But he, too, exhibited some skepticism as to the validity of negative numbers. Symbols for negative numbers and zero also came from the Hindus about this time.

In Italy, Fibonacci, or Leonardo of Pisa (1180–1250), evidently obtained from the Arabs his knowledge of negative quantities, which neither he nor the Arabs considered as roots of an equation. Jerome Cardan (1501–1576) recognized such roots but called them "aestimationes falsae," not attaching any independent significance to them. Michael Stifel (1486–1567) in Germany called negative quantities "numeri absurdi." It is rather interesting to note that the irrationals had met a similar fate. An irrational number was referred to as "analogon," the unutterable.

The Englishman Thomas Harriot (1560–1621) was the first mathematician to give status to negative numbers. Descartes (1596–1650) dared to use the same letter for positive and negative quantities in his

geometry. He did, however, falsely consider a negative number of greater absolute value larger than another of smaller absolute value; to him, (-5) was larger than (-3). Thus it fell to mathematicians of the seventeenth century to deal with operations involving negative numbers.

CLASSIFICATION OF NUMBERS

So far we have given a bird's-eye view of the historic development and some of the extensions of our number system. We might pause a moment to see what numbers we have covered and how they may be classified. The set of natural numbers $\{1, 2, 3, \cdots\}$ has been extended to include zero and the negative integers. The new collection obtained is called the set of integers, and may be designated by

$$\{\cdots -3, -2, -1, 0, 1, 2, 3, \cdots\}.$$

The set of rational numbers constitutes the next enlargement of our set of numbers. Rational numbers are those which can be expressed as the ratio of two integers, that is, as a/b, where a and b are integers and $b \neq 0$. Every integer is a rational number since it is equal to the ratio of itself to one. The other rational numbers are usually written as *common* fractions (fractions in which the numerator is an integer and the denominator is an integer different from 0) or as decimals. The decimal expansion of a rational number is either finite or it is an infinite decimal which, from some point on, has a repeating block of digits. So far then we have the following.

Rational Numbers					
Integers			Fractions Symbolized as		
Negative Integers	Zero	Natural Numbers	Finite Common Fractions*		Periodical Repeating Decimal Fractions*
$\{\cdots, -3, -2, -1\}$	$\{0\}$	$\{1, 2, 3, \cdots\}$	Negative	Zero	Positive

* See p. 58 ff.

There are also numbers that cannot be expressed as ratios of integers; Pythagoras' "scandalous" class episode involved such a number.

These are the so-called "irrationals" or "incommensurables," a very important sub-class of our number system. Archyatas by about 400 B. C. had already classified real numbers as rational and irrational. The number system including both of these sets is called the *real number system*. At this stage, our chart has the following form:

Real Numbers							
Rational Numbers						Irrational Numbers	
Integers			Fractions			Negative	Positive
Negative	Zero	Positive	Negative	Zero	Positive		

MORE ABOUT IRRATIONALS

It is one of the strange quirks in the history of mathematics that as difficult an idea as irrationals should have been chanced upon so early. Euclid (300 B. C.), in his *Elements*, presented an investigation of irrational quantities, thereby establishing the irrationality of $\sqrt{2}$. His reasoning process was like the one previously shown.

Christoff Rudolff (first part of the 16th century) gave a few rules for operations with irrationals or radicals, as these may also be called. Pioneering work of Karl Weierstrass (1815–1897), Richard Dedekind (1831–1916) and Georg Cantor (1845–1918) finally secured a rigorous, purely arithmetic theory for irrational numbers. This new treatment is based upon the theory of functions with considerations of continuity and discontinuity and upon the theory of series with investigations of criteria for convergence and divergence.

However, numbers of the type $\sqrt[n]{a}$ where n and a are natural numbers and $a \neq b^n$, with b another natural number, are not the only irrationals encountered. In 1794, Legendre had suspected a new kind of irrational number which could not be thought of as a root of an algebraic equation. The French mathematician Liouville expressed this idea in 1844 and even realized that these new numbers would constitute a very extensive class. Thus it became necessary to subdivide irrationals into two sets, the algebraic irrationals and the transcendentals. A real number a is said to be algebraic if it is the root of an algebraic equation of the form $a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$

where the coefficients a_i are integers, $a_0 \neq 0$, and n is a natural number. This type of equation is commonly called a polynomial equation. For instance, $\sqrt{2}$ is algebraically irrational because it may be considered one of the solutions of the equation $x^2 - 2 = 0$. Every rational number is obviously algebraic since any number of the form $\frac{a}{b}$ may be considered a root of the equation $bx - a = 0$. Any irrational number which is not algebraic is a transcendental. Transcendentals play a most important role in mathematics, and surprising facts about their density, that is, the frequency of their occurrence, will be discussed later.

One of the transcendentals that a mathematician encounters early in his studies is π, the number indicating the ratio between the circumference of a circle and its diameter. Others are most of the logarithms, the majority of the values of the trigonometric functions and an especially interesting number called e. We shall discuss some of these in Chapter 4.

Transcendentals have a fascinating history. The idea of the number π was first encountered in the ancient Orient. The King James version of the Bible (I Kings 7:23, II Chronicles 4:2) twice gives it the value 3—a far step from recognizing its true status as a transcendental irrational. The Egyptian Rhind Papyrus (about 1650 B. C.) specifies $\pi = \left(\frac{4}{3}\right)^4$, a real rational number. About 240 B. C., Archimedes obtained the inequality $\frac{223}{71} < \pi < \frac{22}{7}$ by a most ingenious method which, computed to two decimals, gives the value $\pi \approx 3.14$. Further attempts to capture the nature of this number were made by Ptolemy of Alexandria, who calculated $\pi \approx 3.1416$ (about 150 A. D.); a Chinese physicist (about 480) who obtained π as ≈ 3.1415929; the Hindu, Aryabhata, who secured the approximation $\pi \approx 3.1416$ (about 530); and Bhāskara, whose interesting values (about 1150) were $\pi = \sqrt{10}$ for ordinary work, $\frac{3927}{1250}$ as an accurate value, and $\frac{22}{7}$ as an approximate one.

In 1767, Johann Heinrich Lambert established the irrationality of π and in 1882 F. Lindemann was able to prove its transcendentality. The Eniac, an electronic calculator, gave—in approximately 70 hours—the value of π to 2035 places. In 1873, William Shanks of England had spent 15 years to compute π to 707 places—a most thankless and perhaps foolish task!

At this stage our table extends itself again, and the mathematician would chart it in the real number system.

The Real Number System

Rational Numbers						Irrational Numbers			
Integers			Fractions			Algebraic		Transcendental	
Negative	Zero	Positive	Negative	Zero	Positive	Negative	Positive	Negative	Positive

COMPLEX NUMBERS

It has been stated that the Hindus already used negative numbers in their computations although they did not regard them as "proper" solutions of an equation. Bhāskara (born 1114 A. D.) attempted to extract a square root of a negative quantity and realized correctly that it does not exist for the real number system.

For a long time, such "imaginary" quantities, as they were later called, received very little attention. But complex numbers forced themselves into the picture in the sixteenth century for the Italian algebraists who attempted to solve equations. Strangely enough, while they were not able to analyze the nature of such quantities, men like Jerome Cardan (1501–1576) and Rafaele Bombelli (born 1530), called them "impossible numbers" and gave several rules for computation with imaginary numbers ($\sqrt{-a}$, where a is a positive real number) and with complex numbers ($a + b\sqrt{-1}$, where a and b are real numbers).

This is an example in the history of mathematics where the pencil was superior to the brain, where results were obtained on formalistic, mechanical grounds without real understanding.

It was the great mathematician Carl Friedrich Gauss (1777–1855) who began an explanation of the nature of imaginary and complex quantities. A new number was introduced whose square equals minus one. Through Gauss's influence its symbol i, suggested by Euler, became generally accepted. This newly created number i aroused great suspicion, and it was considered fictitious, not really existing. Now we consider the name *imaginary number* an unfortunate misnomer, another historical remnant of early hesitations toward abstract and formalistic treatment in mathematics. Complex numbers proved invaluable in man's attempt to use the language of mathematics for a description of

natural phenomena. They are, of course, indispensable from the point of view of pure mathematics.

The chart below is the completed system to date.

The Complex Number System

$a + bi$, a and b are real numbers, i is a number such that $i^2 = -1$

Real Numbers $a \neq 0, b = 0$										Pure Imaginaries $a = 0, b \neq 0$	Complex Numbers (Specific) $a \neq 0, b \neq 0$
Rational Numbers						Irrational Numbers					
Integers			Fractions			Algebraic		Transcendental			
Neg.	Zero	Pos.	Neg.	Zero	Pos.	Neg.	Pos.	Neg.	Pos.		

Logical Development

Capricious and incalculable are the ways in which man developed understanding. The history of the number system follows no logical pattern. Chance elements, fads, and interests were sometimes responsible for advances in a certain direction.

The modern viewpoint involves such rigorous, painstaking analysis that we cannot attempt to do justice to it in this discussion. We will, however, venture to give a somewhat intuitive mathematical development of this structure, and then gaze toward a modern mathematician's demands for a rigorous development.

THE DESIRE TO MAKE OPERATIONS OMNIPOSSIBLE

Let us start with the natural numbers, the ordinary counting numbers $\{1, 2, 3, \cdots\}$. What operations may one perform on them? We are now thinking of the word ***operation***, a very important idea in mathematics, in the elementary way. Moreover, we consider only binary operations, those that combine two numbers in a prescribed manner to obtain a third. In this sense, we have the four fundamental

operations—addition, subtraction, multiplication and division—as well as raising to a power and extracting a root.

Taking any two natural numbers and adding, a third natural number results. Addition is a short process for combining two numbers in a way which could be done by counting. The mathematician says, "addition is omnipossible in the system of natural numbers" or, "the system of natural numbers is closed under addition." Likewise, multiplication and raising to a natural number exponent are omnipossible. However, with any pair of natural numbers chosen in a certain order, subtraction cannot always be performed.

Subtraction is the inverse operation to addition. Thus, to perform the task $a - b$ means to find a number c such that $a = b + c$. Multiplication means repeated addition. To find ab is equivalent to obtaining the sum of a equal addends b.

To make subtraction omnipossible, zero and the negative whole numbers have to be introduced. The enlarged set is the set of integers, written as $\{\cdots -3, -2, -1, 0, 1, 2, 3, \cdots\}$. It is closed under addition, subtraction, and multiplication. Now division causes trouble.

To close the system under division, except by zero, quotients or ratios of integers must be introduced. These are the rational numbers, signified by $\frac{a}{b}$ where a and b constitute an ordered pair $\left(\text{in general } \frac{a}{b} \neq \frac{b}{a}\right)$, a and b are both integers, and $b \neq 0$. These new numbers satisfy the following three conditions:

EQUALITY: Two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ are equal if and only if

$ad = bc$; in symbols, $\frac{a}{b} = \frac{c}{d} \leftrightarrow ad = bc$.

ADDITION: Two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ when added again form a

rational number $\frac{ad + bc}{bd}$; or $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$.

MULTIPLICATION: Two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ when multiplied

result again in a rational number $\frac{ac}{bd}$; or $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

Division is defined as the inverse of multiplication, just as subtraction is the inverse of addition. To divide a by b ($b \neq 0$) means finding a number c such that $a = bc$. To divide two rational numbers $\frac{a}{b}$ by $\frac{c}{d}$ therefore means finding a rational number $\frac{x}{y}$ such that $\frac{a}{b} = \frac{c}{d} \cdot \frac{x}{y}$.

But by the definition for multiplication, $\frac{c}{d} \cdot \frac{x}{y} = \frac{cx}{dy}$. Using the equality principle, this becomes interpreted as $ady = bcx$. One pair of solutions for this statement is $x = ad$ and $y = bc$. This secures a rational number as an answer to the division $\frac{a}{b} \div \frac{c}{d}$ if $y \neq 0$. But $b \neq 0$ by the definition of rational numbers and $c \neq 0$, or else $\frac{c}{d} = 0$; and division by zero has been excluded.

Extraction of roots is, however, still not omnipossible in our extended set, the set of rational numbers. Two difficulties may occur. The first put Pythagoras into a grave dilemma. Even square roots of certain numbers are not to be found in the realm of rational numbers. We have seen that $\sqrt{2}$, for instance, cannot be expressed as $\frac{a}{b}$ with a and b being integers. The numbers necessary to perform this operation are called algebraically irrational numbers. Secondly, not even the introduction of these irrationals secures the possibility of always being able to extract roots. For instance, the square root of a negative number is not an element of the set of real numbers as our extended number realm is called. Complex numbers, that is, numbers of the form $a + bi$, where a and b are real numbers and i is a number such that $i^2 = -1$, had to be introduced. Rules for operating on these numbers were established. These are:

EQUALITY: Two complex numbers $a + bi$ and $c + di$ are equal if the real as well as the pure imaginary components agree, respectively, or: $a + bi = c + di \leftrightarrow a = c$ and $b = d$.

ADDITION: The sum of two complex numbers $a + bi$ and $c + di$ is formed by adding the real as well as the pure imaginary parts separately: $(a + bi) + (c + di) = (a + c) + (b + d)i$.

MULTIPLICATION: The product of two complex numbers $a + bi$ and $c + di$ is obtained in the following manner: $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$.

One can see that the number i behaves like any other number, and that the only condition imposed on it is that $i^2 = -1$.

As explained, the set of complex numbers (the number system of ordinary algebra) is closed with respect to the elementary operations. Yet in this field another remarkable property can be established. This is the famous "fundamental theorem of algebra" first proved by Gauss at the age of 22 in his doctoral dissertation (1799). It states that within the system of complex numbers every polynomial equation has at

least one root. Also, if one counts multiple roots "correctly," the number of roots equals the degree of the equation. What relief this must have been to mathematicians who for generations had explored and investigated algebraic equations as to their solvability and methods for obtaining solutions. To look for a solution without knowing if it is actually there is as bad as digging for oil in a place whose oil content has not been established.

THE DESIRE TO SOLVE POLYNOMIAL EQUATIONS

One might treat the development of the number system from the point of view of polynomial equations altogether. Again we would start with the natural numbers and investigate the solvability of an equation of the form $x + a = b$, where a and b are natural numbers. To assure a solution for this equation, zero and negative integers have to be introduced, the former for the case $a = b$, the latter for $b < a$.

The linear equation $a_0x + a_1 = 0$ (a_0 and a_1 are integers and $a_0 \neq 0$) necessitated the introduction of fractions.

The quadratic equation $a_0x^2 + a_1x + a_2 = 0$ with $a_0 \neq 0$ becomes solvable only if irrationals and complex numbers are recognized.

It was one of the pleasant surprises in the development of mathematics that as one increases the degree of the equation under investigation ($a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$ with $a_0 \neq 0$ and $n \geqq 3$), the number system does not have to be enlarged. Rather, on the basis of Gauss's investigation, all these equations have a root in the complex realm even if the coefficients $a_i (i = 0, 1, \cdots n)$ are not necessarily integers but complex numbers themselves.

It has to be pointed out that this informal sketch, as well as the previous one based on a desire to make the fundamental operations omnipossible, does not explain the occurrence of transcendentals. A discussion like this is a far cry from a modern mathematician's critical standards of rigor. Descriptive, intuitive arguments have to give way to a formalistic, flawless structure. Assumptions and properties, laws and relationships have to be carefully recognized and distinguished, and appropriate proofs given for all deductions made.

ORDERED NUMBER PAIRS

Integers. A useful idea which may now be reviewed is the notion of an ordered number pair, symbolized by (a, b). By *ordered* we mean that there is a first and a second number in the pair in such a manner that $(a, b) \neq (b, a)$ unless $a = b$.

Once again the natural numbers are taken for granted, and it is

designated that both a and b are natural numbers. With this number-pair notation all fundamental properties of integers can be established.

One would first give the basic definitions in the new language as:

EQUALITY: Two number pairs (a, b) and (c, d) are equal if and only if the sum of the first member of the first pair and the second of the second pair equals the sum of the remaining parts, or:

$$(a, b) = (c, d) \leftrightarrow a + d = b + c.$$

ADDITION: Two number pairs (a, b) and (c, d) are added by writing a number pair whose first member equals the sum of the first members of the given pairs and whose second member is the sum of the remaining members, or:

$$(a, b) + (c, d) = (a + c, b + d).$$

MULTIPLICATION: Two number pairs (a, b) and (c, d) are multiplied as follows:

$$(a, b) \cdot (c, d) = (ac + bd, ad + bc).$$

The reader will have recognized that (a, b) signifies what is generally thought of as $a - b$. For instance, $(5, 3)$ would represent the integer 2. All integers may be symbolized in this manner. Zero would be given by any one member of the set $\{(1, 1), (2, 2), (3, 3) \cdots\}$, or, generally speaking, by (a, a). A negative integer would be a number pair (a, b) such that $b > a$.

This, of course, necessitates an order concept among natural numbers, which one may introduce by stipulating that only three possibilities occur in the comparison of two natural numbers a and b:

1. There exists a natural number c such that $a = b + c$, in which case we say that a is greater than b, and write $a > b$.
2. $a = b$.
3. There exists a natural number d such that $a + d = b$. This would mean that a is smaller than b, and would be written as $a < b$.

To illustrate, let us test the fact that zero added to any integer leaves the integer intact. We would write $(a, b) + (c, c) = (a, b)$. Applying the addition rule: $(a, b) + (c, c) = (a + c, b + c)$. But does $(a + c, b + c)$ equal (a, b)? By the definition of equality, this would mean establishing that $a + c + b = b + c + a$, where a, b, and c are natural numbers. Here a new difficulty is encountered. May one be sure that the order of adding is immaterial with natural numbers? Intuitively we are quite convinced of that; has not two plus three always yielded the same result as three plus two? However, this type of thinking would not secure analogous behavior in all circumstances.

Such experiences force the mathematician to stipulate certain properties for operations with natural numbers, often referred to as the five fundamental laws of arithmetic. They are:

THE COMMUTATIVE LAW: The order of elements is immaterial in addition as well as in multiplication.

$$a + b = b + a \tag{1}$$

$$a \cdot b = b \cdot a \tag{2}$$

THE ASSOCIATIVE LAW: The grouping of elements in addition or multiplication may be taken in any manner.

$$a + b + c = (a + b) + c = a + (b + c) \tag{3}$$

$$a \cdot b \cdot c = (a \cdot b)c = a(b \cdot c) \tag{4}$$

THE DISTRIBUTIVE LAW: This combines addition and multiplication.

$$a(b + c) = a \cdot b + a \cdot c \tag{5}$$

Using the commutative law, form (1), we have completed our test.

Here is another one. Is the product of any number other than zero and zero itself really equal to zero? That is, is $a \cdot 0 = 0$ where a is any integer? In the language of number pairs, we wonder if $(a, b)(c, c) = (d, d)$.

Using the definition for multiplication, $(a, b)(c, c) = (ac + bc, ac + bc)$. To establish the equality $(ac + bc, ac + bc)$ with (d, d), investigate to determine whether the equality rule is fulfilled. Since it would give $ac + bc + d = ac + bc + d$, the correctness of the statement is seen immediately.

Let us try the multiplication rule for negative integers, a great trouble-maker in the teaching of mathematics. We wish to establish the fact that the product of two negative integers is a positive integer. To simplify matters, let us first try to prove that $(-1)(-1) = (+1)$. In the language of number pairs, we wonder if $(a, a + 1)(a, a + 1)$ equals $(a + 1, a)$, where a is a natural number. If the left member is evaluated according to the definition of multiplication, it becomes: $[aa + (a + 1)(a + 1), a(a + 1) + (a + 1)a]$, where $a + 1$ and aa are natural numbers, since this system is closed under addition and multiplication. With the use of the distributive law this number pair may be written as $[aa + (a + 1)a + (a + 1)1, a(a + 1) + (a + 1)a]$. Stipulate a further property for natural numbers, namely that $n1 = n$, and use the distributive law again; the new simplification reads:

$$(aa + aa + a + a + 1, aa + aa + a + a).$$

This number pair, however, equals $(a + 1, a)$, the right member of the

above relationship under investigation. The last assertion may be seen by examining to find if the equality condition is satisfied, that is, if $aa + aa + a + a + 1 + a$ equals $aa + aa + a + a + a + 1$. Once again, on the basis of the commutative law for addition, this equality holds. The idea that minus one times minus one equals plus one results as a logical necessity by postulating properties of natural numbers and defining integers, their equality, addition, and multiplication.

We might now attempt the more general proof, namely that the product of any two negative integers is positive. In number pair notation, to say that (a, b) and (c, d) are to be symbols for negative numbers is paramount to conditioning that $b > a$ and $d > c$. In other words, the relationships $b = a + m$ and $d = c + n$, where m and n are natural numbers, must be superimposed on the given number pairs (a, b) and (c, d). Substituting, and using the multiplication rule, we obtain:

$$\begin{aligned}(a, b)(c, d) &= (a, a + m)(c, c + n)\\ &= (ac + ac + mc + an + mn, ac + an + ac + mc),\end{aligned}$$

in which statement the distributive law for natural numbers has, of course, also been utilized. On the basis of the commutative law for addition, the first number of this pair exceeds the second by the product mn and, by the definition of order, is greater than the second number. This means that the product is a positive integer.

Rational Numbers. Rational (not necessarily integral) numbers of the form $\frac{a}{b}$ may also be discussed by means of the ordered number pair notation. Here it must be recognized that (a, b) denotes a number where a and b are integers and $b \neq 0$. In this discussion it will be assumed that the second number of each pair differs from zero. Then we adopt the following definitions:

EQUIVALENCE: $(a, b) = (c, d) \leftrightarrow ad = bc$.
ADDITION: $(a, b) + (c, d) = (ad + bc, bd)$.
MULTIPLICATION: $(a, b)(c, d) = (ac, bd)$.

It may be seen immediately that the integers form a subset of these numbers, namely the set denoted by $(a, 1)$.

Let us establish the rule for dividing fractions. To show that

$$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc},$$

we will have to see that $(a, b) \div (c, d) = (ad, bc)$ or rather that $(a, b) = (ad, bc)(c, d)$. The right side of this statement, by definition, equals

(adc, bcd). To establish the equality between (a, b) and (adc, bcd), the equivalence relation must be fulfilled; that is, one must prove $abcd$ equals $badc$, where a, b, c and d are integers.

The proof would be complete if we could assure ourselves of the commutative and associative laws for multiplication of integers (having previously stipulated these laws for natural numbers). If a return is made to the number pair notation for integers, the task now becomes one of showing that if (a, b) and (c, d) are two integers, then $(a, b)(c, d) = (c, d)(a, b)$. By the definition for multiplication of integers, the left side reads $(ac + bd, ad + bc)$; the right side is $(ca + db, cb + da)$, where a, b, c, and d are natural numbers. Using the commutative law for addition and for multiplication of natural numbers, we settle part of the query. We use a similar argument for establishing the associative laws for integers.

Complex Numbers. Complex numbers also accommodate themselves to the number pair device. We agree on the following definitions. A complex number $a + bi$ is represented by (a, b), where a and b are real numbers, in such a manner that the following relationships hold:

EQUIVALENCE: $(a, b) = (c, d)$ if and only if $a = c, b = d$.

ADDITION: $(a, b) + (c, d) = (a + c, b + d)$.

MULTIPLICATION: $(a, b)(c, d) = (ac - bd, ad + bc)$.

Let us see if the product of two conjugate complex numbers $a + bi$ and $a - bi$ results in a real number; $(a, b)(a, -b)$ is under investigation. On the basis of the definition the product reads $(a^2 + b^2, -ab + ab)$ or $(a^2 + b^2, 0)$. This, however, is a real number.

THE IRRATIONALS

The idea of ordered number pairs cannot be used for the irrationals of the real number set. Nor would ordered n-tuplets of rational numbers suffice. The formation of the set of *all* real numbers from the rationals is the most difficult phase of constructing the entire system of complex numbers from the natural numbers. One method of doing this is based on the partitioning notion, the so-called Dedekind Cut (proposed by Richard Dedekind, 1872). Another way uses the concept of a regular sequence of rational numbers as devised by Georg Cantor (1845–1918). Decimal representations of real numbers offer a third possibility. A fourth approach involves sequences of nested intervals.

The Dedekind Cuts will be used here because they more closely resemble the concept of ordered number pairs. Ordered pairs of classes of numbers instead of number-couplets must be introduced. Objections have been raised against all devices for defining a real number. An

analysis of these in detail goes beyond the scope of this discussion.

Dedekind generated a set of real numbers by classifying the rational numbers in a special way. Symbolize the Dedekind Cut, which separates the rational numbers into two classes A and B, by $(A|B)$. Thus a real number is defined. We stipulate the following conditions on this partitioning into two classes:

1. The set of rational numbers is exhausted, that is, every rational number is in either A or B.
2. A and B each contains at least one rational number.
3. Every element of A is less than every element of B. This property orders the classes.

Three possible types of Dedekind Cuts $(A|B)$ occur in the set of rational numbers.

Type 1	Example	
A has a largest element a and B has no smallest element.	A contains all rational numbers smaller than or $=$ 2.	B contains all remaining rational numbers.
The cut defines the rational number a.	The cut defines the rational number 2.	

Type 2	Example	
A has no largest element and B has a smallest element b.	A contains all rational numbers smaller than 2.	B contains all remaining rational numbers.
The cut defines the rational number b.	The cut defines the rational number 2.	

For a definition of rational numbers, the description of Type 1 or Type 2 may be followed

Type 3	Example	
A has no largest element and B has no smallest element.	A contains all negative rationals, zero, and all positive rationals whose square is smaller than 2.	B contains all remaining rational numbers, that is, all positive rational numbers whose square is larger than 2.
The cut defines an irrational number.	The cut defines the positive square root of two ($\sqrt{2}$).	

Further definitions for the equality of the cuts and the basic operations are introduced. It can be shown that this newly defined real number set satisfies the five basic laws of arithmetic. Thus the set of rational numbers is extended to the set of real numbers by making Dedekind Cuts in the set of rational numbers.

MODERN DEMANDS FOR RIGOR

Recall again that the modern mathematician's approach to an understanding of the notion of number still differs significantly from anything we have done so far. He might create a set of mathematical objects such as $\{1, 2, 3, \cdots\}$ by clearly postulating their behavior or properties.

The natural numbers consist of those elements which may be generated by starting with a first one, called 1, and passing from any element n already generated to its successor $(n + 1)$. Peano's famous five Postulates resulted, in 1889, from this definition.

1. 1 is a natural number.
2. If n is a natural number, then $(n + 1)$ is a natural number.
3. For any two natural numbers n and m, if $n + 1 = m + 1$, then $n = m$.
4. For any natural number n, $n + 1 \neq 0$.
5. $P(n)$ is a meaningful property for natural numbers under the conditions:
   a. 1 has the property $P(1)$.
   b. if n has the property $P(n)$ and $(n + 1)$ has the property $P(n + 1)$; then for every natural number m, $P(m)$ holds. This is the well-known principle of mathematical induction.

With such a start the ensuing theory of a rigorous development of our number system may become rather complex and involved. These remarks were inserted to help the reader glimpse the spirit of the abstractions of modern mathematics in some small measure. Contemporary abstract mathematics presents a form of mentally gyrating gymnastics with enormous consequences, and the creation of abstract models has proved profoundly valuable. A wealth of ready-made understanding about real, concrete objects is obtainable by the establishment of relationships in the structure of an abstract mathematical model.

The Infinities of Numbers

How many numbers are there in our complete number system? Or, for that matter, how numerous are the integers, the fractions, the irrationals? Surely, the uninitiated skeptic says, there is an infinite number of each, and it is meaningless to pose such questions. But a pleasant surprise appears. Cantor has established a hierarchy of "quantitative infinities" and we shall attempt to give you the gist of these thoughts.

The process of counting can conceivably be carried on without

termination. Every natural number is assumed to have a successor. Hence, the infiniteness of the natural numbers is assured. Their plurality may be expressed by the symbol $\aleph_0$ (aleph null). Are all integers (positive and negative integers and zero) more numerous than just the positive integers?

A hasty reply would certainly be made in the affirmative. Yet the infinite has no less a shock for us than the idea that part of a whole may equal the whole.

Man's innate mental curiosity has always been challenged by the infinite, and his imaginative and resourceful attempts to solve the mysteries of the infinitesimally small and the infinitely large play an important role in the history of thought. It is the idea of the infinitely large with which we are here concerned. It permeates the entire realm of mathematics. Since mathematics pursues generality and deals with abstract models and structures, collections of mathematical elements are studied rather than the behavior of a single member. Georg Cantor through his modern theory of sets has strikingly influenced the thinking of contemporary mathematicians. With the help of this theory many trouble-making problems were solved, new insights were gained, "known mathematics" was made more rigorous and precise, and analyses of mathematical situations became clarified and often simplified. In fact the new theory of sets reached the heart of the philosophical foundations of mathematics.

Briefly, and with simplifications, the theory first describes the basic idea of a set or collection or aggregate or assemblage. A set is any collection of elements, with a prescribed rule that determines whether or not a certain element belongs to the given collection. For instance, the reader has surely surmised that the set of natural numbers, the set of integers, and the set of rational numbers are illustrations of such collections.

By an infinite class or set Cantor understands a set with the very property just mentioned. An infinite class is one in which the whole is no greater than some of its parts. This statement, however, is meaningless unless we establish a means for comparing the numerosity of infinite classes. This leads to the next important definition, that of equivalence. Two infinite sets are said to be equivalent (have the same numerosity) if a bi-unique matching procedure may be effected by which to each element of one set there belongs one and only one element of the other and vice versa. The reader will recall the matching principle discussed early in this monograph which was one of the components that led to man's arrival at the notion of number. The

power of this principle is apparent because without its help our knowledge of transfinite cardinals would be impossible.

THE INTEGERS

Having given the symbol $\aleph_0$ to denote the cardinal number of the set of natural numbers, let us try to settle the query we have posed and establish the cardinal number of the set of all integers

$$I\{\cdots -3, -2, -1, 0, 1, 2, 3, \cdots\}.$$

We might arrange the set I as follows: $I\{0, 1, -1, 2, -2, 3, -3, \cdots\}$ and compare it with the set of ordinary counting numbers

$$N\{1, 2, 3, \cdots\}.$$

The following pairing can occur:

$$\begin{array}{cccccccc} 0 & 1 & -1 & 2 & -2 & 3 & -3 & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \end{array}$$

To check the bi-uniqueness of this matching process, let us see it in detail.

1. Let $k \in I$ ($\in$ meaning "is an element of"); then
   if $k = 0$ then $k \leftrightarrow 1$ (k is mated to 1)
   if $k > 0$ then $k \leftrightarrow 2k$
   if $k < 0$ then $k \leftrightarrow 2|k| + 1$.
2. Let $n \in N$; then n is either odd or even, that is,
   $n = 2a + 1$, $(a = 0, 1, 2, \cdots)$ or $n = 2b$, $(b = 1, 2, 3, \cdots)$
   if $n = 2a + 1$ then $n \leftrightarrow -a$
   if $n = 2b$ then $n \leftrightarrow b$.

This establishes the one-to-one correspondence bi-uniquely and we may behold the astonishing result. All integers together are only as numerous as the natural numbers themselves. The cardinal number of I is also $\aleph_0$ or, as the mathematician now says, the set of integers is "countable" or "denumerable."

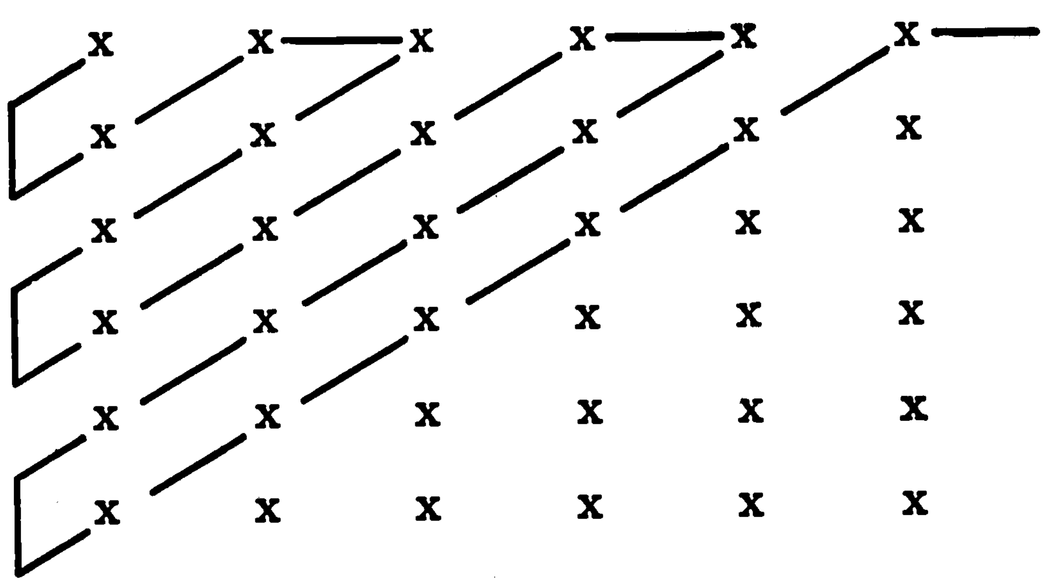
THE SET OF ALL RATIONAL NUMBERS

So-called common sense is in for another jolt. Let us put the set of all rational numbers, symbolized by R, under investigation. This time it is not so easy to arrange our given numbers. What kind of design should we choose? Surely, the rational numbers can't be arranged by size. There is no such thing as the "next larger" fraction, because between any two specified rational numbers an infinite number of fractions could be inserted.

Cantor's famous "weaving procedure" solves the dilemma. Here is a possible arrangement of all positive rationals:

$$
\begin{array}{ccccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & . & . & . \\[2ex]
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & . & . & . \\[2ex]
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & . & . & . \\[2ex]
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & . & . & . \\[2ex]
. & . & . & . & . & . & . \\[2ex]
. & . & . & . & . & . & .
\end{array}
$$

It has been shown that the set of all integers is equivalent to the set of all positive integers. Analogously, the set of all rationals is equivalent to the set of positive rationals. Therefore the set of positive rationals suffices. You will notice that fractions with numerator one, ordered by magnitude, were put in the first row, those with numerator two in the second row, and so on. It may be objected that we allowed duplications to slip into the scheme. But even with this slightly larger collection, the matching will be possible although the array extends *ad infinitum* horizontally as well as vertically. The ingenious device suggested by Cantor now traces these numbers in a diagonally weaving way as suggested by the following guide lines:



This means the pairing:

$$\begin{array}{ccccccccc} \frac{1}{1} & \frac{2}{1} & \frac{1}{2} & \frac{1}{3} & \frac{2}{2} & \frac{3}{1} & \frac{4}{1} & \frac{3}{2} & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \end{array}$$

Behold, we will never run out of natural numbers, and *mirabile dictu*—contrary to so-called common sense—all the rational numbers are not any more numerous than the natural numbers. There are exactly as many fractions as there are counting numbers 1, 2, 3, $\cdots$ despite the fact that between any two such numbers is to be found an infinite number of fractions! Again in *scripta mathematica*: the class of all rational numbers is denumerable (countable). Its numerosity is $\aleph_0$.

THE SET OF IRRATIONALS

Approaching the irrationals, we first select a subclass, the set of all algebraically irrational numbers. These numbers (by definition) are solutions of an algebraic equation whose form is

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0,$$

with n a natural number and a_i $(i = 0, 1, 2, \cdots)$ integers. Introducing a technical thinking procedure and a new concept—the "height" of an algebraic equation—the countability of the algebraic irrationals may be established. (The "height" of an algebraic equation is determined by combining the degree and the coefficients as follows:

$$h = |a_0| + |a_1| + |a_2| + \cdots + |a_n| + n$$

where $|k|$ represents the absolute value of the number k.) By virtue of this definition h must be an integer $\geqq 1$. Since there is a finite number of polynomial equations of a given height h, any value of h yields a finite number of algebraic irrationals. The reasoning process which proves the denumerability of algebraic numbers orders them by successively listing the roots of the equations of height 1, then those of height 2, and so on.

It might seem that all enlarged number domains are as numerous as the original set of natural numbers. Alas, this is not correct!

It can be established that the set of real numbers is not matchable with the natural numbers. Since the set of real numbers includes the rational and irrational numbers as subclasses and since the countability of the rational and of the algebraically irrational numbers has already been shown, it would follow that the transcendental numbers alone are responsible for the increased numerosity. This, in turn, means that there are more transcendental numbers than integers, fractions and algebraically irrational numbers together!

Let us try to establish this. We resort to an especially important and helpful method, indirect reasoning. Let us assume that we have been successful in ordering the entire real number class. Decimal notation is our preference in this argument since any real number may be written as a terminating or a non-terminating decimal fraction. The scheme is:

$$\begin{aligned} R_1 &= I_1.a_1a_2a_3\cdots \\ R_2 &= I_2.b_1b_2b_3\cdots \\ R_3 &= I_3.c_1c_2c_3\cdots \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

A terminating decimal is included in this scheme:

$$\begin{aligned} R_k &= I_k.t_1t_2\cdots t_m \\ &= I_k.t_1t_2\cdots t_m\,000\cdots \end{aligned}$$

When a real number has two infinite decimal expansions, that is, $3.199999\cdots = 3.2000000\cdots$, only one of them need be considered.

The $R_i\,(i = 1, 2, 3, \cdots)$ designate ordered real numbers; $I_i\,(i = 1, 2, 3, \cdots)$ are their integral parts and the small letter sequences are the decimal parts of our numbers. Even though we may not know what values to give these letters, that is, how to order the numbers for whatever arrangement, the scheme is incomplete. This would pose a contradiction to the assumption that the real numbers are countable.

By what sleight of mind can one catch the omission of a certain number if one can't even "see" what numbers are written down? Cantor's elegant diagonal proof demonstrates the mathematician's power of abstract reasoning. However "complete" the array is supposed to be, he asserts, he is always able to write a new number which was not included in the scheme. Joining the decimal digits of the array by a diagonal line, one will encounter the digits $a_1b_2c_3d_4\cdots$ in this order. Write a new number $R = I.k_1k_2k_3\cdots$ in which no k_i is 0 or 9 and such that its first digit $k_1 \neq a_1$, its second digit $k_2 \neq b_2$, $k_3 \neq c_3$, $k_4 \neq d_4$, and so on. This number R will therefore differ from R_1 at least in the first decimal digit, from R_2 in the second decimal digit, and so on. It is a number not contained in the original, supposedly complete array. Adding this number to the scheme would not help either because then the diagonal thinking process can be repeated and a further new number created. The assumption of being able to denumerate all real numbers in an array is untenable. The domain of real numbers, attributable to the numerosity of the transcendentals, exceeds the numerosity of the natural numbers. This new transfinite number is often referred to as c since it may also be established that it is the numerosity of the continuum. There are c points on a straight line or c points on a line segment of however small a length.

2. RECORDING OF NUMBERS

$$a_1 = 2\sum_{n=1}^{\infty} \frac{1}{(2n-1)2^{2n-1}} \quad \text{is a number,}$$

$$a_2 = \int_0^{\infty} e^{-x^2}\,dx \quad \text{is a number, and so are}$$

$$a_3 = \begin{vmatrix} 5 & -2 & 0 \\ 3 & 0 & -4 \\ 1 & 2 & 7 \end{vmatrix} \quad \text{and} \quad a_4 = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n.$$

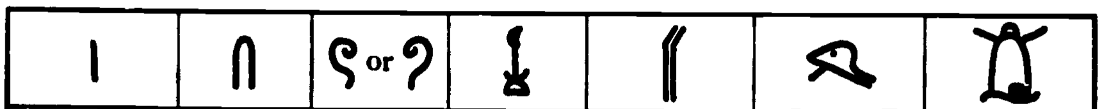
These quantities look very sophisticated. The first one is written as an infinite sum, the second as an improper integral, the third as a determinant, the last as a limit of a function. It requires some mathematical instruction to recognize the first one as $1.0986 \cdots$ (ln 3), the second as $0.88624 \cdots \left(\frac{\sqrt{\pi}}{2}\right)$, the third as 90 and the last as $2.7183 \cdots$ (e). But how did the writing of numbers originate?

Again we return to early civilizations for clues. Man's earliest attempts at recording numbers consisted of a collection of marks and notches. Remnants of these are the first numerals in the Roman system. From such primitive efforts all our different written number systems evolved.

Non-Positional Systems

SIMPLE GROUPING SYSTEMS

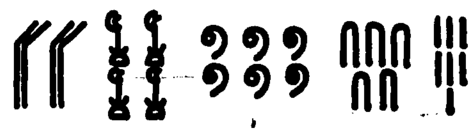
Possibly the earliest type of number system employed elementary grouping. In such a system a natural number b is selected as base and different symbols represent the successive powers b^k of this base ($k = 0, 1, 2, \cdots$). To write a given number, repeating symbols means adding their values.



1	10	10^2	10^3	10^4	10^5	10^6
a vertical staff	a heel bone	a scroll	a lotus flower	a pointing finger	a burbot fish	a man in astonishment

Figure 1

Early Egyptian hieroglyphic numerals (3400 B. C.) illustrate this approach. The symbols (Figure 1) are adopted for the successive powers of 10. For instance, 24,657 would be written as



For writing numbers less than 60, the early Babylonians (2000 B. C. to 200 B. C.) used simple grouping. Like the Egyptians they used the base ten, but they employed a subtractive as well as an additive principle. The symbols are illustrated in Figure 2. Repetition of symbols represented addition; subtraction was indicated by a special symbol . Thus,

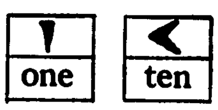
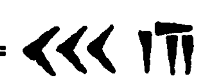


Figure 2

35 =  and 28 (or 30 − 2) = 

One of the Greek systems, the Herodianic or Attic (prior to 300 B. C.), also fits into our description of a simple grouping system. Although ten is again the base, a symbol for five is added to those for the powers of ten. A multiplicative principle is also established and is indicated by joining two symbols. As symbols the initial letters of the respective number names were chosen (see Figure 3).

I	Γ	Δ	H	X	M
1	5	10	10^2	10^3	10^4
	Γ, an old form for Π, the letter pi; initial of ΠENTE (pente)	Δ, the capital delta; initial of ΔEKA (deka)	H, the old Attic breathing, like our h, late represented by '; initial of HEKATON (hekaton)	X, the capital chi; initial of XIΛIOI (chilioi)	M, the capital mu; initial of MΥPIOI (myrioi)

Figure 3

We would write the number 3,786 as XXX𐅅HH𐅄ΔΔΔΓI.

The reader is probably quite familiar with the Roman numerals. Here an additive as well as a subtractive principle is used. The former is indicated by putting a smaller unit after a larger, the latter by reversing the order (there is reason to believe the subtracting idea was not commonly used in ancient times). Symbols for five and the fivefold multiples of powers of ten were introduced beyond those for the powers of ten. The origin of the symbols

I	V	X	L	C	D	M
1	5	10	50	10^2	500	10^3

is uncertain although many explanations have been given. In this system, 1959 is written as MCMLIX.

MULTIPLICATIVE GROUPING SYSTEMS

Upon a chosen base, the multiplicative grouping system imposes two sets of symbols, one for 1, 2, 3, · · · $(b-1)$, the other for $b, b^2, b^3, \cdots$, combining these in a multiplicative manner. It would be as if we used our regular digits 1, 2, 3, · · · 9 and another sequence:

t, h, T, Ŧ, · · ·
tens
hundreds
thousands
tenthousands

Then we would write 6T5h3t2 for 6,532.

The traditional Chinese-Japanese numeral system, which writes numbers vertically, is of this kind. The two sets of symbols are shown in Figure 4, and the number 6,532 is shown in Figure 5.

一	二	三	四	五	六	七	八	九
1	2	3	4	5	6	7	8	9

and

十	百	千
10	10^2	10^3

Figure 4

六
千
五
百
三
十
二

Figure 5

CIPHERED NUMERAL SYSTEMS

A third notion may be referred to as a ciphered numeral system. Here, after selection of a base b, the following sets of symbols are introduced and have their individual signs.

$$
\begin{array}{llllll}
\text{first set:} & 1, & 2, & 3, & \cdots & (b-1) \\
\text{second set:} & b, & 2b, & 3b, & \cdots & (b-1)b \\
\text{third set:} & b^2, & 2b^2, & 3b^2, & \cdots & (b-1)b^2 \\
\cdots & & & & &
\end{array}
$$

The later form of the Egyptian hieroglyphic number writing, the hieratic or demotic, is of this type, as shown in Figure 6.

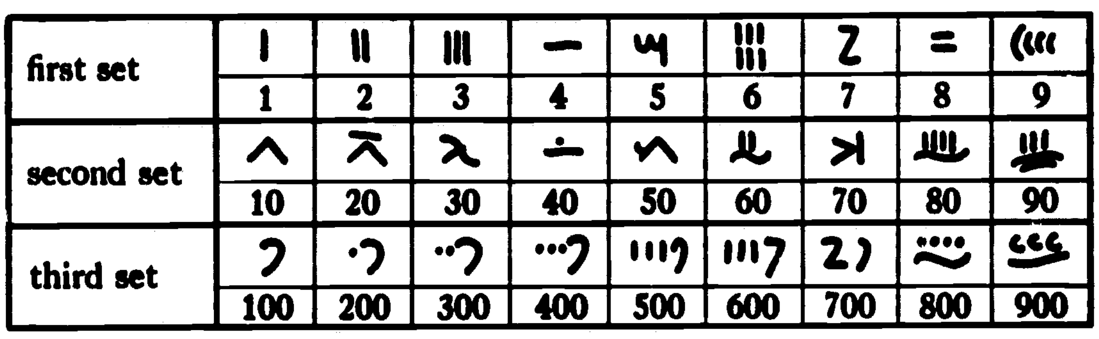
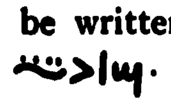


Figure 6

This work requires much memorizing, but the numbers themselves can then be written in a rather compact form. For instance, 875 would be .

A Greek numeral system other than the Herodianic is the Ionic (450 B. C.), another ciphered system. Instead of inventing special signs for all the numbers, the Greeks used the letters of their alphabet supplemented by a few other symbols, as shown in Figure 7.

first set	Α	Β	Γ	Δ	Ε	Ϝ	Ζ	Η	Θ
	1	2	3	4	5	6	7	8	9
second set	Ι	Κ	Λ	Μ	Ν	Ξ	Ο	Π	Ϙ
	10	20	30	40	50	60	70	80	90
third set	Ρ	Σ	Τ	Υ	Φ	Χ	Ψ	Ω	Ϡ
	100	200	300	400	500	600	700	800	900

Figure 7

Our number 875 would now be written as ΩΟΕ.

An alphabetic ciphered system was used by the Hebrews, the Hindu Brahmi, and the Syrians, and in early Arabic and Gothic texts.

Let us digress for just a moment to ponder about the possible remnants of the idea of alphabetic systems. In such a system numbers

and letters become associated; this is the basic idea in many of the still existent superstitions to be found in numerology. In such a "treatise" (still available in our advanced age) a young lady is instructed to use a certain code with whose help the letters of her name and her lover's name may be converted into numbers. Then the numbers are combined by some magic formula and the resulting number translated back into a word. Lo and behold—it may spell love!

The other analogy is part of a highly respectable phase of mathematics—the symbolism of algebra. The power of algebra is to a large extent a function of an effective symbolization of the generalizations sought. The evolution of today's highly sophisticated symbolization was long and tedious. Algebraic notation underwent three basically different stages: the first was rhetoric algebra in which all arguments, thinking procedures, and solutions were written out in detail. Next came syncopated algebra. Abbreviations were resorted to for the most frequently occurring ideas and operations. It was only very gradually that the last stage emerged, a symbolic algebra, and a set of symbols had to be devised to indicate numbers, operations, and functional and many other relationships.

A fundamental concern was a desire to express by some symbol the idea of *any* number. This is one of the basic ideas of algebra, and at the heart of most generalizations. Instead of a whole new collection of wriggles and squibbles, a familiar notation was taken—the alphabet. Thus it was established that in this mathematical shorthand, $\{a, b, c, \cdots\}$ means a set of numbers of any kind, sometimes specified as the set of real numbers or the set of integers.

Most of the symbols encountered in elementary algebra are about 300 years old. But one of the main ideas, the use of letters for numbers, appeared in alphabetic ciphered systems about two millennia ago. There is, however, one important difference. The letters in our algebra generally serve as place holders for *any* number, or for any one member of a set of specified numbers, whereas the early number systems designated certain numbers by certain letters.

Positional Systems

All the systems we have encountered require repeated introduction of new symbols for larger numbers. But in our number system ten digits suffice to write any number. The explanation lies in a positional principle which assigns a place value in addition to a quantitative value for each digit. The history of this positional principle reveals that different civilizations had found it independently.

THE BABYLONIANS

The way in which the ancient Babylonians wrote numbers smaller than sixty has already been mentioned. Later, work in astronomy needed larger numbers. For those a positional type of numeration was adopted and the number sixty was used as a base. Since  and  were symbols for one and ten and repetition of these symbols would mean addition, the number 311,723 would be written as

We recognize that in our symbols this number reads (1)(26)(35)(23) with "digits" enclosed in parentheses. Superimposing base sixty to this representation, the number reads $23 + 35 \cdot 60 + 26 \cdot 60^2 + 1 \cdot 60^3$, which will be found to equal 311,723.

If the position is a decisive factor in the value of a certain digit, what does one do in the absence of a number in a certain place? To indicate, for instance, that one has $26 \cdot 60^2$ and 23 units, one could attempt to write

But that would surely be read as $26 \cdot 60 + 23 = 1{,}583$, or may even be interpreted as $26 \cdot 60^2 + 23 \cdot 60 = 94{,}980$. One can leave a gap but how big must a gap be to be a gap? This poses a serious obstacle in a positional system and in that the Babylonian numeral form was deficient. In Babylonian texts which have been deciphered, the true value of a number can only be conjectured through careful study of the context.

THE MAYAS

In the early sixteenth century Spanish Conquistadores in Yucatán discovered a culture among the Mayan Indians which had many analogies to the Bronze Age in Egypt. While this civilization showed striking similarities with the South American cultures of the time, it differed from them in a unique system of numeration. Its origin is of

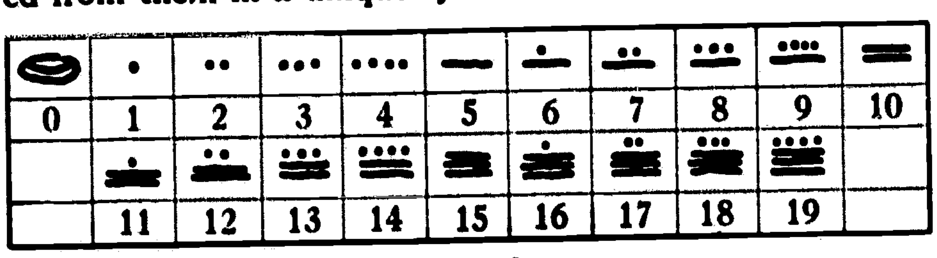
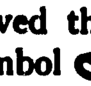


Figure 8

remote but unknown date. The Mayan number system is vigesimal, which means that twenty is the base. This system was carefully

devised, but had a deviation from the usual scheme, which is explained below. The digits are shown in Figure 8.

An example of a Mayan numeral is shown at right. Since numerals were written vertically with the lowest order at the bottom, we interpret this as: $4 + 17 \cdot 20 + 0 \cdot (20 \cdot 18) + 6 \cdot (20^2 \cdot 18) = 43{,}544$. It will be noted that the deviation mentioned before consists in using 18 as one of the factors in the successive powers of twenty so that the local values are of the form $18 \cdot 20^{k-1}$ instead of 20^k. The Mayas had successfully solved the problem of the vacant position by introducing a special symbol  as place holder—which we indicate by our zero.

HINDU NUMERALS

It is, perhaps, disrespectful to the originators of our system that we often refer to it as the Arabic number system. The name might be Hindu-Arabic numerals, since it is to the early Hindus that we owe their existence. The Arabs were instrumental in bringing new ways of writing numbers to Europe.

At first the system was a ciphered one. In the period of approximately 600–800 A. D., a fully developed positional system based on ten and having a symbol for zero was known. Introduction of this system to the Western world was almost completely due to one man, Leonardo Fibonacci of Pisa (1202). The form of the digits changed considerably and was not standardized until the invention of printing.

It often comes as a surprise to hear about the tremendous resistance that the new numerals encountered. Ill feeling existed against the new ways of writing numbers and algorithms for computing with them. Man is very conservative, and Roman numerals and computation on the abacus had become too strongly entrenched. It took longer to extend the idea of positional notation "to the other side"—to make possible the writing of decimal fractions. We owe the first satisfactory exposition on decimal fractions to Simon Stevin (1585 A. D.).

The separation between the integral and the decimal part of a number by a decimal point, or its equivalent, is still not standardized. In our time the American decimal point rests on the base line (5.4); the English put it in the middle $(5\cdot 4)$; the Austrians have it at the top $(5^{\cdot}4)$; and several other countries use a comma (5,4). Following are some examples of earlier decimal expression:

$$5\tfrac{4}{10},\quad 5\;4^{(1)},\quad 5|_4,\quad 5\underline{|4},\quad 5(4,\quad 5{:}4.$$

Another discrepancy occurs in the form of some of the digits—mainly of the one and the seven. The former is written as I or as 1, and the latter as 7 or 7.

Despite this lack of complete agreement, people using the Hindu-Arabic system today have no trouble understanding each other. They know that in this system any number N is expressed as

$$N = a_0 10^n + a_1 10^{n-1} + \cdots + a_{n-1} 10 + a_n$$

where the coefficients a_i ($i = 0, 1, 2, \cdots n$) are taken from the set $\{0, 1, 2, \cdots 9\}$ and n is an integer. Restricting n to positive integers would result in an integral number for N.

It must be emphasized that a positional system involves two aspects: assigning positional value beyond an absolute numerical value to a symbol, and creating a mark for the empty space. The idea of a *zero* ranks among man's most important inventions.

The first actual appearance of the symbol 0 for zero occurred in a Hindu manuscript of 738 A. D. Originally, the zero sign had been introduced as a place holder to indicate the empty column. It was not considered a number.

COMPARISON OF NUMBER SYSTEMS

Studying man's varied attempts at recording numbers highlights several features of a system of numeration. Basic to every such system are (a) the type of symbols to be introduced and (b) the principle upon which these symbols are to be combined. If we met a Roman soldier today and taught him to write the digits 0, 1, 2, $\cdots$ 9, and then tested him by asking him to read the number 52, he would, of course, say seven. We had forgotten to enlighten him about the positional principle underlying our number system.

Another characteristic of different numerals is the fact that the system is either complete or it requires constant introduction of new symbols when larger and larger numbers must be written. In our system, for instance, ten digits suffice to write any number however large. We can even visualize a googol—one followed by one hundred zeros—or a googolplex—one followed by a googol of zeros. Of course we have shorter ways of writing numbers of this kind. Every algebra student knows that googols and googolplexes, if they are defined in the manner explained above, can be written as 10^{100} and 10^{googol}, respectively. He might even know that the logarithm of a googolplex to the base ten would be a googol and that the same kind of logarithm in the case of a googol would be one hundred. The logarithm of a hundred, however, would be two, and all this would be another way of describing these numbers. We could tell the whole story in a compact form by writing log[log(log googolplex)] = 2. We would, however, have to know that the logarithm of a number to the base ten is the exponent to which ten has to be raised to equal that number. Or, in symbols, $\log_{10} N = a$ is equivalent to $10^a = N$.

A further point when comparing different types of numeration concerns simplicity and complexity. The systems differ significantly in the ease and compactness with which numbers can be written, and in the manner in which they lend themselves to computation. About the latter we shall hear more in the next chapter.

OTHER BASE SYSTEMS

The reader has noticed the recurrence of base ten and also the choice of other bases at times. Any natural number greater than one may, in fact, be chosen as a base. The number ten has been favored by the chance element that we are normally born with ten fingers, and our fingers are a natural computing machine.

The general form of a number N, restricted to natural numbers, reads $N = a_0b^n + a_1b^{n-1} + \cdots + a_{n-1}b + a_n$. Here b is the chosen base and is therefore an integer; $b > 1$ and the a_i are taken from the set $\{0, 1, 2, \cdots (b-1)\}$, for $i = 0, 1, 2, \cdots n$.

Let us try out this idea by converting our number 573 into one whose base is seven instead of ten. Then $573 = a_07^n + a_17^{n-1} + \cdots + a_n$. Since the largest power of seven contained in 573 is $7^3 = 343$, n will be 3 and the number will have four digits, a_0 being equal to one; 230 units have yet to be taken care of. This uses the number seven squared four times, and therefore $a_1 = 4$ and 34 units still have to be distributed. But 34 equals four 7's and six, and therefore $(573)_{10} = (1446)_7$ where the subscripts denote the base.

Repeated division, as: $\begin{array}{r}81\\7\overline{)573}\\6\end{array}$, $\begin{array}{r}11\\7\overline{)81}\\4\end{array}$ and $\begin{array}{r}1\\7\overline{)11}\\4\end{array}$

would lead to an abbreviated method of converting a number from our base system to another. Starting with the last quotient and then writing the remainders of the successive divisions in reverse order gives the digits of the new number. A two-column arrangement of the two methods will clarify the abbreviated algorithm.

Basic Form	Abbreviated Form	
$573 = \underline{1} \cdot 7^3 + 230$	$573 = 81 \cdot 7 + \underline{6}$	(1)
$230 = \underline{4} \cdot 7^2 + 34$	$81 = 11 \cdot 7 + \underline{4}$	(2)
$34 = \underline{4} \cdot 7 + \underline{6}$	$11 = \underline{1} \cdot 7 + \underline{4}$	(3)
Combining the above:	By substitution:	
$573 = 1 \cdot 7^3 + 4 \cdot 7^2 + 4 \cdot 7 + 6$	$81 = (1 \cdot 7 + 4)7 + 4$	
	$= 1 \cdot 7^2 + 4 \cdot 7 + 4$	[(3) in (2)] (4)
	$573 = (1 \cdot 7^2 + 4 \cdot 7 + 4)7 + 6$	[(4) in (1)]
	$= 1 \cdot 7^3 + 4 \cdot 7^2 + 4 \cdot 7 + 6$	

Therefore, $(573)_{10} = (1446)_7$.

To check our claim, let us convert $(1446)_7$ into our decadic system. This may be done by interpreting the number $(1446)_7$, as follows: $6 + 4 \cdot 7 + 4 \cdot 7^2 + 1 \cdot 7^3 = 573$.

The least possible value for the base of a positional system of numeration is two. This is called the binary system. Its only digits are zero and one. This was a perfect delight to Leibnitz (1646–1716), who saw the image of creation in writing any number, however large, with the use of these two digits. Zero represented the void and unity was God, the Supreme Being, who drew forth the universe from this void in the same manner in which all natural numbers may be created from zero by the assistance of one.

Although contemporary man has ceased to speculate in this fashion, numbers in the binary system have become amazingly important to him. Mathematical calculations performed by modern electronic digital computers proceed with speeds which tax man's imagination. The layman immediately attributes magic power to these machines and regards them as omniscient and omnipotent "giant brains."

Today we have a variety of mechanical devices, and undoubtedly the next decades will bring further improvements and additional novel constructions. It seems quite certain, however, that creative thinking must still come from human beings who construct the machines, from those who program the problems and feed them into the calculators, from others who set the machines in repair, and from many more, but never from the machines themselves. The superhuman element in the electronic digital computers is their lightning speed, and therein is their tremendous value.

The basic principle underlying the majority of the high speed machines is primarily the binary system of numeration. If it is possible to record any number by using two digits, then an array of vacuum tubes may be laid out which translates this zero-one proposition into an opening and closing of an electric circuit. While the wiring presents many technicalities, the principle by which instructions are prepared and transposed and finally followed by the machine is very simple. This is one of the many cases where a mathematical idea—the idea of a positional system of any base—taken up for mental curiosity and for purely theoretical reasons, was later applied. Unexpected, enormously practical values ensued. We will come back to machines later.

3. METHODS OF COMPUTATION

Historical Considerations

As civilizations started to advance and industry and commerce became more complex, operations with numbers became necessary that could not be done by finger reckoning only. Early number systems were too unwieldy for the required work, so artificial mechanical devices were used. It is wrong to believe that the idea of computational devices was reserved for modern man. These mechanical devices took various forms, but all of them employed much the same principle. The basic notion is that of denoting a unit by a pebble or ball or button or knot or disk or other similar physical object.

MECHANICAL DEVICES FOR COMPUTATION

One type of counting machine invented long ago is the counting board or abacus. A general form of abacus is illustrated in Figure 9. The successive parallel strings, each of which carries ten beads, represent the place value classes. The first one on the right denotes

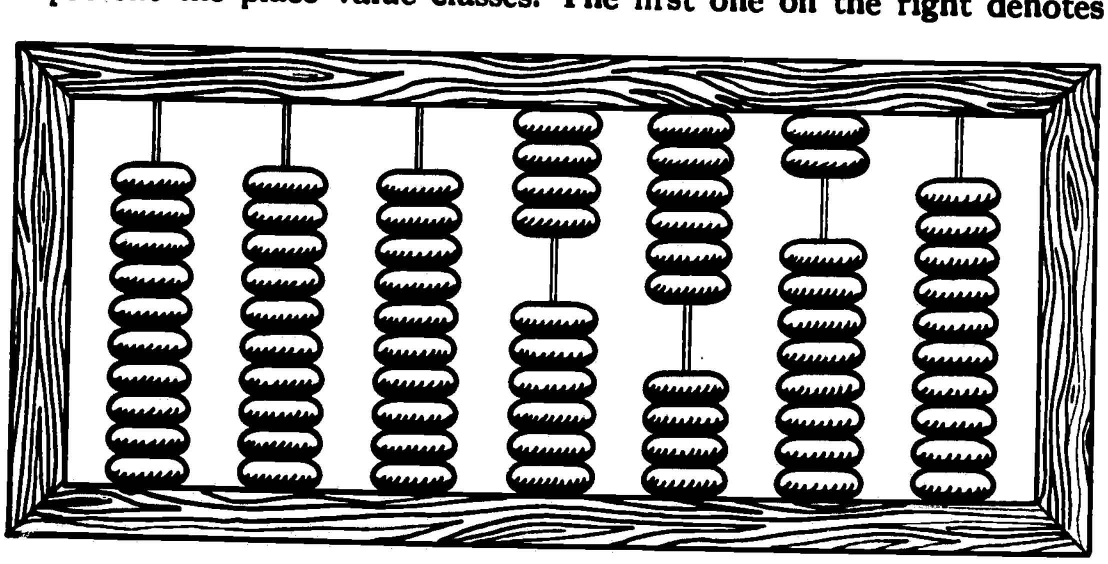


Figure 9

units; the next, tens; and so on. The number represented in the figure is 4,620.

The ancient Egyptians put pebbles in grooves in the sand, each pebble in the first groove representing one unit; each in the second, ten units; and so on.

While the origin of the abacus is a matter of conjecture, the ancient Egyptians, on the testimony of Herodotus, used an abacus. In any collection of Egyptian antiquities are found disks of various sizes which may have served as counters.

The Chinese and the Japanese computing devices, called the Suan-pan and the Soroban respectively, place a bar horizontally across the frame so that five beads are strung on the wire in the lower compartment and one or two in the upper. (The Japanese generally used only one bead above the horizontal bar.) Each of the upper beads represents a value five times as large as each of the lower beads. Whenever five units of the lower column are reached, they are replaced by one of the upper. A number is put on this computing machine by assembling the appropriate number of beads near the cross bar. Figure 10 shows how 67,943 would look.

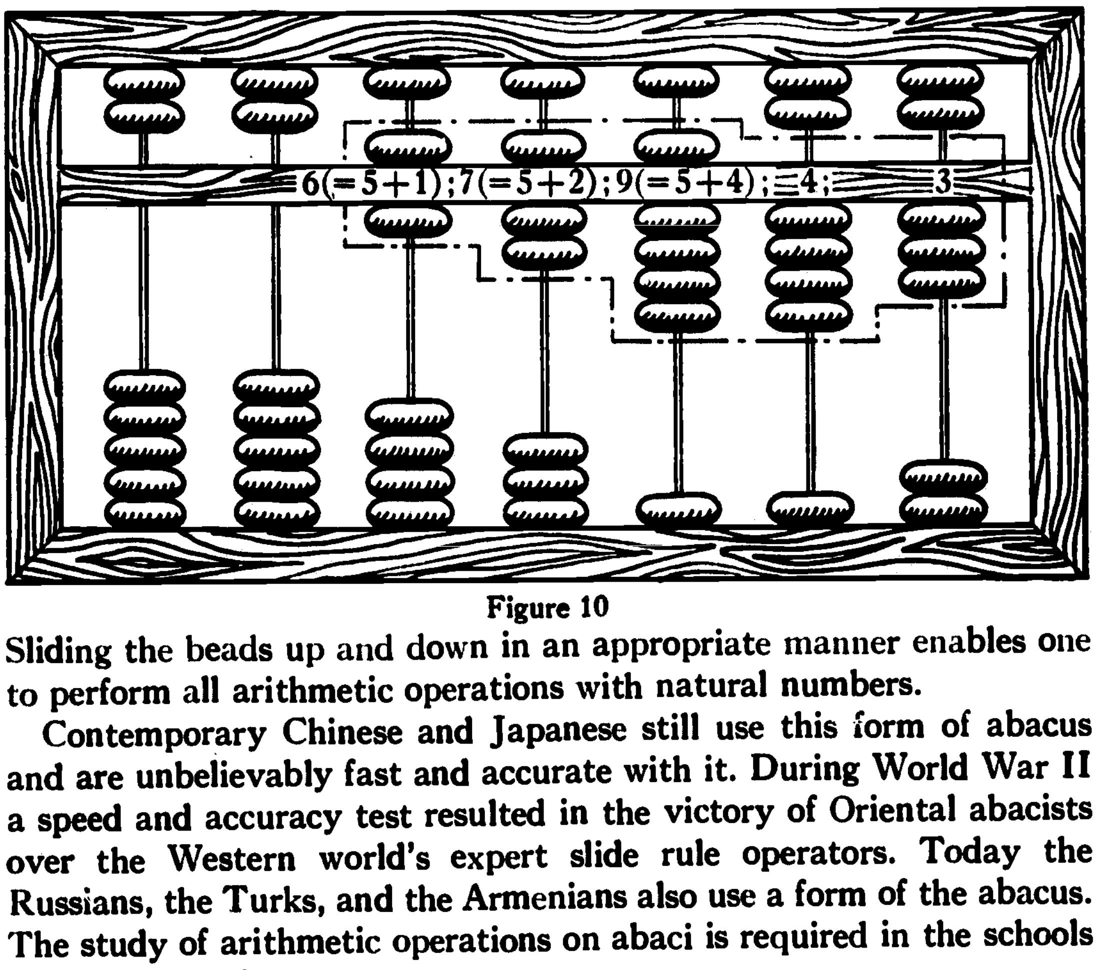


Figure 10

Sliding the beads up and down in an appropriate manner enables one to perform all arithmetic operations with natural numbers.

Contemporary Chinese and Japanese still use this form of abacus and are unbelievably fast and accurate with it. During World War II a speed and accuracy test resulted in the victory of Oriental abacists over the Western world's expert slide rule operators. Today the Russians, the Turks, and the Armenians also use a form of the abacus. The study of arithmetic operations on abaci is required in the schools of these countries.

A Roman abacus of bronze, now in the British museum, has two sets of grooves arranged vertically, one below the other as shown in Figure 11. The upper set contains one pebble in each groove but its value is five times as large as each of the four pebbles in the lower set. The similarity to the Chinese and Japanese abaci is quite striking. The Latin word for pebble is *calculus*, from which our words *Calculus* and *calculate* are derived.

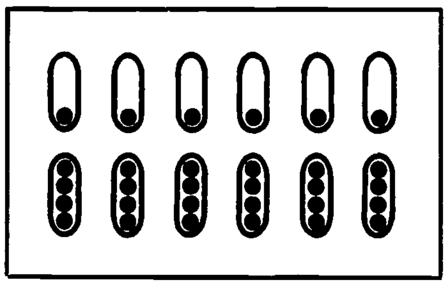


Figure 11

The Greeks often used wax-covered tablets on which figures were marked with a stylus. Sometimes sand was used instead of the wax covering. Sand-reckoning as well as reckoning on wax tablets was difficult because it was necessary to "erase" figures as soon as they were used by smoothing out either the sand or the wax. The type of algorithm used often presents this obstacle and some of the sand reckoners had ingenious methods for checking their work; we will discuss these later.

Another computational device used counters laid out on a counting board. Again, counters on the first line represent units; on the second, tens; and so on. Since paper was not in general use in Europe until about the eleventh century, counters of various other substances were employed. In medieval times the loose counter abacus consisted of a table with lines chalked or painted across. Sometimes even a piece of fabric with a pattern of threads was considered appropriate. Any type of disk, button, or coin was used. The number 269, for instance, would be shown as in Figure 12. In computation, counters were literally carried or borrowed from one position to another; we have a contemporary linguistic remnant of those days. Even as late as the sixteenth century counter reckoning was used side by side with computation by pen for the perfection of certain algorithms to be used in connection with the Hindu-Arabic numerals.

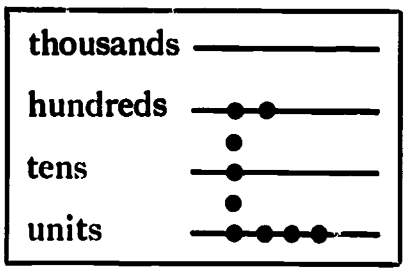


Figure 12

EARLY ALGORITHMS FOR ARITHMETIC OPERATIONS

The Egyptians. The type of arithmetic instruction given in Egypt about 4000 years ago is fairly well known because of the Ahmes Papyrus. Egyptian influence, with algorithms for the fundamental operations, was prevalent for a long time.

Egyptian numerals posed no difficulty in addition and subtraction.

In multiplication with these numerals we find the first approach to a systematic multiplication algorithm. The method employs doubling and halving, and it reduces the number of multiplication facts that must be known.

To multiply 91 by 57 two chains of numbers are formed, the first by successive halving of the numbers involved, the second by doubling. In the halving process remainders are discarded. Now those second column numbers which correspond to even numbers in the first set are crossed out. The required product, 5187, is obtained by adding the remaining values of the second chain.

	Halving	*Doubling*	
	91	57	$= 1 \cdot 57$
	45	114	$= 2 \cdot 57$
Even	(22)	~~228~~	
	11	456	$= 2^3 \cdot 57$
	5	912	$= 2^4 \cdot 57$
Even	(2)	~~1824~~	
	1	3648	$= 2^6 \cdot 57$
		5187	

For the proof of this strange procedure we turn again to the binary system of numeration. The operation $91 \cdot 57$ may be written as

$$(1011011)_2 \cdot (57)_{10} = (1 + 2 + 2^3 + 2^4 + 2^6) \cdot 57.$$

This, however, is equivalent to the above addends. Thus 5187 results.

The generality of this method is seen by the abbreviated process for converting a number from one base system to another (see p. 34). Thus the binary form of a number represents the sum of those powers of two which correspond to those steps whose remainders are one. The addends correspond to the odd numbers which appear in the halving column. This sum of the powers is then multiplied by the multiplier 57. Repeated doubling of 57 and the addition of only those values which line up with the odd numbers in the left column lead to the correct result. To illustrate diagrammatically:

$91 \div 2 = 45$	remainder 1	corresponds to	$1 \cdot 57$
$45 \div 2 = 22$	" 1	" "	$2 \cdot 57$
$22 \div 2 = 11$	" 0	" "	$2^2 \cdot 57$
$11 \div 2 = 5$	" 1	" "	$2^3 \cdot 57$
$5 \div 2 = 2$	" 1	" "	$2^4 \cdot 57$
$2 \div 2 = 1$	" 0	" "	$2^5 \cdot 57$
$1 \div 2 = 0$	" 1	" "	$2^6 \cdot 57$

$(91)_{10} = (1011011)_2.$

Then

$$(91)_{10}(57)_{10} = (1011011)_2(57)_{10}$$
$$= (1 + 2 + 2^3 + 2^4 + 2^6)_{10}(57)_{10} = (5187)_{10}.$$

This old Egyptian technique, we are told, is still being used by Russian peasants.

In the middle ages treatises on computations still dealt at length with duplication (doubling) and mediation (halving). They were considered two more operations in addition to the four fundamental ones.

Division in ancient Egypt was done by using the inverse of the method for multiplication. To divide 5187 by 91, double 91 successively until the double would exceed 5187. One obtains the chain:

91	*	1
182		2
364		4
728	*	8
1456	*	16
2912	*	32

Those partial products which add up to the dividend are determined and the corresponding powers of 2 are added. The answer is $1 + 8 + 16 + 32 = 57$.

If the division is not exact the remainder may be determined by use of the above method. To illustrate, divide 433 by 23.

23		1
46	*	2
92		4
184		8
368	*	16

Those partial products which add up to a number equal to 433 or less than 433 by an amount less than 23 are 368 and 46. The sum of the corresponding powers of two is $16 + 2 = 18$. Therefore the quotient is 18. The remainder is $433 - (368 + 46) = 19$.

The Babylonians. The civilization of ancient Babylonia has been carefully studied and archaeologists have unearthed much information. No arithmetic treatise like that of the Egyptians has been found, but the deciphering of clay tablets, many over 5000 years old, reveals that multiplication was generally done by elaborate tables which were clumsy and difficult to read.

The Hindu-Arabic System. The process of addition did not undergo any changes in the course of time except that it often was done from

left to right. Checking was done by reversing the direction of adding, that is, from the bottom up, then from the top down or vice versa.

A check which may be used in all arithmetic operations—not only in addition—has recently lost some popularity. It is the process of "casting out nines." Let us consider an integer written in our number system. It reads $N = a_0 10^n + a_1 10^{n-1} + \cdots + a_{n-1}10 + a_n$. The coefficients a_i $(i = 0, 1, 2, \cdots)$ are taken from the set $\{0, 1, 2, \cdots 9\}$ and n is a natural number. We will investigate the number N with respect to its divisibility by nine.

First we make the observation that 10^i $(i = 0, 1, 2, \cdots)$ leaves a remainder of one when divided by nine. Let us use a very convenient notation which was first introduced by Gauss. A number a is said to be congruent to a number b with respect to the modulus m if it leaves a remainder of b units when divided by m. Or, in symbols,

$$a \equiv b\ (m) \leftrightarrow a = mt + b$$

where t is an integer. Note that the quotient t does not appear in the congruence notation. Using this symbolism the previous result may now be expressed by saying that $10^i \equiv 1\ (9)$ where i is any natural number or zero.

Second, we establish the behavior of congruences with respect to some forms of addition and multiplication. It is evident that if

$$a \equiv b\ (m)$$

and if c is any integer, then

$$ac \equiv bc\ (m)$$

also. This may be seen by translating the given congruence into the equivalent equation $a = mt + b$ and multiplying each side by c to obtain $ac = mct + bc$. Since mct is a multiple of m, this relationship may be restated as the above congruence. In particular, if $10^i \equiv 1(10)$ then $a_{n-i}10^i \equiv a_{n-i}(10)$.

Furthermore, if two congruences with respect to the same modulus are given, say $a_1 \equiv b_1\ (m)$ and $a_2 \equiv b_2\ (m)$, then the sum of the left members is congruent to the sum of the right members with respect to the same modulus, or: $a_1 + a_2 \equiv b_1 + b_2\ (m)$. In equational form, the data read $a_1 = mt_1 + b_1$ and $a_2 = mt_2 + b_2$. Therefore, $a_1 + a_2 = m(t_1 + t_2) + (b_1 + b_2)$. Since $t_1 + t_2$ is an integer, the claim is established.

Let us apply these principles to our investigation of the divisibility of a general number N by nine. To start with,

$$N \equiv a_0 10^n + a_1 10^{n-1} + \cdots + a_{n-1}10 + a_n \qquad (9)$$

since, if $a = b$, trivially, $a \equiv b(m)$ for any modulus.

$$
\begin{array}{lllll}
\text{But:} & 1 \equiv 1 & (9) & \text{Thus:}\ 1 \cdot a_n \equiv a_n & (9) \\
& 10 \equiv 1 & (9) & 10 \cdot a_{n-1} \equiv a_{n-1} & (9) \\
& 10^2 \equiv 1 & (9) & 10^2 \cdot a_{n-2} \equiv a_{n-2} & (9) \\
& \cdots & \cdot & \cdots\cdots & \cdot \\
& \cdots & \cdot & \cdots\cdots & \cdot \\
& 10^n \equiv 1 & (9) & 10^n \cdot a_0 \equiv a_0 & (9) \\
\hline
\end{array}
$$

$$\text{Adding, } N \equiv a_0 + a_1 + \cdots + a_n \quad (9)$$

In other words, with respect to divisibility by nine, any integral number N corresponds to the sum of its digits. Any integral number is a multiple of nine if the sum of its digits is a multiple of nine. If the sum of the digits divided by nine leaves a remainder R, then the same remainder will be obtained if the original number itself is divided by nine. All multiples of nine may be ignored when finding the sum of the digits.

As an example, let us consider the numbers $a = 76{,}543{,}281$ and $b = 789{,}134{,}563$. When investigating a one would say: $1 + 8$, discard; $2 + 3 + 4$, discard; $5 + 6$, retain a 2; $2 + 7 = 9$. Therefore, a is divisible by nine, or: $a \equiv 0$ (9). For number b the steps are: $3 + 6$, discard; $5 + 4$, discard; $3 + 1 = 4$, retain; omit the 9; $4 + 8 \equiv 3$ (9), retain a 3; $3 + 7 \equiv 1$ (9). Hence, the number b leaves a remainder one when divided by nine, or: $b \equiv 1$ (9).

To use the idea of casting out nines as a check in addition, obtain all respective remainders, often also called excesses. The sum of the excess values of the addends should equal the excess of the sum.

To illustrate, the addition in Figure 13 is to be checked. We proceed by obtaining the excess values of the addends (column A). Their sum leads to the excess seven. The excess of the answer is also seven.

	A	B
567823	4	4
615204	0	0
34901	8	−1
5678	8	−1
234914	5	−4
1,458,520	7	−2

Figure 13

$$
\begin{array}{ll}
5 \equiv -4 & (9) \\
6 \equiv -3 & (9) \\
7 \equiv -2 & (9) \\
8 \equiv -1 & (9)
\end{array}
$$

Figure 14

Incidentally, negative excess values (column B) may also be used to achieve the smallest absolute values of our numbers. Figure 14 establishes these values. Using negative excess values, column B results. It checks again by using this method on the sum.

Casting out nines is not infallible. Correctly done, disagreement in the final excess values would mean an error in the original arithmetic

operation. Agreement in the excess values, however, does not necessarily mean correctness of the solution. If errors had occurred which amounted to multiples of nine, casting out nines would not detect them. A common error of this kind would be transposition of digits.

To counteract the possibility of not detecting an error in casting out nines, casting out elevens may be resorted to. Since

$$\begin{aligned} 1 &\equiv 1 \quad (11) \\ 10 &\equiv -1 \quad (11) \\ 10^2 &\equiv 1 \quad (11) \\ 10^3 &\equiv -1 \quad (11) \\ &\ldots\ldots\ldots \end{aligned}$$

we obtain a fairly simple rule as follows:

$N = a_0 10^n + a_1 10^{n-1} + a_2 10^{n-2} + \cdots + a_{n-1}10 + a_n$ then

$N \equiv a_n - a_{n-1} + a_{n-2} - \cdots + (-1)^n a_0 (11)$ or,

$N \equiv (a_n + a_{n-2} + a_{n-4} + \cdots) - (a_{n-1} + a_{n-3} + a_{n-5} + \cdots)(11).$

Using the addition problem in Figure 13, the following excess values with respect to 11 accrue (negative values may again be used). The sum of these values leaves excess -3. When casting out elevens, 1458520 becomes $11 - 14 = -3$, and the check has been performed.

$$\begin{aligned} 6 - 3 &= 3 \\ 7 - 11 &= -4 \\ 2 - 4 &= -2 \\ 3 - 1 &= 2 \\ 5 - 7 &= -2 \end{aligned}$$

Subtraction was done by various methods. Subtracting a larger digit from a smaller posed the main difficulty. An early suggestion was borrowing. In a fifteenth century work three cases of subtraction were treated, involving subtrahends smaller than, equal to and larger than the minuends. In the first case the difference of the numbers is found. In the second, the answer is zero. The last case will be illustrated by an example. To perform the subtraction 734 minus 528, add the complement of the subtrahend—that is, ten minus that digit—to the minuend and add one to the next digit of the subtrahend. This means adding two (the complement of eight with respect to ten) to four to obtain six in the unit place of the difference. Then add one to two to obtain three. Subtract this from its equal to obtain zero in the tens' digit. The digit in the hundreds' place is found to be two, the difference between seven and five. The answer is 206. Fibonacci suggested a "borrow-and-pay-back" method; the person subtracting 37 from 65 would say: seven from fifteen leaves eight, four from six leaves two.

$$\begin{array}{r} 527 \\ -385 \\ \hline 142 \end{array}$$

These are among the forerunners of our present algorithms for subtraction. An additive or a subtractive principle is employed, and in each case a borrowing or equal complement form may be selected. In the case of the subtraction at left, the following possibilities exist:

1. additive, borrowing: 5 and what is 7
   8 and what is 12
   3 and what is 4
2. additive, equal complements: 5 and what is 7
   8 and what is 12
   4 and what is 5
3. subtractive, borrowing: 5 from 7
   8 from 12
   3 from 4
4. subtractive, equal complements: 5 from 7
   8 from 12
   4 from 5

	A	B
5 2 7	5	−1
−3 8 5	7	0
1 4 2	−2	−1

Here are the schemes for casting out nines and elevens. Column A applies to nines, column B to elevens.

Multiplication procedures also used many different devices. They varied mainly in the manner of recording the partial products. In the *Treviso Arithmetic* (1478) Gelosia Multiplication made its appearance. The name *Gelosia* indicates a resemblance to a window grating. With this method we multiply 786 by 534 as follows:

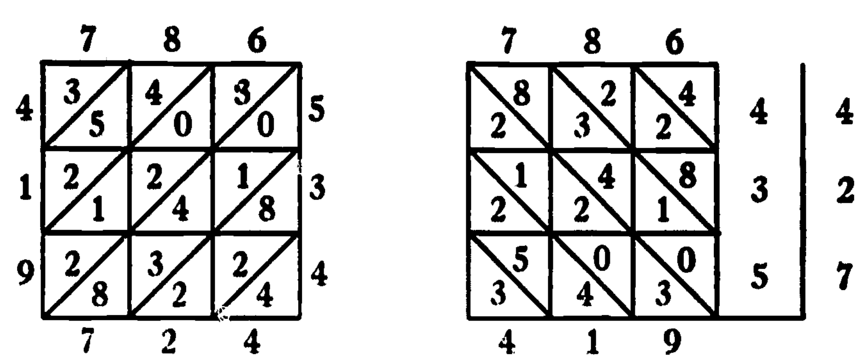


Figure 15

The answer is 419724. The diagrams are self-explanatory, illustrating two main forms of multiplication. In this country multiplication is almost always performed by starting with the digit of lowest place value. In Europe both methods are used.

To check by casting out nines, the excess values of multiplicand and multiplier, that is, three and three, are multiplied to obtain 0(9) which agrees with the excess value of the product. In casting out elevens, one would find five and six as the respective excess values of the factors and, correctly, minus three as the excess of the product.

~~1~~
~~1~~ 7 8
~~9~~ ~~8~~ ~~9~~
~~1~~ ~~0~~ ~~2~~ ~~2~~ 3
7 ~~3~~ ~~4~~ ~~8~~ ~~5~~ (214; 343
~~2~~ ~~1~~ ~~4~~ ~~4~~ ~~4~~
~~2~~ ~~1~~ ~~1~~
~~2~~

Division undoubtedly caused confusion then as it does now. Fibonacci in the fifteenth century introduced a *scratch* or *galley* method (so named perhaps because it looked like a ship of his day), which was still used more than 300 years later. We will show this method by dividing 73485 by 214. The completed work would look like the illustration at left. The answer is 343 and the remainder is 83.

Part I. The form of the quotient is 3 _ _ (3 digits by observation).

Step 1	*Step 2*	*Step 3*
1 7 3 4 8 5 ~~2~~ 1 4 Three 2's are 6 plus 1 is 7.	1 0 7 ~~3~~ 4 8 5 ~~2~~ ~~1~~ 4 Three 1's are 3 plus 0 is 3.	9 ~~1~~ ~~0~~ 2 7 ~~3~~ ~~4~~ 8 5 ~~2~~ ~~1~~ ~~4~~ Three 4's are 12 plus 92 is 104.

Part II. The quotient now becomes 34 _.

Step 1	*Step 2*	*Step 3*
1 ~~9~~ ~~1~~ ~~0~~ 2 7 ~~3~~ ~~4~~ 8 5 ~~2~~ ~~1~~ ~~4~~ 4 ~~2~~ 1 Four 2's are 8 plus 1 is 9.	~~1~~ ~~9~~ 8 ~~1~~ ~~0~~ ~~2~~ 7 ~~3~~ ~~4~~ 8 5 ~~2~~ ~~1~~ ~~4~~ 4 ~~2~~ ~~1~~ Four 1's are 4 plus 8 is 12.	~~1~~ 7 ~~9~~ ~~8~~ ~~1~~ ~~0~~ ~~2~~ 2 7 ~~3~~ ~~4~~ ~~8~~ 5 ~~2~~ ~~1~~ ~~4~~ ~~4~~ ~~2~~ ~~1~~ Four 4's are 16 plus 72 is 88.

Part III. The quotient is 343.

Step 1	*Step 2*	*Step 3*
1 ~~1~~ 7 ~~9~~ ~~8~~ ~~1~~ ~~0~~ ~~2~~ 2 7 ~~3~~ ~~4~~ ~~8~~ 5 ~~2~~ ~~1~~ ~~4~~ ~~4~~ 4 ~~2~~ ~~1~~ 1 ~~2~~ Three 2's are 6 plus 1 is 7.	~~1~~ ~~1~~ 7 ~~9~~ ~~8~~ 9 ~~1~~ ~~0~~ ~~2~~ ~~2~~ 7 ~~3~~ ~~4~~ ~~8~~ 5 ~~2~~ ~~1~~ ~~4~~ ~~4~~ 4 ~~2~~ ~~1~~ ~~1~~ ~~2~~ Three 1's are 3 plus 9 is 12.	~~1~~ ~~1~~ 7 8 ~~9~~ ~~8~~ ~~9~~ ~~1~~ ~~0~~ ~~2~~ ~~2~~ 3 7 ~~3~~ ~~4~~ ~~8~~ ~~5~~ ~~2~~ ~~1~~ ~~4~~ ~~4~~ ~~4~~ ~~2~~ ~~1~~ ~~1~~ ~~2~~ Three 4's are 12 plus 83 is 95.

To check division by casting out, rewrite the proposed relationship in a multiplicative form so as to investigate $73485 = 343 \cdot 214 + 83$. Casting out nines, one checks the corresponding relationship among the remainders which here would read $0 \equiv 1 \cdot 7 + 2$. Since $7 + 2 \equiv 0(9)$, casting out nines checks the work. The corresponding relationship with respect to eleven would be $5 \equiv 2 \cdot 5 + 6$. But $16 \equiv 5(11)$, which concludes this check.

Although the arithmetic algorithms put an ease and elegance into the art of written reckoning, the transition to these new methods was far from rapid. Indeed, the struggle between the Algorists and the Abacists lasted several centuries. It was not until the beginning of the sixteenth century that the new rules for operation were fully adopted.

Computation Today

The seventeenth century brought a great step forward in computational work. John Neper (or Napier), a Scotch nobleman, politician, and magician, turned to mathematics for amusement and relaxation. Napier's Bones, an ingenious device for multiplying, dividing and extracting roots was popular for a long time. A set of rods or bones was made for the multiplication tables as shown in Figure 16.

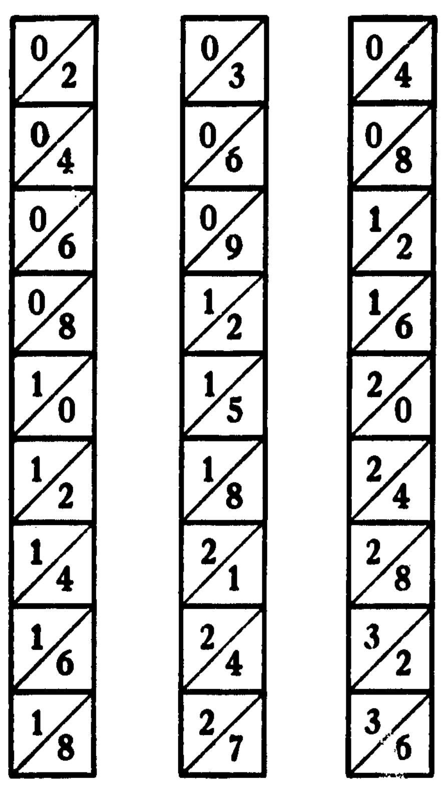


Figure 16

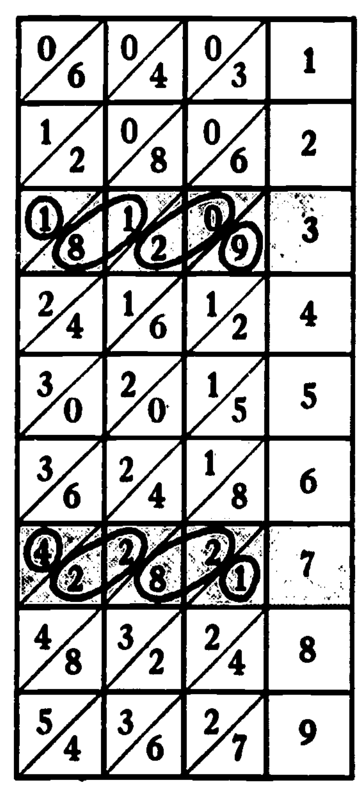


Figure 17

There was also a guiding rod which contained the successive digits. To multiply 643 by 37, the "bones" would be laid out as in Figure 17.

The partial results 4501 and 1929 are then easily read off, and the final value of 23791 is obtained by the addition of 4501 and 19290. (Note that in reading off the partial products, a diagonal form of addition is used.)

Napier contributed other important computational aids. By studying the relationships between points moving on two separate straight lines with different velocities, he originated logarithms.

In today's notation a logarithm is defined as an exponent. In other words, the logarithm of a number N to a base b is the exponent a which has to be attached to base b to obtain N, or, in symbols:

$$\log_b N = a \leftrightarrow b^a = N.$$

Any positive number $b > 1$ may be chosen as a base. The two numbers most frequently taken are ten and e, a most interesting transcendental number we shall discuss later. Logarithms with base ten are called ***common logarithms***; those with base e are called ***natural logarithms.***

By means of logarithms every arithmetic task of multiplication or division may be reduced to one in addition or subtraction. The operations of raising to a power and extracting roots are reduced to very simple multiplications and divisions. Logarithms saved years of drudgery for mathematicians and those employed in computational work. Anyone using a slide rule or logarithmic tables of any kind has benefited from these inventions.

Blaise Pascal, at the age of 19, invented an adding machine in 1642. Any automatic counter, the cash register, the speedometer of an automobile, the fare register on a bus, the automatic recorder on a Geiger counter, and innumerable other mechanisms are the modern descendants of this idea.

About 1673, Leibnitz improved Pascal's machine by a device which makes it possible to perform repeated additions, or to multiply. When a machine can add and multiply, however, it can also do the inverse operations of subtraction and division simply by reversing the process. Electrified and streamlined, such calculating machines are now found in banks and other business establishments.

Greater developments were yet to come. Electronic computers replaced Pascal's notched wheel with a tube or a circuit. This did not mean a basic change in the principle of the computing machine; it did mean tremendous increases in speed. The tube or the circuit counts about 1,000,000 times faster than the toothed wheel!

Fundamentally there are two types of such machines, the analog computers and the digital computers (arithmetical machines). The former is constructed analogously to the mathematical structure of the problem it is designed to solve. It is therefore not adaptable in a

general sense, and is mathematically limited somewhat in its use.

The digital computers not only perform arithmetic operations, they store information, make comparisons, select numbers on the basis of them, and follow a sequence of instructions. The programmer has to break a problem down into a series of very simple basic arithmetic steps which the machine keeps repeating. While the mathematician in his work aims at simplification, abbreviation and shortcuts, the machine "likes" tediousness, step by step approximation, trial and error selections and repetition.

Suppose we compile a table of squares with the help of the electronic digital computer. The programmer seeks a repetitive arithmetic procedure. He finds that the squares of the natural numbers may be obtained by arithmetic progressions according to the rule

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2,$$

as an algebra student may verify. To find the successive squares, the machine computes:

$$\begin{aligned} 1^2 &= 1 &&= 1 \\ 2^2 &= 1 + 3 &&= 4 \\ 3^2 &= 1 + 3 + 5 &&= 9 \\ 4^2 &= 1 + 3 + 5 + 7 &&= 16 \\ &\cdots\cdots\cdots\cdots\cdots \\ 7431^2 &= 1 + 3 + 5 + \cdots + 14{,}861 &&= 55{,}219{,}761 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

This is a very cumbersome procedure.

The new machines offer a means of handling a massive amount of data, especially in statistics. We can now program social and economic problems whose computation in the past would have overtaxed our human calculating powers. Many new uses have been found, and additional uses will undoubtedly be forthcoming. And yet may we rightly call these machines *giant brains?* Do they think?

Electronic machines do more than compute; they are developing into logical machines rather than mere calculating ones. It is correct, therefore, to say that the machine can do logical "thinking." This is possible because deductive logic may be formalized and the formalization can be built into the machine.

Although machines can use deductive logic, man's reasoning powers exceed this type of thinking. Seeking unifying principles, searching for simplifications, exhibiting insight, having intuition and powers of imagination, the very essence of man's distinctly human qualities transcend the machine's ability. Man has not been able to construct a machine which will imitate this precious gift of the mind.

4. OUR NUMBERS TODAY

The Real Numbers

The real numbers are perhaps the most important in mathematics. Most elementary functions are defined on sets of real numbers. Nearly all mathematics leading to and including the calculus is a study of their properties. The real numbers may always be thought of as those that are used to label all the points on a scaled line.

THE RATIONAL NUMBERS

Integers. What we call *arithmetic* today was named *logistica* by the Greeks. *Arithmetica* was the study of the theoretical aspects of number. This was more highly regarded than *logistica* since the latter dealt with practical considerations and was considered slave's work.

Number Mysticism. The mystic number worship of the Pythagoreans with all its magic and superstition was the forerunner of the theory of numbers, one of today's most difficult and fascinating branches of mathematics. The theory of numbers is a study of the properties of natural numbers. In the same sense other sciences may often be traced back to man's interest in the occult and the mysterious. Was not astrology the forerunner of astronomy; alchemy the father of chemistry?

Pythagorean number speculations created many strange notions about numbers. "One" is not a number but the source of numbers. It means reason. Even numbers are feminine; odd ones, masculine. Lest the female reader rejoice she must be told that the reason for this distinction lay in the fact that even numbers are divisible by two; thus (in analogy to the females) they are soluble, ephemeral and earthy. The odd numbers, lacking this divisibility, are indissoluble, celestial, and, indeed, masculine. "Two" represents opinion; "four," justice. Perfect marriage would be "five," since this is the union between the first female and the first male number. "Six" is a *perfect* number, since it equals the sum of its factors (except itself): one, two, and three.

There are also *abundant* and *deficient* numbers. Incidentally, until recently only 15 perfect numbers had been found, all of which are even numbers. One of the many unsolved problems in the theory of numbers concerns itself with an investigation of the existence of odd perfect numbers. Perhaps female readers may rejoice after all!

The number 50 also has an interesting past. It was regarded as "the supreme principle of the production of the world." It expressed the perfection of a right triangle since the sum of the squares of the three simplest numbers which can be used as the measure of its sides, three, four, and five, equals 50.

"Master, what is friendship?" Pythagoras was asked. "Friendship, indeed, is signified by the pair of numbers, 220 and 284," was his sinister reply. The uninitiated may see the resemblance to friendship if he lists the factors of both numbers; the sum of the aliquot parts of each number equals the other number. To associate number characteristics with friendship and to consider this a phenomenon to be marvelled at is one thing. To see a number property from the mathematical point of view, and—as in this case—to seek other numbers which exhibit the same behavior is another. It was not until the seventeenth century that other such number pairs were added to the list. This is an example of number worship and numerology bequeathing to us a problem which stimulated investigations and research in the theory of numbers.

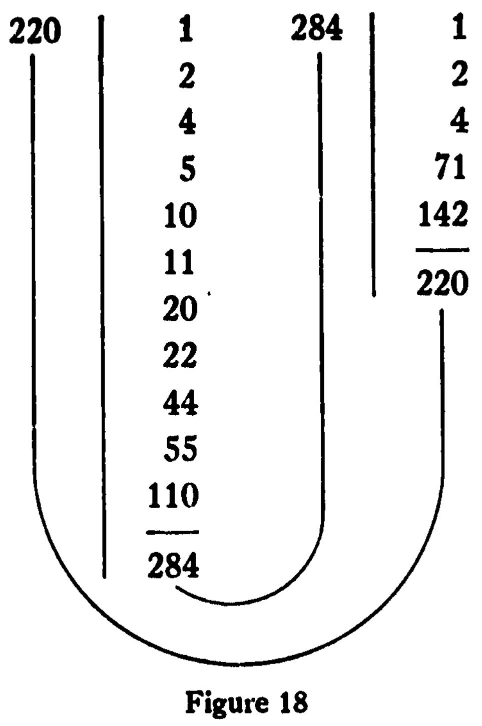


Figure 18

Just as today's fraternities and sororities have special emblems as keys or pins of their societies so also the Pythagorean brotherhood had recognition symbols. Only if a pentagram was shown and some mystic words spoken did the secret door open. The words were: "See, what you thought to be four was really ten and a complete triangle and our password." This idea was due to another strange Pythagorean conception—representing numbers by dots. These "figurate numbers" were subdivided into triangular and square numbers, pentagonals, and others. They are illustrated in Figure 19. The password explains itself when we observe the fourth triangular number.

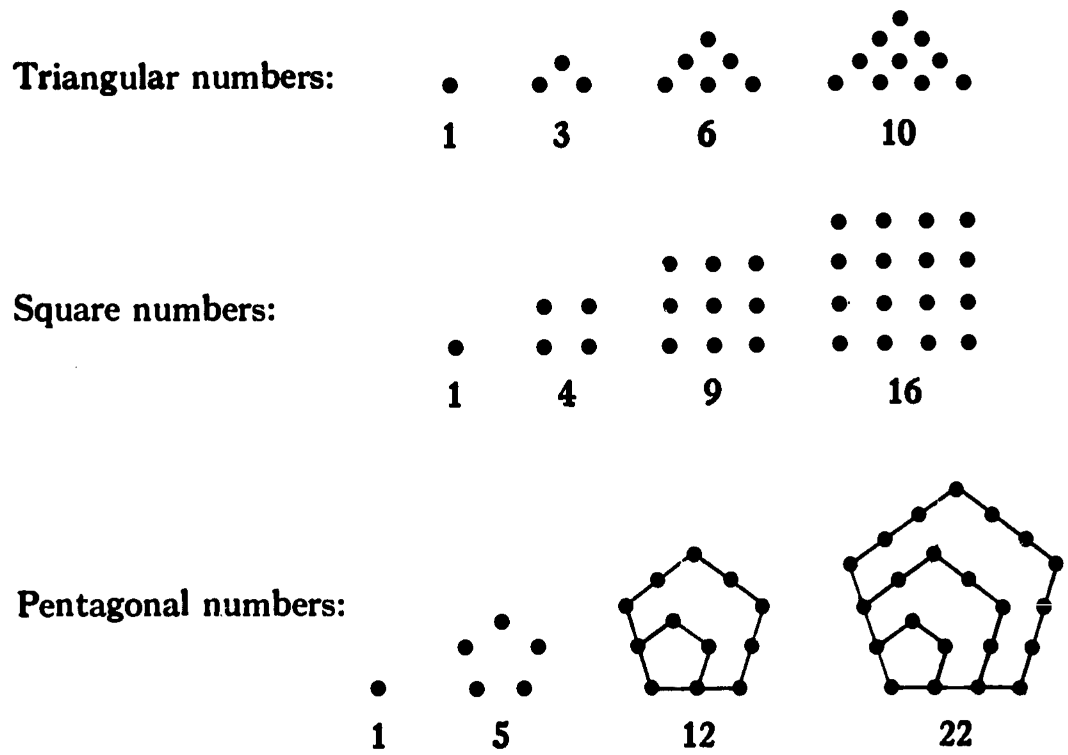


Figure 19

Although figurate numbers as such have been long discontinued, many interesting number properties may be established in this geometric fashion. For instance, the relationship which shows that the square of any integer may be obtained as the sum of an appropriate number of odd integers starting with one may be seen from the diagram in Figure 20. This relationship was used in connection with our example showing how an electronic digital computer would obtain squares of numbers. One other remnant of figurate numbers is our use of the words *square* and *cube* in connection with numbers, and probably also the term *figures* in referring to numbers, and *to figure.*

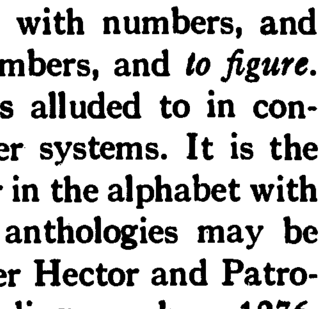


Figure 20

Another direction which numerology took was alluded to in connection with our discussion of alphabetic number systems. It is the so-called "Gematria" which associates every letter in the alphabet with a number. Numerous such examples in Greek anthologies may be pointed to. For instance, Achilles' superiority over Hector and Patroclus was seen to be the result of the corresponding numbers 1276, 1225, and 87.

Number mysticism was by no means confined to the ancient Greeks, although their influence in this area was great. "Three" has a transcendent meaning in many languages. The phrase *thrice blessed* indicates

a very special blessing. Jupiter's thunderbolt and Neptune's trident were triple pronged. Cerberus, the watchdog of Hades, had three heads. Trinities do not occur in Christian religions only.

The other most popular number is seven. There were the seven heavenly bodies of the Babylonians, the origin of our seven days in the week. There was the seventh day of rest propagated by the Hebrews. The seven wonders of the world, the seven liberal arts, the seven wise men of Greece, and sabbatical leave are other examples.

Gematria was also used by Christian theologists. There is the well-known number 666 which was to be the Beast of the Apocalypse. Numerous interpretations of this number have been given through the ages—not long ago Hitler was added to the list. Devout Hebrew scholars still consider Gematria a part of their study.

Some Ideas from the Theory of Numbers. Gradually, in the course of centuries, the natural numbers lost their associations with mysticism. But their special appeal to man has never waned. Number theory has always been a favorite among mathematicians. Gauss's remark is well known: "Mathematics is the queen of the sciences and arithmetic is the queen of mathematics." The theory of numbers is also the one branch of mathematics that has a very intrinsic fascination for the layman. Even amateurs have at times been able to make useful suggestions and conjectures.

Very basic in this branch of mathematics is the class of prime numbers. A prime is defined as an integer $n > 1$ which is not divisible by any other integer a unless $a = 1$ or $a = n$. Divisibility here means divisibility without a remainder. Thus, with the help of the symbolism we have already introduced, if for $n > 1$, $n \not\equiv 0(a)$, $a \neq 1$ and $a \neq n$ then n is a prime number.

Numerous questions come to mind. How many prime numbers are there? Is there an infinite number of them? How does one determine if a given large number is prime? Is there a formula which yields only primes? Is there one which gives all prime numbers? Do prime-twins, that is, primes which differ only by two as 17 and 19, or 29 and 31, occur in the realm of very large numbers? These are a few of the many questions that naturally arise. Several have not yet been answered. There remain many unsolved problems in this part of mathematics.

That the number of primes is infinitely large was known by the ancient Greeks. Euclid's proof is considered very elegant and a model of indirect reasoning in mathematics. By a tentative assumption he argued the contradiction of the theorem, and considered the class of primes finite. Symbolizing this set by $\{p_1, p_2 \cdots p_n\}$, he then constructed a new number N as follows: $N = p_1p_2p_3 \cdots p_n + 1$. That is,

γ

he added one to the product of "all" primes. Since this new number N is larger than any of the primes, it cannot be prime by the above assumption. Being composite, it must have one of the above p_i $(i = 1, 2, 3, \cdots)$ as a factor. But $N \equiv 1(p_i)$ for $i = 1, 2, 3, \cdots n$ and therefore the assumption is untenable. The number of primes must be infinite!

Complete tables of primes have been computed at least within the range $(1, 10^7)$. Certain much larger primes have also been found. Recently one such number was seen to have 386 digits! It is nothing less than the number $2^{1279} - 1$, and the calculation was not accomplished by an industrious man; it was the work of a digital computer.

To find the primes within a reasonable range, for instance, from one to a hundred, a very simple device, the "sieve of Eratosthenes" (about 200 B. C.) may be used. One would write down the set of integers from one to one hundred in order. Considering two as the first prime number, all even numbers are crossed out. The first remaining number, three, is the next prime. Now all numbers divisible by three are deleted. The first remaining one in this series, five, constitutes the next prime. The process is repeated in this manner.

Many attempts to find a law governing the distribution of primes proved futile. No formula was found to yield all primes. None could be established to give the exact number of primes contained in the set of integers $\{1, 2, 3, \cdots, (n-1), n\}$ where n is any integer. The distribution of individual primes appeared exceedingly irregular. For instance, there are 25 primes between 1 and 100, 11 between 1300 and 1400, five primes will be found over the 20-unit range from 1420 to 1440, but over the equally long interval between 2558 and 2578 there is none.

Finally, attention was drawn to a study of the average distribution of primes. Empirically, Gauss saw an amazing resemblance between the behavior of primes and that of natural logarithms! He introduced the idea of density of primes, defining it as the ratio of A_n, the number of primes in the set $\{1, 2, 3, \cdots n\}$, to n, the numerosity of the set. And he noticed that this ratio $\frac{A_n}{n}$ approaches $\frac{1}{\ln n}$ more and more closely with larger and larger values of n. In sophisticated symbolism, we would say:

$$\lim_{x \to \infty} \frac{A(x)}{\frac{x}{\ln x}} = 1.$$

Gauss's insight could not be verified for another century. Powerful concepts of modern mathematics were necessary to prove his theorem.

This is one of the many cases in the history of mathematics where ideas which at first are considered totally unrelated and isolated are later found to be closely intertwined or to be different aspects of one all-embracing theory.

Number Peculiarities. In the last section we glanced at some ideas that would be met in the theory of numbers. We cannot do justice to the many worthwhile number peculiarities that might be treated.

A little earlier we noticed how one may check the divisibility of a number by nine or by eleven without actually doing the division. Maybe this will delight only the mathematically minded since there is not too much practical value in it. All these rules may be derived by investigating the general form of an integer N, as previously done:

$$N = a_0 10^n + a_1 10^{n-1} + \cdots + a_{n-1}10 + a_n.$$

The following divisibility rules are derived. They are grouped according to similarities.

$$\begin{array}{lcr}
N \equiv 0(2) & \xleftrightarrow{\text{if and only if}} & a_n \equiv 0(2) \\
N \equiv 0(4) & \longleftrightarrow & 10a_{n-1} + a_n \equiv 0(4) \\
N \equiv 0(8) & \longleftrightarrow & 100a_{n-2} + 10a_{n-1} + a_n \equiv 0(8)
\end{array}\Bigg\}$$

$$\left.\begin{array}{lcl}
N \equiv 0(3) & \longleftrightarrow & \sum_{i=0}^{n} a_i \equiv 0(3) \\
N \equiv 0(9) & \longleftrightarrow & \sum_{i=0}^{n} a_i \equiv 0(9)
\end{array}\right\}$$

$$\left.\begin{array}{lcl}
N \equiv 0(5) & \longleftrightarrow & a_n = 0 \text{ or } a_n = 5 \\
N \equiv 0(10) & \longleftrightarrow & a_n = 0
\end{array}\right\}$$

$$N \equiv 0(6) \quad \longleftrightarrow \quad \begin{array}{c} N \equiv 0(2) \\ \text{and} \\ N \equiv 0(3) \end{array}$$

Translating these into English, we have: an integer is divisible by 2, 4, or 8, if and only if the last digit, the last two digits read as a number, or the last three digits read as a number are, respectively, divisible by 2, 4, or 8. For instance, if $a = 56798$, $b = 56796$, and $c = 56792$ then a is divisible by 2; b by 2 and 4; and c by 2, 4, and 8.

An integer is divisible by 3 or by 9 if the sum of the digits is divisible by either 3 or 9. If $a = 5271$ and $b = 5274$ then a is divisible by 3 but not 9; b may be divided by 3 or 9 without a remainder.

Divisibility by 5 and 10 is determined by the last digit which in the first case has to be 5 or 0; in the second, 0.

For divisibility by 6, criteria for both 2 and 3 must be fulfilled.

The factor 7 does not show simple behavior in this routine. An especially impractical procedure would be the following. If one wishes to see if a number, say 57813, is divisible by 7 without performing the division, one may regard the number as if it were written to base three—disregarding the fact that such a number could not have digits equalling or exceeding 3. Here it would mean interpreting the number as

$$3 + 1 \cdot 3 + 8 \cdot 9 + 7 \cdot 27 + 5 \cdot 81 = 3 + 3 + 72 + 189 + 405 = 672.$$

Now divisibility of this number with respect to seven has to be investigated. The previous procedure is repeated:

$$(672)_3 = 2 + 7 \cdot 3 + 6 \cdot 9 = 2 + 21 + 54 = 77.$$

Continuing, we have $(77)_3 = 7 + 7 \cdot 3 = 7 + 21 = 28$. Then,

$$(28)_3 = 8 + 2 \cdot 3 = 14 \text{ and } (14)_3 = 4 + 3 = 7.$$

Therefore, 57813 is divisible by seven. Any grade school pupil may have exceeded our speed, but would he have had as much fun?

Now let us take an integer other than zero and increase it by one. Divide this sum by a number obtained by increasing the reciprocal of the original integer by one. Lo and behold—the given integer results. Try another one. We make the same observation. Start with a fraction instead of an integer. The rule still holds. The initiated, of course, translating this idea into algebra, and finding that

$$\frac{a+1}{\frac{1}{a}+1} = a$$

regardless of the choice of a, does not consider this mysterious.

Let us explore some more. Take number 8 and write:

$$\begin{array}{llll}
1 \cdot 8 = 8 & \text{and } 8 + 0 = 8 & & \\
2 \cdot 8 = 16 & \text{and } 6 + 1 = 7 & & \\
3 \cdot 8 = 24 & \text{and } 4 + 2 = 6 & & \\
4 \cdot 8 = 32 & \text{and } 2 + 3 = 5 & & \\
5 \cdot 8 = 40 & \text{and } 0 + 4 = 4 & & \\
6 \cdot 8 = 48 & \text{and } 8 + 4 = 12 & \text{and } 2 + 1 = 3 \\
7 \cdot 8 = 56 & \text{and } 6 + 5 = 11 & \text{and } 1 + 1 = 2 \\
8 \cdot 8 = 64 & \text{and } 4 + 6 = 10 & \text{and } 0 + 1 = 1 \\
9 \cdot 8 = 72 & \text{and } 2 + 7 = 9 & & \\
10 \cdot 8 = 80 & \text{and } 0 + 8 = 8 & & \\
11 \cdot 8 = 88 & \text{and } 8 + 8 = 16 & \text{and } 6 + 1 = 7
\end{array}$$

and so on. The answers keep "rejuvenating" themselves; this is a way to get younger on paper as you grow older in years.

Maybe all digits behave in this manner. What about 9?

$$\begin{array}{rll}
1 \cdot 9 = 9 & \text{and} & 9 + 0 = 9 \\
2 \cdot 9 = 18 & \text{and} & 8 + 1 = 9 \\
3 \cdot 9 = 27 & \text{and} & 7 + 2 = 9 \\
4 \cdot 9 = 36 & \text{and} & 6 + 3 = 9 \\
5 \cdot 9 = 45 & \text{and} & 5 + 4 = 9 \\
6 \cdot 9 = 54 & \text{and} & 4 + 5 = 9 \\
7 \cdot 9 = 63 & \text{and} & 3 + 6 = 9 \\
8 \cdot 9 = 72 & \text{and} & 2 + 7 = 9 \\
9 \cdot 9 = 81 & \text{and} & 1 + 8 = 9 \\
10 \cdot 9 = 90 & \text{and} & 0 + 9 = 9 \\
11 \cdot 9 = 99 & \text{and} & 9 + 9 = 18 \quad \text{and} \quad 8 + 1 = 9
\end{array}$$

and so on. This permits you to remain the same age, *ad infinitum.*

Here is a basic notion for those who teach the multiplication pairs. A finger reckoning method obtains the product of any two digits from 5 to 10. It is necessary to know the multiplication table to 5 times 5. Suppose we multiply 8 by 7. On one hand raise as many fingers as the multiplicand exceeds 5, which in this case is 3. On the other hand, in a similar manner, indicate with raised fingers the excess of the multiplier over 5, which is now 2. The upraised fingers or digits (5) are units of (10) whose product is therefore 50. The clenched fingers on the one hand (3) and on the other (2) are multiplied equalling 6. The sum of 50 and 6 which equals 56 is the correct result. The general proof of this finger method may make an interesting exercise or project for the better students of an algebra class.

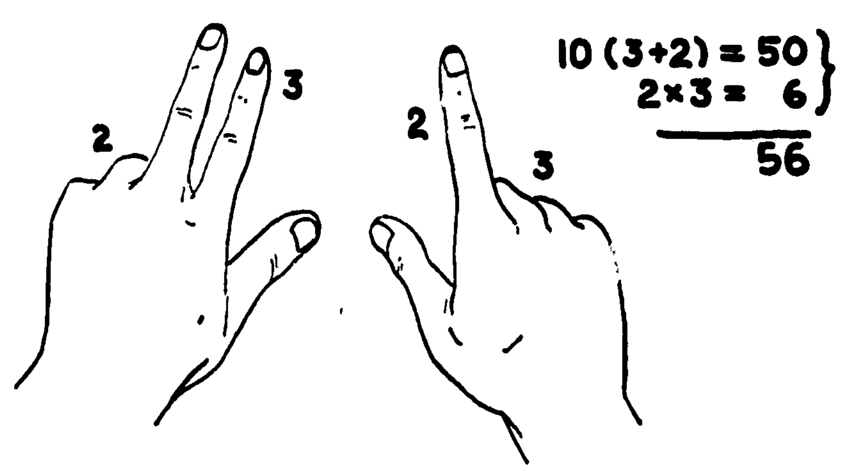


Figure 21

What about the cubes of the first ten counting numbers? We make another amazing observation as we check on the unit digits. All ten digits occur, and each of them occurs just once! The corresponding squares do not show the same peculiarity. However, duplications occur very regularly as shown by the guide lines. (See Figure 22.)

$1^3 = 1$	$1^2 = 1$
$2^3 = 8$	$2^2 = 4$
$3^3 = 27$	$3^2 = 9$
$4^3 = 64$	$4^2 = 16$
$5^3 = 125$	$5^2 = 25$
$6^3 = 216$	$6^2 = 36$
$7^3 = 343$	$7^2 = 49$
$8^3 = 512$	$8^2 = 64$
$9^3 = 729$	$9^2 = 81$
$10^3 = 1000$	$10^2 = 100$

Figure 22

This behavior of squares and cubes may be used for quick extracting of cube or square roots provided the radicands are perfect cubes or squares. Let us obtain $\sqrt[3]{185193}$. We know the answer must be a two-digit number such that the tens digit is 5 (the largest cube contained in 185). The units digit must be 7 since it is the only digit whose cube ends in 3. Therefore, the required number is 57.

The same procedure may be used for extracting square roots provided one takes precautions regarding the repetitions noted. To extract $\sqrt{5476}$, the tens digit must be 7, the units digit could be 4 or 6. We tentatively try 74 but have to verify this result by squaring 74. Incidentally, for this procedure the algebraic idea of squaring a binomial may be used since a number consisting of two digits may be thought of as the binomial $10t + u$, where t is the tens and u the units digit. The square of this binomial would follow the pattern of

$$(a + b)^2 = a^2 + 2ab + b^2.$$

However, if the result be written from right to left, then place value takes care of itself and we may think as follows: 74^2 equals (writing right to left) 4^2 or 16, put down 6, carry 1, twice the product of the two terms means $2 \cdot 7 \cdot 4 = 56$ and 1 gives 57. Write down 7, carry 5. $7^2 = 49$ and 5 is 54. The answer reads 5476, and hence 74 was the correct solution. You could also square 74 by taking

$$70 \times 78 + 16 = 5476.$$

Common Fractions and Decimal Fractions. It does not require much mental effort to recognize a common fraction as a rational number. For instance, $\frac{1}{2}$ could be interpreted as any one member of the set $\left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \cdots \frac{a}{2a}, \cdots\right\}$ and may therefore be thought of as the ratio between the numerator of any of these fractions and its denominator.

That a terminating decimal fraction is rational is also easily seen; 5.678, for instance, could be stated as $5\frac{678}{1000}$ or as $5\frac{339}{500}$. This is plainly a rational number.

Among the non-terminating decimals we must, however, distinguish two types, periodically repeating ones and irregular ones that do not show any periodicity. The latter are always irrational, but that the former are rational will now be shown. It is, of course, evident that $.3333 \cdots$, or $.\dot{3}$ as it is often abbreviated, is the equivalent of 1/3 and is therefore rational. We will now present a simple procedure whereby any non-terminating periodically repeating decimal may be converted into a common fraction or mixed number. Several other methods are well known.

The given number N is symbolized as

$$N = (I.a_1a_2 \cdots a_rb_1b_2 \cdots b_nb_1b_2 \cdots b_nb_1b_2 \cdots b_n \cdots)$$

or as $N = (I.a_1a_2 \cdots a_r\overline{b_1b_2 \cdots b_n})$.

$$\text{Then} \quad N = I + \frac{(a_1a_2 \cdots a_rb_1b_2 \cdots b_n) - (a_1a_2 \cdots a_r)}{\underbrace{99 \cdots 9}_{n}\,\underbrace{00 \cdots 0}_{r}}$$

where the symbolism $(k_1k_2 \cdots k_m)$ indicates a number whose successive digits are $k_1, k_2, \cdots k_m$.

In words, to convert a periodically repeating decimal into its rational equivalent, write a fraction whose denominator has as many nines as the period has digits followed by as many zeros as the anteperiod has digits. In the numerator, state the number obtained by reading up to and including the first period. From this value subtract the anteperiod. Simplify and reduce if necessary.

To illustrate, we turn $a = 3.2345678567856\overline{78} \cdots$ (or $3.234\overline{5678}$) into a common fraction. In accordance with the foregoing, we obtain:

$$a = 3\frac{2345678 - 234}{9999000} = 3\frac{2345444}{9999000} = 3\frac{586361}{2499750}.$$

To check this value, we divide 586361 by 2499750. This shortcut may be derived in two possible ways. To verify the general procedure, it is necessary to use the general notation of the algebraic form of a number.

Method I (In terms of geometric progressions).

$$
\begin{aligned}
a &= 3 + \frac{234}{10^3} + \frac{5678}{10^7} + \frac{5678}{10^{11}} + \frac{5678}{10^{15}} + \cdots \\
&= 3 + \frac{234}{10^3} + \frac{5678}{10^7}\left(1 + \frac{1}{10^4} + \frac{1}{10^8} + \cdots\right) \\
&= 3 + \frac{234}{10^3} + \frac{5678}{10^7}\left(\frac{1}{1 - \frac{1}{10^4}}\right) \\
&= 3 + \frac{234}{10^3} + \frac{5678}{10^3(10^4 - 1)} \\
&= 3 + \frac{234 \cdot 10^4 + 5678 - 234}{10^3(10^4 - 1)} \\
&= 3\frac{2345678 - 234}{9999000} \\
&= 3\frac{2345444}{9999000} \\
&= 3\frac{586361}{2499750}.
\end{aligned}
$$

Method II (In terms of algebra).

Let d be the decimal part of the number a. Then

$$
\left.\begin{aligned}
d &= .234567856785678\cdots \\
10^4 d &= 2345\,.678567856785678\cdots
\end{aligned}\right\}
$$

Subtracting, we obtain:

$$
\begin{aligned}
10^4 d &= 2345.678567856785678\cdots \\
d &= .234567856785678\cdots \\
\hline
d(10^4 - 1) &= 2345.444 \\
d &= \frac{2345444}{10^3(10^4 - 1)} = \frac{2345444}{9999000} = \frac{586361}{2499750} \\
a &= 3\frac{586361}{2499750}
\end{aligned}
$$

If a periodic decimal does not have an anteperiod, the arithmetic is simpler. For instance, $b = .\overline{648}$ would equal $b = \frac{648}{999} = \frac{24}{37}$. Incidentally, terminating decimal fractions may be considered a special case of

periodically repeating, non-terminating decimals. Thus 5.678 would be equal to 5.67800000 · · ·, or to 5.6779999 · · ·.

Let us see what happens if we permit zero to become part of our fraction. We are considering the number $\frac{a}{b}$. This, by definition, constitutes a value c such that $a = bc$. If $a = 0$ and $b \neq 0$ then the result reads $\frac{0}{b} = c$ or $0 = bc$. This means that c must be zero. Zero divided by a non-zero number is zero.

What about $\frac{a}{b}$ with $a \neq 0$ and $b = 0$? One would have $a = bc$ or $a = 0$. This, however, leads to a contradiction; there is no number which answers the task of dividing a number by zero. Division by zero is not permitted—it does not lead to a number.

If we tried $\frac{a}{b}$ with $a = 0$ and $b = 0$, then $a = bc$ would just read $0 = 0$ regardless of the choice for c. This case does not lead to a contradiction but to a peculiarity. From this point of view $\frac{0}{0}$ could be any number. So it is agreed that $\frac{0}{0}$ is not a symbol for a definite number.

THE IRRATIONALS

Algebraically Irrational Numbers. As we mentioned previously, algebraic irrationals were first encountered in connection with line segments. The problem is that of obtaining a measure for the length of the diagonal of a square whose side measurement is given. We could use the connection with line segments as an interesting and simple method for obtaining successive square roots graphically. See the spiral in Figure 23. One starts with a right triangle whose sides have unit length. The next right triangle has the hypotenuse of the previous one as one of its legs. The process may be continued indefinitely. The successive hypotenuse measures of this set of right triangles is the set of successive square roots $\{\sqrt{2}, \sqrt{3}, \cdots\}$.

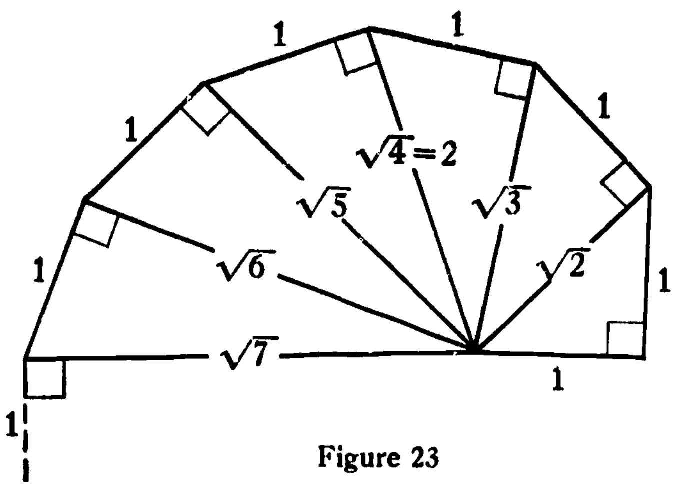


Figure 23

An irrational number, as its name indicates, cannot be converted into a fraction. Let us try a new idea—continued fractions. Any

rational fraction, say $\frac{79}{31}$ may be converted into a continued fraction as follows:

$$\frac{79}{31} = 2 + \frac{17}{31} = 2 + \cfrac{1}{\frac{31}{17}} = 2 + \cfrac{1}{1 + \frac{14}{17}} = 2 + \cfrac{1}{1 + \cfrac{1}{\frac{17}{14}}}$$

$$= 2 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{3}{14}}} = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\frac{14}{3}}}} = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \frac{1}{2}}}}}$$

Eventually, this operation must terminate by a process called the Euclidean Algorithm. Thus a rational number may always be expressed as a *finite* continued fraction.

The Euclidean Algorithm could be set up as follows:

$$\begin{aligned} 79 &= 31 \cdot \underline{2} + 17 \qquad & 14 &= 3 \cdot \underline{4} + 2 \\ 31 &= 17 \cdot \underline{1} + 14 \qquad & 3 &= 2 \cdot \underline{1} + 1 \\ 17 &= 14 \cdot \underline{1} + 3 \qquad & 2 &= 1 \cdot \underline{2} + 0 \end{aligned}$$

A zero remainder ends the process. The structure of the Euclidean Algorithm for a fraction $\frac{a}{b}$ shows that every subsequent step involves a smaller dividend than the preceding one. Since we may assume that a and b are relatively prime, a remainder of 1 must eventually appear. The next step gives the remainder 0. Also, note the role that the underlined quotients in the Euclidean Algorithm have in the continued fraction. In order, they become the integral part and the successive denominators.

Irrationals would naturally lead to infinite continued fractions. Square root expressions, the class of algebraically irrational numbers whose degree is two, may be converted into infinite continued fractions in a simple manner. These continued fractions display great regularities whereas the decimal expansions of these numbers remain irregular in all circumstances. In fact, any irrational number which is the solution of a quadratic equation with integral coefficients can be represented by a periodically repeating continued fraction. We are indebted to Euler for this theorem. Conversely, as Lagrange has shown, any periodically continued fraction is a number which may be considered a real irrational solution of a quadratic equation.

While the proofs of the above theorems need not be given here, the following illustrations may help.

$$\sqrt{2} = 1 + (\sqrt{2} - 1) = 1 + \cfrac{1}{\cfrac{1}{\sqrt{2} - 1}} = 1 + \cfrac{1}{\sqrt{2} + 1}$$

$$= 1 + \cfrac{1}{2 + (\sqrt{2} - 1)} = 1 + \cfrac{1}{2 + \cfrac{1}{\cfrac{1}{\sqrt{2} - 1}}} = 1 + \cfrac{1}{2 + \cfrac{1}{\sqrt{2} + 1}}$$

$$= 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}}$$

To check, we call the value of the continued fraction x. Then

$$x = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}} = 1 + \cfrac{1}{1 + \boxed{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}}}$$

or, $x = 1 + \dfrac{1}{1 + x}$.

Since the original continued fraction reappears, $x^2 = 2$ and $x = \sqrt{2}$.

Here is a different one:

$$\sqrt{3} = 1 + (\sqrt{3} - 1) = 1 + \cfrac{1}{\cfrac{1}{\sqrt{3} - 1}} = 1 + \cfrac{1}{\cfrac{\sqrt{3} + 1}{2}}$$

$$= 1 + \cfrac{1}{1 + \cfrac{\sqrt{3} - 1}{2}} = 1 + \cfrac{1}{1 + \cfrac{1}{\sqrt{3} + 1}}$$

$$= 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \ddots}}}}}}$$

The check reads
$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + x}}$$
or, $x^2 = 3$ and $x = \sqrt{3}$.

Another fascinating number in mathematics is G, the golden number. According to legend, a beauty contest took place in ancient Greece. The charming competitors, alas, were the differently-shaped rectangles. Which was most appealing to the eye? The consensus was that the best-proportioned rectangle would be obtained in the following manner. Take a rod AB of unit length. Bend it at a point X in such a manner that $\frac{x}{1} = \frac{1-x}{x}$ (see Figure 24). This relationship is called *the*

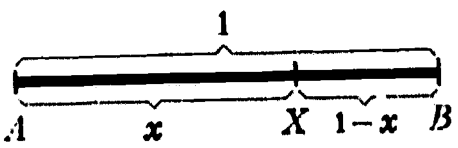


Figure 24

divine proportion or *the golden section*. Choosing x as the length and $(1 - x)$ as the width of the rectangle, the most pleasing form will result. The reader is encouraged to check modern ideas of beauty by comparing the dimensions of the American flag, the postal card, *The Mathematics Teacher*, *The Arithmetic Teacher*, index cards, and others with this proportion. (See Figure 25.)

The above equation leads to $x^2 + x - 1 = 0$ which has the golden number $G = \frac{-1 + \sqrt{5}}{2}$ as its positive solution. To convert G into a continued fraction, one derives

$$G = \cfrac{1}{\cfrac{2}{\sqrt{5} - 1}} = \cfrac{1}{\cfrac{\sqrt{5} + 1}{2}} = \cfrac{1}{1 + \cfrac{\sqrt{5} - 1}{2}} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}}.$$

This is also a most beautiful continued fraction in terms of simplicity. The regular pentagon and pentagram are proportioned in such a manner that comparisons between different parts almost invariably lead to the occurrence of the *golden number*. This regularity and beauty may have induced the Pythagoreans to pick the pentagram as the emblem for their fraternity pin.

The continued-fraction representation of the golden number has another interesting feature. If a continued fraction is terminated at

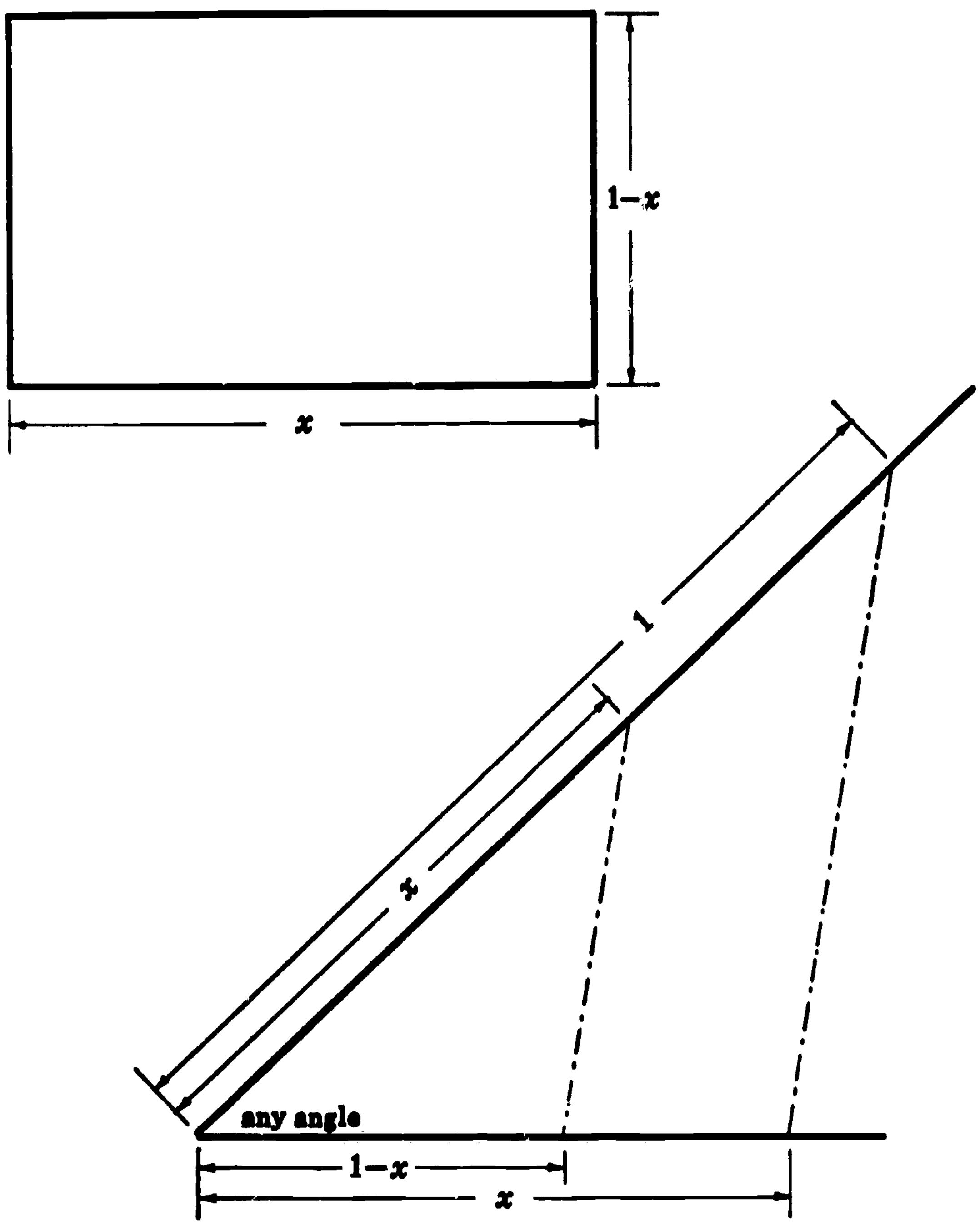


Figure 25

any stage, the result is an approximation of the number to which the continued fraction amounts. Every subsequent fracturing produces a closer approximation than the preceding.

$$\text{Since } G = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}}$$

the first approximation, $G_1 = \frac{1}{1} = 1$

$$G_2 = \cfrac{1}{1 + \cfrac{1}{1}} = \frac{1}{2}$$

$$G_3 = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}} = \frac{2}{3}$$

. . . .

The series of convergents, as these approximations are known, is

$$\left\{\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \cdots\right\}.$$

From the above it is evident that beginning with the third ratio, each succeeding numerator equals the sum of the two immediately preceding numerators. The same happens with the denominators. You will also notice that beginning with the second ratio, each numerator is the same as the denominator of its immediate predecessor. The set of numbers made up of $\{1, 2, 3, 5, 8, 13, \cdots\}$ is called a Fibonacci series.

Fibonacci numbers are found in nature. Take a twig perpendicular to its base and select a bud on it. Then spiral around the twig until a bud is reached which is vertically above the originally selected bud. Not only will the number of buds encountered be a Fibonacci number but the number of revolutions around the stalk will be another such number. The particular number of the established group is a characteristic property of that species of plant. The golden number emerges not only in mathematics but also in architecture, phyllotaxis, morphology, chemistry, and music.

Transcendentally Irrational Numbers. When a new number is under investigation, for instance $3^{\sqrt{2}}$, it may sometimes be important for the mathematician to determine if it is irrational, and especially if it is transcendentally so. Both investigations become very intricate. The establishment of the transcendentality of a given number does not follow any repeatable pattern and new and sometimes very clever devices have to be resorted to in each case. Indeed, until recently only few of the mathematically interesting numbers could be shown to be transcendental. Lest we forget, the transcendentals are more numerous than all rational and algebraically irrational numbers together.

Three well-known problems, the "famous problems of antiquity," have been thrust upon us. One of these concerns the task of constructing the side of a square whose area should equal the area of a given circle. This is the familiar problem of squaring the circle. Even now, we sometimes call a person a circle-squarer if we want to indicate that he is one who always attempts the allegedly impossible. Contrary to the wishful thinking of the amateur who never acknowledges defeat, it has been proven mathematically that it is impossible to solve this problem by construction. The term *construction* is a technical one in mathematics and means the perfecting of a geometric configuration with only a straight-edge and compass—a Platonic restriction.

Today an elementary student becomes acquainted with the relationship $C = \pi d$ for the circumference of a circle whose diameter is d and with $A = \pi r^2$ for the area of a circle whose radius is r. He could solve the problem immediately by letting the side of the square have x units and equating the measure of its area, x^2, to πr^2, thus obtaining $x = r\sqrt{\pi}$. This is not what we mean by obtaining the length of x by construction. The constructional aspect has plagued mathematicians for 2000 years. The nature of π was the crux of the problem.

A means for obtaining the value of π to any desired accuracy was known to Archimedes (240 B. C.). He noted that the length of the circumference of a circle lies between the lengths of an inscribed and a circumscribed polygon. By successively increasing the number of sides of these polygons, closer and closer approximations were reached. An approximate value would be $\pi \approx 3.1415926535898$.

It is one thing to have some idea of the value of a number, but quite another to understand its nature. It was not until 1882 that Lindemann found a procedure to prove the transcendentality of π. This should have killed all the dreams of the circle-squarers. Straight-edge and compass constructions can be used only if line segments have lengths that are represented by numbers which are roots of first or second degree polynomial equations.

The fact that π is transcendental means that it cannot be the solution of any polynomial equation with integral coefficients. Still some most surprising connections between π and the integers have been found in the form of nested roots, infinite series, continued fractions, and infinite products. Their proofs require complicated mathematical machinery, but a few of them will be given here for the interest of the mathematically curious.

$$\pi = \frac{2}{\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots}$$

$$\frac{\pi}{2} = \underbrace{\frac{2}{1} \cdot \frac{2}{3}} \cdot \underbrace{\frac{4}{3} \cdot \frac{4}{5}} \cdot \underbrace{\frac{6}{5} \cdot \frac{6}{7}} \cdot \underbrace{\frac{8}{7} \cdot \frac{8}{9}} \cdots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

$$\frac{\pi}{4} = 4\left(\frac{1}{5} - \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} - \frac{1}{7 \cdot 5^7} + \cdots\right) + \cdots$$
$$- \left(\frac{1}{239} - \frac{1}{3 \cdot 239^3} + \frac{1}{5 \cdot 239^5} - \frac{1}{7 \cdot 239^7} + \cdots\right)$$

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + 7^2}}}} \ddots$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

$$\frac{\pi}{2} = 1 + \frac{1}{3}\left(\frac{1}{2}\right) + \frac{1}{5}\left(\frac{1 \cdot 3}{2 \cdot 4}\right) + \frac{1}{7}\left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right) + \frac{1}{9}\left(\frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}\right) + \cdots$$

One of the most important of natural phenomena is growth. The rate at which a whole population increases is related to the size of that population. The rate at which a specific amount of a radioactive substance diminishes is dependent on its original amount. The rate of many chemical reactions is a function of the quantity of the participating reacting substances. The rate of growth of a plant depends upon the size of the plant and so does the growth of capital depend on the principal. In each of these cases, the dependency encountered is one of direct proportionality. This is the law of growth which applies equally to organic and to inorganic processes. The rate of growth is proportional to the growing quantity; the rate of decay is proportional to the decaying quantity.

Translating this law into mathematical symbols, one obtains a certain kind of equation, called a differential equation, which yields the exponential function, written e^x, as a solution. Careful investigation of this function reveals its nature and properties. Since the exponential function describes as important a natural phenomenon as that of growth, it is omnipresent. Its use in the natural sciences, economics, statistics, sociology, psychology, and the theory of games shows its wide applicability today.

Those who are acquainted with the calculus may be able to translate this idea into the differential equation $D_x y = ky$ where k is the proportionality factor and is therefore constant. If $k > 0$, a growth function

is encountered; $k < 0$ leads to a decay function; $k = 0$ will be eliminated since it would mean a constant function. The above relationship is equivalent to $\int \frac{dy}{y} = k \int dx$ which becomes $\ln y = kx + \ln c$ if we wish to express the constant of integration as $\ln c$. Taking antilogarithms of each side we obtain $y = ce^{kx}$, the exponential function.

A special value of the exponential function is obtained by considering $k = 1$ and $x = 1$. This leads to the famous Euler constant e (given by Euler in 1748) whose approximate value is $2.7182818284590 \cdots$. The number e serves also as the basis for natural logarithms, whose name indicates the close relationship with natural phenomena. In other words, we have $\ln N = a \leftrightarrow e^a = N$ as the definition for natural logarithms with the corollary that $\ln e = 1$. The conversion principles of natural and common logarithms are:

Natural Logarithms ($\ln N$) to Common Logarithms ($\log N$)	Common Logarithms ($\log N$) to Natural Logarithms ($\ln N$)
$\log N = a$ $10^a = N$ $a \ln 10 = \ln N$ $\log N \ln 10 = \ln N$ $\left.\begin{matrix}\therefore \log N = \dfrac{\ln N}{\ln 10} \\ (\ln 10 \approx 2.303)\end{matrix}\right\}$	$\ln N = b$ $e^b = N$ $b \log e = \log N$ $\ln N \log e = \log N$ $\left.\begin{matrix}\therefore \ln N = \dfrac{\log N}{\log e} \\ (\log e \approx 0.4343)\end{matrix}\right\}$

Nine years before Lindemann's proof for the transcendental nature of π, Hermite established e as a transcendental. As in the case of the number π, many relationships between e and integers can be found—some of them exceedingly regular and simple—with infinitely continued procedures. We'll write some of these:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$$

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cdots}}}}}}}}$$

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \ddots}}}}}$$

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

$$\frac{1}{e} = \lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n$$

$$\frac{1}{e} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots$$

$$\sqrt{e} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{9 + \ddots}}}}}}}}$$

$$e = \lim \left[\left(1\frac{1}{2}\right)^2, \left(1\frac{1}{3}\right)^3, \left(1\frac{1}{4}\right)^4, \cdots \right]$$

Before departing from the numbers π and e, we must give a most striking mathematical relationship. While space limitations prevent us from establishing it, or even from giving some of its implications, let us just quote it as $e^{i\pi} + 1 = 0$. This is an equation that combines the most important symbols of mathematics in an amazingly simple form. Much mysticism has been read into it. Perhaps zero and one represent arithmetic; i, algebra; π, geometry; and e, analysis!

Complex Numbers

Complex numbers met with distrust on their appearance. First acquaintance with this idea still perplexes our students and leaves them with a feeling of discomfort and mystery. As great a mathematician as Euler—although he did not hesitate to apply complex numbers—felt this way about them: "All such expressions as $\sqrt{-1}$,

$\sqrt{-2}$, etc., are consequently impossible or imaginary numbers, since they represent roots of negative quantities; and of such numbers we may truly assert that they are neither nothing, nor greater than nothing, nor less than nothing, which necessarily constitutes them imaginary or impossible."

The other extensions of the number system, at least the domain of the non-negative numbers, could be linked up with reality. They could be seen as measures of real objects. Even the negative numbers finally became generally accepted inasmuch as physical meanings could also be attached to them. The mathematician, of course, does not care about a connection with reality. For him, all numbers are abstractions and their justifications are given by a formal structure of definitions and derivations. But philology and ontology reveal the slow acceptance of abstractions by mankind and by each individual. How does one attach physical existence to a notion which introduces a new number whose square would equal minus one?

Anticipated by Wessel, a Norwegian surveyor, and Argand, a Parisian, Gauss (1777–1855) resorted to a graphic device to help the concrete-minded. These are his own words (1831):

> Our general arithmetic, so far surpassing in extent the geometry of the ancients, is entirely the creation of modern times. Starting originally from the notion of absolute integers it has gradually enlarged its domain. To integers have been added fractions, to rational quantities the irrational, to positive the negative, and to the real the imaginary. This advance, however, had always been made at first with timorous and hesitating steps. The early algebraists called the negative roots of equations false roots, and this is indeed the case, when the problem to which they relate has been stated in such a form that the character of the quantity sought allows of no opposite. But just as in general arithmetic no one would hesitate to admit fractions, although there are so many countable things where a fraction has no meaning, so we ought not to deny to negative numbers the rights accorded to positive, simply because innumerable things admit of no opposite. The reality of negative numbers is sufficiently justified since in innumerable other cases they find an adequate interpretation. This has long been admitted, but the imaginary quantities—formerly and occasionally now improperly called impossible, as opposed to real quantities—are still rather tolerated than fully naturalized; they appear more like an empty play upon symbols, to which a thinkable substratum is unhesitatingly denied even by those who would not depreciate the rich contribution which this play upon symbols has made to the treasure of the relations of real quantities.
>
> The author has for many years considered this highly important part of mathematics from a different point of view, where just as objective an existence can be assigned to imaginary as to negative quantities. . . .

Geometric interpretations of real numbers as points on a line always aided the visualization of numbers. One could see that a certain arbitrary point on a line could represent zero, that the natural numbers would be placed equidistantly on one side of the line (arbitrarily chosen on the right side of it) and negative integers in the opposite direction. If fractions, or even square roots, had to be located on this number

line, the positions of the corresponding points could be determined by construction. Even though this would not be true of higher algebraic irrationalities nor indeed of transcendentals, it seemed intuitively clear and understandable that there would be a point on the line which would be π units away from zero in the positive direction, and could therefore serve as the image of π and vice versa. Although many details of this correspondence have plagued mathematicians, the intimate relationship between the real number domain and the continuum of points on a line was finally accepted. To every real number belongs a point on a straight line whose distance from an arbitrarily selected zero point equals that number, and for every such point there is a real number which states its distance from that zero point.

Could complex numbers be interpreted in the same manner? The reader recalls a theorem in plane geometry which states that the height of a right triangle is the geometric mean between the two adjacent segments of the hypotenuse. In Figure 26 this relationship reads: $h^2 = \overline{AD} \cdot \overline{DB}$. Returning to the number line for real magnitudes, one may draw a right triangle choosing as its hypotenuse the line segment whose terminal points are the numbers (-1) and (1). According to the theorem the square of the distance $\overline{OI}$ will then be (-1). Since we wish to interpret a number (called i) which displays the property that its square equals (-1), the point I corresponds to the number i, or line segment $\overline{OI}$ is i units long.

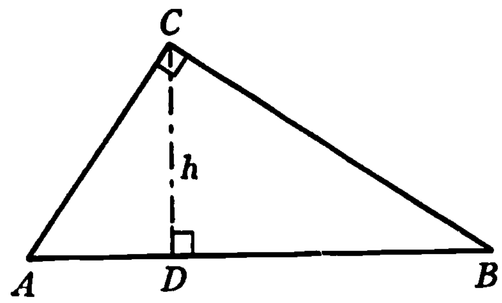


Figure 26

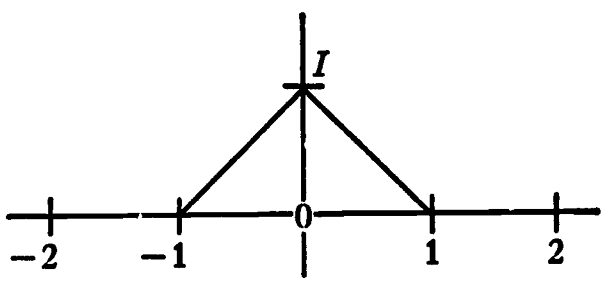


Figure 27

This was the basic idea on which Gauss explained the entire complex plane. Any number $a + bi$ becomes represented by a point whose coordinates are (a, b), and vice versa. (Compare this also with the number pair notation of a complex number.) This means that to locate the geometric image of a complex number one would again start from an arbitrarily selected zero point, called the origin. Then one moves a units on the previously discussed real number line, and from that position one travels perpendicularly b units. To every complex number belongs a point in the plane, and to every point in the

plane belongs a complex number. The real numbers are seen to constitute a subset of the set of the complex numbers.

Instead of assigning the point (a, b) to the complex number $a + bi$, one may designate the directed line segment from the zero point to (a, b) as its image. In this case we talk about the vector corresponding to $a + bi$. By *vector* we mean a mathematical object which has both magnitude and direction.

To add the complex numbers $a + bi$ and $c + di$, a new complex number $(a + c) + (b + d)i$ is found. If geometry is used, a striking similarity between the behavior of vectors and forces is exhibited. As seen from Figure 28, the sum of the two vectors corresponds to the resultant in the appropriate parallelogram of forces.

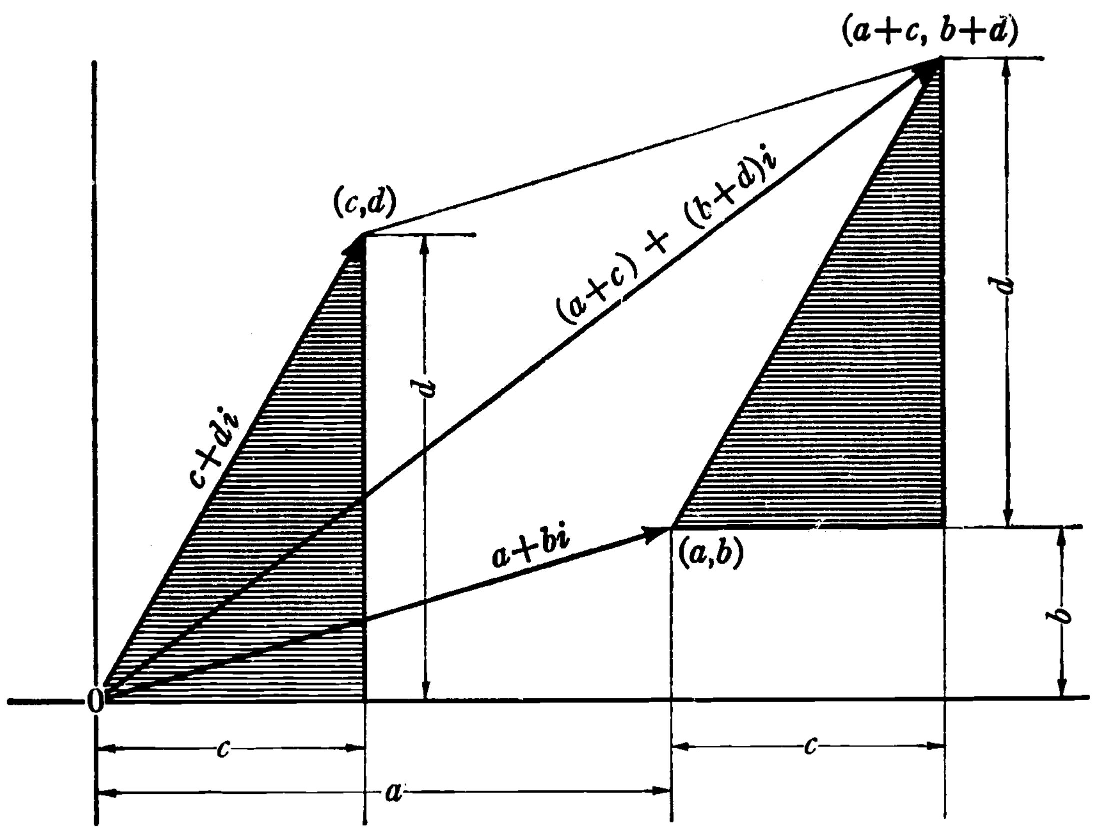


Figure 28

Multiplying $a + bi$ by i to obtain $-b + ai$ also has a simple geometric interpretation. The vector (a, b) has been rotated counterclockwise by a right angle. See Figure 29.

These ideas initiated a very important new branch of mathematics, vector analysis (19th century). It was seen that such basic quantities as forces and velocities could be presented by vectors. Combining of physical quantities and investigating their relationships was reduced to operating on geometric entities in a geometric manner.

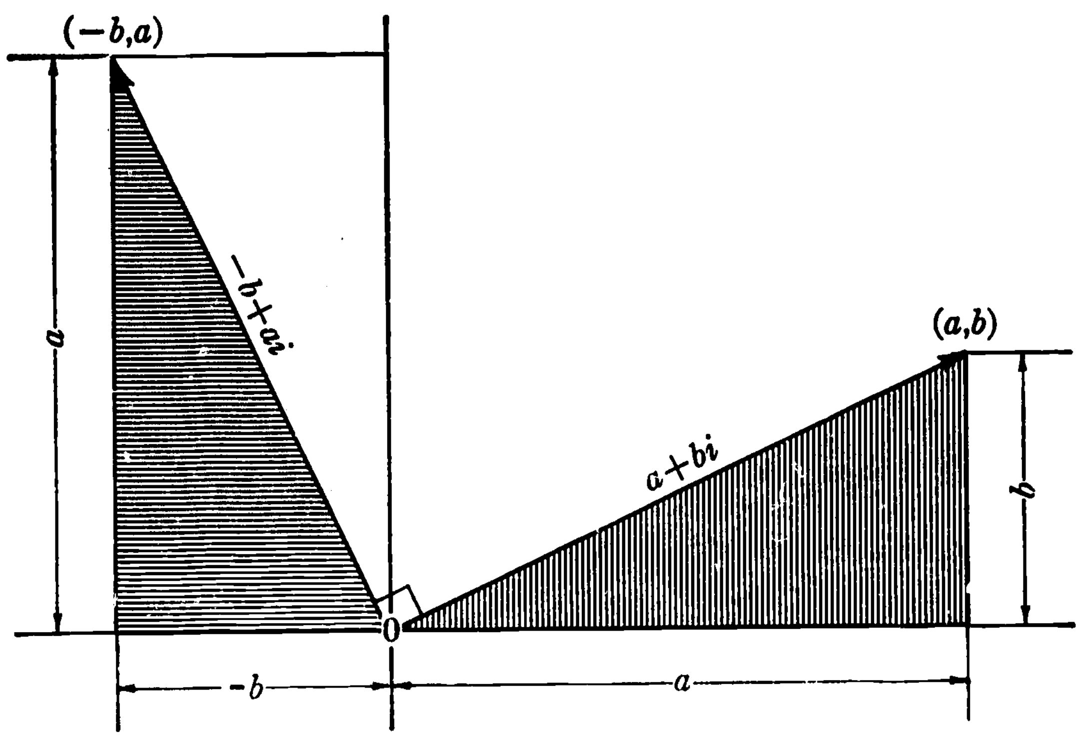


Figure 29

Hypercomplex Numbers

As has already been pointed out, no further generalizations of the number concept are necessary to solve ordinary polynomial equations. The fundamental theorem of algebra, one of the many gems in mathematics, was bestowed on mankind by the "prince of mathematicians," Karl Friedrich Gauss. It states that in the domain of complex numbers every polynomial equation—even if the coefficients themselves are complex numbers—has a root, and as many roots as the degree indicates. Real numbers are, of course, a subset of complex numbers.

If, however, solvability of equations is not the motive for an extension of the number concept, then several avenues may be followed to arrive at new kinds of numbers. A study of these would run beyond the scope of our discussion. We might, however, mention one such extension, the introduction of quaternions. It will be recalled that the complex numbers may be considered as ordered number pairs such that (a, b) corresponds to the number $a + bi$. For clarity we will list the basic properties of these number pairs again:

EQUALITY: $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.
ADDITION: $(a, b) + (c, d) = (a + c, b + d)$.
MULTIPLICATION: $(a, b)(c, d) = (ac - bd, ad + bc)$.

The reader might guess that our extension will now consist in

considering number triplets rather than pairs. This could indeed be done, but we will now jump a step further and consider quadruplets of numbers to define our quaternions. Modelling our basic assumptions after the structure of the complex numbers, we have:

EQUALITY: $(a, b, c, d) = (e, f, g, h)$ if and only if $a = e$, $b = f$, $c = g$, and $d = h$.

ADDITION: $(a, b, c, d) + (e, f, g, h) = (a + e, b + f, c + g, d + h)$.

MULTIPLICATION: $(a, b, c, d)(e, f, g, h) = (ae - bf - cg - dh, af + be + ch - dg, ag + ce + df - bh, ah + bg + de - cf)$.

The last relationship looks especially formidable. Should we let $b = c = d = f = g = h = 0$, the real numbers are seen to be a subset of the quaternions. If only $c = d = g = h = 0$, the ordinary complex numbers emerge. Now the complex number (a, b) would appear as $(a, b, 0, 0)$ just as the real number a is written as $(a, 0, 0, 0)$. The imaginary unit i which in number pair notation reads $(0, 1)$ becomes $(0, 1, 0, 0)$ in the new language, and the real unit $(1, 0)$ now reads $(1, 0, 0, 0)$. However, there are two other such units among the quaternions, namely $(0, 0, 1, 0)$, called j and $(0, 0, 0, 1)$, named k. Using the multiplication rule, one obtains not only $i^2 = -1$ (as with complex numbers) but also $j^2 = -1$ and $k^2 = -1$ without i, j, and k equalling each other, as can be seen from the definition of equality.

Stranger surprises are in store. To compute the product $i \cdot j$ we have: $(0, 1, 0, 0)(0, 0, 1, 0) = (0, 0, 0, 1) = k$. To obtain $j \cdot i$ the relationship becomes: $(0, 0, 1, 0)(0, 1, 0, 0) = (0, 0, 0, -1)$. This answer could be established as equalling $-k$ but whatever it is, it differs from k, so that we can conclude that $i \cdot j \neq j \cdot i$. The commutative law does not hold in this number system!

This short glimpse toward further generalizations of our number concept is included to help the reader see that what seems a "common sense truism" need not be correct. Such an understanding is real insight in mathematics.

5. PANORAMA

We have finished the journey. We have explored the evolution of number from the first step taken by prehistoric man because of his curiosity about quantity. Many millenia were needed to ripen the concept of natural number. Then began the march of further and increasingly rigorous generalizations and abstractions that characterize mathematics. No human being has ever "seen" a "three" any more than he has "seen" a geometric "point" or any other mathematical idea. But the mind of man does not stand still. The number system was stretched to include zero, negative integers, fractions, irrational quantities, and complex magnitudes. We have tried to show how these generalizations came about and how the mathematician orders them into a logical system.

Symbolization of numbers was also a tedious process. The fact that we have varying number systems is an indication of man's ingenuity.

It is one thing to conceive of a number and write its numeral. It is quite another to understand how to operate with it. Computational algorithms are, of course, closely interlinked with the type of numeration used. A Roman or a Greek of old, dreaming of calculations on modern high speed computers, would probably have beseeched his gods for an explanation.

We have tried to tell the captivating story of number in very limited space. In Chapter 4 we selected certain ideas connected with each subset of our number system, and tried to highlight them. Many other ideas were reluctantly omitted.

Somehow, we turn back to Pythagoras. "Number rules the universe." "The essence of all things is number." The man who was first to recognize the importance of generality in mathematical reasoning was also first to divine the role of number as an aid to interpreting the laws of nature. Now, more than two millennia afterward, what are modern man's sentiments about number?

It has been estimated that more has happened in the world of mathematics in the last 50 years than throughout the preceding 2000 years. The science of mathematics in its modern development is a way

of thinking in a realm of complete abstraction and generalization. Strangely, this fact has not generally taken hold. The much-alluded-to "certainty of mathematics" is attributed to its "concreteness." It is, in fact, the result of rigorous abstraction which thereby accomplishes absolute generalization. Mathematics may be briefly described as that field of knowledge which devotes itself to explorations in number, form, abstract structures, and order relationships.

It is interesting to note that the more abstract mathematics became the more widely were its ideas applied. The concept of functional interrelatedness, a central idea in mathematics, found a "real" counterpart in the laws of nature. The use of mathematical models which analyze the structure of a given situation and explain it as an ordered system has become more prominent. Indeed, only if a phenomenon under investigation is translatable into a mathematical law is it considered explained and the underlying problem solved. The natural sciences which traditionally used the language of mathematics, employ it more and more. The emergence of nuclear physics presented additional demands.

Mathematical ideas dominate not only the exact sciences. Our present century has witnessed a tremendous increase in their power. Mathematics became essential in the study of economic problems; linear programming was developed; planning prices for maximum profit and minimum cost became mathematicized; the theory of probability and games was applied to merchandizing. To manufacture a product which could stand up well against competitors but would not be so efficient as to hinder continued production, quality control borrowed methods from mathematics. Advances in engineering have necessitated additional aid from mathematics. In biometrics, mathematics is applied to biological data. More and more mathematical models had to be developed for "operations research." In the field of communication, information theory profited from mathematical research. Mainly under the mantle of probability and statistics whose methods had spread noticeably, mathematics has penetrated psychology, sociology, medicine and advertising. The age of electronics made many impositions on mathematics. Numerical analysis received a tremendous impetus through the advent of automation and high speed electronic computers.

Does number rule the universe? We leave this question to the theologian and the metaphysician. It may be safe to assert one thing: to construct a seeming representation of the conditions involved in the order of the universe, number still seems all-important. In molding the keys that man has so far been able to manufacture to unlock the secrets of the universe an everpresent ingredient is ***the notion of number***.