DOCUMENT RESUME

ED 034 683 SE 007 647

AUTHOR White, Lee J.
TITLE Minimum Covers of Fixed Cardinality in Weighted Graphs.
INSTITUTION Ohio State Univ., Columbus. Computer and Information Science Research Center.
SPONS AGENCY National Science Foundation, Washington, D.C.
REPORT NO TR-69-2
PUB DATE Mar 69
NOTE 17p.

EDRS PRICE EDRS Price MF-$0.25 HC-$0.95
DESCRIPTORS College Mathematics, Higher Education, *Information Processing, Information Theory, *Mathematics, *Technical Reports

ABSTRACT

Reported is the result of research on combinatorial and algorithmic techniques for information processing. A method is discussed for obtaining minimum covers of specified cardinality from a given weighted graph. By the indicated method, it is shown that the family of minimum covers of varying cardinality is related to the minimum spanning tree of that graph. (RP)

SE

Received in RSP 5-8-69
No. of copies 7
Grant (Contract) No. 534.1

F-NSF
GN-534.1
TR-69-2

TECHNICAL REPORT 69-2

MINIMUM COVERS OF FIXED CARDINALITY

IN WEIGHTED GRAPHS

by

Lee J. White

March, 1969

Computer and Information Science
Research Center

The Ohio State University

Columbus, Ohio

MINIMUM COVERS OF FIXED CARDINALITY IN WEIGHTED GRAPHS

by

Lee J. White

TECHNICAL REPORT NO. 69-2

On Work Performed Under

Grant No. GN-534.1, National Science Foundation

March, 1969

Computer and Information Science Research Center
The Ohio State University
Columbus, Ohio 43210

PREFACE

This report is the result of research on combinatorial and algorithmic techniques for information processing supported in part by Grant Number GN 534.1 from the Office of Scientific Information Service of the National Science Foundation to the Computer and Information Science Research Center, The Ohio State University.

A version of this paper was presented at the Fall Meeting of the Society for Industrial and Applied Mathematics (S.I.A.M.), Philadelphia, Pennsylvania in October 1968.

The research was administered and monitored by The Ohio State University Research Foundation as Project RF 2218.

MINIMUM COVERS OF FIXED CARDINALITY IN WEIGHTED GRAPHS*

Lee J. White+

Given a weighted graph, a method is discussed for obtaining minimum covers of specified cardinality. It is shown that the family of minimum covers of varying cardinality is related to the minimum spanning tree of that graph.

1. Introduction

Consider a finite weighted graph [G, c] where E and V are the sets of edges and vertices of G respectively, and c_i is the weight of edge $e_i \varepsilon E$, where edge weights are arbitrary real numbers. A cover is a subset of E such that each vertex of V is incident to at least one edge of the subset.

The minimum cover problem is to find a cover of minimum weight sum:

$$\text{Min } \underline{c}^T \underline{x} \text{ subject to } A\underline{x} \geq 1, \; x_i = 0 \text{ or } 1,$$

where A is the vertex-edge incidence martrix of the graph G, $\underline{x}$ is a vector corresponding to the edges of G, $\underline{c}$ a vector of edge weights, and T indicates a vector transpose.

* This research was supported in part by the Systems Engineering Laboratory of the University of Michigan through contract AF 30 (602) - 3546, and in part by NSF GRANT GN 534.

+ Computer and Information Science, Ohio State University

Norman and Rabin [6] utilized the concept of reducing paths to solve the minimum cardinality cover problem, but solution effort of the algorithm grows exponentially with N, the number of vertices of the graph. Based on "matching" techniques of Edmonds [1,2], the author [7] developed an algorithm to solve the minimum cover problem for which solution effort grows as N^4. Edmonds [3] has developed an efficient method to solve the degree - constrained subgraph problem

$$\text{Min } \underline{c}^T \underline{x} \text{ subject to } \underline{b_1} \leq A\underline{x} \leq \underline{b_2},\ x_i = 0 \text{ or } 1,$$

which obtains the minimum cover as a special case.

2. k-Covers

Given a weighted graph [G, c], consider the minimum <u>k-cover</u> problem:

$$\text{Min } \underline{c}^T \underline{x} \text{ subject to } A\underline{x} \geq \underline{1},\ \sum_{e_i \varepsilon G} x_i = k,\ x_i = 0 \text{ or } 1.$$

Transform the graph [G, c] to graph $[G_\lambda, c - \lambda]$ by subtracting λ from each edge weight. The parameter λ may assume any real value, and may be interpreted as a dual variable or Lagrange multiplier as discussed by Everett [4], corresponding to the constraint

$$\sum_{e_i \varepsilon G} x_i = k ,$$

Define W (C) as the weight of cover C in [G, c], and W_λ(C) as the weight of C in $[G_\lambda, c - \lambda]$. |C| denotes the cardinality of set C.

Lemma 1

For any real λ, if C is a minimum cover in $[G_\lambda, c - \lambda]$ and |C| = k, then C is a minimum k - cover in [G, c].

Proof

Let C' be any k-cover in G. Then for any λ,

$$W_\lambda(C) = W(C) - k\lambda,$$

$$W_\lambda(C') = W(C') - k\lambda,$$

and thus $W(C) \leq W(C')$.

Lemma 2

Given a weighted graph [G, c] and λ_1, λ_2, such that $\lambda_2 > \lambda_1$. Let C_1 be a minimum cover in $[G_{\lambda_1}, c - \lambda_1]$, where $|C_1| = k_1$, and C_2 a minimum cover in $[G_{\lambda_2}, c - \lambda_2]$, where $|C_2| = k_2$. Then $k_2 \geq k_1$.

Proof

By assumption,

$$W_{\lambda_1}(C_1) \leq W_{\lambda_1}(C_2) \text{ and } W_{\lambda_2}(C_2) \leq W_{\lambda_2}(C_1) \text{ or}$$

$$W(C_1) - k_1\lambda_1 \leq W(C_2) - k_2\lambda_1$$

$$W(C_2) - k_2\lambda_2 \leq W(C_1) - k_1\lambda_2.$$

Adding these two inequalities, and rearranging yields

$$k_1(\lambda_2 - \lambda_1) \leq k_2(\lambda_2 - \lambda_1)$$

or $\qquad k_1 \leq k_2$, since $\lambda_2 > \lambda_1$.

Although the parameter k has been shown to be monotonic with λ, we must ensure that all values of k will be obtained as continuous values of λ are examined. Theorem 1 resolves this question, and the proof is presented in the appendix.

Theorem 1

Given a weighted graph $[G, c]$, where the minimum cardinality of any cover is m. Then given k, $m \leq k \leq |E|$, there exists a λ such that some minimum cover C in $[G_\lambda, c - \lambda]$ is of k - cardinality.

3. Vertex and Edge Partitions

For arbitrary values of λ, such that $\lambda > \underset{e_i \varepsilon G}{\mathrm{Min}} [c_i]$, partition the vertices in $[G_\lambda, c - \lambda]$ into two sets as shown in Figure 1:

1) V_N, vertices which are incident to an edge of nonpositive weight.

2) $V_p = V - V_n$

Define an edge partition of $[G_\lambda, c - \lambda]$:

1) P_λ, all edges in $[G_\lambda, c - \lambda]$ with at least one endpoint in vertex set V_p. Define a cover of V_p as a subset of the edges of P_λ such that each vertex of V_p is incident to at least one edge of this subset.

2) N_λ, all edges in $[G_\lambda, c - \lambda]$ with both endpoints in V_N.

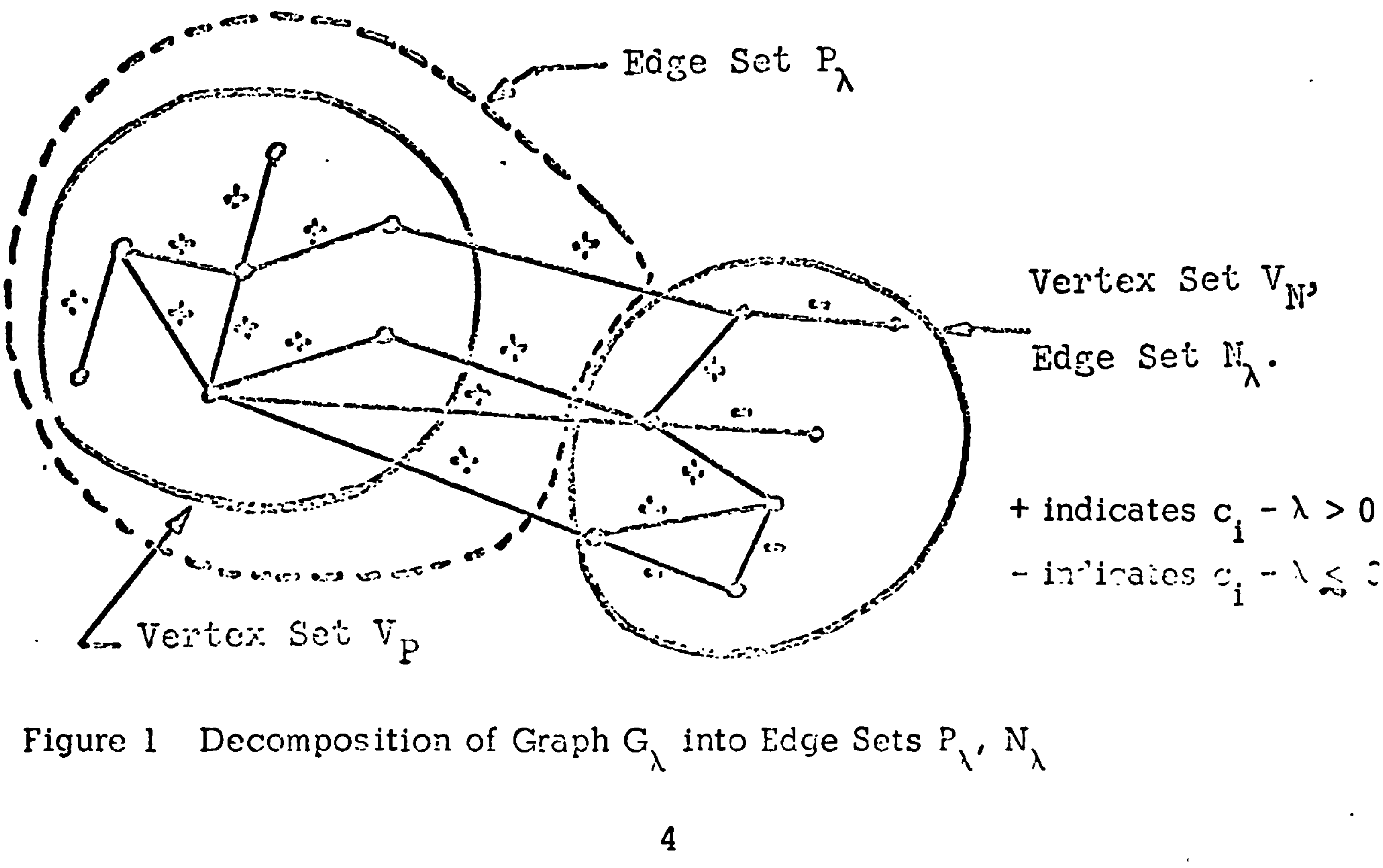

Figure 1 Decomposition of Graph G_λ into Edge Sets P_λ, N_λ

Theorem 2

Given $[G, c]$, and any λ, form $[G_\lambda, c - \lambda]$. Define C as the edges of a minimum cover of V_p, together with all the nonpositive edges of G_λ. If $|C| = k$, then C is a minimum k - cover in $[G, c]$.

Proof

Let D be any k - cover in G. Then

$$W_\lambda(C) = W(C \cap P_\lambda) + W(C \cap N_\lambda)$$

$$W_\lambda(D) = W(D \cap P_\lambda) + W(D \cap N_\lambda)$$

But since we found a minimum cover of V_p,

$$W_\lambda(C \cap P_\lambda) \leq W_\lambda(D \cap P_\lambda),$$

and since C uses all nonpositive edges in $[G_\lambda, c - \lambda]$,

$$W_\lambda(C \cap N_\lambda) \leq W_\lambda(D \cap N_\lambda) \text{ , and } W_\lambda(C) \leq W_\lambda(D).$$

Application of Lemma 1 yields the desired result.

The algorithm suggested by Theorem 2 decomposes the minimum k-cover problem into an "easy" problem in N_λ, and a "hard" cover problem for node set V_P. When an edge becomes negative, this ensures it will not only be in the next larger cardinality minimum cover, but in every subsequent minimum cover of greater cardinality. Exclusion of some zero weighted edges in N_λ may be necessary in order to attain minimum k - covers for all feasible values of k, and an efficient technique exists for this process.

When λ becomes sufficiently large such that all nodes are in set V_N, the minimum k - cover problem becomes easy for all k above the corresponding value. The critical value for which this occurs is:

$$\bar{\lambda} = \underset{v_j}{\text{Max}} \left\{ \underset{\substack{\text{all } e_i \text{ incident} \\ \text{to vertex } v_j}}{\text{Min}} [c_i] \right\}$$

The cardinality at which the minimum k - cover will contain a cycle (simple closed path) can be seen clearly in edge set N_λ: it occurs at the lowest value of λ for which all edges of a cycle become negative in $[G_\lambda, c - \lambda]$.

4. Minimum Spanning Trees

Define a <u>tree</u> of a graph G as a connected subgraph which contains no cycles. A <u>spanning tree</u> is a tree which contains all the vertices V of G. A <u>forest</u> is a subgraph which contains no cycles, and thus is a union of disjoint trees.

Kruskal [5] developed and proved the following algorithm to obtain a spanning tree of minimum weight for a connected weighted graph [G, c].

1) Well order the edges of G as $e_1, e_2, \ldots, e_n$ so $i < j \rightarrow c_i \leq c_j$.

2) Initially select $T = \{e_1, e_2\}$.

3) If the union of e_3 with T forms a cycle, permanently discard e_3; otherwise add e_3 to T.

4) Continue adding minimum elements to T in a similar fashion, such that T remains a forest in G until the number of edges in T is $(N - 1)$.

5) T is a minimum spanning tree of graph G.

Edmonds introduces the notion of a "greedy algorithm". Define an algorithm to obtain an optimal subset of a finite set as greedy if after well ordering the elements of the set by weight, each element requires examination only once, and upon examination, can either be placed in the solution set or permanently discarded.

The minimum spanning tree procedure is clearly a greedy algorithm, and obtains a cover of the given graph. Also the "easy" part of the minimum k - cover algorithm in Theorem 2 is greedy. Thus we inquire as to a relationship between the minimum spanning tree algorithm and the approach of Section 3 to obtain minimum k - covers.

5. Minimum Forest k - Covers

The question of the relationship between minimum spanning trees and minimum k - covers is complicated by the fact that the latter configuration may contain cycles. Define a forest cover in a graph G as a cover of G which contains no cycles. If the graph is connected, we can demand a forest k-cover configuration,

$$m \leq k \leq (N-1)$$

where m is the minimum cardinality of all covers.

Theorem 3

Given a weighted graph $[G, c]$, and any λ, form $[G_\lambda, c - \lambda]$. Let C be the edges of a minimum cover of V_P, together with a minimum forest cover of V_N, using edge set N_λ. If $|C| = k$, then C is a minimum forest k - cover in $[G, c]$.

Proof

Let D be any forest k - cover in [G, c], and consider the weights of C and D in $[G_\lambda, c - \lambda]$.

$$W_\lambda(C) = W_\lambda(C \cap P_\lambda) + W_\lambda(C \cap N_\lambda),$$

$$W_\lambda(D) = W_\lambda(D \cap P_\lambda) + W_\lambda(D \cap N_\lambda).$$

But since we found a minimum cover of V_P,

$$W_\lambda(C \cap P_\lambda) \leq W_\lambda (D \cap P_\lambda).$$

The minimum forest cover of $(C \cap N_\lambda)$ of V_N can be found by applying the minimum spanning tree algorithm to edges N_λ. This algorithm terminates when no further edges in N_λ can be added without forming a cycle. Neither edge sets $(C \cap N_\lambda)$ nor $(C \cap P_\lambda)$ contain a cycle. There exists no path in $(C \cap P_\lambda)$ between vertices of V_N as this would contradict the assumption that $(C \cap P_\lambda)$ is a minimum cover of V_P, so C contains no cycles. $W_\lambda(C \cap N_\lambda) \leq W_\lambda (D \cap N_\lambda)$, so $W_\lambda(C) \leq W_\lambda(D)$, and application of Lemma 1 completes the proof.

As in Section 3, the problem of finding a minimum forest k - cover is decomposed into "hard" and "easy" parts, and the latter is greedy. It can be shown that for all $m \leq k \leq (N-1)$, there is a λ such that the minimum forest k - cover is given by the indicated construction, together with a simple technique for breaking ties when zero weighted edges occur in N_λ.

As λ exceeds the value

$$\bar{\lambda} = \underset{v_j}{\text{Max}} \left\{ \underset{\substack{\text{all } e_i \text{ incident} \\ \text{to vertex } v_j}}{\text{Min}} [c_i] \right\},$$

the algorithm becomes equivalent to the minimum spanning tree algorithm. It is precisely as the edge of weight $\bar{\lambda}$ enters Kruskal's solution set that the edges of this set form a cover of the nodes V in graph G.

APPENDIX

The purpose of this appendix is to provide a proof of Theorem 1. First consider the following definitions.

A <u>subgraph</u> E_1 of a graph G with edges E and vertices V is defined as a graph whose edges are $E_1 \subset E$, and whose vertices are the set of endpoints $V_1 \subset V$ of the edges E_1. For convenience, we refer to a subgraph by its set of edges. A <u>path</u> in a graph is a sequence of edges $P = \{e_1, e_2, \ldots, e_n\}$ together with an associated sequence of vertices $\{v_1, v_2, \ldots, v_{n+1}\}$ such that consecutive edges e_i and e_{i+1} in the path have a common vertex v_{i+1} and each edge appears only once in the edge sequence P.

Given two sets of edges, E_1 and E_2, in a graph G, define the <u>symmetric difference</u> as

$$E_1 \oplus E_2 = (E_1 - E_2) \cup (E_2 - E_1).$$

An <u>alternating path</u> P (relative to the sets E_1 and E_2) is a path whose edges are alternately in $(E_1 - E_2)$ and $(E_2 - E_1)$.

Given a graph G, define a <u>reducing path</u> P relative to covers C_1 and C_2 of G as a path P such that:

(1) P alternates in subgraph $C_1 \oplus C_2$ with respect to edges in C_1 and C_2, and

(2) $C_1 \oplus P$ and $C_2 \oplus P$ are both covers in G.

Lemma A.1

Given a graph G, and two arbitrary covers C_1 and C_2 of G, subgraph $C_1 \oplus C_2$ can be decomposed into edge-disjoint reducing paths.

This result was proved by Norman and Rabin [6]. Construct a maximal alternating path P in subgraph $C_1 \oplus C_2$, removing the edges of this path. This produces a new subgraph in which another maximal alternating path can be removed. Since C_1 and C_2 are both covers in in G, $C_1 \oplus P$ and $C_2 \oplus P$ are both covers in G, and P is a reducing path.

Theorem 1

Given a weighted graph [G,c] where the minimum cardinality of any cover is m. Then given k, $m \leq k \leq |E|$, there exists a λ such that a minimum cover C in $[G_\lambda, c - \lambda]$ is of k - cardinality.

Proof

Suppose there exists a λ, and $\delta > o$ for which all ε such that

$$o < \varepsilon < \delta,$$

C_1 is a minimum cover in $G_{\lambda - \varepsilon}$ where $|C_1| = k_1$, and C_2 is a minimum cover in $G_{\lambda+\varepsilon}$, where $|C_2| = k_2$. Further, suppose that k given in the hypothesis is such that

$$k_1 < k < k_2.$$

Consider the subgraph $C_1 \oplus C_2$. By Lemma A.1, this subgraph decomposes into edge-disjoint reducing paths. Partition these paths into three classes:

Class I	$\lvert C_1 \cap P_I \rvert = \lvert C_2 \cap P_I \rvert + 1$
Class II	$\lvert C_2 \cap P_{II} \rvert = \lvert C_1 \cap P_{II} \rvert + 1$
Class III	$\lvert C_1 \cap P_{III} \rvert = \lvert C_2 \cap P_{III} \rvert$

Class I

Consider a reducing path P_I in Class I.

If

$$W_\lambda (P_I \cap C_2) - W_\lambda (P_I \cap C_1) \geq 0,$$

this contradicts the assumption C_2 is a minimum cover in $G_{\lambda+\varepsilon}$, for $\varepsilon < \delta$, since $C_2 \oplus P_I$ is a cover with $(k_2 + 1)$ edges with smaller weight sum than C_2 in $G_{\lambda+\varepsilon}$.

If

$$W_\lambda (P_I \cap C_1) - W_\lambda (P_I \cap C_2) > \alpha > 0,$$

this contradicts the assumption that C_1 is a minimum cover in $G_{\lambda-\varepsilon}$, for $\varepsilon < \alpha$, since $C_1 \oplus P_I$ is a cover with $(k_1 - 1)$edges with smaller weight than C_1 in all such $G_{\lambda-\varepsilon}$.

Thus reducing paths of type P_I cannot exist in subgraph $C_1 \oplus C_2$.

Class II

If

$$W_\lambda (P_{II} \cap C_2) - W_\lambda (P_{II} \cap C_1) > \alpha > 0,$$

then this contradicts the assumption that C_2 is a minimum cover in

$G_{\lambda+\varepsilon}$, for $\varepsilon < \alpha$, since $C_2 \oplus P_{II}$ is a cover with $(k_2 - 1)$ edges with smaller weight than C_2 in $G_{\lambda+\varepsilon}$. If $W_\lambda (P_{II} \cap C_1) - W_\lambda (P_{II} \cap C_2) > \alpha > 0$, then this contradicts the assumption that C_1 is a minimum cover in $G_{\lambda-\varepsilon}$, for $\varepsilon < \alpha$, since $C_1 \oplus P_{II}$ is a cover with $(k_1 + 1)$ edges with smaller weight than C_1 in $G_{\lambda-\varepsilon}$. Thus only reducing paths P_{II} of zero weight can exist in G_λ.

Class III

Apply the same arguments as for Class II, except extend ε to $\varepsilon < \delta$, to show that only reducing paths P_{III} of zero weight can exist in G_λ.

Thus there must be exactly $(k_2 - k_1)$ reducing paths of type P_{II} in G_λ, all of weight zero, and while any number of paths of type P_{III} may exist in G_λ, all must have weight zero.

Thus in graph G_λ, the weights of C_1 and C_2 are identical, but by implementing combinations of reducing paths of types P_{II} and P_{III}, different covers of all k-cardinalities, for

$$k_1 \leq k \leq k_2$$

can be obtained, and will have this same minimum weight in G_λ.

REFERENCES

1. J. Edmonds, "Paths, Trees, and Flowers", Canadian J. Math, 17, pp 449-467 (1965).

2. _____________, "Maximum Matching and Polyhedron with o,1-Vertices", J. Res NBS, Vol. 69B, Nos 1 and 2, pp 125-130 (1965).

3. _____________, "Optimum Degree-Constrained Subgraphs", to appear.

4. H. Everett III, "Generalized Lagrange Multiplier Method for Solving Problems of Optimum Allocation of Resources", Op Res, Vol. 11, pp 399-417 (1963).

5. J. B. Kruskal, Jr., "On the Shortest Spanning Subtree of a Graph and the Traveling Salesman Problem", Proc. Amer Math Soc, 7, pp 48-50 (1956).

6. R. Z. Norman and M.O. Rabin, "An Algorithm for a Minimum Cover of a Graph", Proc Amer Math Soc, Vol 10, pp 315-319 (1959).

7. L. J. White, "A Parametric Study of Matchings and Coverings in Weighted Graphs", Tech. Rpt. 06920-11-T, The Systems Engineering Laboratory, Department of Electrical Engineering The University of Michigan (Ph.D Thesis - 1967).