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The lack of tested models of the instructional process is the prime hindrance to significant management science in educational research. The nature of education, with long timelags between ultimate performance and present activity, makes the modeling most difficult. This paper reports on one of several efforts which have been made to develop quantitative models of the educational process. The model is developed for future inclusior 'o a school district budgeting and program-planning model. The history of the el's development and some of the problems encountered are discussed. Practic implications for further management science in educational research are alsc c awn. (Author/HW)

DEVELOPMENT OF QUANTITATIVE MODELS OF THE EDUCATIONAL PROCESS

by

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ABSTRACT

Development of Quantitative Models of the Educational Process

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The lack of tested models of the instructional process is the prime hindrance to significant management science in educational research. The nature of education, with long time lags between ultimate performance and present activity, makes the modeling most difficult. There are several efforts which have been made to develop quantitative models of the educational process. This paper reports on only one of them. The model is being developed for future inclusion into a school district budgeting and program planning model. The history of the model's development as well as some of the problems encountered are discussed. Practical implications for further management science in educational research are also drawn.

Introduction:

There is a missing link in the application of management science to problems of educational managers. This missing link is a set of tenable models which relate management controllables or decision variables to educational outcomes. The existence of the missing link causes a gap between what management science should do in educational research and what it is being done.

I feel that the largest contributions to be made by management scientists will come through working with top management of educational enterprise. We should be helping them control their systems more effectively. This implies that we should be formulating their decision problems so that a form of analysis can be applied. If we are to analyze the decision and control problems of top managers, much work has to be done on developing an understanding of how the controllable portions of the educational process relate to measureable objectives.

Just imagine for a moment that industrial operations were carried out without the notion of profit and that least cost operations are not always justified. How many of our standard management science applications would still hold? Very few, I believe. Perhaps the work in information systems would stand but little else. Formulations which would still work in this poorly defined situation would be quite minor in terms of their importance to a whole enterprise.

Without filling in the missing link the impact of management science on educational systems is going to be limited. In the meantime, managers of educational systems can get some indirect benefit from operations research. Application of management science in education can help in drawing better school attendence boundaries, in improving forecasts of population and enrollment or in building more effective planning systems. I believe, however, that these contributions will not satisfy those educators who really need much more fundamental help.

The reasons for the difference between what should be done and what is being done are many. Some of them are common to the practice of management science in any organization, and some are, I believe, unique to the area of educational management. Problems in carrying out management science in education range from lack of support in terms of well-qualified personnel and funds to lack of data and a lack of belief that the models developed are good models.

In the remainder of this paper I would like to review the development of a particular model of the educational process. With that case as a background we have a better basis on which to discuss some of the problems which hinder management science in educational research

A Case History of an Attempt at Quantitative Modeling of the Education Process

For several years a group at the University of Pennsylvania headed by Roger Sisson has been trying to use operations analysis in educational systems. Because of this experience we are enthusiastic about the possibilities of operations analysis (management science) in educational systems but we do not expect a revolution in the near future. I believe that we can identify many problems which are impeding management science in educational systems. Some of these problems are discussed later.

Background and the Starting Point....

Our interest in educational performance models started as a result of a simulation study which we did for the School District of Philadelphia. * We built and tested an aggregate simulator of the operating and financial side of the system. Then we ran experiments on the simulator to estimate the sensitivity of district wide costs and other factors to changes in operating policy and capital budget parameters. The result of this effort was a series of presentations to planning and financial personnel in the system.

The reaction to the model and the studies which we ran was highly favorable at first. But after management became familiar with the model they began to ask questions which went completely beyond it. Generally the questions were about the quality or educational benefits which would result from alternative operating and capital policies. These questions

*Roger L. Sisson: "Some Results of a Simulation of An Urban School District," University of Pennsylvania, Management Science Center Report #042467, March, 1967.

were the starting point in our search for the missing link.

Roger Sisson has discussed modeling of the educational process. ** He describes the research and modeling process in terms of three levels: the experimental or "no theory" approach, the "thermodynamic" approach and the "real theory" or basic approach. The "no theory" approach consists of experimentation with prototype systems. "Thermodynamic" modeling consists of stating hypothesized relationships between inputs and outputs and evaluating parameters by regression. The real theory approach consists of building models based upon concepts which apparently explain a wide class of phenomena. The successive levels also involve progressively higher levels of abstraction.

We began our attempt to build a quantitative model of the education process at the "thermodynamic" level. (The name "thermodynamic" really came after we had the model built.) The educational research literature abounds in empirical studies which are the basis for proposing relationships of a "thermodynamic" model. Enough has been learned from these studies to enable a level of abstraction and modeling above that of plain experimentation.

The starting point was a definition of education as a communication process. Each of the major controllable and environmental variables which were hypothesized to be relevant to the process were included in a non-linear mathematical model. The final relationship involved changes of

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** Roger L. Sisson: "Can We Model the Education Process?" Address before the National Conference on Operations Analysis in Education, U. S. Office of Education, Washington, D. C., November, 1967. To be published in Socio-Economic Planning Sciences.

composite achievement test scores as a function of teacher time per student, books per student, space per student, and level of parental education of students. We refer to this as the SD 1. 5 model.

Tests of the model are hard to obtain because there is little variation in the variables from school to school and we were working with aggregate data on a school by school basis. In spite of these difficulties this first model was generally a better predictor of achievement test scores changes than linear regressions on the same data.

Toward a Theoretical Model....

The next step in the effort resulted from building a very detailed simulator of school district operations from the educational program point of view. * The achievement prediction process is to be contained explicitly in this simulation. We wanted a much richer model of the educational process because of this simulator's detailed representation of educational program activity.

A second feature of this model is that we have tried to base it as much as possible on the available theories of learning and intellectual development. This model which we call "ACHIEV" represents a step away from the "thermodynamic" level of modeling toward the "basic level." It is not yet an entirely theoretical model, however. ACHIEV incorporates some elements of theory but also relies on hypothesized functional

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*Miguel Szekely, Martin Stankard and Roger Sisson: "Design of a Planning Model for an Urban School District." Socio-Economic Planning Sciences, Vol. 1, 1968, Pergamon Press (in preparation).
A program is the creation of an instructional environment for a group of students.

relationship among some variables.

The ACHIEV model is still tentative. We have not yet programmed it for the computer. The educational system simulator which it complements (SDTwo) is programmed and running. The ultimate goal is to use the completed simulator (with performance evaluation included) in program planning for large school districts or collections of smaller districts.

Requirements for the New Model ....

The present form of the model provides

a distribution of achievement for each group of students whose program mix has been simulated during a simulated year. The program mix is the group of educational programs which the student actually experienced that year.

This is based upon:

. The achievement distribution for that group of students at the end of the previous simulated school year.

. Information about the student group's grade level, socio-economic background, home and school areas (in the event of voluntary transfers or busing) and other students data from the simulator.

. Information from the simulator about programs given to the students and the extent to which each program was fully implemented; (pctapp) a variable from the simulator which equals the percentage of required funds applied to the target group in each program.
  - teachers required vs. teachers available
  - space required vs. space available.

In the past, the "measure of performance" has been the change of score on an achievement type test over a period of time. * We have used this mainly by default. There does not appear to be any practical alternative to

*Martin Stankard and Roger L. Sisson, "On the Modeling of Relationships between Performance and Resource Management in an Urban School District," presented at the TIMS-ORSA Joint Meeting, San Francisco, May 1968.

this at present. Achievement test scores do not provide any information on students' attitudes or on the "affective domain" of performance. Many educators whom we have contacted claim that the affective or attitudinal performance considerations are the more important performance areas.

The approach to performance prediction which we are proposing here does not include the affective domain. We are not specifying a particular performance instrument as being "the" performance measure. Instead we have a general achievement index in mind, something on the order of a grade point average. The desire is to gauge how close a particular simulated policy (group of programs) comes to allowing each group of students to achieve their best.

The SDTwo model provides a much richer background for making these performance evaluations than did our first Simulator SDOne or SD 1. 5. Continuing with the SD 1. 5 approach, we view education as a communication process. Now the simulation of individual educational programs for each group of students provides more information on which to base a prediction of expected student performance. To use the information available we must (1) describe each program in terms of the communication which it is expected to produce, (2) describe how the communication is absorbed by the students and (3) relate this finally to an achievement or performance index.

An Outline of the ACHIEV Model ....

The best way to describe the ACHIEV Model is to outline the steps necessary for applying it. A more detailed discussion is included in Appendix I. The model is referred to a specified group of students. The steps in modeling the process of education for this group are:

1) The characteristics of this group of students are defined in terms of variables relevant to estimating the group's probable interaction with the educational programs applied. This list includes the group's age, socio-economic background and attendance behavior, etc.

2) Define all programs to be applied to the group. Ideally the objective of each program is specified first. Then the process of the program is defined. (How much student time is allotted to the program and how much of that allotted is spent in lectures, films, class discussion, with tutors, etc?)

3) Based upon the characteristics of the students and the proposed process of each program a breakdown of program activity is estimated. This consists of estimating the proportion of program time spent in each of several states.

4) The proportion of program time in each state is weighted and then summed across states within a program. The resulting index for each program is aggregated across programs.

5) The index of participation in the educational process is then aggregated over time by a smoothing function. The final index summarizes the program exposure of the group of students. We call this index "Richness."

6) We assume that the achievement index for a student in the group is correlated from year to year. We also assume that two consecutive year's achievement index will be jointly distributed with the normal probability distribution function. The information about the educational experience of the students in this group (the Richness variable) is then used to modify the parameters of this distribution.

The approach to modeling the educational process represented by the ACHIEV Model is only an indication of the direction in which quantitative modeling of the educational process can develop.

The model as it stands now is primarily useful as a generator of hypotheses and as a guide to the types of data which should be gathered for validating models of the educational process. At its present stage of development it is hard to class the model as quantitative. Until all of the data required is available I suppose that we should refer to this type of model as "semi-quantitative."

Problems in Quantitative Modeling of the Education Process

The major problems which we have encountered in trying to model the education process may be grouped into three classes:

1) problems in the development of theory and model structure

2) difficulty in validating any models produced and

3) problems in implementing models to improve system performance.

Problems encountered in structuring models of the educational process generally depend on finding a definition of the process which will be acceptable to the ultimate user of a model. It becomes apparent after you try to model the education process that the business of education is not very well defined. One manager may accept a definition while his colleague will not accept it at all. In part this is due to a lack of operationally defined goals and objectives and in part it is due to the inherent complexity of the educational process.

The second area where trouble lurks is in validating models. The principal problem here is lack of data. These first two problem areas are discussed more fully below.

Finally, problems in implementing models are encountered. These are mainly the product of failures because of the first two trouble spots mentioned. Decision-makers commonly do not believe in the models either because they do not conform to some definition of the education process or because they are not fully validated.

1) Development Problems ....

a. Lack of Operational Goals and Objectives ....

Some educational managers have not decided "What business they are in." Education then becomes many things to many people. If someone is bold enough to state a set of operational criteria for an educational unit, for example, a school district, this is the signal for a good deal of criticism and controversy.

The problem of setting up performance measures gets harder to handle as we look at larger and larger segments of an educational system. It is one problem to evaluate the performance of the process at the classroom level, and a more complicated problem to do so adequately at the school or district level. The quantities which we measure on one student may not properly be aggregated across groups. The summarization as we move from individual to class to school may actually average out the information on performance of the process.

b. The Complexity of the Educational Process ....

Part of the problem arises from the nature of the process of education. First, public education is dealing with a special subpopulation, the children. The intellectual capabilities of children and the ways in which these capabilities change are not really understood. Guilford, * for example, has gathered evidence which suggests that there may be over 120 distinguishable types of intellectual capability. When we add to this a time dimension we have a very complicated situation. Some capabilities may be best influenced at different times in an individual's growth.

Formal public education for a student usually lasts for about twelve years. Except for the continuity of parental influence, the individual student is not exposed to a single twelve year process but to a sequence of twelve or more one year processes. Each of these is under the tactical control of a different manager (the teachers). The management of a student's education as a single twelve year trajectory is not common. Actually the educational process is often managed as though there were two educational processes, "primary education" and

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* Guilford, J. P., "Intelligence Has Three Facets;" Science, May 10, 1968.

"secondary education", or even as thought there were twelve, "first grade education, second grade education, . . . " and so on. Many real dynamic factors are ignored in this way and while the management job is feasible, it is probably not nearly optimal.

2) Validation Problems

a. Lack of Data . . . .

One of the more frustrating problems in quantitative modeling of the educational problems is the lack of good data on process variables together with related data on input and output. Process data such as text utilization, teaching time breakdowns, utilization of various audio-visual devices, amounts of student-teacher and student-student interaction, etc., must be used in order to test and revise models of the process. As data becomes available I believe that we can continuously improve our ability to explain output (however it is described) given measurements on input.

It is important to develop sources of data which include not just input or process, or output variable, but all three. Often school districts maintain elaborate records on output (as measured by achievement tests) but fail to record the process conditions which presumably account for the changes in achievement scores. In other cases, input is monitored but it cannot be related to output because process data is not collected.

Judging from their data sources educational managers in the past have not appreciated the value of good information in decision making. They have not developed the types of data collection and reporting systems which are usually found in larger industrial firms. It is doubtful that this situation will change much in the near future.

We can draw a useful analogy here between economics and education. Before the advent of macro-economics based Keynesian theory the data gathered on economic processes were nearly hopeless as explicators of economic phenomena. The theories of macro-economics gave Simon Kuznets the basis on which to establish the national income accounting system. This data base has greatly aided the testing and development of models of economic processes at work.

Perhaps in the educational processes we need a similar rationale for deciding on relevant variables. We need some macro-theory of education upon which we could build a data base of educational information. Perhaps not on a national level, however.

Insufficient Resources Devoted to Educational Research ....

The continuing lack of stable and substantial support for all educational research is one of the largest barriers to significant management science in educational research. The support must be in terms of qualified personnel, and adequate and stable financial support for extended periods of time. Research appears to be one of the few areas of education where we are likely to find economies of scale. It appears that present educational research support and talent is so diversified that the realization of scale economies is not very likely.

A Concluding Remark

Recently I have been working with Computer Assisted Instruction. I feel that this medium can be a great help in the job of developing the types of models of the educational process which we need. The data gathered as a by-product of C.A.I. is very useful in modeling, even though it does not cover the entire process.

I believe that C.A.I. can be used much more effectively as a research tool if closer attention is paid to the need for research data. I propose a longitudinal application of C.A.I.

Briefly, this research project would involve installing suitably programmed C.A.I. in grades 1, 2, 3, 5, 7, 9, 11, and 12. This would be done in a sample of schools or a whole school district and maintained for a period of at least twelve years. The project would be conducted by a consortium consisting of research representatives of universities, the school district involved, the computer hardware and software suppliers, and the research sponsor.

The essential feature of this proposal is that detailed data would be gathered on a population of student over their entire educational history. This would provide a base for research inquiry which is unrivaled. Careful longitudinal studies could then be conducted.

A less sophisticated proposal, but possibly just as useful in that school district management and research teams should begin to broaden their disciplinary backgrounds. I advocate a wider use of industrial engineers and other trained investigators in studying classroom interactions. There is entirely too little understanding of how the process of education actually is carried on at the classroom level. Much of the present understanding is anecdotal in nature or relies on memories of personal experiences in years gone by.

Appendix I

In this appendix we discuss the structure of the ACHIEV model in more detail. Mr. Charles Goldman of the University of Pennsylvania has participated in the development of the model to this point. We are still working to refine the model and expect to have it programmed for experimentation and ultimately for validation.

The simulator of which this model will ultimately be a part uses educational programs as its basic package of education process. As a result the effect of education on the performance of a student group is modeled on a program by program basis. In proposing this model we have considered the problems of dependence among programs in their effects on achievement and the question of the validity of aggregating certain quantities across programs. We do not have satisfactory answers to these particular problems at this time.

The discussion of this model begins with the relevance of programs to performance. The representation of achievement as a function of accumulated educational process is proposed and finally the bivariate normal model of year to year achievement is suggested.

Programs and Their Relationship to Achievement ....

A program is the creation of an instructional environment for a group of students. Within the environment the students may experience stimuli designed to instruct or motivate them. The organized presentation of stimuli and the process of providing students with a reservoir of experiences is the basis for modeling their expected educational performance.

Within a particular environment certain responses are permitted or required of students. The responses may be spontaneous answers in a classroom putting answers on a test paper, being required to answer questions by a teacher, etc. The question of the quality of the responses (whether right or wrong, good or bad, etc., is neglected in this note: we are also neglecting the qualitative aspect of the stimuli). The only possible justification for this omission is that we are proposing a first approach. The main explanatory variables upon which we are trying to base performance forecasts are levels of stimulus and response encountered by students as a result of a period of education. The underlying premise is that the stimuli and responses are indicative of the amount of communication in which the students participate.

The performance of a simulated group of students will depend not only upon the communication which goes on in a particular simulated school year, but also it will depend upon their past educational experiences. Before beginning to represent what we have said up to now we will sketch a procedure for assessing the relationship between a program and performance.

Steps to Relate Programs to Performance ....

An idealized procedure for relating programs to performance would proceed as follows:

1. Identify and specify the target group of students. This consists of stating their socio-economic backgrounds, past achievement grade and age. The rest of the procedure is referred strictly to this group of students.

2. Specify the ideal educational process involved in this program. Beside resource usage information such as the student teacher ratio, use of audio-visual devices, and field trips and special lectures the learning activity of the program should be specified. Based upon the ideal process the program activity should be broken

down two ways:

a) teacher controlled

b) student controlled

Activity breakdown:

3. The breakdown of program activity is a judgemental process. It would have to be done by an experienced teacher who is familiar with the target group of students. This expert teacher would estimate the proportion of the time that the target group of students spends in state i under ideal process conditions (2 above).

ACTIVITY BREAKDOWN

STUDENT CONTROLLED

		HIGH response	LOW response
Teacher Controlled	HIGH stimulus	p_1 (i=1) e. g. small group tutoring computerized instruction	p_2 (i=2) e. g. Film, Good Lecture, Good educational TV
	LOW stimulus	p_3 (i=3) e. g. testing, quiz, Lab experiment	p_4 (i=4) e. g. Recess unsupervised class

4. Estimate a weight or "value" associated with each of the proportions identified in the preceding step. We propose that the weight be estimated from answers to the following questions. "Given that the target group is in one of the four states, what is the proportion of the students in the target group who are <u>inactive</u>?" Call this proportion π_i, i=1, 2, 3, 4.

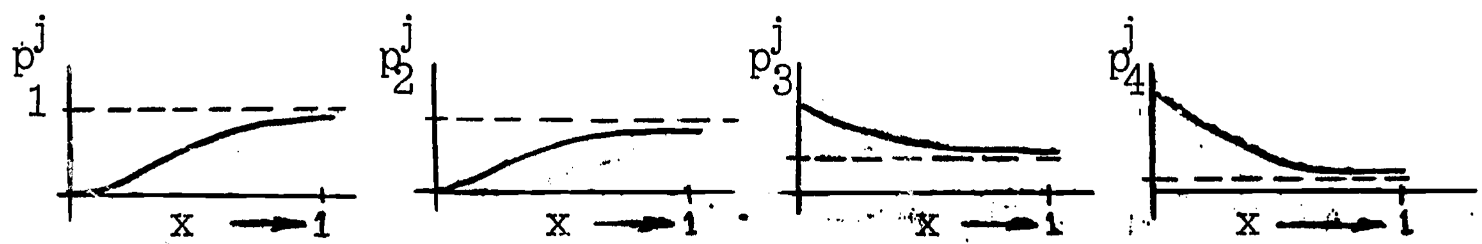
5. Determine the proportion of the target group's time spent in each

program, Φ^j. Here j = 1,2,..., NPROG (number of programs). As a convention the superscript j will always indicate that the variable is referred to program j.

6. For the target group of students compute their "Δ Richness"

$$\Delta \text{ Richness} = \sum_{j=1}^{NPROG} \left(\sum_{i=1}^{4} p_i^j \left[1 - \pi_i^j \right] \Phi^j \right)$$

7. The Δ Richness calculated in this way represents the impact which the programs have on the students' experience if all programs are implemented according to their ideal specification. Because of limited resources this is unlikely. The effect of insufficient staffing or funding in a program will be to change the proportions of time which the target group spends in each of the activity classes. Let x be the % of staffing requirements actually met for a program; program j.



The graphs show an hypothetical example of how these p_i^j might change with the proportion of staffing requirements met in program j. Another independent variable (pctapp)* might be included. We specify the p_i for each program as a function of staffing and pctapp for that program

$$p_i^j \text{ (staffing, pctapp)}; \quad \sum_i p_i^j (\cdot, \text{ pctapp}) = 1.0$$

*pctapp is the percentage: (funds actually spent in the program) ÷ (funds required to carry out the program to ideal specifications)

So Δ Richness becomes

$$\Delta \text{Richness} = \sum_{j=1}^{NPROG} \Phi^j \left\{ \sum_{i=1}^{4} p_i^j (\text{staffing, pctapp}) \cdot (1 - \pi_i^j) \right\}$$

The Nature of Achievement over time

In order to represent the dependence of present performance for the target group on past performance we absorb ΔRichness, computed for a target group at the end of a simulated year, into a quantity R, total richness. At year n let

$$R(n) = \alpha \cdot R(n-1) + \beta \cdot \Delta \text{Richness } (n)$$

where α and β are parameters for each target group. (They may be time dependent within a target group)

Let the distribution of achievement for the target group in question be normally distributed with parameters $\mu_{(n-1)}$, $\sigma^2_{(n-1)}$

These parameters are computed from the past achievement distribution for that group. For the year in question (year n) the corresponding parameters are μ_n, σ^2_n

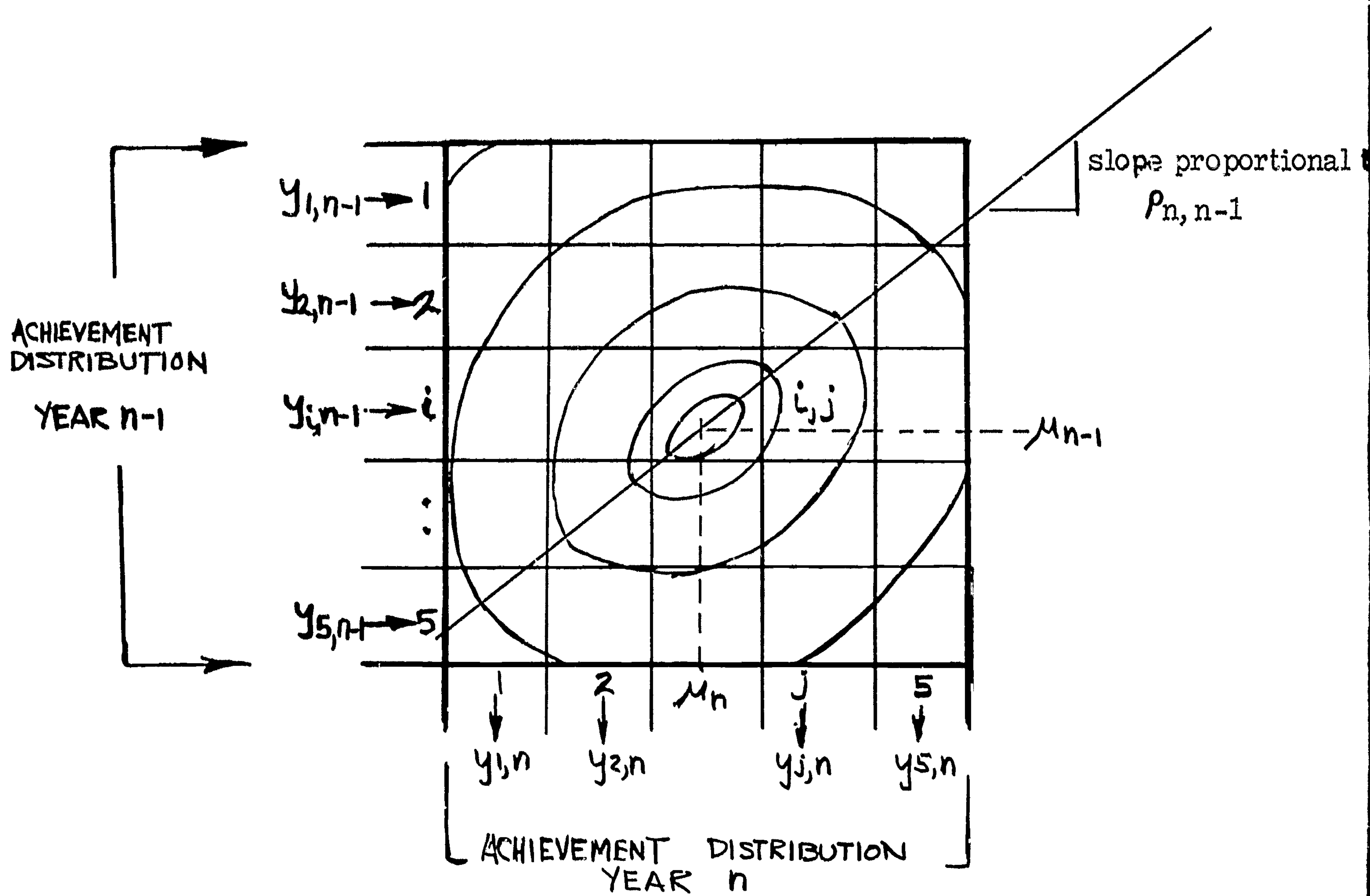
Let $$\frac{\sigma^2_n}{\sigma^2_{n-1}} = f_1 \text{ [\# of different programs, } R(n) \text{]}$$

$$\mu_n - \mu_{(n-1)} = C_o + (R_{(n)}) \cdot f_2 (\text{Space / Stu.}) \cdot f_3 (\text{Socio. Econ.})$$

$$\sigma_{n,n-1} = \rho_{n,n-1} \quad \sigma_n \qquad \sigma_{(n-1)}$$

$\rho_{n,n-1}$ is a measure of the "coupling" between an individual student's relative position in the achievement distribution for year n-1, and year n.

$$\rho_{n,n-1} = f_4 (\text{Socio-Econ. background, } R_{(n)})$$



Let $\underset{\sim}{\mu} = \begin{bmatrix} \mu_n \\ \mu_{n-1} \end{bmatrix}$ and $\underset{\sim}{\Sigma} = \begin{bmatrix} \sigma_n^2 & \sigma_{n,n-1} \\ \sigma_{n-1,n} & \sigma_{(n-1)}^2 \end{bmatrix}$

Then for year n,

$$p_{ij} = \underset{\boxed{i,j}}{\int\!\!\int} N(z_n \, z_{n-1}) \, dz_n \, dz_{n-1} = p_{ij}(\underset{\sim}{\mu}, \underset{\sim}{\Sigma}) \quad i, j = 1, \ldots, 5$$

and $y_{i,n} = \sum_j p_{ij} \, y_{j,n-1}$ $\quad i = 1, \ldots, 5$

$\quad j = 1, \ldots, 5$

Where $\boxed{i, j}$ indicates that the integration is to be taken over the range of the two variables (Z_n, Z_{n-1}) given by the i, jth cell.

Notice that it is not possible to interpret the p_{ij} as Markov transition probabilities. The bivariate normal formulation, however, has a terrific advantage over the Markov Model in as much as it requires that many fewer parameters be estimated. The hypothesis of normality is also testable.

We can compute Markov type transition probabilities, however, if the parameter $\rho_{n,n-1}$ does not change for n=1, 2, 3... etc. If we let r_{ij} be the probability that a randomly selected student is observed in state i at year n-1 and state j in year n, then

$$r_{ij} = P[j \mid i] = P[i, j] / P[i]$$

or

$$r_{ij} = \frac{\iint_{\boxed{ij}} N(z_n, z_{n-1})\, dz_n\, dz_{n-1}}{\sum_j \int_{\boxed{i,j}} N(z_n, z_{n-1})\, dz_n}$$

To describe the Markov process we also need the probability distribution of initial states of the system, namely

$$\left[\left(\sum_j \int_{\boxed{1,j}} N\, dz_n\, dz_{n-1}\right), \left(\sum_j \int_{\boxed{2,j}} N\, dz_n\, dz_{n-1}\right), \ldots, \left(\sum_j \int_{\boxed{5,j}} N\, dz_n\, dz_{n-1}\right) \right]$$

Note that the model is non-Markovian when we allow ρ to change from year to year.*

The crucial question about the model is whether or not the functions $f_1(\cdot), \ldots, f_4(\cdot)$ can be specified. Without really detailed data on educational process it will be impossible to establish these relationships with real confidence. We propose to specify these functional forms by subjective means where necessary.

*William Feller, Introduction to Probability Theory and its Applications, Vol. II, Chap. III, John Wiley and Sons, New York 1966.