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ON THE CONSTRUCTION OF LATIN SQUARES COUNTERBALANCED FOR IMMEDIATE SEQUENTIAL EFFECTS.

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THIS REPORT IS ONE OF A SERIES DESCRIBING NEW DEVELOPMENTS IN THE AREA OF RESEARCH METHODOLOGY. IT DEALS WITH LATIN SQUARES AS A CONTROL FOR PROGRESSIVE AND ADJACENCY EFFECTS IN EXPERIMENTAL DESIGNS. THE HISTORY OF LATIN SQUARES IS ALSO REVIEWED, AND SEVERAL ALGORITHMS FOR THE CONSTRUCTION OF LATIN AND GRECO-LATIN SQUARES ARE PROPOSED. THE REPORT IS DISCUSSED IN TWO PARTS--(1) ORTHOGONAL SQUARES AND (2) SEQUENTIAL COUNTERBALANCING. THE RESULTS ARE OF PARTICULAR APPLICATION TO RATING STUDIES AND TO DESIGNS REQUIRING THE "ROTATION" OF TEACHERS IN CLASSROOM RESEARCH. (RS)

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WISCONSIN RESEARCH AND DEVELOPMENT

CENTER FOR COGNITIVE LEARNING

RESEARCH
COGNITIVE LEARNING
DEVELOPMENT

Technical Report No. 25

ON THE CONSTRUCTION OF LATIN SQUARES COUNTERBALANCED FOR IMMEDIATE SEQUENTIAL EFFECTS

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Based on a master's thesis under the direction of

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2	9	4
7	5	3
6	1	8

--A magic square,
symbolic of the planet Saturn,
found in a medieval Arabic MS.
Cited in (16)

PREFACE

This Technical Report is based on the master's thesis of Tom R. Houston. Members of the thesis committee were Julian C. Stanley, Chairman; Frank B. Baker; Henry B. Mann; and J. Marshall Osborn.

Extending knowledge about, and improving educational practices related to, cognitive learning in children and youth is the primary goal of the Wisconsin R & D Center for Cognitive Learning. The Laboratory of Experimental Design, part of the technical section of the R & D Center, provides valuable assistance to project directors in the design of experiments and also in the analysis of data. Further, the staff of the LED are charged with extending knowledge about experimental designs, scaling procedures, data analysis and the like.

This Technical Report is the fifth in a series describing new developments in the methodological area. It deals with Latin squares as a control for progressive and adjancy effects in experimental designs. The history of Latin squares is reviewed, and several algorithms for the construction of Latin and Greco-Latin squares are proposed. Results are of particular application to rating studies, and to designs requiring the "rotation" of teachers in classroom research.

Herbert J. Klausmeier
Co-Director for Research

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INTRODUCTION

Latin squares, variously called "magic squares" and "Eulerian squares," are of considerable antiquity. The epigraph of this paper gives an example from astrology that may go back to Hellenistic times, and the extension of this use into ritual magic is well known; e.g., in Marlowe's (10) play The Tragedy of Doctor Faustus (1604), reference is made to

> ...Jehovah's name
> Forward and backward anagrammatized (3.9,10)

probably alluding to a Latin square, k = 4, constructed by (19.2) below, whose types were the Tetragrammaton. It is reported (10) that this square summons the Devil, for which reason it will not be exhibited here.

In recent times Latin squares have been found to have further practical application in experimental design. Fisher (6) places the use of the Knut Vik square (a permutation by rows of the square described in (6.2) below) in agricultural experimentation as early as 1872. It is convenient to categorize applications of Latin squares in experimental design under three headings:

I. To increase the number of experimental units. A field subjected to k treatments can be divided by restricted randomization into k^2 plots by a Latin square design, giving a k-fold increase in experimental units (and distributes soil variability) for the same acreage.

II. For fractional factorial designs. Latin and hyper-Latin squares are sometimes used to sample possible treatment combinations within k^{-n} fractional designs. Since such designs involve k^n aliases for every effect by the confounding of interactions, their use is questionable unless there is assurance, rarely available in the behavioral sciences, that the interactions are close to zero.

III. For counterbalancing repeated measures designs. When each experimental subject receives k successive treatments, it is often feasible and desirable to equalize progressive effects (warm-up, fatigue, practice, etc.) by Latin squares. These insure that every treatment occurs, on the average, in the $\frac{1}{2}(k+1)$th ordinal position, and in every position with equal frequency.

Note that the last application is a case of the second, where ordinal position is considered as a non-interactive factor with k levels. A difficulty can arise in such cases, as the following hypothetical example illustrates:

Suppose that a study of diabolism were to be conducted, to determine the relative effectiveness of four methods of exorcism: Bell (T_1), Book (T_2), Candle (T_3), and Team Exorcism (T_4). The dependent variable could be the response to the question "Do you renounce the Devil?", dichotomously scored (affirmative = 1, no response or negative = 0). Suppose further that the investigator wished to apply all treatments to each S (Ss being 4N demoniacs), employing a Latin square design of the form of (9.1) below, so that non-ultimate T_1 was always followed by T_2, T_2 by T_3, T_3 by T_4, and T_4 by T_1. Then if T_2 elicited a significantly higher mean response than T_1, there would still remain several plausible alternative hypotheses to the interpretation that T_2 was more potent than T_1. Perhaps T_4, which preceded T_1 in 3N cases, left Ss unable to respond until after T_2; or perhaps T_2 is comparatively ineffective unless an immediately prior T_1 has rendered Ss susceptible to its effects.

Such "residual effects" are a general problem in repeated measures designs which an injudicious choice of Latin squares can aggravate, as in the above example. Randomization (i. e., random linear permutations by row and column of a randomly selected square) is often suggested as a remedy, but in dealing with small numbers there is considerable risk to the investigator of substituting bad luck for bad judgment and obtaining a design with unfortunately biased residuals contaminating main effects. Complete sets of permutations avoid this, but there may not be enough Ss for this to be possible.

As Grant (8) and others have observed, such effect cannot be entirely controlled by Latin squares. It is often worthwhile, however, to control for the most potent of these, the immediate sequential effects, by constructing Latin squares in such a manner that every treatment is immediately followed by every other an equal number of times. Existing procedures for accomplishing this will be reviewed in this paper, and a general solution offered for the previously unsolved case of k odd, where k is the order of the Latin square. This solution also provides for the counterbalancing within (but not across) alphabets of Greco-Latin squares for k odd. It should be noted that the given procedure (23.1) counterbalances only within rows; the extension to within-column counterbalancing is trivial, but seems to be without application in the behavioral sciences. In the diabolism example given above, incidentally, Faustus' square (19.1) will counterbalance immediate sequential effects.

PART I
ORTHOGONAL SQUARES

1. Throughout this discussion, k will be the symbol for the order of a Latin square (defined below). For some natural number c, $k' = 2c + 1$, and $k'' = 2c$. The notation e_{ij} (sometimes with a comma in the subscript) will represent a cell in rectangular array E, at the intersection of the ith row and the jth column, i and j ranging from 0 to (k-1). E_{ij} represents the type occupying cell e_{ij}, and E_n the type in $e_{n,0}$. Thus $e_{2,5} = E_7$ signifies that the same type is in cell $e_{2,5}$ as $e_{7,0}$.

Definition 1: Rectangular array E is a Latin square if:

(1.1) $E_{iq} \neq E_{ij} \neq E_{pj}$ for all $i \neq p$, $j \neq q$.

(1.2) There are $k = i(max)+1 = j(max)+1$ types.

(1.3) Exactly one type occupies each cell.

The number k is called the order of square E. An example of a Latin square of order k = 5 is:

$$(1.4) \quad \begin{matrix} 1 & 3 & 4 & 0 & 2 \\ 4 & 2 & 0 & 3 & 1 \\ 2 & 1 & 3 & 4 & 0 \\ 0 & 4 & 2 & 1 & 3 \\ 3 & 0 & 1 & 2 & 4 \end{matrix} \quad = (1.4)E.$$

2. A Latin square will be called a standard form (SFL) if the types in its first row and column are in natural order. Any Latin square can be put into the form of a SFL by pre- and post-multiplication with appropriate permutation matrices. For example, the square given above may be standardized by the operation:

$$(2.1) \quad \begin{matrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{matrix} \cdot (1.4)E \cdot \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{matrix} = \begin{matrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 4 & 2 & 3 \\ 2 & 4 & 3 & 0 & 1 \\ 3 & 2 & 1 & 4 & 0 \\ 4 & 3 & 0 & 1 & 2. \end{matrix}$$

Transformations such as the above, changing the order of the rows and/or columns of a Latin square, will be called linear permutations. Call all the SFLs obtainable from a given Latin square a linear set (a term not in general use). A number of higher-order groupings are available for the classification of Latin squares [see Norton (11), Fisher and Yates (7)], including transformation sets, species, families, domains, only the first of which will be discussed here. An interesting feature of all classificatory systems, however, is that as of the present, the cardinality of the set of all SFLs, linear sets, species, etc. is unknown for any $k > 7$.

3. Two Latin squares of the same order, R and G, are said to be orthogonal to each other if, when juxtaposed by corresponding elements to form a k order array of double-entry cells (R,G), every paired combination of types R_m, G_n occurs exactly once. (Latin squares are so named, incidentally, because Roman letters are often used to represent the several types of such squares. Similarly, squares of the form of (R,G) are called "Greco-Latin squares" because Greek letters are often used to represent the types of one of the orthogonal squares.) An example of a Greco-Latin square for $k = 3$ is:

$$\begin{matrix} 00 & 12 & 21 \\ 11 & 20 & 02 \\ 22 & 01 & 10. \end{matrix} \qquad (3.1)$$

Orthogonal squares cannot be constructed for $k = 2$ or 6. Gunther (14) cites an early proof of the latter (1842) by Clausen by exhaustive examination of 6x6 squares. Akar (12), also cited in Norton's extensive review of the literature before 1939, gave a proof in 1895 that orthogonal squares existed for any k' odd, doubtless along lines similar to those followed in Algorithms I or II below. The even numbers proved more refractory. Bose (3) and Stevens (17) independently and concurrently proved the existence of orthogonal squares for $k = 2^n$ and $k = p'^n$ in general, for p' a prime and n a natural number greater than one. Mann (9) provided for the construction of non-cyclic (see section 6) orthogonal squares for a large class of values of k; but not until 1959 did Bose and Shrikhande (4) disprove Euler's (13) conjecture that no orthogonal squares exist for $k = 4t+2$, for t an integer.

4. For $k = 5$ there exist 56 SFLs. These fall into sixteen linear sets, ten of five SFLs and six of one. Appendix A lists these SFLs in lexicographic order, indicating the transformation set to which they belong. To illustrate the concept of a transformation set, define Q(E), called a recoding of a Latin square E, as the square constructed by substituting throughout E Q_0 for E_0, Q_1 for $E_1, \ldots, Q_{k-1}$ for E_{k-1}, where $Q_0, Q_1, \ldots, Q_{k-1}$ is any permutation of the first k non-negative integers. Call \$(w) any linear permutation of a SFL (for $k = 5$) designated in Appendix A as a member of linear set (w). Then the notation

(4.1) SFL(1) 02413 = \$(6)

indicates that any SFL from set (1), when recoded by the permutation 02413, gives a square whose SFLs are in set (6). The reader can easily verify that:

(4.2)

SFL(1) 13024 = \$(7)	SFL(1) 01234 = \$(1)
SFL(1) 24130 = \$(8)	SFL(1) 12340 = \$(2)
SFL(1) 30241 = \$(9)	SFL(1) 23401 = \$(3)
SFL(1) 41302 = \$(10)	SFL(1) 34012 = \$(4)
	SFL(1) 40123 = \$(5).

For any integer $m \neq 0$, if $x = am + r$, for integer a and $0 \leq r < m$, x is said to be congruent to r modulo m, written $x \equiv r(\text{mod. } m)$, and $x = r$ (reduced modulo m). Define $\underline{V}$ as the union of linear sets (1) through (5), and $\underline{V}'$ as the union of linear sets (6) through (10), v and v' being members of these sets. The reader can verify that

(4.3)
$$\begin{aligned} &SFL(v)\ (0+y, 1+y, 2+y, 3+y, 4+y)\ (\text{mod. } 5) = \$(v) \\ &SFL(v')\ (0+y, 1+y, 2+y, 3+y, 4+y)\ (\text{mod. } 5) = \$(v') \\ &SFL(v)\ (0+y, 2+y, 4+y, 1+y, 3+y)\ (\text{mod. } 5) = \$(v') \\ &SFL(v')\ (0+y, 2+y, 4+y, 1+y, 3+y)\ (\text{mod. } 5) = \$(v), \end{aligned}$$

and analogous statements regarding $\underline{V}$ and $\underline{V}'$ can be made for similar sets of permutations, within each of which relationships such as occur in (4.1, 2) exist. Since by definition, Latin squares within the same linear set may be permuted into each other, by appropriate recodings any square in the union of $\underline{V}$ and $\underline{V}'$ can be transformed into any other. A transformation set is defined as a set of all Latin squares that by recodings and linear permutations can be changed into each other.

5. It is not difficult to show that the six 5x5 SFLs in Appendix A not designated as members of linear sets (1)-(10) are in fact not members of the transformation set $\underline{V}$ union $\underline{V}'$. Consider the squares

(5.1)
$$\begin{matrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 3 & 4 & 2 \\ 2 & 3 & 4 & 1 & 0 \\ 3 & 4 & 0 & 2 & 1 \\ 4 & 2 & 1 & 0 & 3 \end{matrix} = T, \qquad \begin{matrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 4 & 0 & 2 & 3 \\ 2 & 0 & 3 & 4 & 1 \\ 3 & 2 & 4 & 1 & 0 \\ 4 & 3 & 1 & 0 & 2 \end{matrix} = C,$$

where T is a member of the union of linear sets (1)-(10), while C is not. It is claimed that no recoding or linear permutation of T will give C.

Note that in T there exist "rectangles" of the form:

(5.2)
$$\begin{matrix} t_{i,j} & t_{i,j+n} \\ t_{i+m,j} & t_{i+m,j+n} \end{matrix}$$

such that $T_{i,j} = T_{i+m,j+n}$ and $T_{i+m,j} = T_{i,j+n}$ for $n, m \neq 0$. An example of such a rectangle in T is where $t_{1,2} = t_{0,3}$, $t_{0,1} = t_{2,3}$; thus $i = 0$, $j = 1$, $m = n = 2$. Clearly in any recoding Q(T) of T, $q(t)_{0,1} = q(t)_{2,3}$ and $q(t)_{1,2} = q(t)_{0,3}$. Similarly the rectangles persist over linear permutations, which merely add constants (modulo k) to i, j, m, and n. The square C has no such rectangles, so C is outside the transformation set of $\underline{V}$ union $\underline{V}'$. By Algorithm I below, it can be demonstrated that the six 5x5 Latin squares outside of $\underline{V}$ union $\underline{V}'$ are all members of a second transformation set.

6. For k = 5, the SFLs having no such rectangles (<u>intercalates</u> is another name for such configurations) can be recoded and linearly permuted into the SFL:

(6.1) $e_{ij} = (i + j)\ (\text{mod. } k)$.

Such squares are called cyclic Latin squares (CLS), and can be constructed for any natural number k. For any columnwise linear permutation of a CLS, it immediately follows from this definition that

(6.2) if $e_{ij} = E_n$ and $e_{i,j+p} = E_{n+q}$,
then $e_{i+m,j+p} = E_{n+m+q}$

where all subscripts (as throughout this paper) are reduced modulo k.

7. Note that the orthogonal squares in (3.1) are both columnwise permutations of the CLS for k = 3. Greco-Latin squares of any order k' can always be obtained from CLSs by the following construction:

Algorithm I: Prepare an empty k'xk' matrix, k' = 2c+1. Index the first c columns $a_1, a_2, \ldots, a_c$ from left to right and the last c columns $b_c, b_{c-1}, \ldots, b_1$ from left to right. Index the central column C_m. Assign k values v_j to these column indices so that these values form a permutation of the first k non-negative integers with the restriction

(7.1) $(a_j - b_j) \neq (a_n - b_n) \neq (b_j - a_j)$ for all $j \neq n$,

all values reduced modulo k. In every column j let

(7.2) $e_{ij} = (i - v_j)$ (mod. k).

Call the square thus constructed A. Construct square M:

(7.3) $m_{ij} = a_{i,k-1-j}$

where $a_{i,j} = e_{ij}$ in (7.2). It is claimed that A and M are orthogonal Latin squares, and that any M constructed from Q(A) is orthogonal to Q(A).

8. Example of Algorithm I for k' = 7: Let v_i be the permutation 1564302, which satisfies (7.1). Constructing A by (7.2), and recoding by the permutation (randomly chosen) 3401265 gives Q(A) =

(8.1)

3	1	0	2	6	4	5
4	2	1	6	5	0	3
0	6	2	5	3	1	4
1	5	6	3	4	2	0
2	3	5	4	0	6	1
6	4	3	0	1	5	2
5	0	4	1	2	3	6

·

0000001
0000010
0000100
0001000 = Q(M).
0010000
0100000
1000000

The reader can verify that in (8.1) Q(A) $\perp$ Q(M)

9. That A will always be a Latin square (from which it follows that M, Q(A), and Q(M) are also) follows immediately from (7.2) and from the specification that the values of v_i be a permutation of the first k non-negative numbers; A will always be a columnwise linear permutation of the CLS, as for example:

(9.1)

0	k-1	k-2	...	1
1	0	k-1	...	2
2	1	0	...	3
.	.	.		.
.	.	.		.
.	.	.		.
k-1	k-2	k-3	...	0.

The same construction (9.1) serves as a proof that (7.1) always has at least one solution for any k'. In (9.1), v_i forms the permutation 0, 1, 2, ..., k-1, so that

(9.2) $(a_j - b_j) = (2j - 1)$ (reduced modulo k)

(9.3) $(b_j - a_j) = (1 - 2j)$ (reduced modulo k).

Since $k' = 2c + 1$, and $j \leq c$, the right-hand term in (9.2) is an increasing function of j, and in (9.3) a decreasing function of j as j varies from 1 to c. And since these are of opposite parity (the first odd, the second even) it follows that (7.1) is satisfied by (9.1).

10. To prove that $A \perp M$, for all A constructed by Algorithm I, it shall be shown that (Q(A), Q(M)) is always a Greco-Latin square. In the first c columns of (Q(A), Q(M)) entries are of the form:

(10.1) $(A_n, A_n + a_j - b_j)$,

the subscripts referring to the first column of Q(A). In the last c columns the corresponding expression is:

(10.2) $(A_n, A_n + b_j - a_j)$.

In the median column the expression is:

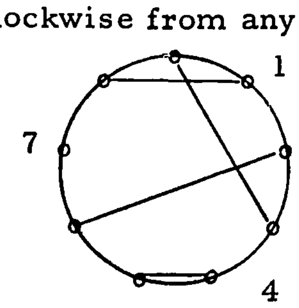
(10.3) (A_n, A_n).

By (7.1), for a given n, (10.1, 2, 3) assume k different values in the k columns of (Q(A), Q(M)). But n has k different values; therefore (Q(A), Q(M)) has k^2 different entries, which proves that it is a Greco-Latin square.

Note that for $k = k''$, $m_{ij} \neq a_{ij}$ by (7.3), so in (Q(A), Q(M)) entries of the form of (10.3) cannot occur. This proves that the algorithm fails for k even. Euler (13) has in fact shown that for k'' no Latin square orthogonal to a CLS (or any permutation or recoding thereof) exists.

11. Any assignment of v_i values satisfying (7.2, 3) can be represented as a circle having k' equally spaced points on its perimeter, 2c of these being connected by c chords of c different lengths. These points are identified with the first k non-negative integers, numbered consecutively clockwise or counterclockwise from any point. For k' = 9,

(11.1)



is such a figure. Identify each chord with a different number j, so that the values of j form a permutation of the first c natural numbers, and assign to one endpoint of every chord the symbol a, and b to the other endpoint. Thus each chordal endpoint has a unique name a_j or b_j. Assign to the point lying on no chord the name C_m. By construction (7.1) is satisfied by a permutation of the first k non-negative integers. For k' < 50, this is in fact a convenient method of solution, alternative to (9.1). In experimental designs, (A,M) can be randomly permuted and lineraly recoded, without affecting orthogonality.

12. For k' = ab, Algorithm I can be iteratively used to construct squares like:

(12.1)

A	B	C	D	E	F	G	H	I				
B	C	A	E	F	D	H	I	G		(R)	(S)	(T)
C	A	B	F	D	E	I	G	H				
D	E	F	G	H	I	A	B	C				
E	F	D	H	I	G	B	C	A	=	(S)	(T)	(R)
F	D	E	I	G	H	C	A	B				
G	H	I	A	B	C	D	E	F				
H	I	G	B	C	A	E	F	D		(T)	(R)	(S).
I	G	H	C	A	B	F	D	E				

Call M constructed by (7.3) the <u>mirror image</u> of A, for any Latin square A. The reader can verify that the square given in (12.1) is orthogonal to its mirror image. For the case of double use of Algorithm I, if k' = gh, it will be possible to partition k types into g sets of h; call these sets $\underline{A}$, $\underline{B}$, ..., $\underline{G}$. Using the types in each set, construct by Algorithm I g Latin squares A, B, ..., G of order h, using the same values for v_i in each square.

Now treating A, B, ..., G as elements in a super-column construct by Algorithm I a Latin super-square letting k' = g and $c = \frac{1}{2}(g-1)$. Note that these constructions are always possible, since k' = gh = 2c+1 implies $g, h \equiv 1 (\text{mod. } 2)$. The resulting super-square is not in general a CLS permutation, and can easily be changed into k' order squares of different transformation sets having orthogonal mirror images by the judicious rearrangement of types within one or more of the h-order squares.

13. To digress at some length, let it be remarked that by a construction quite similar to Algorithm I orthogonal Latin squares can be obtained for any k' odd, one of which is not necessarily in the same transformation set as the CLS. <u>Definition</u>: A diagonal of square array E is any set of cells in E of the form:

(13.1) $e_{i,0}, e_{i+1,1}, e_{i+2,2}, \ldots, e_{i+k-1,k-1}.$

In this section, unless otherwise indicated, all values are reduced modulo k. From section 1 it follows that a Latin square has k diagonals of k cells each. Note that any CLS put into the form of (9.1) has only one type on each diagonal, while its mirror image has k types on each diagonal. It follows at once that (9.1) is orthogonal to its mirror image.

<u>Algorithm II</u>: Prepare an empty k'xk' matrix, and obtain some permutation $D_0, D_1, \ldots, D_{k-1}$ of the first k non-negative integers, such that:

(13.2) $(D_i - D_q) \neq (i - q)$ for all $i \neq q$.

Write this permutation into the first column of matrix D. In every column j, let:

(13.3) $d_{ij} = (D_{i-j}) + (j)$.

This implies that:

(13.4) if $D_{ij} = D_n$, then $D_{i+y, j+y} = (D_n) + (y)$.

If C is a CLS in the form of (9.1), it is claimed $D \perp C$.

Since orthogonality follows from the construction, it shall only be shown that D is a Latin square, or specifically, that (1.1) holds, or that

(13.5) $D_{iq} \neq D_{ij} \neq D_{pj}$ for all $i \neq p$, $j \neq q$.

Suppose $D_{iq} = D_{ij} = D_z$ for some z, i, with $j \neq q$. Then

(13.6) $D_{i-j, j-j} = (D_z) - (j)$ by (13.4), and
$D_{i-q, q-q} = (D_z) - (q)$; therefore
$D_{i-j} - D_{i-q} = (q - j)$

which contradicts (13.2). Therefore $D_{iq} \neq D_{ij}$. Suppose $D_{ij} = D_{pj} = D_z$ for some z, j with $i \neq p$. Then by (13.4)

(13.7) $D_{p-j, j-j} = (D_z) - (j)$; by (13.6)
$D_{i-j} = D_{p-j}$

which contradicts the definition of $D_0, D_1, \ldots, D_{k-1}$. Then (13.5) is proved, and D is a Latin square.

For k' = 7, Algorithm II gives the square:

(13.8)

$$\begin{matrix} 2 & 1 & 5 & 0 & 3 & 6 & 4 \\ 5 & 3 & 2 & 6 & 1 & 4 & 0 \\ 1 & 6 & 4 & 3 & 0 & 2 & 5 \\ 6 & 2 & 0 & 5 & 4 & 1 & 3 \\ 4 & 0 & 3 & 1 & 6 & 5 & 2 \\ 3 & 5 & 1 & 4 & 2 & 0 & 6 \\ 0 & 4 & 6 & 2 & 5 & 3 & 1 \end{matrix} = D \perp C = \begin{matrix} 0 & 6 & 5 & 4 & 3 & 2 & 1 \\ 1 & 0 & 6 & 5 & 4 & 3 & 2 \\ 2 & 1 & 0 & 6 & 5 & 4 & 3 \\ 3 & 2 & 1 & 0 & 6 & 5 & 4 \\ 4 & 3 & 2 & 1 & 0 & 6 & 5 \\ 5 & 4 & 3 & 2 & 1 & 0 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 & 0. \end{matrix}$$

It can be proven that D in (13.8) is not in the CLS transformation set. Note that $d_{2,1} = d_{4,4}$, and $d_{2,4} = d_{4,1}$, a rectangle of the form of (5.2). If such a rectangle could exist in a a CLS, by (6.1, 2) $E_{n+q} = E_{n+m}$ and $E_{n+m+q} = E_n$ for some $q \neq 0$, E a CLS. This implies that $m = q$, and that $m + q = k$, or that $2q = k$. Since q is an integer and k odd, this is impossible.

It is stated without proof that (13.2) is satisfied for any k' by:

(13.9) $D_i = (yi - z) \pmod{k}$

for any z and any $y > 1$ where $(y, k) = 1$. This rule always gives a D in the CLS transformation set. In (13.8) the permutation 2516430 gave a non-CLS D, but no general method of obtaining such permutations is known to this writer. Appendix B lists several non-CLS permutations.

It is natural to ask whether Algorithm II gives orthogonal squares for k even; herein might lie a simple counterexample to Euler's conjecture (see section 3). But like Algorithm I, Algorithm II can be shown to fail for all $k = k''$. Suppose $D_0, D_1, \ldots, D_{k''-1}$ were a permutation of the first k non-negative integers satisfying (13.2). Define $b_i = i - D_i$ for $i = 0, 1, 2, \ldots, k''-1$. It follows from (13.3) that $b_i = b_q$ for $i \neq q$. Since b_i is reduced modulo k'', this implies that:

(13.10) $$\sum_{i=0}^{i=k''-1} b_i = \sum_{i=0}^{i=k''-1} i = c(k''+1)-k'' \equiv c(\text{mod. } k'').$$

But by the definition of b_i and by the fact that the D_i's are a permutation of the first k non-negative integers,

(13.11) $$\sum_{0}^{k''-1} b_i = \sum_{0}^{k''-1} (i-D_i) = \sum_{0}^{k''-1} i - \sum_{0}^{k''-1} D_i = 0,$$

which contradicts (13.10), so Algorithm II fails for k''.

Despite this result, Euler's conjecture can be disproved by a simple combination of techniques used in Algorithms I and II. For example, Parker's (15) 10x10 Greco-Latin square, constructed by an extension of the methods of Bose and Shrikhande (4), is composed of a 7x7 square with ten types from each alphabet orthogonal by diagonals; a 7x3 and a 3x7 rectangle with cyclically orthogonal rows and columns respectively, each with seven types from each alphabet; and a 3x3 square of three types from each alphabet, orthogonal by any method. Appendix C gives an 18x18 square analogously constructed, and the outline of a general method for obtaining anti-Eulerian squares. See Barra and Guerin (2) for $k'' \not\equiv 6(\text{mod. } 12)$.

PART II
SEQUENTIAL COUNTERBALANCING

14. Every row i of a Latin square has k-1 pairs of consecutive elements of the form $E_{i,j}$ $E_{i,j+1}$. E.g., row

(14.1) 4 1 0 2 3

from a square where k = 5, has four pairs: 4 1; 1 0; 0 2; and 2 3. Since there are k rows, there are k(k-1) pairs in E of order k. Defining $E_i E_n \neq E_n E_i$, it can be seen that k(k-1) is the number of possible pairs of types in E. Definition: A Latin square containing all possible consecutive pairs in its rows is said to be counterbalanced for immediate sequential effects (abbreviated SQCB).

15. Theorem 1: A square in the CLS transformation set is SQCB if and only if the set of all values d(j) is a permutation of the first k-1 natural numbers, where d(j) = (s - r) (mod. k), $E_{i,j} = E_r, E_{i,j+1} = E_s$.

From (6.1) it follows that d(j) is constant for any i, despite recodings or linear permutations by rows or columns (although these introduce new constant values). Suppose E were a CLS such that d(j) = d(j*) for j ≠ j*. Then $E_{i,j}$ (= E_m) is followed by $E_{i,j+1} = E_{m+d(j)}$. But in some row i* ≠ i, $E_{i*,j*} = E_m$ is followed by $E_{i*,j*+1} = E_{m+d(j*)} = E_{m+d(j)}$. So $E_m E_{m+d(j)}$ occurs twice in E. Then if E were SQCB the remaining k(k-1)-1 pairs of types would occur in only k(k-1)-2 pairs of cells, which is impossible. Thus the theorem is necessary.

Proof of sufficiency follows from the theorem's requirement that type E_m be followed by types $E_{m+1}, E_{m+2}, \ldots, E_{m+k-1}$ for every E_m occurring in the first k-1 columns of E. Reduced modulo k, this gives k-1 different types following E_m. Therefore all k types in E are followed by k-1 types exactly once.

Corrolary: No CLS-transformation set square is SQCB for k = k'.

In such a square, if it existed, by Theorem 1:

(15.1) $$\sum_{j=0}^{j=k-2} d(j) = \sum_{i=i}^{i=k-1} i = \frac{1}{2}k(k-1) = kc \equiv 0 (\text{mod. } k).$$

But then by (6.1,2), in such a square where $E_{i,0} = E_m$, then by the definition of d(j), $E_{i,k-1} = E_{m+kc} = E_m$, which contradicts (1.1).

16. Bradley (5) gives a method of constructing SQCB permutations of CLSs for k'' that satisfies the restriction of Theorem 1. He specifies that $d(j) = (-1)^j(j)$, by letting the first row be of the form:

(16.1) $E_0, E_{k-1}, E_1, E_{k-2}, E_2, \ldots, E_{c-1}, E_{c+1}, E_c$,

and setting the types equal to the subscripts in (16.1),

(16.2) $e_{i,j} = (E_{0,j}) + (i)$

describes the rest of the cells. Since for $j' \equiv 1(\text{mod. } 2)$, $d(j')$ is a monotonically decreasing function of j, and for $j'' \equiv 0(\text{mod. } 2)$, $d(j'')$ is monotonically increasing, then Bradley's d(j) forms a permutation of the first k-1 natural numbers, $d(j'')$ and $d(j')$ being of opposite parity.

17. For k = p'-1, another SQCB CLS is available:

(17.1)

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 1 & 3 & 5 \\ 3 & 6 & 2 & 5 & 1 & 4 \\ 4 & 1 & 5 & 2 & 6 & 3 \\ 5 & 3 & 1 & 6 & 4 & 2 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{matrix} = S; \; S^* = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 2 & 5 & 1 & 4 \\ 2 & 4 & 6 & 1 & 3 & 5 \\ 6 & 5 & 4 & 3 & 2 & 1 \\ 4 & 1 & 5 & 2 & 6 & 3 \\ 5 & 3 & 1 & 6 & 4 & 2; \end{matrix}$$

S is an example for p' = 7 (p' must be a prime). S* is a linear permutation of S by rows, by which it can be seen that S is a recoded and linearly permuted CLS. In S, the SFL, such squares are symmetrical around both diagonals and have the peculiarity that if $e_{ij} = E_n$, $e_{i,j+1} = E_m$, and $E_{i+r,j+q} = E_n$, then $e_{i+r,j+q+1} = E_{n+r}$, where all subscripts are reduced modulo p'. S is constructed by the rule

(17.2) $e_{ij} = E_{(j \cdot i)}$,

again reducing by modulo p', where i and j range from 1 to k, and are in fact simply modular multiplication tables for p'. Alimena (1) was apparently unaware of this when he described his procedure for constructing them (which might also serve as a masterplan for a spiderweb). Alimena asserts that squares of the form of (17.2) are superior for experimental work, in terms of sequential properties, to those described by Bradley (16.1, 2).

18. Define the separation between two types in a row of a Latin square as the absolute difference between the column indices of the cells in which they appear. The mean separation is the summed separations for two letters divided by k. (Since this is to be an ad hoc means of comparing (17.2) and (16.1, 2), the directional insensitivity of these definitions is unimportant; by both constructions every row is the mirror image of some other row.) For k = 10, mean separations by types are tabled in (18.1). Entries above the main diagonal have reference to (16.1, 2); those below, to (17.2), or rather to Alimena's construction, which has identical separation values. Note that for an ideally counterbalanced square, mean separations would be constant across all types.

(18.1)

	E_0	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9
E_0	-	1.8	3.2	4.2	4.8	5.0	4.8	4.2	3.2	1.8
E_1	3.0	-	1.8	3.2	4.2	4.8	5.0	4.8	4.2	3.2
E_2	3.2	3.8	-	1.8	3.2	4.2	4.8	5.0	4.8	4.2
E_3	3.2	3.0	4.0	-	1.8	3.2	4.2	4.8	5.0	4.8
E_4	4.0	3.8	4.0	3.2	-	1.8	3.2	4.2	4.8	5.0
E_5	3.0	3.2	3.0	3.8	5.0	-	1.8	3.2	4.2	4.8
E_6	3.8	4.0	3.0	5.0	3.8	3.2	-	1.8	3.2	4.2
E_7	3.8	3.2	5.0	3.0	3.0	4.0	4.0	-	1.8	3.2
E_8	4.0	5.0	3.2	4.0	3.2	3.8	3.0	3.8	-	1.8
E_9	5.0	4.0	3.8	3.8	3.0	4.0	3.2	3.2	3.0	-

The average absolute deviation from the grand mean is 0.5 for (17.2), 1.0 for (16.1,2). For a 10x10 square, the greatest possible mean separation is 5.0, the least 1.8; (16.1,2) achieves this range, and (17.2) the upper limit. Since it is generally less detrimental that successive treatments tend to be remote than that they tend to follow one another closely, (18.1) supports the view that modular tables give "better" squares than (16.1,2); but the difference may not be of practical importance, and it must be remembered that (17.2) is only applicable when k = p'=1.

19. It can be shown that (17.2) will not generalize to

(19.1) $e_{ij} = E_{(j \cdot i)}$,

with the subscript reduced modulo $k+1 \neq p'$. If E were a square so constructed, then $k+1 = ab$ for some a, $1 < a \leq k^{\frac{1}{2}}$, and in some cell e_{ab}, by (19.1), where i, j range from 1 to k, $e_{i,j} = e_{a,b} = E_0$. But $i \cdot 1 \not\equiv 0 (\text{mod.} (k+1))$ in the range of i, so $E_0 \neq E_{n,1}$ so E is not a Latin square.

20. By the corrolary to Theorem 1, no SQCB Latin square exists in a CLS transformation set for k'. No SQCB Latin square in any transformation set is known for k', and it is conjectured that none exists.

It will be shown, however, that for k' it is always possible to construct two Latin squares E and F such that each pair of k(k-1) possible pairs of types occurs exactly twice within the rows of E and F. Such squares will be said to be complementary.

<u>Algorithm III:</u> For k' = 2c + 1, squares E and F where

(20.1) $e_{ij} = (-1)^j \left[(j+1)/2\right] + i$

$f_{i,j} = (-1)^j \left[(k-j)/2\right] + i$

can be constructed, where $[x]$ represents the integral part of x, such that E is complementary and orthogonal to F.

It is evident that each column of E and F conforms to (6.2). Then E and F are cyclic Latin squares if for constant i, $e_{i,j}$ and $f_{i,j}$ form permutations of the first k non-negative integers as j ranges from 0 to k-1. Note that for j'' even, $e_{i,j''}$ in (20.1) is a monotonically increasing function of j''; and for j' odd, $e_{i,j'}$ is a monotonically decreasing function of j'. But

(20.2) $c = e_{i,j''(max)} = e_{i,j'(min)} = c + 1;$

an analagous statement holds for $f_{i,j}$. Then by monotonicity, E and F are Latin squares.

To show $E \perp F$, set

(20.3) $$b(j) = \begin{cases} j + c + 1 & \text{for } j \text{ even;} \\ c - j & \text{for } j \text{ odd.} \end{cases}$$

Then $E_{ij} - F_{ij} \equiv b(j) \pmod{k}$. Now if $b(2t) \equiv b(2t'+1) \pmod{k}$, then $2t + c + 1 \equiv c + 2t' - 1 \pmod{k}$, so $2t + 2t' + 2 \equiv 0 \pmod{k}$. But since $2t + 2t' + 2 < 2k$, we must have $2t + 2t' + 2 = k$, which is impossible, since k is odd. Since (20.3) gives two monotonic functions, this proves that $E \perp F$.

Furthermore, for d(j) in E, as defined in section 15 above, d(j) assumes the values 2, 4, ..., k-1 twice. Then every pair of types $E_{i,j}\ E_{i,j+1}$ is of the form

(20.4) $E_m\ E_{m+2y}$ for some natural number y.

It follows that kc ordered pairs of types occur in E. Similarly, F contains kc pairs of the form

(20.5) $E_m\ E_{m+2y-1}$

since d(j) assumes c different odd values twice in any row. Since (20.4) ≠ (20.5), it follows that kc + kc = k(k-1) different pairs occur twice in E and F, so E and F are complementary.

21. Example of Algorithm III, for k = 7:

(21.1)

0	6	1	5	2	4	3			3	4	2	5	1	6	0		
1	0	2	6	3	5	4			4	5	3	6	2	0	1		
2	1	3	0	4	6	5			5	6	4	0	3	1	2		
3	2	4	1	5	0	6	=	E;	6	0	5	1	4	2	3	=	F.
4	3	5	2	6	1	0			0	1	6	2	5	3	4		
5	4	6	3	0	2	1			1	2	0	3	6	4	5		
6	5	0	4	1	3	2			2	3	1	4	0	5	6		

The extension of this construction to within-column counterbalancing is accomplished by permuting the last k-1 rows until the first column is the transpose of the first row in both squares. The construction will also give pairs of Greco-Latin squares counterbalanced for sequential effects within (but not between) alphabets.

APPENDIX A
THE 5 x 5 SQUARES

The k = 5 SFLs in Lexicographic Order:
First rows and columns are deleted; linear sets indicated.

(1) 0 3 4 2 3 4 0 1 4 1 2 0 2 0 1 3	(10) 0 3 4 2 3 4 0 1 4 0 2 1 2 1 0 3	(5) 0 3 4 2 4 0 1 3 2 4 0 1 3 1 2 0	(9) 0 3 4 2 4 1 0 3 2 4 1 0 3 0 2 1	(9) 0 4 2 3 3 0 4 1 4 1 0 2 2 3 1 0	(5) 0 4 2 3 3 1 4 0 4 0 1 2 2 3 0 1	(1) 0 4 2 3 4 3 0 1 2 1 4 0 3 0 1 2
(10) 0 4 2 3 4 3 1 0 2 0 4 1 3 1 0 2	(4) 2 0 4 3 3 4 0 1 4 1 2 0 0 3 1 2	(3) 2 0 4 3 3 4 0 1 4 1 0 2 0 3 2 1	(8) 2 0 4 3 4 3 0 1 0 4 1 2 3 1 2 0	(7) 2 0 4 3 4 3 1 0 0 4 2 1 3 1 0 2	(8) 2 3 4 0 0 4 1 3 4 0 2 1 3 1 0 2	(7) 2 3 4 0 0 4 1 3 4 1 0 2 3 0 2 1
CLSa 2 3 4 0 3 4 0 1 4 0 1 2 0 1 2 3	(9) 2 3 4 0 4 0 1 3 0 4 2 1 3 1 0 2	(6) 2 3 4 0 4 1 0 3 0 4 1 2 3 0 2 1	(10) 2 3 4 0 4 1 0 3 0 4 2 1 3 0 1 2	(4) 2 4 0 3 0 3 4 1 4 0 1 2 3 1 2 0	(3) 2 4 0 3 0 3 4 1 4 1 2 0 3 0 1 2	(5) 2 4 0 3 3 0 4 1 4 1 2 0 0 3 1 2
(2) 2 4 0 3 3 1 4 0 4 0 1 2 0 3 2 1	(1) 2 4 0 3 3 1 4 0 4 0 2 1 0 3 1 2	CLSb 2 4 0 3 4 3 1 0 0 1 4 2 3 0 2 1	CLSc 3 0 4 2 0 4 1 3 4 1 2 0 2 3 0 1	(8) 3 0 4 2 4 1 0 3 0 4 2 1 2 3 1 0	(7) 3 0 4 2 4 1 0 3 2 4 1 0 0 3 2 1	(10) 3 0 4 2 4 3 0 1 2 4 1 0 0 1 2 3
(6) 3 0 4 2 4 3 1 0 0 4 2 1 2 1 0 3	(9) 3 0 4 2 4 3 1 0 2 4 0 1 0 1 2 3	(4) 3 4 0 2 0 1 4 3 4 0 2 1 2 3 1 0	(2) 3 4 0 2 0 3 4 1 4 1 2 0 2 0 1 3	(6) 3 4 0 2 4 0 1 3 2 1 4 0 0 3 2 1	(8) 3 4 0 2 4 3 0 1 2 0 4 1 0 1 2 3	(3) 3 4 2 0 0 1 4 3 4 0 1 2 2 3 0 1
(1) 3 4 2 0 0 3 4 1 4 0 1 2 2 1 0 3	(5) 3 4 2 0 0 3 4 1 4 1 0 2 2 0 1 3	(2) 3 4 2 0 4 0 1 3 0 1 4 2 2 3 1 0	CLSd 3 4 2 0 4 1 0 3 2 0 4 1 0 3 1 2	(4) 3 4 2 0 4 3 0 1 0 1 4 2 2 0 1 3	CLSe 4 0 2 3 0 3 4 1 2 4 1 0 3 1 0 2	(4) 4 0 2 3 3 1 4 0 0 4 1 2 2 3 0 1
(3) 4 0 2 3 3 1 4 0 2 4 0 1 0 3 1 2	(5) 4 0 2 3 3 4 0 1 0 1 4 2 2 3 1 0	(2) 4 0 2 3 3 4 0 1 2 1 4 0 0 3 1 2	(1) 4 0 2 3 3 4 1 0 0 1 4 2 2 3 0 1	(8) 4 3 0 2 0 1 4 3 2 4 1 0 3 0 2 1	(9) 4 3 0 2 0 4 1 3 2 0 4 1 3 1 2 0	(10) 4 3 0 2 0 4 1 3 2 1 4 0 3 0 2 1
(6) 4 3 0 2 3 0 4 1 2 4 1 0 0 1 2 3	CLSf 4 3 0 2 3 1 4 0 0 4 2 1 2 0 1 3	(7) 4 3 0 2 3 4 1 0 2 0 4 1 0 1 2 3	(7) 4 3 2 0 0 1 4 3 2 4 0 1 3 0 1 2	(6) 4 3 2 0 0 4 1 3 2 0 4 1 3 1 0 2	(2) 4 3 2 0 3 0 4 1 0 4 1 2 2 1 0 3	(3) 4 3 2 0 3 4 0 1 0 1 4 2 2 0 1 3.

APPENDIX B

NON-CLS PERMUTATIONS SATISFYING (13.2)

For the permutations below, any integer y can be added to all terms, provided that reduction modulo k follows. Any term can be taken as the initial term, if followed by the k-1 terms to the right in order; thus each solution below implies k^2-1 additional solutions.

k = 7:	0 2 6 5 3 1 4	0 2 5 1 6 4 3
k = 9:	0 2 5 7 1 3 8 6 4	0 2 6 1 7 4 8 3 5
	0 3 6 2 1 8 7 5 4	0 4 1 7 6 3 2 8 5
	0 2 7 5 1 8 4 6 3	0 2 4 8 7 3 1 6 5
	0 3 8 2 7 6 1 5 4	0 2 7 1 6 8 5 4 3
	0 2 5 7 1 4 8 3 6	0 2 6 1 7 2 5 8 4
	0 6 3 1 8 7 5 4 2	0 4 1 8 6 2 7 5 3
	0 3 6 1 7 4 2 8 5	0 2 5 8 1 7 4 6 3
	0 3 7 2 8 6 4 1 5	0 3 1 8 7 6 4 2 5
k = 11:	0 2 7 6 1 9 5 3 10 4 8	0 2 4 9 7 3 10 1 5 8 6
	0 2 9 8 6 3 10 4 7 1 5	0 3 10 9 8 4 7 5 2 1 6
	0 2 10 7 6 3 9 1 4 8 5	0 2 7 1 6 9 3 10 4 8 5
	0 2 6 10 9 4 8 5 3 1 7	0 4 1 8 2 9 7 3 10 6 5
	0 2 5 10 3 9 8 1 6 4 7	0 2 5 1 10 9 8 6 4 3 7
	0 2 8 5 1 10 9 6 4 7 3	0 2 4 7 1 10 9 6 3 5 8
	0 2 10 6 9 7 4 3 1 8 5	0 2 4 9 7 10 3 5 1 8 6
	0 2 6 9 7 1 5 4 10 3 8	0 3 8 2 1 9 7 10 6 5 4
	0 2 8 5 9 1 4 10 7 6 3	0 2 8 5 9 3 10 4 7 1 6
	0 4 9 5 3 10 7 2 1 6 8	0 4 10 9 5 1 8 6 2 7 3
	0 2 8 1 6 9 3 10 7 5 4	0 2 6 10 9 7 5 4 3 1 8
	0 2 6 1 8 5 3 10 9 4 7	0 2 4 7 10 1 9 6 5 3 8
k = 13:	0 2 9 1 12 8 10 6 4 11 7 3 5	0 2 10 12 8 4 11 9 5 7 3 1 6
	0 3 8 7 12 2 11 5 4 10 9 1 6	0 3 8 7 12 2 9 1 6 5 11 10 4
	0 3 7 4 12 2 10 1 11 8 6 9 5	0 2 6 9 1 4 11 3 10 12 8 5 7
	0 3 7 11 8 1 12 5 9 6 4 10 2	0 4 9 8 2 11 5 3 12 10 7 6 1
	0 4 3 11 10 7 5 1 12 6 2 9 8	0 2 4 6 9 1 12 11 3 8 7 5 10
	0 2 8 10 7 3 5 12 4 11 1 6 9	0 2 5 1 12 11 10 9 7 3 6 8 4
	0 3 1 6 12 10 7 11 4 2 8 5 9	0 5 1 8 2 11 3 10 9 4 12 7 6

0 3 1 11 8 12 7 10 5 2 6 9 7   0 2 8 1 7 9 3 12 4 11 5 10 6
0 3 1 9 12 2 10 8 11 7 6 5 4   0 2 4 6 11 9 12 5 3 1 7 10 8

k = 15:
0 4 10 1 14 7 5 13 12 6 11 3 2 9 8
0 2 11 13 7 3 10 12 5 8 1 4 14 9 6
0 4 11 10 2 1 8 13 7 6 14 12 5 3 9
0 2 11 8 3 13 1 9 12 5 7 14 10 4 6
0 3 12 11 1 6 9 5 14 13 2 10 8 7 4
0 2 5 13 10 12 4 9 7 3 14 1 6 8 11
0 2 6 11 9 1 3 14 10 8 13 5 7 4 12
0 3 8 13 12 6 9 5 2 1 14 7 11 10 4
0 4 10 7 13 3 8 12 5 1 6 2 11 14 9

k = 17:
0 2 14 10 3 13 8 12 4 7 16 15 6 5 11 1 9
0 2 11 9 1 15 4 12 3 8 14 13 6 16 10 5 7
0 2 12 14 9 3 13 6 5 11 16 7 15 4 1 10 8
0 2 10 1 8 14 13 4 3 12 15 7 11 6 16 9 5
0 4 1 8 6 15 12 16 9 3 14 2 7 11 10 5 13
0 2 16 12 8 15 5 13 10 14 6 9 3 7 4 1 11
0 4 1 10 8 15 12 16 3 11 6 5 9 14 2 13 7
0 2 8 1 15 12 16 10 13 5 9 6 14 4 11 7 3
0 3 6 10 15 14 11 8 1 5 16 9 7 4 13 12 2
0 2 11 15 1 9 16 13 4 8 12 14 10 7 5 3 6
0 3 1 8 7 16 13 11 4 15 2 12 9 6 5 10 14
0 2 6 10 1 15 5 13 16 3 12 14 8 11 9 7 4

k = 19:
0 2 16 8 14 1 12 18 7 17 13 4 10 15 11 9 6 5 3
0 2 6 9 14 10 5 15 1 11 4 18 8 3 17 7 13 12 16
0 2 17 15 14 11 9 5 10 16 18 7 13 3 8 1 6 12 4
0 2 5 9 8 14 4 18 13 3 17 10 1 6 16 11 7 12 15
0 2 13 17 11 1 9 16 14 7 12 8 6 18 5 4 15 10 3
0 2 16 5 8 13 9 17 1 14 7 18 6 3 12 11 15 4 10
0 2 18 11 6 17 16 3 15 13 9 14 7 6 12 1 10 4 8
0 2 11 17 6 10 9 18 15 3 14 7 1 4 12 8 13 16 5
0 2 6 14 13 1 4 15 11 8 16 5 17 10 7 3 18 12 9
0 2 14 11 10 1 5 12 17 3 7 15 4 16 9 13 18 8 6
0 2 12 9 3 18 14 11 4 16 5 13 10 6 17 1 8 7 15
0 2 15 13 3 8 12 5 17 6 14 18 4 9 16 1 11 10 7.

APPENDIX C

AN 18 x 18 GRECO-LATIN SQUARE

B j	F l	H j	I m	L a	A n	N b	M i	J o	O c	D p	P d	C q	Q h	K e	G g	R f	E r
C k	G m	I l	J a	M b	F r	B n	N c	A j	K o	O d	E p	P e	D q	Q i	L f	H h	R g
D l	H a	J m	K b	A c	R h	G r	C n	N d	B k	L o	O e	F p	P f	E q	Q j	M g	I i
E m	I b	K a	L c	B d	J j	R i	H r	D n	N e	C l	M o	O f	G p	P g	F q	Q k	A h
F a	J c	L b	M d	C e	B i	K k	R j	I r	E n	N f	D m	A o	O g	H p	P h	G q	Q l
G b	K d	M c	A e	D f	Q m	C j	L l	R k	J r	F n	N g	E a	B o	O h	I p	P i	H q
H c	L e	A d	B f	E g	I q	Q a	D k	M m	R l	K r	G n	N h	F b	C o	O i	J p	P j
I d	M f	B e	C g	F h	P k	J q	Q b	E l	A a	R m	L r	H n	N i	G c	D o	O j	K p
J e	A g	C f	D h	G i	L p	P l	K q	Q c	F m	B b	R a	M r	I n	N j	H d	E o	O k
K f	B h	D g	E i	H j	O l	M p	P m	L q	Q d	G a	C c	R b	A r	J n	N k	I e	F o
L g	C i	E h	F j	I k	G o	O m	A p	P a	M q	Q e	H b	D d	R c	B r	K n	N l	J f
M h	D j	F i	G k	J l	K g	H o	O a	B p	P b	A q	Q f	I c	E e	R d	C r	L n	N m
A i	E k	G j	H l	K m	N a	L h	I o	O b	C p	P c	B q	Q g	J d	F f	R e	D r	M n
N n	O p	P r	Q o	R q	M e	A f	B g	C h	D i	E j	F k	G l	H m	I a	J b	K c	L d
O o	P q	Q n	R p	N r	H f	I g	J h	K i	L j	M k	A l	B m	C a	D b	E c	F d	G e
P p	Q r	R o	N q	O n	E b	F c	G d	H e	I f	J g	K h	L i	M j	A k	B l	C m	D a
Q q	R n	N p	O r	P o	D c	E d	F e	G f	H g	I h	J i	K j	L k	M l	A m	B a	C b
R r	N o	O q	P n	Q p	C d	D e	E f	F g	G h	H i	I j	J k	K l	L m	M a	A b	B c

The above 18 x 18 square was constructed by the following strategy:

(1) Obtain some odd number a, $k/4 < a < k/3$.

(2) Partition k x k array S into rectangles R_1, R_2, R_3, R_4:

i) R_1 = first k-a rows, last k-a columns of S.

ii) R_2 = first k-a rows, first a columns of S.

iii) R_3 = last a rows, last k-a columns of S.

iv) R_4 = last a rows, first a columns of S.

(3) Let P(F) and P(f) be permutations $F_1, F_2, \ldots, F_{k-a}$, and $f_1, f_2, \ldots, f_{k-a}$ of 2(k-a) types altogether.

(4) Let P(E) and P(e) be permutations $E_1, E_2, \ldots, E_a$ and $e_1, e_2, \ldots, e_a$ of 2a types different from the types in (3).

(5) Write F_1 into A, where A is some randomly chosen set of k-a cells lying on k-a different rows, columns, and diagonals (as defined in (13.1)) of R_1.

(6) Write P(F) into the diagonals of R_1 as in (13.4).

(7) Write P(E) into the unoccupied cells of the first row of R_1.

(8) Write E_i into every cell of R_1 with E_i on the same diagonal by (7).

(9) Write F_1 into some cell of every row of R_2 not having an F_1 in R_1 by (5), so F_1 lies in every column of R_2.

(10) Write F_1 into some cell of every column of R_3 not having an F_1 in R_1 by (5), so F_1 lies in every row of R_3.

(11) Write P(F) cyclically into the columns of R_2 and the rows of R_3, from top to bottom, left to right.

(12) Call the last k-2a rows, first k-2a columns of R_1 "R*."

(13) Find z(c) sets C_n of k-2a cells in R* such that

i) every cell in C_n lies on a different row, column, and diagonal of R_1;

ii) every type in P(E) is contained once in C_n;

iii) k-3a different types in P(F) occur in C_n.

(14) Call the set of types $C_n \cap P(F)$ "X_n."

(15) Let the first a rows and a columns of R_2 be called R**.

(16) Let the last a rows and columns of R_3 be called R***.

(17) Find some set B_1 of 2a cells, a in R** and a in R***:

i) every cell in B_1 lies on a different row, column of S.

ii) B_1 contains 2a different types from P(F).

(18) Call the types in P(F) not in B_1 "Y_1."

(19) If $X_n = Y_1$ for some C_n, go on to (24).

(20) If for n = 1,2,...,z(c) $X_n \neq Y_n$, go back to (17) and find $B_2 \neq B_1$ satisfying definition of B_1. If necessary, go on to $B_3, B_4, \ldots, B_m$. If $X_n = Y_m$, for some B_m, C_n, go on to (24).

(21) If for m = 2,3,...,z(b) no $X_n = Y_m$, go back to step (5) and write F_1 into $A_2 \neq A$, where A_2 defined as A. If necessary, go on to $A_3, A_4, \ldots, A_u$. If $X_n = Y_m$ for some A_u, B_m, C_n, go on to (24).

(22) If for u = 2,3,...,z(u) $X_n \neq Y_m$, go back to (1) and obtain $a_2 \neq a$, a_2 defined as a. If necessary, go on to $a_3, a_4, \ldots, a_v$. If $X_n = Y_m$ for some a_v, A_u, B_m, C_n, go on to (24).

(23) If for v = 2,3,...,z(v) $X_n \neq Y_m$, give up.

(24) If $X_n = Y_m$, write f_1 into the cells of B_m and C_n.

(25) Write P(f) cyclically into the diagonals of R_1 containing f_1, as in (6).

(26) Write P(e) into the cells of the first row of R_1 unoccupied by f_i, as in (7).

(27) Write e_i constantly into diagonals of R_1 containing e_i, as in (8).

(28) Write P(f) cyclically into the columns of R_2 and into the rows of R_3 as in (11).

(29) Construct by Algorithm I or II an a-order square having P(E) in the cells as types, orthogonal to a square having P(e) as types. Write both squares into the cells of R_4.

(30) Now S is a Greco-Latin square of order k''.

For the 18x18 square exhibited above, the solution was obtained when v = 1, u = 1, m = 4, z(c) = 12. For k'' = 18, a is uniquely = 5. This solution took an hour by hand. Note that the values z(b), z(u), and z(v) are arbitrary limits, set as numbers of trials or as amounts of computer time. Of these four, z(v) will tend to have the smallest limiting values and z(u) the largest. In the opinion of this writer, it would be worthwhile to test z(v) and z(c) exhaustively, if possible, contenting oneself with a fraction of the possible values for z(u) and z(b).

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