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The use of symbols and abbreviations in mathematics

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The language of mathematics is unique and complex. One feature of the mathematical register is the use of symbols and abbreviations. Whilst it may be possible for a student to think mathematically in the absence of symbols, the written communication of mathematical ideas cannot be achieved concisely without the use of mathematical symbols. Further, it is possible that the fear and dislike of algebra can be attributed to the failure to understand fully the symbols inherent in this area of mathematics.

This paper examines some of the complexities of the symbolic aspects of mathematical language, where possible using the 2010 National Assessment Program: Literacy and Numeracy (NAPLAN) (ACARA, 2010) numeracy test items as examples.

It is important to clarify what is meant by mathematical symbols and abbreviations. Many symbols are familiar to even the youngest students: they include the ten numerals (0 through to 9) and the 26 letters of our alphabet. Others, such as the symbols used for equality, currency, and the four arithmetic operations (=, $, ¢, +, −, ×, ÷) are introduced at an early stage of schooling. Others, such as the letters of the Greek alphabet, and the symbols used to represent more complex mathematical ideas (%, √, <, >, ±, ∞) are encountered only in the later years of schooling. The recall or recognition of the symbols themselves is not complex. It is the semantics or meanings that we assign to the symbols or the concepts that they represent, that makes them challenging for students. Further, the syntax, or the way in which the symbols are used, introduces additional complexities for students.

The meaning of mathematical symbols and abbreviations

Classification

Students must be able to decode and verbalise the full range of mathematical symbols relevant to their stage of mathematical studies. They can be classified into seven groups:

- numerals: the Hindu-Arabic numerals (0, 1, 2, 3, 4, 5, 6, 7, 8, and 9) and also those of other numeration systems such as Roman (I, V, X, D, C, L, and M) that are used, in various combinations, to represent numbers of all types;
- operators: the arithmetic operators such as +, –, ×, ÷ and their 'synonyms' such as · (dot) and / (vinculum), but also operators such as √ and ! and the generic operator * that are introduced in later years;
- comparatives: the symbols used to denote equality and inequality and other relationships: =, <, >, ≤, ≥, ≡, ≈, ∝, ⊂ and their converse forms (usually involving a line through the relevant symbol) ≠, ⊄, ≮, ≉;
- grouping symbols such as parentheses ( ), braces { }, brackets [ ] and also the vinculum;
- pronumerals that can be variables, unknowns, or parameters; they can be represented by any symbol, but most commonly by the upper and lower case forms of the 26 letters of our alphabet (italicised) and the 24 letters of the Greek alphabet; included in this group are the 'standard' pronumerals used to represent certain concepts such as gradient (m), radius (r), mass (m), area (A);
- geometric symbols such as Δ, ⊥;
- shortened forms which may be abbreviations or symbols, which can be further subdivided as *mathematical* (%, ∴, ∞, ƒ( ), ±, ∫, ∃, ∀); *units of measurement*, where any letters used are *not* italicised ($, ¢, km, cm^2, m^3, L, g, mL, °C, s, h); and *common use* (3D, N, S, E, W, am, pm).

This classification can assist students to understand the wide variety of different symbols used in mathematics, and their different purposes. Classification and matching games (such as concentration, dominoes, and card shuffles) and investigation of similarities and differences can be used to teach the meaning of these symbols. Younger students may benefit from a place mat—a laminated sheet attached to the top of their desk summarising the symbols and their meanings.

Use of symbols in NAPLAN numeracy tests

Of course, not all symbols and abbreviations appear in NAPLAN numeracy tests. For example, the use of symbols and abbreviations in the 2010 NAPLAN numeracy tests (ACARA, 2010) is summarised in Table 1.

The use of symbols and abbreviations in the NAPLAN numeracy tests indicates the variety and complexity of the symbolic language of mathematics

Table 1. The use of symbols and abbreviations in NAPLAN numeracy tests (ACARA, 2010)

Year 3	*Year 5*	*Year 7*	*Year 9*
Units such as m, cm Four operations (+, –, ÷, ×) =, $	Units such as g, kg, and ml Decimal point Quotients written as 12 ÷ 4 or $\frac{2}{5}$ > N for north C4 for a grid reference, 3D for three dimensional	Units such as km, mm, mg ( ), %, ° Times such as 4:20 am, pm N, S, E, W for directions	Units such as m^2, cm^2 Variables such as p, q, y, x Negative numbers such as –18° C Ratios such as 1:2 Intervals such as 145–149 min for minute √, exponents such as 10^3 5 | 2 meaning 52
Note. It is implied that symbols and abbreviations used at a lower level were also used in higher levels.			

and demonstrates the increasing use of symbols and abbreviations in more senior years of schooling. Notably, symbols and abbreviations are sometimes written as words or as a combination of words and symbols. For example, in Years 7 and 9, words were used when writing “minutes,” “cents,” “cubic metres,” and “metres per minute” and a combination of words and symbols was used in “6-sided,” “6 metres,” “3 times,” and “litres per 100 km.” “Square centimetres,” “grams,” “litres,” “kilograms,” “dollars” and numbers such as “seventy-five” were written as either words or symbols. Large numbers were often written in words. In general, words were used more commonly in the lower grades and symbols in the higher grades. Examples can be seen in questions 17 and 30 of the Year 9 Calculator NAPLAN numeracy test (ACARA, 2010). In question 30, the word “centimetre” and the symbol “cm” were both used, and in question 17, both “min” and “minute” were used.

Difficulties in decoding symbols

An immediate difference between symbols and prose is the direction of reading. Prose is read from left to right, top to bottom. While this does not apply to all symbols, it is not possible to substitute a different rule; students must use judgement or experience in deciding how to read a symbol. This could be why some students manage to grasp the subtleties of decoding symbols, while others remain perplexed.

Different symbols can have the same meaning; for instance, *twelve divided by three* can be represented by

$$12 \div 3,\ \frac{12}{3},\ {}^{12}\!/_{3}\ \text{ or }\ 3\overline{)12}$$

There are a variety of symbols used to represent approximately equal to, including $\approx$, $\doteqdot$ and $\triangleq$.

Some symbols are very similar in appearance. The subtle differences between some symbols can be the shape or size of otherwise identical symbols, the use (or not) of bold type or italics, the selection of upper or lower case, or additional marks such as dashes or dots. These must be made explicit to students so that they can decode and use them correctly. For example: the symbol for zero (0) and the upper or lower case symbols for the letter O can be confused; the symbol which represents multiplication ($\times$), can be confused with the upper or lower case letters x and the variable x; the symbol for the number 1 can be confused with the upper case letter I.

Some abbreviations and symbols such as those found in the measurement strand of mathematics can have similarly subtle differences. An example of the complexity of symbols and abbreviations is that of the physical quantity *length* which is abbreviated l (in italics) and has a unit of measurement metre which uses the symbol m (not in italics). However the letter m, when italicised can be used to represent mass or gradient and l representing length, can be confused with the number 1. The symbols and abbreviations used in the Standard International (SI) system are specific in terms of the use of upper case or lower case letters (APA, 2010). Most units are in lower case except the units named after people such as Watt, W; Newton, N and others such as mega-, M (to distinguish it from m for the prefix milli-), litre, L (to distinguish it from the number 1), giga- G, and tera-, T (APA, 2010). At higher levels of schooling, more complex abbreviations are used such as m/s, which can also be written $\mathrm{m.s^{-1}}$, and $\mathrm{m^2}$, which is verbalised as “square metres” and not “metres squared.” Because SI units (of length, time, etc.) are treated as *symbols* (although they may appear to be abbreviations), unit plurals are the same as the singular form: for example 4 cm, not 4 cms.

A space is left between a number and symbol with the exception of measures of angles (APA, 2010). The correct units of time (symbols) are s for second, min for minute, h for hour and d for day (Bureau International des Poids et Mesures, 2004).

The complexity of decoding mathematical symbols and abbreviations is especially evident in the strands of statistics and probability in which there are specific rules for the use of italics, bold print, and upper and lower case. Symbols and abbreviations for just these strands take a full four pages in the American Psychological Association manual (APA, 2010, pp. 119–122).

Mathematics text is often dense and precise (Spanos et al., 1988). This means that vast amounts of information can be given in a small amount of text, for instance the term

$$\left[(2+6)\times\sqrt{4}\right]^2$$

is lexically dense as it would take a whole sentence to describe it in words. Whereas everyday English text can be understood despite inaccuracies in spelling and word usage, comparable errors in the use of mathematical symbols can have significant impact on meaning. Small changes in the use of symbols can cause major changes in meaning. For instance,

$$\left[2+6\times\sqrt{4}\right]^2$$

has an entirely different meaning (value) to the similar term

$$\left[(2+6)\times\sqrt{4}\right]^2$$

Although not necessary for test purposes, teachers and students need to be able to make meaning of some non-metric or non-SI units and alternative shortenings which are sometimes used in television or movies and foreign textbooks and websites. Examples of such units are bar and mmHg (millimetre of mercury) for pressure, M for nautical mile (Bureau International des Poids et Mesures, 2004) and lb for pound, ft for foot, mi for mile, but mph for miles per hour, and F for Fahrenheit. Some abbreviations and symbols vary from text to text; for example sec is often used for second and cc for cubic centimetre instead of cm^3, depending on conventions used.

The meaning of some symbols can vary from country to country. The symbol used for "nine" in Mexico can be confused with a lower case g (Van De Walle, Karp & Bay-Williams, 2010). The decimal point and comma can also have different meanings in different countries (New South Wales Department of Education, 1989). In Australia, the recognised standard is that the decimal point is a dot (full stop) and spaces are used to segment thousands in large numbers such as 3 243 685 (note that numbers less than 10 000 do not use a space!), although the US standard is to use commas instead of spaces (3,243,685). In some other countries, the comma is used as a decimal point. Symbols for money such as $ for dollars, R for rand, £ for pound, and € for euro can vary from country to country. Abbreviations such as N for north and 3D for three-dimensional may not be familiar to children from non-English speaking countries.

The effect of syntax on the meaning of mathematical symbols and abbreviations

Most mathematics teachers will have experienced the confusion caused by symbols and abbreviations which are similar in appearance, although different in meaning. Examples are the symbol for subtraction in 4 – 3, the symbol for a negative number in –3, and the symbol used to represent an interval in 5–9. Thus, a student may read "within the next 5–10 years" as "five minus ten years". The symbol used to represent a negative number and the symbol used to represent an interval can be seen in questions 9 and 22 of the Year 9 Calculator NAPLAN numeracy paper respectively (ACARA, 2010). The difference in the meaning of the symbol – is caused by the syntax, that is, the three different contexts in which the symbol is used. Similarly, the dot used to represent multiplication can be confused with the dot used to represent a decimal point. A colon can be used to write a ratio or to separate hours and minutes in a written record of time. Examples of these can be seen in question 8 of the Year 9 Calculator NAPLAN numeracy paper and in question 16 of the Year 9 Non-calculator NAPLAN numeracy papers respectively (ACARA, 2010). In question 20 of the Year 9 Non-calculator NAPLAN numeracy test, the key of the stem and leaf diagram uses the symbols 5|2 to represent the number 52, but it could be confused with the fraction 5/2 otherwise written as 5 ÷ 2. In all of these examples, other elements of the questions give clues to the intended meaning of the symbols.

The symbolic language used in mathematics can be used differently in other contexts, for example, A3 might represent a question number, a grid reference on a map or a paper size, and 3D can refer to three-dimensional whereas the similar D3 can be a grid reference or a question number. In another example, 4/133 and 6.2 overs in cricket have very different meanings to division and a decimal place used in mathematics.

Sometimes symbols can be implied although they may not be included. The absence of a sign in $7n$ implies multiplication. However, while $7n$ implies $7 \times n$, terms such as Σx or dx are not products (Orton, 2004). Symbols were inferred in question 24 of the Year 9 Non-calculator NAPLAN numeracy test (ACARA, 2010) where $2hb$ implied $2 \times h \times b$. Another example can be seen in question 15 of the Year 9 Calculator NAPLAN numeracy test (ACARA, 2010) in which $\sqrt{9.5+6.5}$ inferred the use of brackets, thus either the sum must be calculated first or brackets must be included when using a calculator √(9.5+6.5). In question 17 of the Year 9 Non-calculator NAPLAN numeracy test (ACARA, 2010) the term

$$\frac{n^2-10}{2}$$

implied that $n^2 - 10$ was in brackets.

Since the power of mathematics is to solve real-world problems, learners need to be able to bridge the gap between the real world and the symbols and skills of mathematics (Ellerton & Clements, 1991). Many educators have discussed the difficulties that learners can experience when converting word problems into mathematical terms. Often a variety of words or phrases can be used to represent a single symbol; for example, "subtract," "take away," "minus," "difference" and the – symbol can all represent the same concept. The NSW Department of Education (1989) referred to twenty ways of verbalising "9 – 3" including "nine minus three," "three less than nine," "decrease nine by three," and so on. Examples of these can be found in questions 7 and 11 of the Year 7 Non-calculator NAPLAN numeracy test and question 28 of the Year 9 Non-calculator NAPLAN numeracy test.

The subtraction example also demonstrates that the order of the symbols and words is not always the same, which can lead to *reversal errors*. Spanos et al. (1988) gave examples of reversal errors; for instance, the term "nine less than a number" can lead a student to write "$9 - n$" rather than "$n - 9$". Reversal errors can also occur with the statement, "There are six times as many boys as girls" that may lead a student to write $g = 6b$ rather than

$b = 6g$. Mestre (1988) linked this kind of error to confusion between the use of symbols as labels and variables. In another example, "divide 4 into 20," "divide 20 by 4," "divide 20 into 4 equal parts" all have the same meaning, although the order of the words is different. Students often attempt to deal with this confusion by dividing by the smaller number regardless of the order in which they are presented. Although order is important in some instances, in other situations it is not crucial. For example if a number is multiplied by seven it can be represented as $7 \times \mathrm{n}$ or $n \times 7$ or even $7n$.

Similar wording can be used for different functions; for instance: "square" and "square root" or "divided by" and "divided into" (Spanos, Rhodes, Dale & Crandall, 1988). A number of words may represent one symbol; for instance $\leq$ represents the phrase "smaller than or equal to."

The strategy of directing children to key words may be misleading. Words do not always translate to the same mathematical symbol (Zevenbergen, 2000). For instance, the word "altogether" has been "used in many questions in NAPLAN numeracy tests. *Altogether* is often taken to imply addition. However, this is not always the case which can be seen in questions 12, 13, 27, 30, 33 of the Year 3, 2010 numeracy paper" (Quinnell, 2011, p. 19).

Mestre's (1988) research, which focused on solving algebraic word problems, indicated that problems which made use of negatives and those which did not clearly state the unknown, or make use of the word "equal", were especially difficult for learners. It is possible that students who are faced with different statements containing the word "is" find it difficult to tell which comparative symbol is needed. Examples are the statements "y is five times a number" and "y is more than six". In these cases, the students need to take note of the words following the word "is". Mestre (1988) noted that problems that referred to two variables were sometimes written as $2n$ or students used two different variables, even if the variables were dependent on each other. Cocking and Chipman (1988), referring to previous research, stated that algebraic word problems were solved more successfully when teaching focused on word meanings and their translation into symbols.

Learners need to be given opportunities to write and verbalise symbols, and explanations need to aid learning. For example, a visual understanding of "cubic centimetres" needs to be linked with the use of the symbol cm^3. Symbols and abbreviations need to be written very carefully, taking into account the size, position and order (Rubenstein & Thompson, 2001). Links need to be made between symbolic, written, graphic and oral language. Rubenstein and Thompson (2001) advocated strategies which could be used to introduce symbols to learners. They suggested that learners should draw examples and counter examples of statements such as $\overline{PQ} \perp \overline{EF}$, or write symbolic statements that apply to certain diagrams, or practice by reading and writing statements containing symbols.

Word processing of mathematical symbols and abbreviations

Teachers need to be adept at using symbols and abbreviations correctly for test and text purposes when using word processing software. As discussed above, standard abbreviations and symbols have specific rules about the use of upper or lower case text. At higher levels, some symbols should be typed in italics and others should be in bold print as per the American Psychological Association manual (APA, 2010, pp. 119–122). Basic symbols can be found on the computer keyboard. When using Microsoft Word, most other symbols can be found under Insert>Symbol or by using add-on software packages such as MathType or its cut-down version Equation Editor (which comes with Microsoft Word); go to Insert>Object>Microsoft Equation or Insert>Equation, depending on the version. In these software packages, variables such as x, y, z are italicised by default. Examples can be seen in questions 21 and 24 of the Year 9 NAPLAN Calculator numeracy test and question 25 of the Year 9 Non-calculator numeracy test (ACARA, 2010). Failure to reproduce symbols correctly in teacher-generated materials may lead to confusion amongst students.

Conclusions

The examples above paint a picture of the linguistic complexity posed by the use of symbols and abbreviations in mathematics, especially the complexity of mapping words into symbols. Understanding the difficulties which students face when working with this unique part of the mathematics register can enable a teacher to pre-empt problems and assist learners to develop competence in this area of mathematics.

"Focusing on the use of language is a crucial strategy in good mathematics teaching and a teacher's guidance can assist students to master the language of mathematics" (Quinnell & Carter, 2011, p. 49). One step in this process is for teachers to encourage learners to develop knowledge of the use of mathematical symbols and abbreviations. The result may be the easing of the linguistic load for learners of the subject.

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