ABSTRACT

Reported is the result of research on combinatorial and algorithmic techniques for information processing. A method is discussed for obtaining minimum covers of specified cardinality from a given weighted graph. By the indicated method, it is shown that the family of minimum covers of varying cardinality is related to the minimum spanning tree of that graph. (RP)
MINIMUM COVERS OF FIXED CARDINALITY

IN WEIGHTED GRAPHS

by

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Computer and Information Science
Research Center
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MINIMUM COVERS OF FIXED CARDINALITY
IN WEIGHTED GRAPHS*

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Given a weighted graph, a method is discussed for obtaining minimum covers of specified cardinality. It is shown that the family of minimum covers of varying cardinality is related to the minimum spanning tree of that graph.

1. Introduction

Consider a finite weighted graph \([G, c]\) where \(E\) and \(V\) are the sets of edges and vertices of \(G\) respectively, and \(c_i\) is the weight of edge \(e_i \in E\), where edge weights are arbitrary real numbers. A cover is a subset of \(E\) such that each vertex of \(V\) is incident to at least one edge of the subset.

The minimum cover problem is to find a cover of minimum weight sum:

\[
\text{Min } c^T x \quad \text{subject to } Ax \geq 1, \quad x_i = 0 \text{ or } 1,
\]

where \(A\) is the vertex-edge incidence matrix of the graph \(G\), \(x\) is a vector corresponding to the edges of \(G\), \(c\) a vector of edge weights, and \(T\) indicates a vector transpose.

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Norman and Rabin [6] utilized the concept of reducing paths to solve the minimum cardinality cover problem, but solution effort of the algorithm grows exponentially with \( N \), the number of vertices of the graph. Based on "matching" techniques of Edmonds [1,2], the author [7] developed an algorithm to solve the minimum cover problem for which solution effort grows as \( N^4 \). Edmonds [3] has developed an efficient method to solve the degree - constrained subgraph problem

\[
\min c^T x \quad \text{subject to } b_1 \leq Ax \leq b_2, \quad x_i = 0 \text{ or } 1,
\]

which obtains the minimum cover as a special case.

2. \( k \)-Covers

Given a weighted graph \([G, c]\), consider the minimum \( k \)-cover problem:

\[
\min c^T x \quad \text{subject to } Ax \geq 1, \quad \sum_{e \in E} x_i = k, \quad x_i = 0 \text{ or } 1.
\]

Transform the graph \([G, c]\) to graph \([G_\lambda, c - \lambda]\) by subtracting \( \lambda \) from each edge weight. The parameter \( \lambda \) may assume any real value, and may be interpreted as a dual variable or Lagrange multiplier as discussed by Everett [4], corresponding to the constraint

\[
\sum_{e \in E} x_i = k.
\]

Define \( W(C) \) as the weight of cover \( C \) in \([G, c]\), and \( W_\lambda(C) \) as the weight of \( C \) in \([G_\lambda, c - \lambda]\). \(|C|\) denotes the cardinality of set \( C \).

Lemma 1

For any real \( \lambda \), if \( C \) is a minimum cover in \([G_\lambda, c - \lambda]\) and \(|C| = k\), then \( C \) is a minimum \( k \)-cover in \([G, c]\).
Proof

Let \( C' \) be any \( k \)-cover in \( G \). Then for any \( \lambda \),

\[
W_{\lambda} (C) = W(C) - k \lambda,
\]

\[
W_{\lambda} (C') = W (C') - k \lambda,
\]

and thus \( W(C) \leq W(C') \).

Lemma 2

Given a weighted graph \([G, c]\) and \( \lambda_1, \lambda_2 \), such that \( \lambda_2 > \lambda_1 \).

Let \( C_1 \) be a minimum cover in \([G_{\lambda_1}, c - \lambda_1]\), where \( |C_1| = k_1 \), and \( C_2 \) a minimum cover in \([G_{\lambda_2}, c - \lambda_2]\), where \( |C_2| = k_2 \). Then \( k_2 \geq k_1 \).

Proof

By assumption,

\[
W_{\lambda_1} (C_1) \leq W_{\lambda_1} (C_2) \quad \text{and} \quad W_{\lambda_2} (C_2) \leq W_{\lambda_2} (C_1) \quad \text{or}
\]

\[
W(C_1) - k_1 \lambda_1 \leq W(C_2) - k_2 \lambda_1
\]

\[
W(C_2) - k_2 \lambda_2 \leq W(C_1) - k_1 \lambda_2.
\]

Adding these two inequalities, and rearranging yields

\[
k_1 (\lambda_2 - \lambda_1) \leq k_2 (\lambda_2 - \lambda_1)
\]

or

\[
k_1 \leq k_2, \quad \text{since} \quad \lambda_2 > \lambda_1.
\]

Although the parameter \( k \) has been shown to be monotonic with \( \lambda \), we must ensure that all values of \( k \) will be obtained as continuous
values of \( \lambda \) are examined. Theorem 1 resolves this question, and the proof is presented in the appendix.
Theorem 1

Given a weighted graph \([G, c]\), where the minimum cardinality of any cover is \(m\). Then given \(k, m \leq k \leq |E|\), there exists a \(\lambda\) such that some minimum cover \(C\) in \([G_\lambda, c - \lambda]\) is of \(k\) cardinality.

3. Vertex and Edge Partitions

For arbitrary values of \(\lambda\), such that \(\lambda > \min_{e \in G} c_i\), partition the vertices in \([G_\lambda, c - \lambda]\) into two sets as shown in Figure 1:

1) \(V_N\), vertices which are incident to an edge of nonpositive weight.
2) \(V_P = V - V_N\)

Define an edge partition of \([G_\lambda, c - \lambda]\):

1) \(P_\lambda\), all edges in \([G_\lambda, c - \lambda]\) with at least one endpoint in vertex set \(V_P\). Define a cover of \(V_P\) as a subset of the edges of \(P_\lambda\) such that each vertex of \(V_P\) is incident to at least one edge of this subset.
2) \(N_\lambda\), all edges in \([G_\lambda, c - \lambda]\) with both endpoints in \(V_N\).

Figure 1 Decomposition of Graph \(G_\lambda\) into Edge Sets \(P_\lambda, N_\lambda\)
Theorem 2

Given $[G, c]$, and any $\lambda$, form $[G_\lambda, c - \lambda]$. Define $C$ as the edges of a minimum cover of $V_p$, together with all the nonpositive edges of $G_\lambda$. If $|C| = k$, then $C$ is a minimum $k$-cover in $[G, c]$.

Proof

Let $D$ be any $k$-cover in $G$. Then

$$W_\lambda(C) = W(C \cap P_\lambda) + W(C \cap N_\lambda)$$

$$W_\lambda(D) = W(D \cap P_\lambda) + W(D \cap N_\lambda)$$

But since we found a minimum cover of $V_p$,

$$W_\lambda(C \cap P_\lambda) \leq W_\lambda(D \cap P_\lambda),$$

and since $C$ uses all nonpositive edges in $[G_\lambda, c - \lambda]$,

$$W_\lambda(C \cap N_\lambda) \leq W_\lambda(D \cap N_\lambda),$$

and $W_\lambda(C) \leq W_\lambda(D)$.

Application of Lemma 1 yields the desired result.

The algorithm suggested by Theorem 2 decomposes the minimum $k$-cover problem into an "easy" problem in $N_\lambda$, and a "hard" cover problem for node set $V_p$. When an edge becomes negative, this ensures it will not only be in the next larger cardinality minimum cover, but in every subsequent minimum cover of greater cardinality. Exclusion of some zero weighted edges in $N_\lambda$ may be necessary in order to attain minimum $k$-covers for all feasible values of $k$, and an efficient technique exists for this process.

When $\lambda$ becomes sufficiently large such that all nodes are in set $V_N$, the minimum $k$-cover problem becomes easy for all $k$ above
the corresponding value. The critical value for which this occurs is:

$$\bar{\lambda} = \max \sum_{j=1}^{2} \min_{i \in \text{all } e_i \text{ incident to vertex } v_j} c_i$$

The cardinality at which the minimum $k$-cover will contain a cycle (simple closed path) can be seen clearly in edge set $N_{\lambda}$: it occurs at the lowest value of $\lambda$ for which all edges of a cycle become negative in $[G_{\lambda}, c - \lambda]$.

4. Minimum Spanning Trees

Define a tree of a graph $G$ as a connected subgraph which contains no cycles. A spanning tree is a tree which contains all the vertices $V$ of $G$. A forest is a subgraph which contains no cycles, and thus is a union of disjoint trees.

Kruskal [5] developed and proved the following algorithm to obtain a spanning tree of minimum weight for a connected weighted graph $[G, c]$.

1) Well order the edges of $G$ as $e_1, e_2, \ldots, e_n$ so $i < j \Rightarrow c_i \leq c_j$.

2) Initially select $T = \{e_1, e_2\}$.

3) If the union of $e_3$ with $T$ forms a cycle, permanently discard $e_3$; otherwise add $e_3$ to $T$.

4) Continue adding minimum elements to $T$ in a similar fashion, such that $T$ remains a forest in $G$ until the number of edges in $T$ is $(N - 1)$.

5) $T$ is a minimum spanning tree of graph $G$. 
Edmonds introduces the notion of a "greedy algorithm". Define an algorithm to obtain an optimal subset of a finite set as greedy if after well ordering the elements of the set by weight, each element requires examination only once, and upon examination, can either be placed in the solution set or permanently discarded.

The minimum spanning tree procedure is clearly a greedy algorithm, and obtains a cover of the given graph. Also the "easy" part of the minimum k - cover algorithm in Theorem 2 is greedy. Thus we inquire as to a relationship between the minimum spanning tree algorithm and the approach of Section 3 to obtain minimum k - covers.

5. Minimum Forest k - Covers

The question of the relationship between minimum spanning trees and minimum k - covers is complicated by the fact that the latter configuration may contain cycles. Define a forest cover in a graph G as a cover of G which contains no cycles. If the graph is connected, we can demand a forest k-cover configuration,  

\[ m \leq k \leq (N-1) \]

where m is the minimum cardinality of all covers.

Theorem 3

Given a weighted graph \([G, c]\), and any \(\lambda\), form \([G_\lambda, c - \lambda]\). Let C be the edges of a minimum cover of \(V_p\), together with a minimum forest cover of \(V_N\), using edge set \(N_\lambda\). If \(|C| = k\), then C is a minimum forest k - cover in \([G, c]\).
Proof

Let $D$ be any forest $k$-cover in $[G, c]$, and consider the weights of $C$ and $D$ in $[G_\lambda, c - \lambda]$.

\[
W_\lambda(C) = W_\lambda(C \cap P_\lambda) + W_\lambda(C \cap N_\lambda), \\
W_\lambda(D) = W_\lambda(D \cap P_\lambda) + W_\lambda(D \cap N_\lambda).
\]

But since we found a minimum cover of $V_P$,

\[
W_\lambda(C \cap P_\lambda) \leq W_\lambda(D \cap P_\lambda).
\]

The minimum forest cover of $(C \cap N_\lambda)$ of $V_N$ can be found by applying the minimum spanning tree algorithm to edges $N_\lambda$. This algorithm terminates when no further edges in $N_\lambda$ can be added without forming a cycle. Neither edge sets $(C \cap N_\lambda)$ nor $(C \cap P_\lambda)$ contain a cycle. There exists no path in $(C \cap P_\lambda)$ between vertices of $V_N$, as this would contradict the assumption that $(C \cap P_\lambda)$ is a minimum cover of $V_P$, so $C$ contains no cycles. $W_\lambda(C \cap N_\lambda) \leq W_\lambda(D \cap N_\lambda)$, so $W_\lambda(C) \leq W_\lambda(D)$, and application of Lemma 1 completes the proof.

As in Section 3, the problem of finding a minimum forest $k$-cover is decomposed into "hard" and "easy" parts, and the latter is greedy. It can be shown that for all $m < k < (N - 1)$, there is a $\lambda$ such that the minimum forest $k$-cover is given by the indicated construction, together with a simple technique for breaking ties when zero weighted edges occur in $N_\lambda$. 

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As $\lambda$ exceeds the value

$$\bar{\lambda} = \max \left\{ \min \left[ c_i \right] \right\}_{j \in v_j}$$

all $e_i$ incident
to vertex $v_j$

the algorithm becomes equivalent to the minimum spanning tree algorithm.

It is precisely as the edge of weight $\bar{\lambda}$ enters Kruskal's solution set

that the edges of this set form a cover of the nodes $V$ in graph $G$. 
APPENDIX

The purpose of this appendix is to provide a proof of Theorem 1.

First consider the following definitions.

A subgraph $E_1$ of a graph $G$ with edges $E$ and vertices $V$ is defined as a graph whose edges are $E_1 \subseteq E$, and whose vertices are the set of endpoints $V_1 \subseteq V$ of the edges $E_1$. For convenience, we refer to a subgraph by its set of edges. A path in a graph is a sequence of edges $P = \{e_1, e_2, \ldots, e_n\}$ together with an associated sequence of vertices $\{v_1, v_2, \ldots, v_{n+1}\}$ such that consecutive edges $e_i$ and $e_{i+1}$ in the path have a common vertex $v_{i+1}$ and each edge appears only once in the edge sequence $P$.

Given two sets of edges, $E_1$ and $E_2$, in a graph $G$, define the symmetric difference as

$$E_1 \oplus E_2 = (E_1 - E_2) \cup (E_2 - E_1).$$

An alternating path $P$ (relative to the sets $E_1$ and $E_2$) is a path whose edges are alternately in $(E_1 - E_2)$ and $(E_2 - E_1)$.

Given a graph $G$, define a reducing path $P$ relative to covers $C_1$ and $C_2$ of $G$ as a path $P$ such that:

1. $P$ alternates in subgraph $C_1 \oplus C_2$ with respect to edges in $C_1$ and $C_2$, and
2. $C_1 \oplus P$ and $C_2 \oplus P$ are both covers in $G$. 

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Lemma A.1

Given a graph $G$, and two arbitrary covers $C_1$ and $C_2$ of $G$, subgraph $C_1 \oplus C_2$ can be decomposed into edge-disjoint reducing paths.

This result was proved by Norman and Rabin [6]. Construct a maximal alternating path $P$ in subgraph $C_1 \oplus C_2$, removing the edges of this path. This produces a new subgraph in which another maximal alternating path can be removed. Since $C_1$ and $C_2$ are both covers in $G$, $C_1 \oplus P$ and $C_2 \oplus P$ are both covers in $G$, and $P$ is a reducing path.

Theorem 1

Given a weighted graph $[G,c]$ where the minimum cardinality of any cover is $m$. Then given $k$, $m \leq k \leq |E|$, there exists a $\lambda$ such that a minimum cover $C$ in $[G_\lambda, c - \lambda]$ is of $k$-cardinality.

Proof

Suppose there exists a $\lambda$, and $\delta > 0$ for which all $\epsilon$ such that $0 < \epsilon < \delta$, $C_1$ is a minimum cover in $G_{\lambda - \epsilon}$ where $|C_1| = k_1$, and $C_2$ is a minimum cover in $G_{\lambda + \epsilon}$, where $|C_2| = k_2$. Further, suppose that $k$ given in the hypothesis is such that $k_1 < k < k_2$. 

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Consider the subgraph $C_1 \oplus C_2$. By Lemma A.1, this subgraph decomposes into edge-disjoint reducing paths. Partition these paths into three classes:

**Class I**

\[ |C_1 \cap P_I| = |C_2 \cap P_I| + 1 \]

**Class II**

\[ |C_2 \cap P_{II}| = |C_1 \cap P_{II}| + 1 \]

**Class III**

\[ |C_1 \cap P_{III}| = |C_2 \cap P_{III}| \]

**Class I**

Consider a reducing path $P_I$ in Class I.

If

\[ W_\lambda(P_I \cap C_2) - W_\lambda(P_I \cap C_1) \geq 0, \]

this contradicts the assumption $C_2$ is a minimum cover in $G_{\lambda+\epsilon}$, for $\epsilon < \delta$, since $C_2 \oplus P_I$ is a cover with $(k_2 + 1)$ edges with smaller weight sum than $C_2$ in $G_{\lambda+\epsilon}$.

If

\[ W_\lambda(P_I \cap C_1) - W_\lambda(P_I \cap C_2) > \alpha > 0, \]

this contradicts the assumption that $C_1$ is a minimum cover in $G_{\lambda-\epsilon'}$ for $\epsilon < \epsilon'$, since $C_1 \oplus P_I$ is a cover with $(k_1 - 1)$ edges with smaller weight than $C_1$ in all such $G_{\lambda-\epsilon'}$.

Thus reducing paths of type $P_I$ cannot exist in subgraph $C_1 \oplus C_2$.

**Class II**

If

\[ W_\lambda(P_{II} \cap C_2) - W_\lambda(P_{II} \cap C_1) > \alpha > 0, \]

then this contradicts the assumption that $C_2$ is a minimum cover in
$G_{\lambda+\varepsilon}$, for $\varepsilon < \varepsilon'$, since $C_2 \oplus P_{II}$ is a cover with $(k_2 - 1)$ edges with smaller weight than $C_2$ in $G_{\lambda+\varepsilon}$. If $W_\lambda(P_{II} \cap C_1) - W_\lambda(P_{II} \cap C_2) > \varepsilon > 0$, then this contradicts the assumption that $C_1$ is a minimum cover in $G_{\lambda+\varepsilon}$.

$G_{\lambda-\varepsilon}$, for $\varepsilon < \varepsilon'$, since $C_1 \oplus P_{II}$ is a cover with $(k_1 + 1)$ edges with smaller weight than $C_1$ in $G_{\lambda-\varepsilon}$. Thus only reducing paths $P_{II}$ of zero weight can exist in $G_{\lambda}$. 

**Class III**

Apply the same arguments as for Class II, except extend $\varepsilon$ to $\varepsilon < \delta$, to show that only reducing paths $P_{III}$ of zero weight can exist in $G_{\lambda}$.

Thus there must be exactly $(k_2 - k_1)$ reducing paths of type $P_{II}$ in $G_{\lambda}$, all of weight zero, and while any number of paths of type $P_{III}$ may exist in $G_{\lambda}$, all must have weight zero.

Thus in graph $G_{\lambda}$, the weights of $C_1$ and $C_2$ are identical, but by implementing combinations of reducing paths of types $P_{II}$ and $P_{III}$, different covers of all $k$-cardinalities, for

$$k_1 \leq k \leq k_2$$

can be obtained, and will have this same minimum weight in $G_{\lambda}$. 

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REFERENCES


