DOCUMENT RESUME

ED 033 034

SE 007 214

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Informal Geometry for Young Children; Cambridge Conference on School Mathematics; Feasibility Study No. 34b.
Cambridge Conference on School Mathematics, Newton, Mass.

Pub Date [69]

Note-37p.

EDRS Price MF-$0.25 HC Not Available from EDRS.

Descriptors-*Elementary School Mathematics, *Geometric Concepts, *Geometry, Grade 1, Grade 6, *Instruction, *Instructional Materials, Mathematical Concepts, Resource Materials

These materials were written with the aim of reflecting the thinking of The Cambridge Conference on School Mathematics (CCSM) regarding the goals and objectives for school mathematics. These materials are intended to provide children with a variety of informal activities in intuitive geometry in the elementary school. Opportunities are provided for children to gain experience with many types of rigid motions - namely translations, rotations, and reflections. The type of work described in this report gave students the opportunity to become familiar, by direct experience and experiment, with important geometrical concepts before they were to be studied theoretically. Included are descriptions of a number of activities. Comments by teachers concerning the effectiveness of various activities and procedures are also included. This document is the best available copy. [Not available in hard copy due to marginal legibility of original document]. (RP)

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INFORMAL GEOMETRY FOR YOUNG CHILDREN

MARION WALTER

The Cambridge Conference on School Mathematics
Educational Services Incorporated
June 1965

SE 007 214

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INFORMAL GEOMETRY FOR YOUNG CHILDREN

Introduction

The need for students to experience a variety of informal intuitive activities has been emphasized recently by psychologists, physicists and mathematicians. For example, in his forthcoming book, Experimenters in the Classroom, Professor Philip Morrison writes:

> The mathematicians joy in precision and generality has tended to lead school mathematics a little away from the emphasis on the intuitive experimental beginnings of number, geometry, calculus and even algebra . It seems to us that the significance and excitement of many such mathematical structures is muted unless a premathematical basis in experience has preceded or accompanied the more formal elaborations.[1]

The need for intuitive experiences before a topic in mathematics is studied formally by a student is stressed in The Report of the Cambridge Conference on School Mathematics:

> The importance of a suitable background of experience, involving diversified sensory input, in developing clear mathematical concepts suggests that a full use be made of general heuristic cognitive patterns ...called..."pre-mathematics". ...an intuitive or pre-mathematics approach offers the opportunity of an early introduction of important concepts...[2]

This emphasis is particularly strong in the field of geometry.

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1. Philip Morrison, Experimenters in the Classroom (to be published).

2. Educational Services Incorporated, Goals for School Mathematics: The Report of The Cambridge Conference on School Mathematics (Boston, Houghton Mifflin Co., 1963), p. 13.

Work requiring pupils to visualize patterns in two or three dimensions, especially in the earliest grades (K-3), is also recommended in The Cambridge Report.[3] The writer has always been impressed by the large number of college students who have difficulty with problems which require the visualization of figures for their solution. Not only is it useful for mathematics students to have some facility in this direction, but it is becoming increasingly necessary for students in other fields and professions - architecture, crystallography, biochemistry, to mention but a few.

The work described in this paper was originated and developed by the writer partly in an attempt to provide sample projects which might fulfill the needs described above, partly to meet the requirements of a particular situation (described later in this paper), and partly in the hope of devising some playful, exciting, and at the same time mathematically useful program.

It is clear that little intuitive premathematical material is available, and the writer's association with Educational Services Incorporated has made the need of such material even more apparent. Professor Marshall Stone, despite the otherwise critical tone of his evaluation of The Cambridge Report concedes:

> "So far as the basic elementary school curriculum is concerned, I believe that it should comprise arithmetic, physical (or intuitive) geometry... The emphasis should be on the concrete operational aspects of mathematics..."[4]

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3. I bid., pp. 33,34.

4. Marshall H. Stone, Review of Goals for School Mathematics: The Report of the Cambridge Conference on School Mathematics, The Mathematics Teacher, VOl. LVIII, No. 4, (April, 1965), p. 358.

The main task undertaken by the writer has been to devise appropriate materials and use them with youngsters. The problem has been approached in the spirit of a pilot study. There has been no attempt to control the variables which might account for the apparent success to date of this project. It is stressed, therefore, that this is a first attempt, and that it should, of course, be followed by further development and study.

The guiding principle throughout has been to afford the children an intuitive grasp of mathematical ideas which could, at a later stage, be taught more formally.

The mathematical significance of this work, i.e., what are some of the sophisticated mathematical ideas which underlie this premathematical experience will be discussed at the end of this paper.

Description of the Work Carried Out in the Summer of 1964 [5]

When this work was begun, the writer did not know and could not control the number of class meetings. A great many different visual problems were considered before the first class meeting, but there was time to develop only a few of them. It is hoped that more of them can be incorporated into a more comprehensive program in the future.

The lessons were planned on a day-to-day basis, and what was done on any day depended heavily on the ideas generated in working with the students during the previous session, on their reactions and on their questions. For example, the games described below were not planned initially, but were devised by the writer to meet specific needs which arose as work with the children progressed. Clearly the material needs to be brought back to the classroom and perhaps expanded further.

Work Carried Out With a Sixth Grade Class

Twenty-four students who would enter the sixth grade in September of 1964 took part in the work described here. They were a heterogeneous group, coming from all school districts of Cambridge, Mass. Sixteen sessions of about twenty to thirty minutes duration were held.

The work required the children to visualize two and three dimensional figures. Often they had to decide how a figure would look after it had been moved or folded in a certain way. Part of the time these students

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5. This part of the work was carried out while the writer was working with the Cambridge Conference on School Mathematics under the auspices of Educational Services Incorporated. I should like here to thank all the members of the conference for their help and encouragement, particularly Professor Gleason of Harvard University and Professor Lomon of M.I.T.

used actual paper models, but much of the work was done in their heads. A paper model was always available, however, for checking or other purposes. Thus the children did not need to resort to the authority of the teacher, but could check their results by actually manipulating a concrete model.

Over and beyond its initial mathematical usefulness in giving them experience in visualization, this work introduced the students, in a natural way, to the idea of a geometric transformation, to the concept of symmetry, and to some basic ideas of group theory. The naturalness with which it led to these concepts can be seen from the fact that they arose in the classroom although the writer had originally intended to work only on the visualization of figures (since some members of the group were experimenting with ideas leading to the basic concepts of group theory with other grades).

In this paper only work which was actually tried is described. Many of the questions raised or topics touched upon can be greatly expanded. Many ideas suggested by the trial teaching could not be followed up because of lack of time.

In the description which follows, some diagrams are drawn solely to facilitate reading. This does not necessarily imply that the students were given the diagrams or that they drew such diagrams. The degree to which the work is done with concrete objects and actual motion depends, of course, on the previous experience and ability of the students.

Obtaining the Different Patterns

To begin with, the students were asked to visualize the following:

a box,

a box all of whose sides were equal,

a box without a top,

a box with square sides and no top. (Fig. 1)

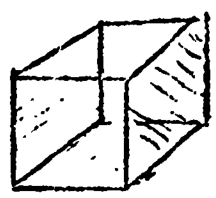

Fig. 1.

They were asked, "What does such a box look like flattened out? Can you draw it?"

Many drew Fig. 2.[6]

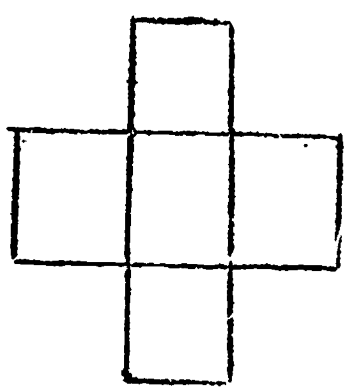

Fig. 2.

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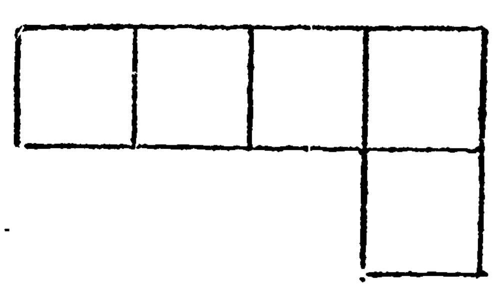
6. When this question was asked of a group of college students, all except one drew Fig. 2. Only one drew the figure below, which is, of course, also correct, and one of the eight possibilities.



This led to an examination of the "essentially different" patterns one can obtain by using five squares. Because the problem was concerned with a box, laying down the squares as in Fig. 3, for example, was not tried or suggested by any student at this time, and all kept the edges vertical and horizontal without being told to do so.

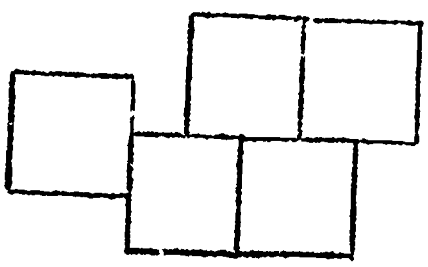


Fig. 3

What was meant by "essentially different"?[7] Was the pattern in Fig. 4 different from that in Fig. 5?

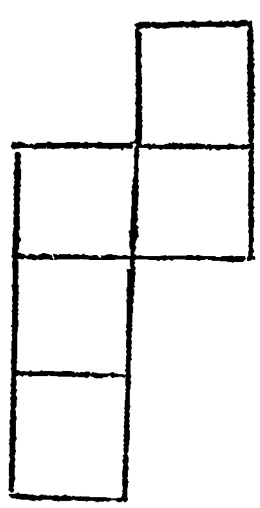


Fig. 4.

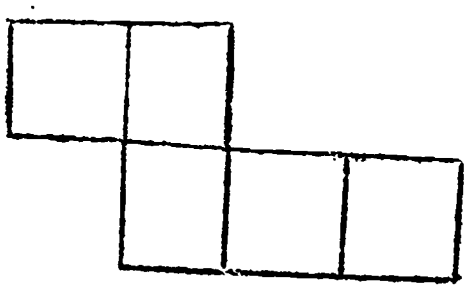


Fig. 5.

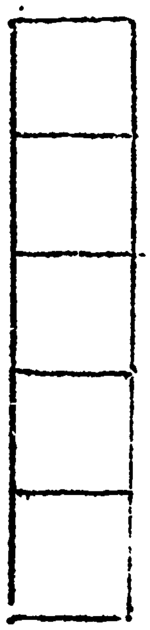


Fig. 6.

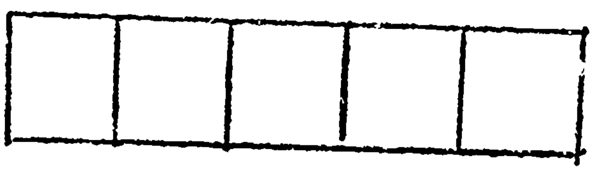


Fig. 7.

7. The answer depends, of course, on whether the paper is the same color on both sides. Working with paper which had two differently colored sides would help bring out the difference between a rotation and a reflection.

The students quickly realized that Fig. 6 and 7 were "essentially" the same pattern. At this stage they checked whether two patterns were "essentially" the same by seeing whether a paper cut-out of one would fit on the other.

Eventually the students obtained all twelve different patterns of five squares. (Fig. 8)

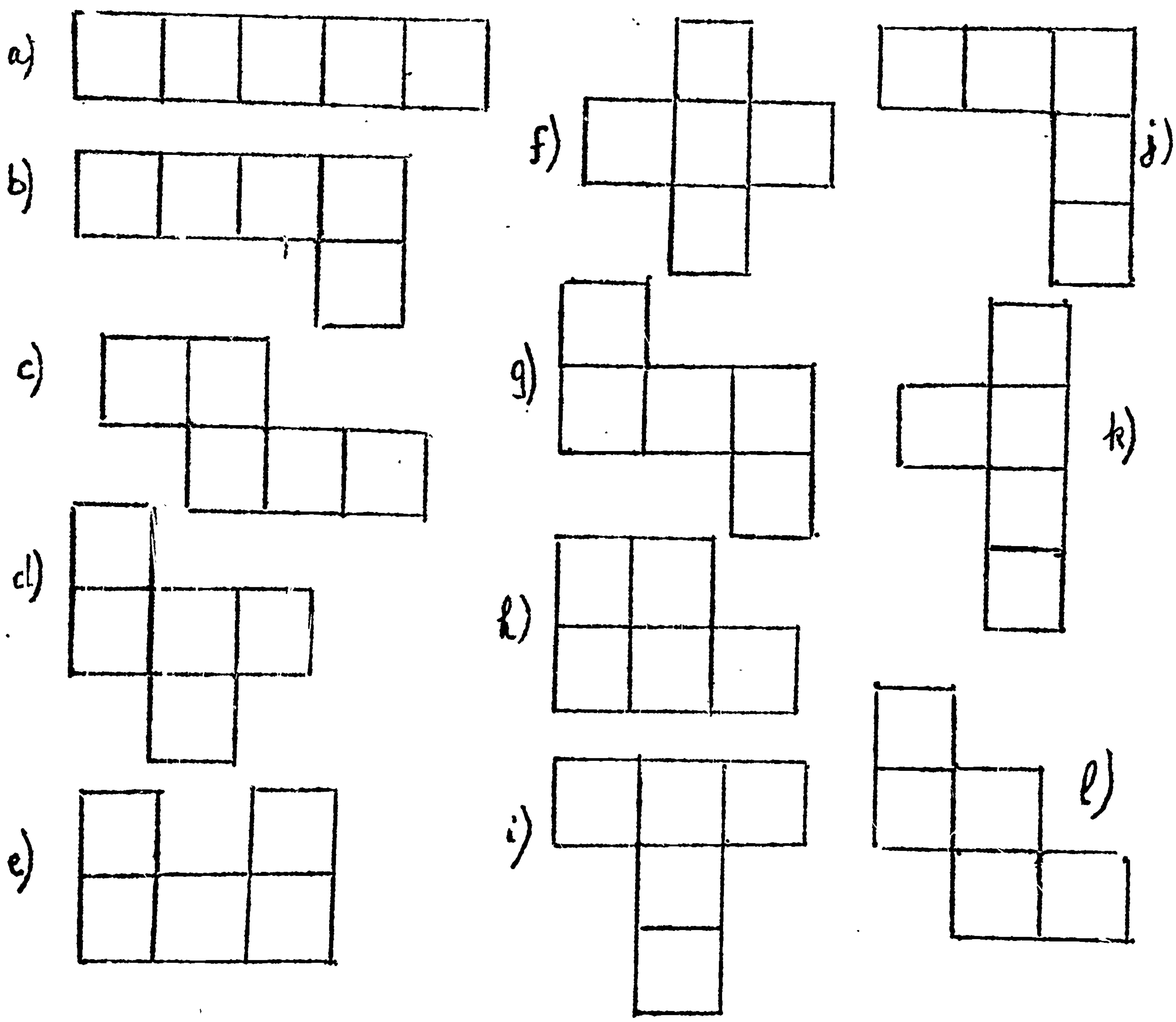


Fig. 8.

Some patterns fold into boxes without a top and some do not. Which ones do so was not fully investigated at this time in this class, although for several patterns the question was raised.[8] Some students made models to decide --others decided just by looking at the pattern. One boy felt pattern a) (Fig. 8) would fold into a box, but most thought it would not. When asked if he would like to try with a model, he answered, "Yes."

"How long would you like to have to try?" he was asked.

"Oh, about two years!"

The next day he came back jubilant and said, "It would make a box if you could cut a piece off."

The possibility of finding an algorithm to decide which patterns fold into a box and which do not was not investigated at this time.

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8. This problem is mentioned by Franz Hohn (see bibliography) in an article which was brought to my attention after the work of the summer was over. There are also many books on polyominoes (patterns made up of connected squares). The writer came to think of using the five-square patterns after being given a game of "HEXIT". This is a 10" x 6" box into which one is supposed to fit the twelve different arrangements of five squares which are provided (a frustrating game!). At that time the writer was unaware that books and articles on polyominoes had appeared in the mathematical literature. (For references, see the Martin Gardner entry in bibliography.)

The Game of Listing Patterns

After everyone had found all twelve patterns, the following game was played:

Two teams were formed. (Luckily there were twelve students on each side!) Each team worked at its own blackboard. Each player in turn could draw a new pattern of five squares until each team had drawn (hopefully) twelve "different" patterns on the board. Then each team examined the drawings of the other team for duplicates, which were, of course, not allowed.

For example patterns a) and b) of Fig. 9 appeared on the same team, as did a) and b) of Fig. 10.

Fig. 9.

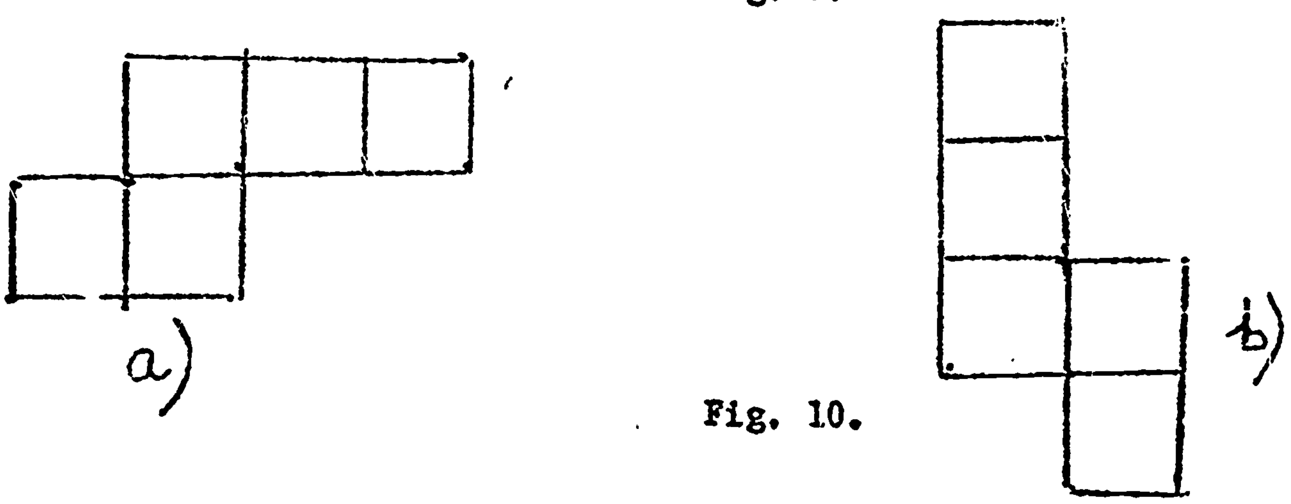


Fig. 10.

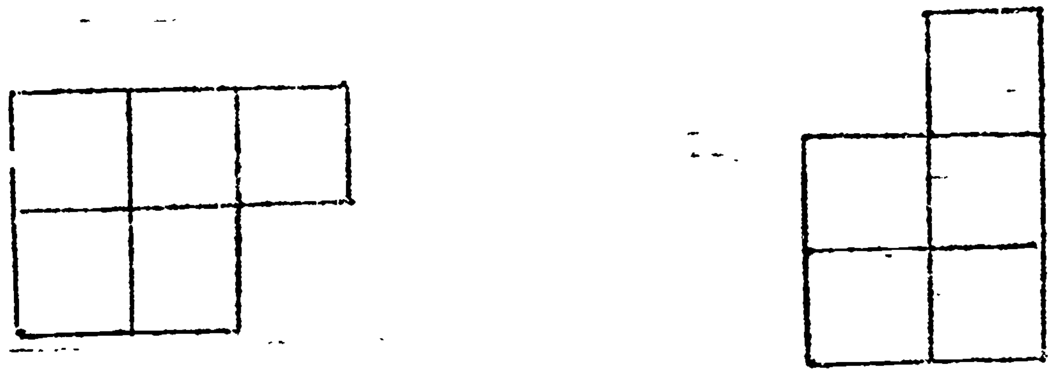
When a student found a duplicate on the opposing team, he had to demonstrate with a paper model that pattern a) could be made to coincide with pattern b), and hence that a) and b) were "essentially" the same. The game was played several times on different days, always with much excitement. Duplication occurred each time, but to a lessening degree. It is interesting to note that the students were

so enthusiastic about finding the twelve patterns each time and discovering the duplicates that no one ever asked which team had won.

No attempt was made at this time to find a systematic way of listing all twelve patterns. There was discussion, however, on why it was easier for the first players on each team to find a new pattern than it was for the last players.

After the game had been played several times, it became clear that some patterns were quite "safe" and others "dangerous", in the sense that they invited duplication. For example, no one ever drew Fig. 2 twice. (See p. 6.) Nor did anyone every draw Fig. 6 when Fig. 7 had already been drawn. (See p. 7)

On the other hand, pattern a) in Fig. 11 was often put down twice, first as a), then as b), let us say.



a) b)

Fig. 11

Other dangerous patterns were Fig. 12 and Fig. 13.

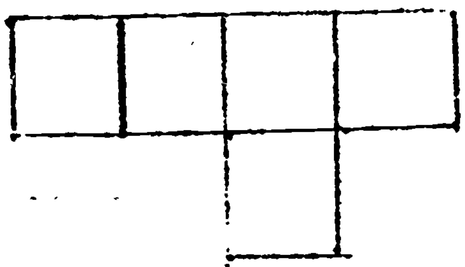


Fig. 12.

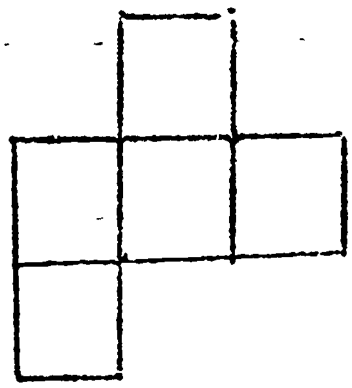
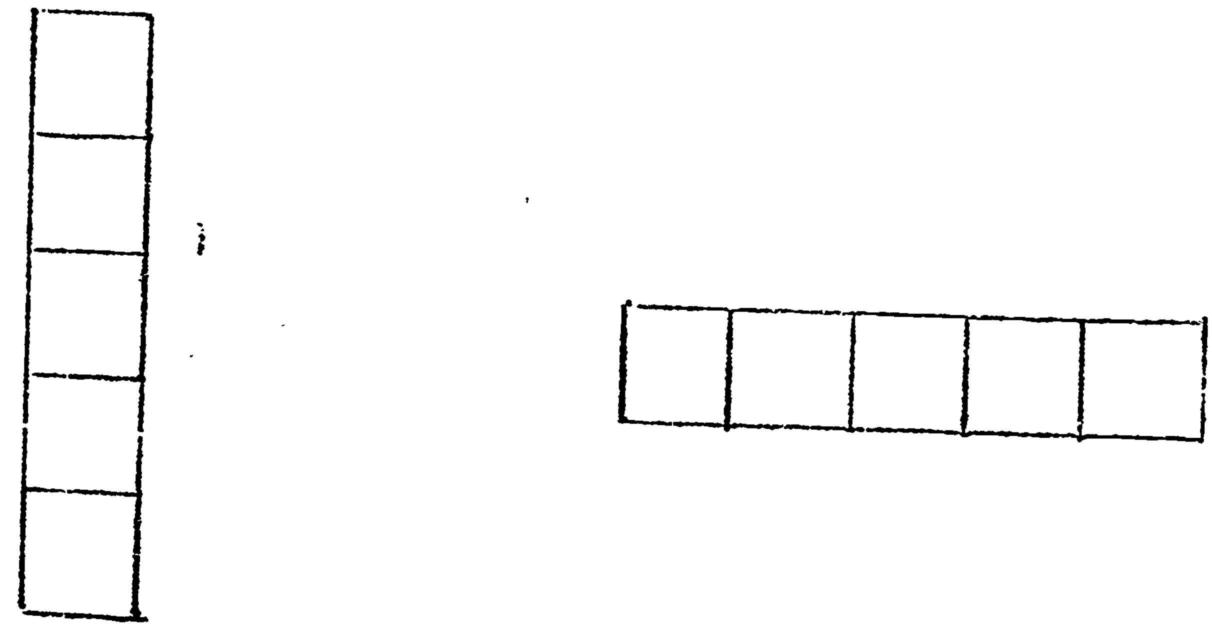


Fig. 13.

Examining the Patterns

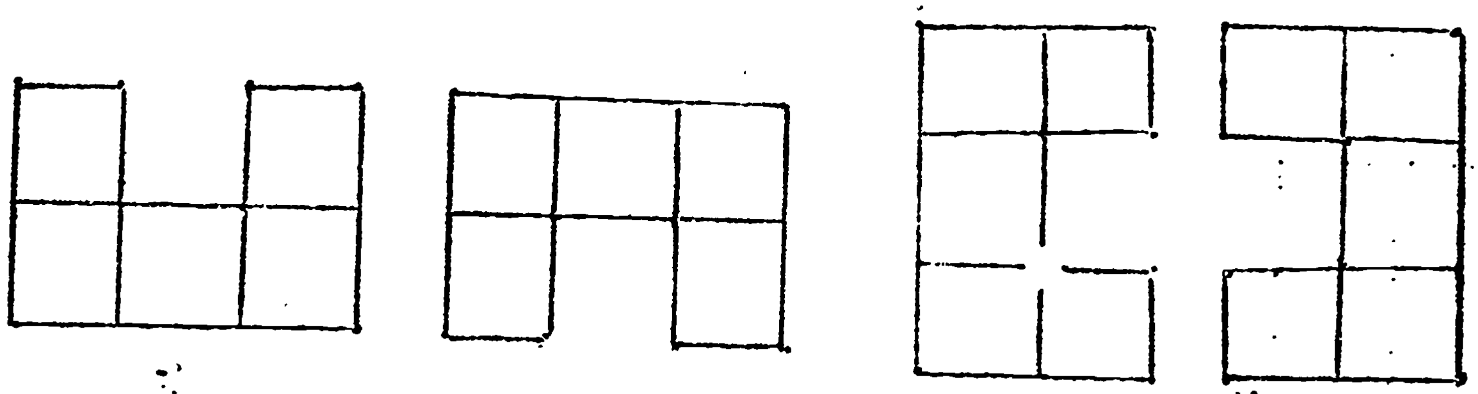
Why were some patterns "dangerous"? This question led naturally into a discussion of how many different positions each pattern had. It was found that the "safe" patterns had few positions, while the "dangerous" ones had more. Keeping lines horizontal and vertical, the students found that the patterns had one, two, four, or eight different positions. Pattern a) of Fig. 14 has two different positions, for example.



a) b)

Fig. 14.

Pattern a) of Fig. 15 has four different positions.



a) b) c) d)

Fig. 15.

This led to the question, "Why do some patterns have eight, others only four or two positions, and some only one?"

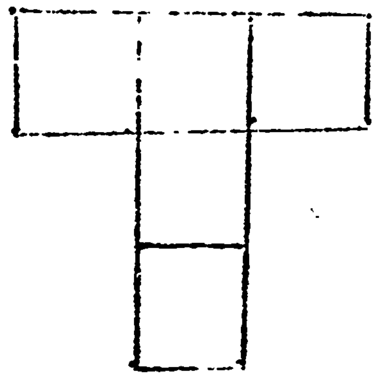
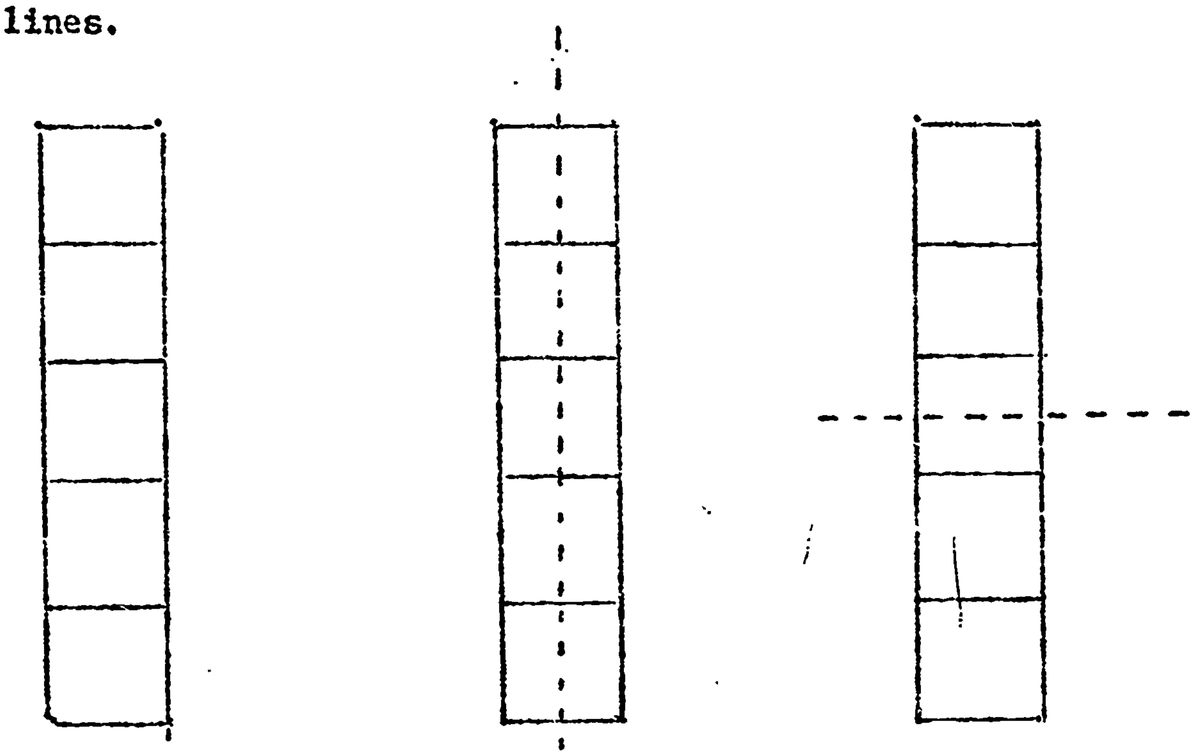


Fig. 16.

Fig. 16. was examined. The students noticed that a "flip" in the vertical line gave them the same position back again. It was not immediately seen that if this pattern were folded down the vertical line, both halves would match. At this point a little extra work was done on "special folding lines" (lines such that a pattern folded along one would be divided into matching halves.) Some students tried using a mirror on the pattern at the teacher's suggestion. Mirror Cards (see below) were not tried because of the great shortage of time. (It would be interesting to see whether prior play with the Mirror Cards would enable the students to spot the line of symmetry sooner.) They did, of course, see that flipping would do the trick, but they did not connect it with the special folding line.

It was found that pattern a) of Fig. 17 had two special folding lines.



a) b) c)

Fig. 17.

These were later called vertical and horizontal flip lines. They saw that a half turn about the center also gave them back the same position, as did a whole turn. (Fig. 18) (About which point the figure was to be rotated was not discussed.)

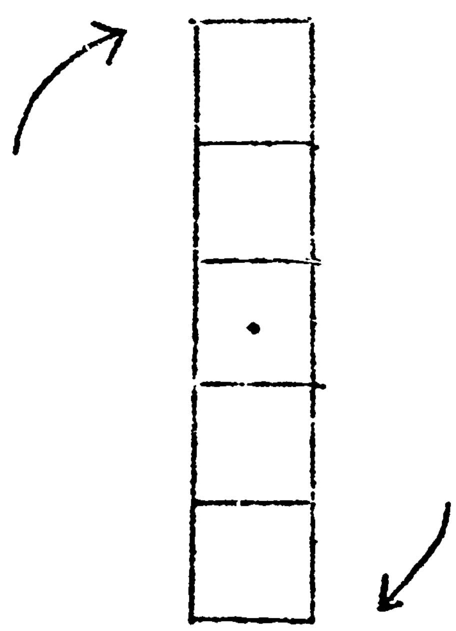


Fig. 18.

They discovered quickly that Fig. 19 had four special flip lines.

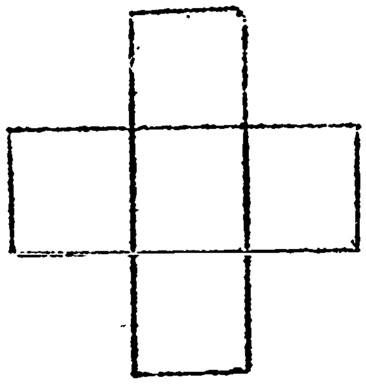
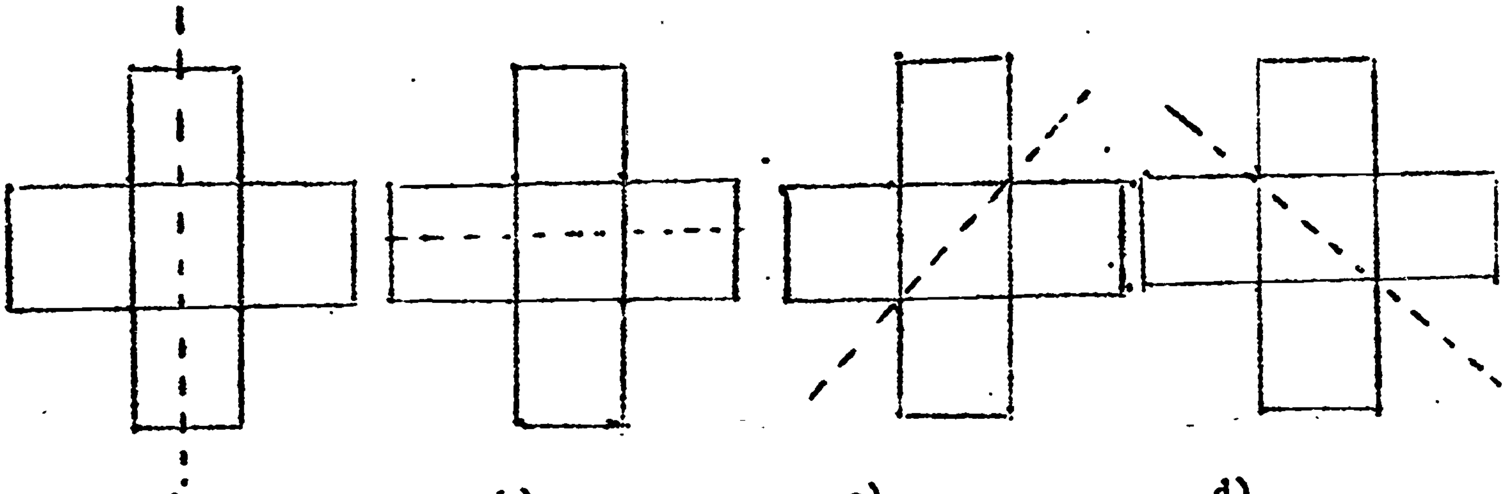


Fig. 19.

These lines were the V line, H line, R line (right slant) and L line (left slant.)



a) b) c) d)

Fig. 20.

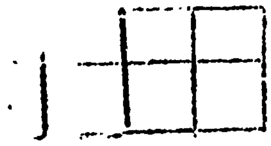
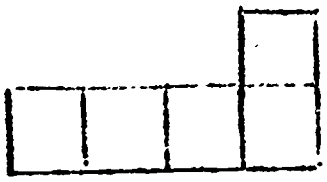
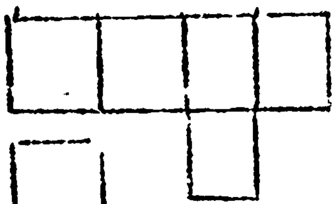
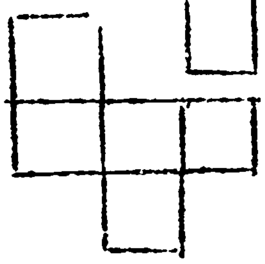
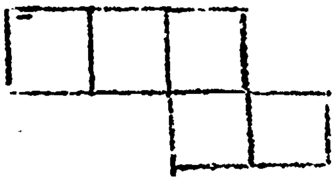
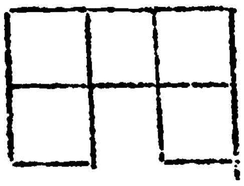
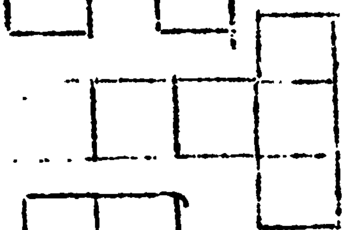
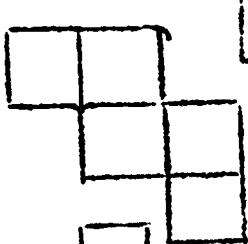
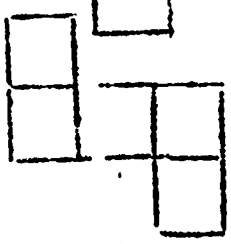
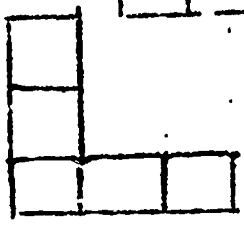
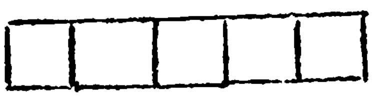
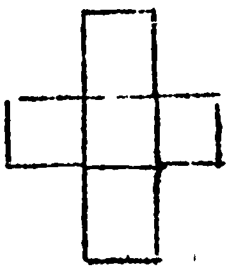
They saw that a quarter, half, three quarter, and full turn also gave back the same position.

They saw that the "dangerous" patterns (as those in Fig. 21, for example) had no special folding lines, nor would any rotation other than the full turn put them back in the same position. In fact each of the "dangerous" patterns had eight different positions.

a) b)

Fig. 21.

The following information was obtained:

Pattern	Motions That Give Back Original Position	Number of Different Positions Possible
	1	8
	1	8
	1	8
	1	8
	1	8
	V, 1 (OR H, 1)	4
	H, 1 (OR V, 1)	4
	R, 1 (OR L, 1)	4
	$\frac{1}{2}$, 1	4
	R, 1 (OR L, 1)	4
	H, V, $\frac{1}{2}$, 1	2
	$\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, 1, H, V, R, L.	1

The Game "What Motions Were Made?"

A different game was then played. The students were asked to consider Fig. 22.

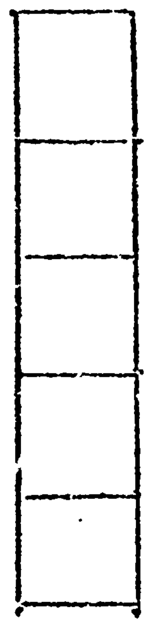


Fig. 22.

Which motions could one make to keep the position the same? Clearly, H,V, ½, and full turn. (H was used to denote both the horizontal flip line on a pattern and the motion "flip about horizontal flip line." Perhaps these should be distinguished in future play.) Could the students tell which motion had been made if their eyes were closed at the time it was made? The need for marking the figure was realized. They suggested marking it on one side as in Fig. 23.

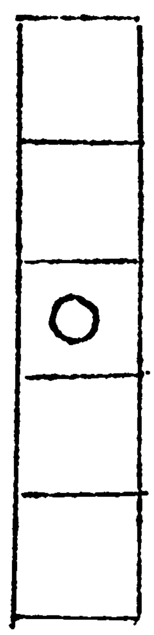


Fig. 23.

Half the class kept their eyes closed while the game was played once. Fig. 24 shows what they saw before they closed their eyes and after they opened them.

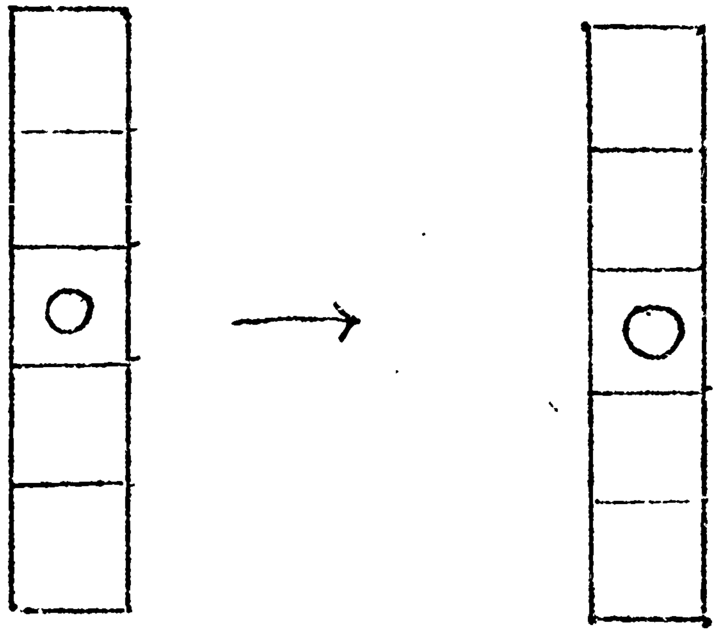


Fig. 24.

They realized that they could not tell whether a half or a full turn had been made. It was suggested that the figure be marked as in Fig. 25.

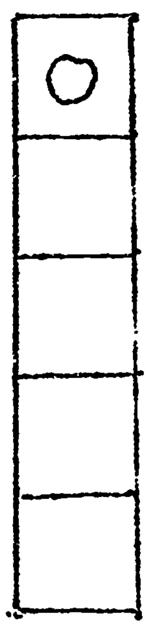


Fig. 25.

The game was played again. Again half the class kept their eyes closed. Fig. 26 shows the move.

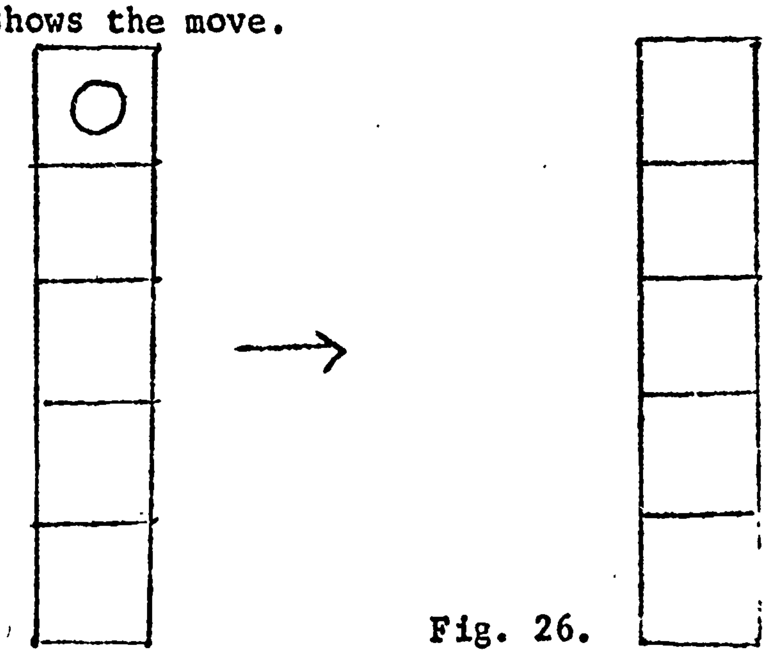


Fig. 26.

Now they realized they had to mark the back! It was suggested that the back be marked with another little circle directly behind the one in front.

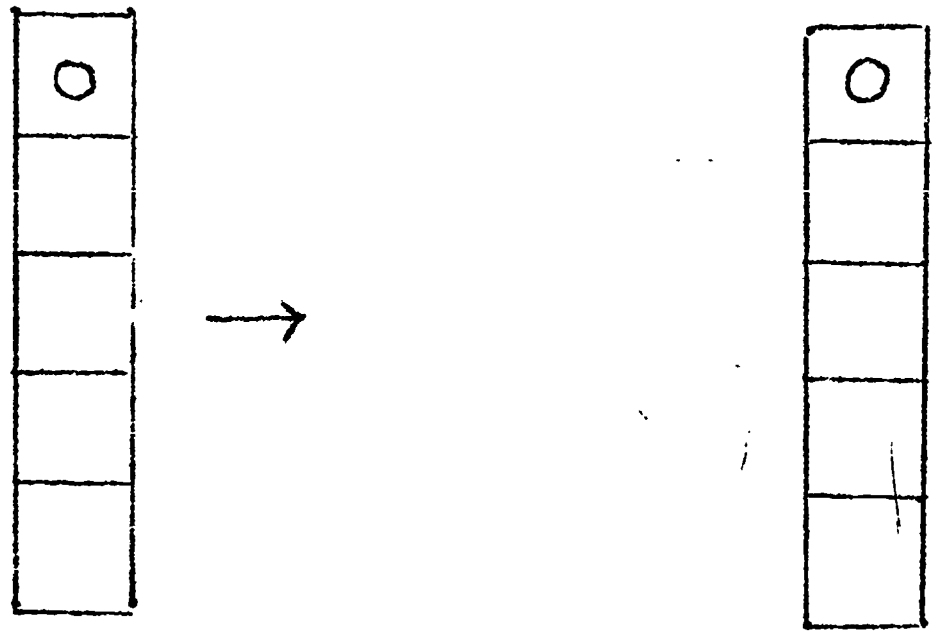
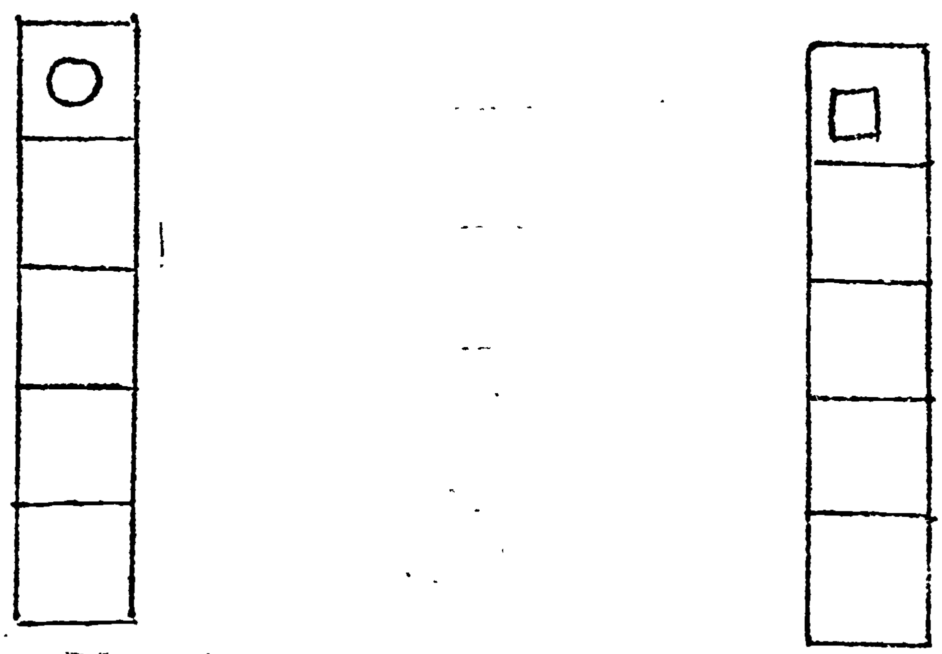


Fig. 27.

Playing the game shown in Fig. 27 aroused amusement and the realization that the marking was still not adequate. It was suggested that a little square be used on the second side. Thus finally the piece was marked with the square directly behind the circle mark. (Fig. 28 shows both sides.)



a) b)

Fig. 28.

Now the game was played several times. Each time half the class kept their eyes closed, and the other half watched while the motion was made. They were very quick at seeing, for example, that the motion in Fig. 29 was perhaps done by a horizontal flip.

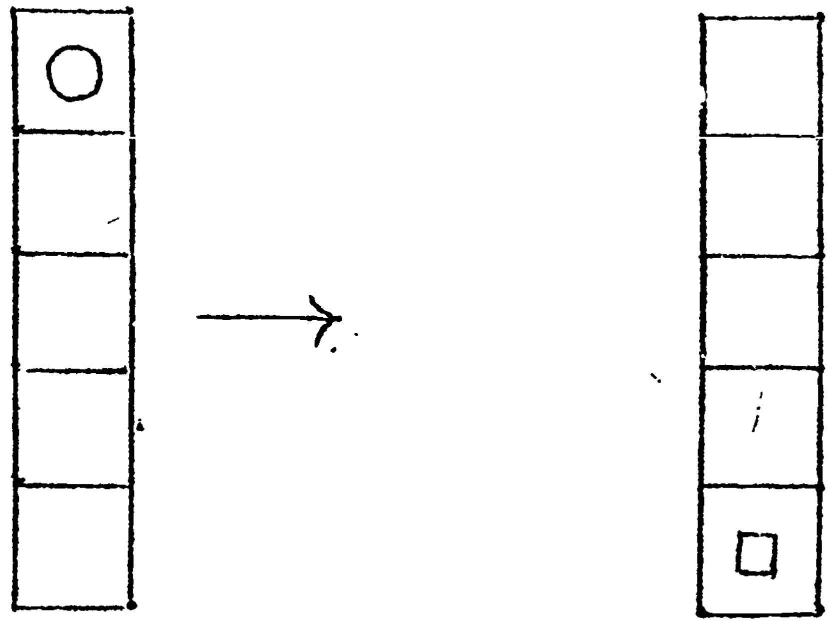


Fig. 29.

"Perhaps" became necessary because students soon began to give the product of more than one motion. It was realized that HHHHHHV was equivalent to V. They found the rule that an even number of half turns, horizontal flips, vertical flips, right slant flips or left slant flips was equivalent to not moving the piece.

The game was played again, starting with a) Fig. 28. The students realized that if only one motion was made, they could tell for certain which motion had been made even when their eyes had been closed.

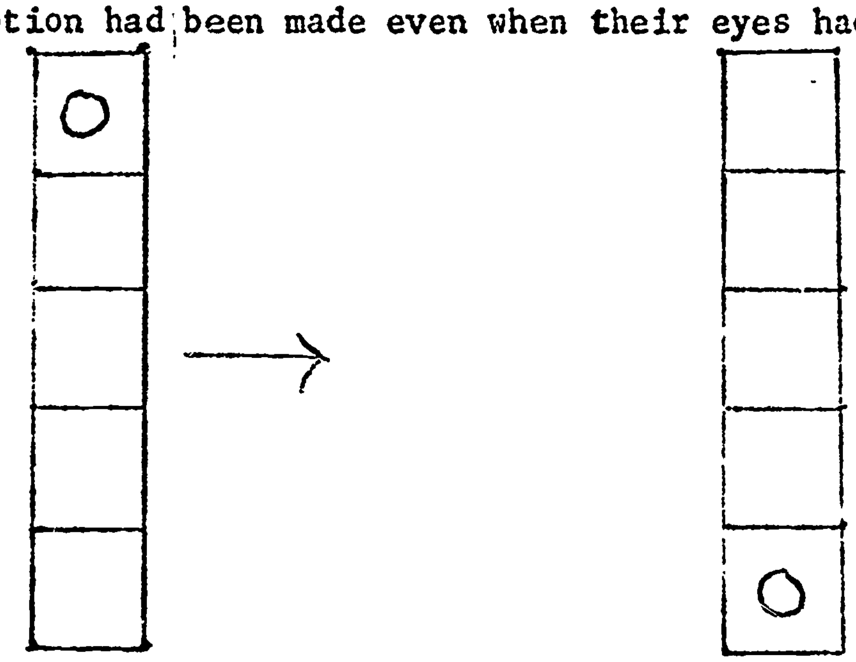


Fig. 30.

For example in Fig. 30, the result must have come from a half turn if one motion were used. Since the students spontaneously began to give

the product of two or more motions, however, the game was played using two. Now there was considerable excitement. When Fig. 30 was done again, the students who had had their eyes shut gave the possible motions with almost frightening speed! HV, VH, 1½, ½1 were suggested. (1 stands for full turn) They saw now that it was a matter of luck whether they picked the two that had actually been carried out. They found that there were four possible ways of getting each new position in two motions. For example, Fig. 30 could be done in one move using ½ turn, but by using two motions it could be done by HV, VH, 1½, ½1.

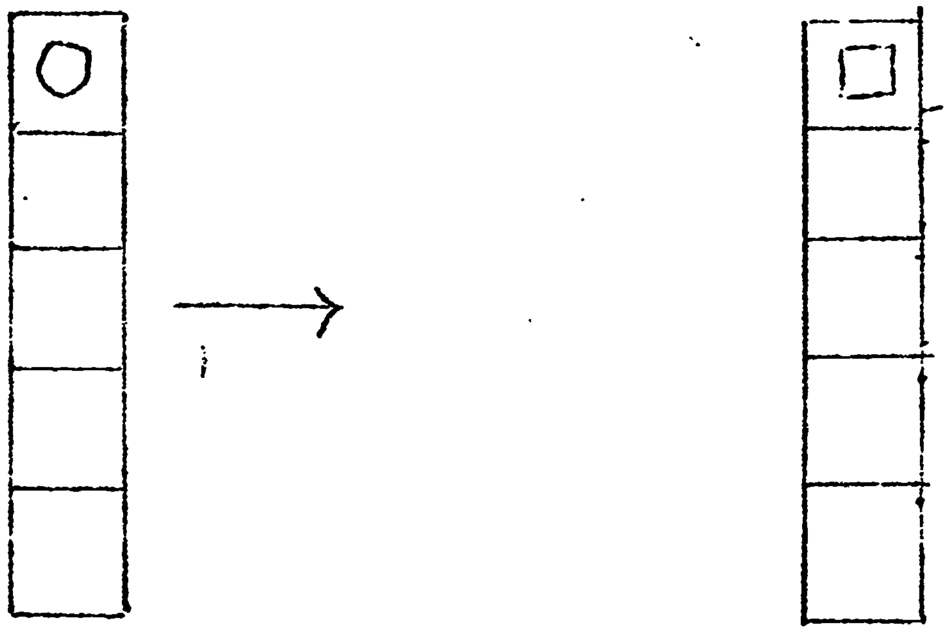


Fig. 31.

Fig. 31 could be done in one motion by V
in two motions by 1V, V1, H½, ½H

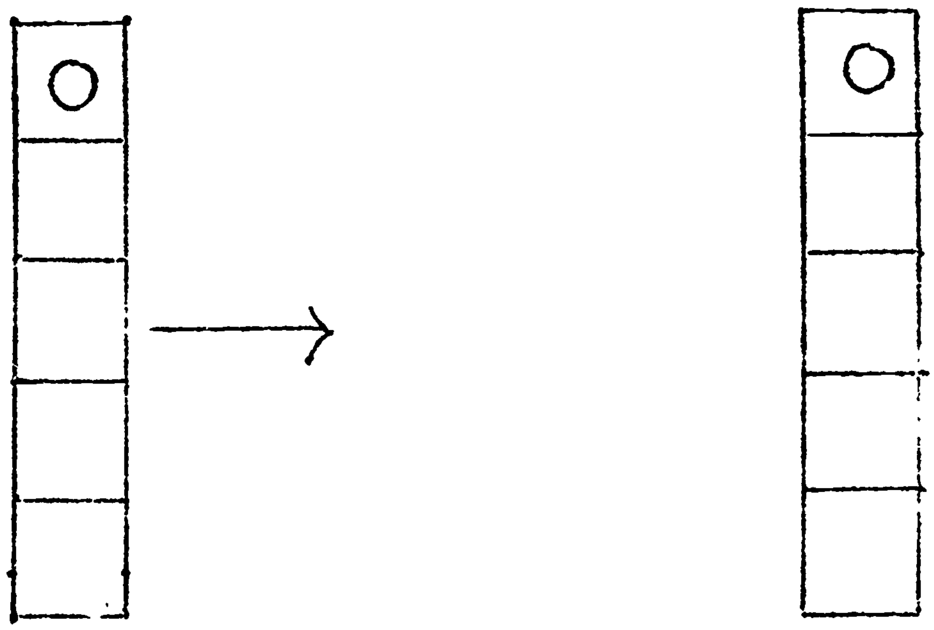


Fig. 32

Fig. 32 could be done in one motion by 1
in two motions by ½½, HH, VV, 11

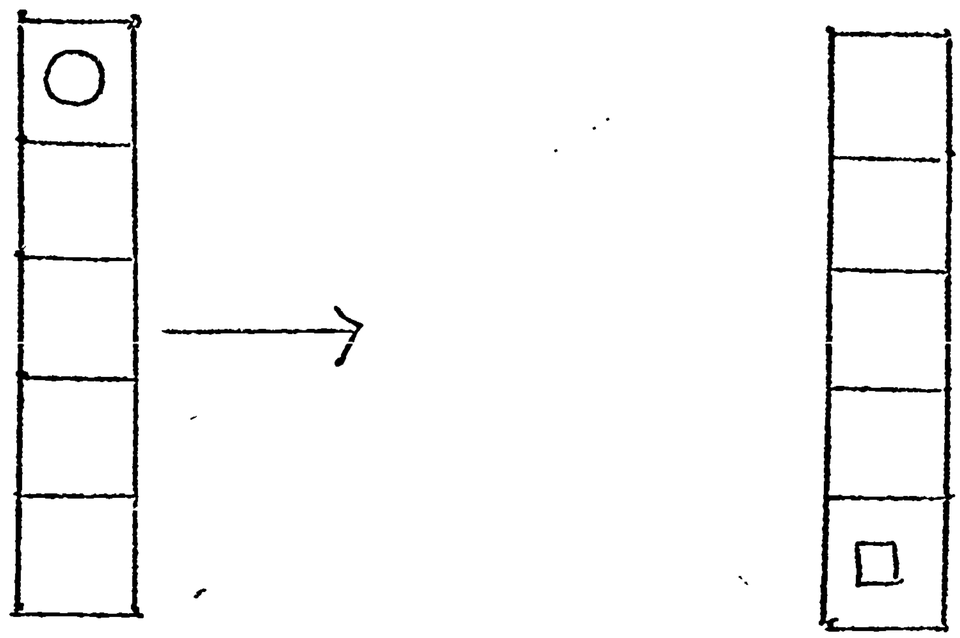


Fig. 33

Fig. 33 could be done in one motion by H

in two motions by 1H, H1, $V\frac{1}{2}$, $\frac{1}{2}V$.

Making a Table

From here it was an easy matter to go on to make the table below:

	1	$\frac{1}{2}$	H	V
1	1	$\frac{1}{2}$	H	V
$\frac{1}{2}$	$\frac{1}{2}$	1	V	H
H	H	V	1	$\frac{1}{2}$
V	V	H	$\frac{1}{2}$	1

The students could read the table more easily in the direction of finding out what HV is equivalent to, than how to get the effect of V starting with $\frac{1}{2}$. But only a ridiculously short time was available for this work. Neither of the words "commutativity" or "group" was introduced at this time. When a student was asked, however, whether he could erase part of the table and then put it back without having to think very hard, he realized that he could erase the top triangular entries since they matched the lower part.

The game was then played with the pattern in Fig. 34.

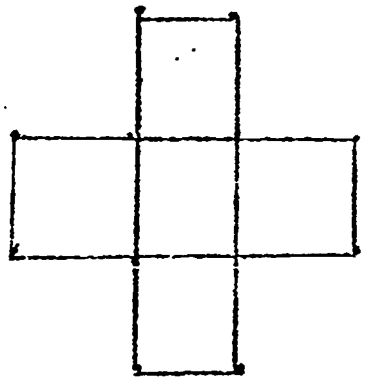


Fig. 34.

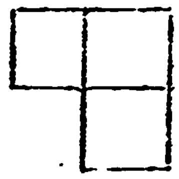
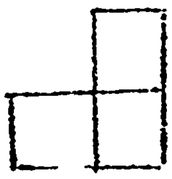
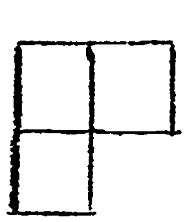
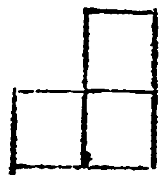
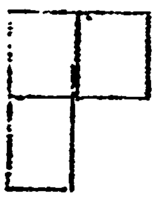
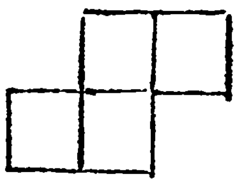
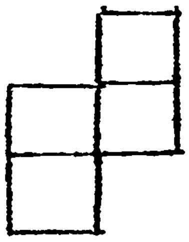
Again there was a careful discussion of how to mark it. 1/4, 1/2, 3/4, 1, V, H, R, L were the abreviations used for the eight motions leaving the pattern in the same position. Again two motions were performed while half the students had their eyes shut. The writer cannot emphasize enough the speed and accuracy with which the students produced the possible eight products for the various new positions. Again the table showing the products of two motions was made for this pattern. Because this was the last day of classes, the temptation to start the table with 1, ½, H, V was not resisted.

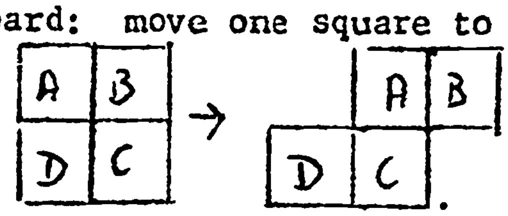
	1	1/2	H	V	1/4	3/4	R	L
1	1	1/2	H	V	1/4	3/4	R	L
1/2	1/2	1	V	H	3/4	1/4	L	R
H	H	V	1	1/2	L	R	3/4	1/4
V	V	H	1/2	1	R	L	1/4	3/4
1/4	1/4	3/4	R	L	1/2	1	V	H
3/4	3/4	1/4	L	R	1	1/2	H	V
R	R	L	1/4	3/4	H	V	1	1/2
L	L	R	3/4	1/4	V	H	1/2	1

The fact that the top 4 x 4 corner was the same as the previously obtained table was noted. This is the point at which, unfortunately, classes had to stop. It is interesting to note that the students who were interviewed for other purposes did not think they had been learning <u>mathematics</u> during this time.

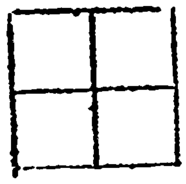
Work Carried Out With A First Grade Class

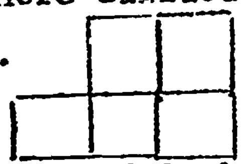
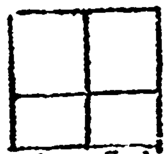
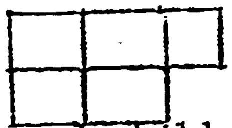
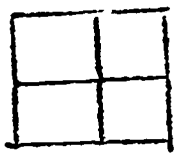
About twenty students who had just finished kindergarten took part in the work described below. Again they were a heterogeneous group. There were only nine sessions of about half an hour each. The description of the work done will be given in somewhat briefer form than above, since the work is of the same type as described there only on a more elementary level.

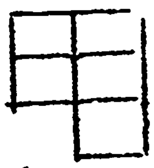
The work involved visualizing patterns formed by three, four and five squares, three or four equilateral triangles and some of the three dimensional figures built from these shapes. The children were each given three paper squares and asked to make different patterns - with whole edges touching. When they laid out  or  using their separate squares, they were given patterns of that shape. When  was suggested by various children as being a different pattern from , the children showed how one could turn or flip the pattern so that  fitted on . The same type of work was carried out with four squares. They learned quickly to recognize when a pattern was duplicated (in the sense described previously) each time actually moving the paper pattern to show that  and  for instance, could be covered by the same pattern.

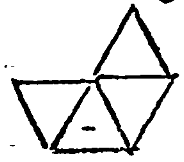
A game was played on a felt board: move one square to make a "different" pattern. For example: 

. The children were able to do this so accurately and quickly that it soon turned out to be too easy a game. They made parts of boxes from 3 and 4

squares and on experimenting found that  for instance was not useful for making part of a box. Some children combined various 4-patterns or asked for extra squares.

The five-squares patterns were obtained at the felt board - the students in turn coming to it to put up a new pattern. More difficulty was encountered here in avoiding making the same pattern.  was duplicated particularly often. After putting  on the felt board, the children realized they had no way out for the 5th square if they did not want to duplicate  and only one child thought of removing the 4th square. Several children then realized and some verbalized why there was no way out at that point, by enumerating all the possibilities for the position of the 5th square once  had been put on the felt board.

They built boxes without tops out of some of the shapes and found that   among others were not box-makers.

The same type of work was carried out with 3 and 4 equilateral triangles. Their increased facility here showed the effect of their previous experience with the squares. They found out quite quickly that there was only one 3-triangle pattern and explained well why this was so. They also found the three 4-triangle patterns quickly. Here the one most often duplicated was . Three dimensional models were built from all the patterns - pyramids and tetrahedrons were among more unconventional and imaginative shapes obtained.

Thus the children experimented with different arrangements of 3, 4, 5 triangles and squares which gave them experience with the symmetries of the shapes and patterns. It also gave them practice in making different arrangements of 3, 4, 5 objects. Taking turns to make different patterns at the felt board gave them practice in visualizing various shapes in different positions before trying to place them on the board.

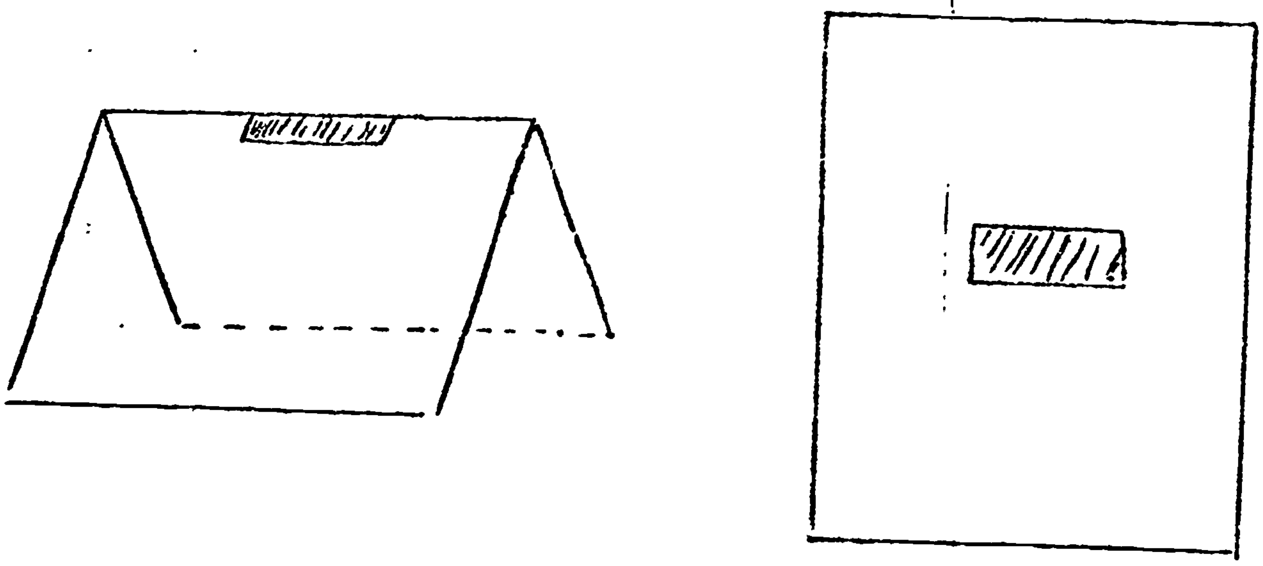
Throughout the children were enthusiastic and eager to make 2 dimensional patterns and 3 dimensional models. As with the sixth grade students, much more could have been done in extending the classes along lines suggested by the children's work, but time did not allow this.

The Mirror Cards[9]

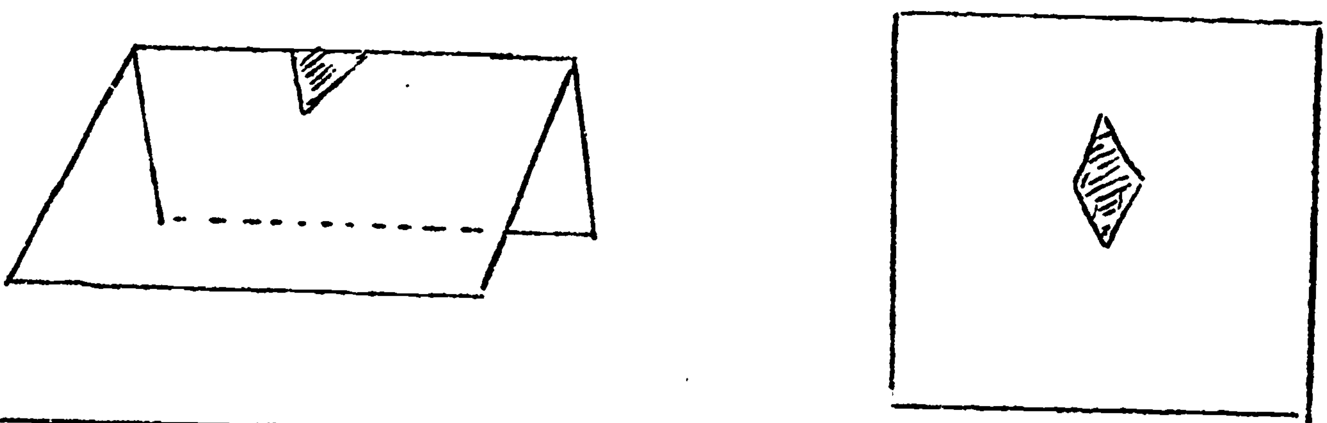
Why The Cards Were Invented

In the summer of 1963, a class of seventh graders were learning about the eclipse of the sun. They were students at a summer school held at the Peabody School in Cambridge, Massachusetts under the sponsorship of E.S.I.

For the purposes of their study they were making "pinholes" in sheets of paper. To make a rectangular "pinhole" in a sheet of paper with rather blunt scissors was not much of a problem. The students saw easily that folding over the piece of paper and then cutting a small rectangle out of it would do the trick. (See illustration below.)



Then they were asked to make a triangular pinhole. Without hesitation some of the children folded the paper and cut out a small triangular shape. They were surprised when they unfolded the paper to find that the hole was not a triangle. (See illustration below.)



9. This work was begun while the writer was working with the Elementary Science Study of Educational Services Incorporated, during the summer of 1963. I should like here to thank the members of this group and the group in optics of the previous year for their help and encouragement. I am especially grateful to Professor P. Morrison P. Singer, and L. Rasmussen.

This led the writer to investigate ways of giving the students more experience with the idea of bilateral symmetry. Possible games using paperfolding and cutting were experimented with.

The previous summer the optics group at E.S.S., of which the writer was a member, had worked with mirrors and semi-transparent mirrors, and some of these mirrors were still on hand. It was a short step from cutting paper patterns to using a mirror. The expected image of a pattern was drawn behind a semi-transparent (two-way) mirror and the mirror put down on the pattern to check. This procedure still seemed somewhat cumbersome and led, eventually to the development of the cards themselves.

The attached pamphlet on the Mirror Cards is a trial version of the teacher's guide which will be sent out with the cards. It is addressed to elementary school teachers without special training in mathematics and it does not, therefore, discuss the mathematics of symmetry. The writer is preparing an accompanying guide which will provide the formal mathematical background.

The cards have now been produced by E.S.I. and should be ready for distribution in time to accompany this paper. Since they are difficult to produce by hand in large numbers, they have not yet been tested on a broad scale. They were used by small groups of children at the Peabody School in the summer of 1963 and at the Morse School in the summer of 1964. The Cambridge School teachers who attended the summer school in 1964 used them for several sessions, and many wanted them for their students.

Two sets were introduced at the Estabrook School in Lexington, Massachusetts, this year, and a short report by Professor Lomon appears in the appendix to this paper. The writer has talked about them and introduced them to elementary school teachers or teachers-in-training at Wheelock College, Emmanuel College, and the Harvard Graduate School of Education. All seemed enthusiastic about the cards and very involved while working with them.

No formal evaluation has been undertaken. Production of the cards has been supported not only by E.S.I., but also by mathematicians and psychologists. A large number of requests for them has already been received. Further plans for more extensive and integrated use of the Mirror Cards are now being considered.

What Are The Mirror Cards

(It would be helpful at this point to read through the teacher's guide and try some of the sets.)

The Mirror Card Sets range from simple problems for the youngest children to more complex ones for older children and adults. Each card of a set has a colored design on it.

The basic problem is the same for all the sets. Can the player by using the mirror on one of the cards obtain from that design and its mirror image the design on another card. Some designs are possible to make, others are not. There is variation in the ways the rules are set up and this is stated in the guide provided with the cards. In addition, the instruction for each of the 14 sets of cards is printed on the lead card for each set, for those children who can read. The Mirror Cards have been used by children from 5-75 although one exceptional child of $3\frac{1}{2}$ was able to do quite a number of them.

Some Benefits of The Mirror Cards

A few of the advantages of the Mirror Cards for teaching are summarized here. The approach is non-verbal.[10] They are highly visual. They are free of mathematical notation.

Some other benefits are these: The material is of a highly playful nature; the children can work almost entirely on their own. Even children who cannot read the instructions soon learn how to play from a few children to whom the teacher introduces the cards. The children can check their own work without resorting to the authority of the teacher. They can see whether they are correct or not when they use the mirror.

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10. Even children with severe language difficulty can participate effectively and can excel in the use of the cards. At the Morse School one Chinese-speaking student who had great difficulty with English was absolutely fascinated with the cards and insisted on having her own set to take home. She excelled in working with the cards, and this was the first time during her classes that she had held her own.)

Not only do the children have a chance to check their own work but they also have an opportunity to predict and immediately amend their prediction if it is not correct. This seems to give them a great sense of power.

The children evidence pleasure in another way. Them seem to relish their control over their decisions whether to make a whole rabbit out of a half rabbit, whether to increase the distance between two patterns or not, whether to double the number of circles or not.

The children can work at their own rate - the contemplative child may linger a long time over a card even if he has done it - experimenting with the change in pattern as the mirror is moved. Harder sets are available for those who are ready and they can always make up their own.

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11. It is interesting that the elementary school teachers who have used the cards tend to have the immediate reaction of asking if they are right or not. They seem not to trust their judgement at all. The writer has always pointed out to them that they really do not need to ask, since they can check themselves, but this seems to be an idea foreign to them. No wonder the children don't learn to operate independently in their other work.

The Mathematics of the Mirror Cards

The students do not connect their work immediately with mathematics, and thus the student who may be jaded or otherwise badly motivated with respect to mathematics may make a fresh start.

Eventually the Mirror Cards can be integrated into a general mathematical context.[12]

Primarily the work with the cards concerns the physical effect of a mirror and the mathematical concept of reflection in a line as contrasted with rotations and translations.[13]

The theory of reflection or the laws of the mirror are not verbalized (necessarily) but the student will come away with a comparatively clear and complete set of operational concepts about what a mirror does to a pattern, and the relative positions of a pattern and its mirror image - i.e., the laws of reflection. The student will also find out what a mirror will not do - it will not push or turn the image with it as it is pushed or turned - that is, it cannot combine reflection with a translation (glide reflection) or reflect and rotate.

These ideas may be incorporated into the study of the mathematics of transformations later on, or into more formal study of the mirror.[14]

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12. This integration will be tried for the first time at the Estabrook School in the fall of 1965.

13. If three dimensional objects are used, then reflection in a plane is involved. As a matter of fact one of the kindergarten children using the cards decided to try to see what would happen when little wooden blocks were used instead of the cards. He investigated the mirror images of quite complicated three dimensional patterns.

14. Bernice Hoffman, author of a physics text to be published shortly by Addison Wesley, plans to use the Mirror Cards as soon as they are distributed, with her own ninth grade students at the Riverdale Country School for Girls to introduce them to the study of the mirror.

Why Is This Work Useful

Now that the work and material has been briefly described, one may ask again what is the value of doing this type of work with young children? An answer may conveniently be given in two parts, The first relating to general reasons, the second to specific mathematical ones. One may add that the next time children work with this material new questions will be raised by them, new approaches will be discovered. Above all an adequate amount of time needs to be available.

General Reasons

Broadly speaking this material will give children the following:

a) Experience in visualizing static two and three dimensional patterns.
b) Experience in visualizing the movements (such as rotations, reflections, folding and unfolding) of two and three dimensional figures.
c) Experience in carrying out these motions physically.
d) Experience in predicting the final position of a figure after a certain motion has been carried out.
e) Experience in checking their own predictions and answers by themselves and if necessary correcting or amending them.[15]
f) Experience in raising and investigating their own questions.[16]
g) Experience in using notation and other shorthand devices involving ideas with which they have had firsthand, concrete experience.
h) Experience in recognizing symmetry and in general, learning to be observant about figures.
i) Experience in dealing with congruent figures.

Mathematical Reasons

The concept of a transformation plays an important role throughout mathematics. Certain transformations play an especially important role in geometry. In different geometries those properties which are left invariant under a particular transformation are investigated. In plane and solid geometry those properties which are left invariant under the

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15. This is recommended in the Cambridge Report, p. 35.

16. This is suggested in the Cambridge Report, Appendix D, p. 80.

particular transformation called a rigid motion are studied. All of the transformations discussed in this paper except those which involved the folding and the unfolding of the patterns are rigid motions. In further work it might be possible to include non-rigid motions, by use of concave and convex mirrors or materials such as rubber sheets which would lend themselves to stretching and shrinking, and thus arrive at some topological ideas.

Since the idea of a rigid motion plays a leading role in the concept of congruence, it is important that the children gain firsthand experience with it before having to apply it. This work gives the children an intuitive feeling for rigid motions by the manipulations of the patterns before they formally meet the idea of congruent triangles.

This work gives the children some experience with many types of rigid motions - namely translations, rotations, and reflections. No attempt was made at this stage to show that <u>every</u> rigid motion can be obtained by a combination of these, although surely an intuitive feeling for this was already emerging.

Much of the work with the patterns gives the children practice with rotations, especially those of 90°, 180°, 270° and 360°.[17]

Much of the work involved reflections about a line. This work also gives the children practice in using and finding lines of symmetry. The Mirror Cards especially give the students firsthand intuitive non-verbal experience with a line of symmetry and its connection with a reflection about that line. When the children can predict whether a card can or cannot be done without their actually using the mirror, they will have learned that not only must the pattern have two congruent parts, one of which must match part of the pattern on the Mirror Card, but also that these parts must be symmetric with respect to a line. Thus they learn that congruency of two parts is a necessary but not a sufficient condition for bilateral symmetry. Because of the many cards that cannot be matched, the children learn to recognize non-symmetry thus strenghtening their understanding of symmetry.

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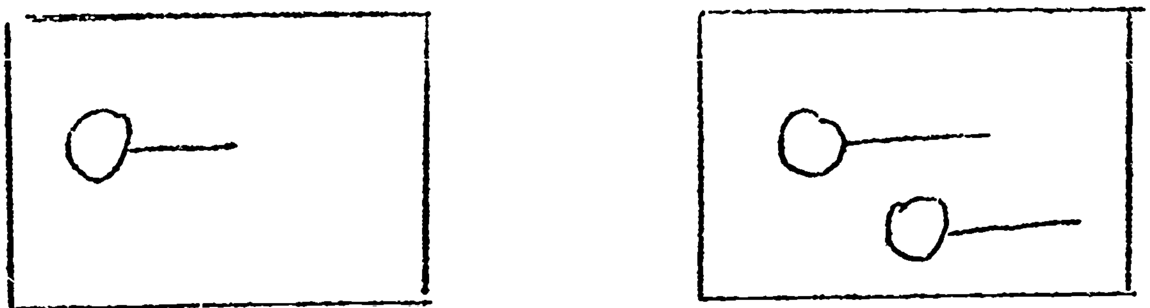
17. The fact that the rotations were carried out about a fixed point was not brought out.

The students did not specifically work with translations but did discover that the mirror does not perform a translation. The children often pushed the mirror in the hope that the image would move with it and thus that it would carry out the translation required to obtain figure b) from figure a)



a) b)

The children studied those motions that leave a figure invariant. They noticed that different figures can have different motions leaving them invariant. The question as to what a figure might look like if the motions leaving it invariant are known was not investigated. However, this part of the work led naturally to the basic idea of a group although of course no formal definition or terminology was used. The children worked, in fact, with a commutative group of order 4 and a non-commutative group of order 8 - all on an intuitive level.

If children are to understand the power of commutativity when they come to the study of algebra, they must have examples of non-commutativity to be able to appreciate it.

The writer feels that several of the fundamental theorems of group theory would emerge if this work were carried further. It must be stressed again that this, as the work carried out, would be done in an intuitive experimental way. In other words, one would not necessarily have to give formal definitions or use fancy terminology. For example, the student who realized that half the table on page 23 could be erased was perhaps on the way to understanding commutativity but he certainly did not know the word "commutative."

Perhaps it is in order to end this paper with a remark a senior in college recently made. "Oh, now I understand what an isomorphism is- Last semester while I took modern Algebra all I knew was the definition of it!" The type of work described in this paper gives the student opportunity to become familiar, by direct experience and experiment with important concepts before they have to be studied theoretically.

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