VISUALIZING DECIMAL MULTIPLICATION WITH AREA MODELS: OPPORTUNITIES AND CHALLENGES

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Abstract

This article describes a pilot study in which pre-service elementary teachers (PSTs) used rectangular area models on base-10 grid paper to begin making sense of multiplication of decimal fractions. Although connections were made to multi-digit whole number multiplication and to the distributive property, the PSTs were challenged by interpreting geometric aspects of the models they produced. In the context of these activities, the PSTs confronted concepts of dimensionality, unit labeling, and unit coordination that were not at issue in the domain of whole numbers. Tasks that help learners distinguish between one- and two-dimensional measurements are suggested as potential remedies.

Key Words: Preservice Teacher Education; Content Knowledge; Mathematical Knowledge for Teaching; Decimal Multiplication; Area Measurement; Area Models.

Introduction

The process standards of the Principles and Standards for School Mathematics (NCTM, 2000) emphasize that learning programs integrate problem solving, mathematical communication and multiple representations to connect ideas. Research has indicated that helping students achieve learning in an environment such as the NCTM envisions requires that teachers know the mathematics they will teach with depth and flexibility (CBMS, 2001). This presents a challenging goal to mathematics teacher educators in that prospective elementary teachers appear to lack profound mathematical understanding in several areas of elementary mathematics, including number, operations, and measurement (Ball, 1990; Graeber, Tirosh & Glover, 1989; Ma, 1999; Menon, 1998; Simon & Blume, 1994; Sowder, Armstrong, Lamon, Simon, Sowder, & Thompson, 1998; Tirosh & Graeber, 1989).

Although there exists a growing body of knowledge about teaching and learning of mathematics for children, less is known about how to support future teachers' facility with the mathematical knowledge they will need for teaching (Ball, Hill, & Bass, 2005; Ball, Lubienski, & Mewborn, 2001; Hiebert, Morris, Berk, & Jansen, 2007). To begin to address concerns about PST mathematical understanding, two content courses were developed at a small state university in the western United States. The curriculum for the first of these courses placed an emphasis on connecting students' procedural knowledge of numbers and operations with the concepts that underlie the procedures.

One area of noted difficulty for teachers was in trying to explain why the standard algorithm for multi-digit whole number multiplication works the way it does (Ma, 1999). As an extension of this difficulty, prospective teachers struggle as they try to understand how decimal fraction multiplication relates to the operation on whole numbers (Izsak, 2008; Thipkong & Davis, 1991). Instead, they rely on a rule that requires moving the decimal point in the product according to the number of decimal places in the factors. Mathematics educators, instructors for the courses in which this paper is situated, conjectured that these adult learners did not possess mental models of decimal numbers as fractions with base-10 structure. Further, the PSTs lacked interpretations of multiplication that were robust enough to accommodate decimal factors. Several authors have recommended the use of base-10 models such as base-10 blocks or grid paper to demonstrate the size relationships among decimal numbers (Bassarear, 2011; Owens & Super, 1993; Van de Walle, Karp, & Bay-Williams, 2010). As a co-designer of our first course for PSTs in number and operations, I helped to create and implement several tasks that utilized these materials to model decimal quantities.

Based on the idea that rectangular arrays can demonstrate properties of multiplication, reveal equivalent problems, and foster higher level multiplicative thinking, educators also suggest that these same base-10 materials be used to model multiplication of whole numbers and decimal fractions (Albert & McAdam, 2007; Bassarear, 2011; Englert & Sinicrope, 1994; Ross & Kurtz, 1993; Young-Loveridge, 2005). In our course for PSTs, base-10 rectangular area models were introduced as a robust interpretation for whole number and decimal fraction multiplication (Rathouz, 2011).

This article provides a glimpse into some of the adult learners' area representations as they try to make sense of multiplication of decimal numbers. What issues arise as adult learners use rectangular area models to make connections among procedures they know and the concepts that underlie those procedures? Which aspects do learners struggle with as they construct and interpret the models? These are questions we began to learn more about through observation and analysis of PST written work. Within this article, I present potential advantages and challenges of utilizing base-10 area representations with adult learners. Further, I describe a mathematical activity to address some of the noted challenges, all with the goal of helping PSTs to make sense of decimal numbers and the multiplication concepts and processes they may one day teach.

Whole Number Multiplication Models

During the semester-long pilot study, 35 PSTs were introduced to rectangular area models constructed with base-10 blocks and drawn on base-10 grid paper to visualize multi-digit whole number multiplication. The models were intended to help link the calculations of the standard algorithm for multi-digit multiplication to the distributive property and reveal the partial products as areas of smaller rectangles, understandings that Ma (1999) found lacking in teachers from the United States. The participants produced area models for multi-digit whole-number problems such as 57 x 14, shading and labeling four partial products in the model (500 + 200 + 70 + 28) (see Figure 1 as an example). During class discussions, several recognized that the distributive property was at work, noting the similarity to multiplying binomial algebraic expressions such as (5x + 7) (1x + 4) using "F.O.I.L.," (first, outside, inside, last).

Notably missing from Figure 1 are labels to indicate the dimensions of the rectangle. As I observed PSTs creating such diagrams, I noticed that they often "counted boxes" along the bottom of the grid and on the side of the grid to find the "length" and "width" of the rectangle. It was as if they were visualizing rows and columns of boxes in an array rather than thinking about the multiplicative relationship between the dimensions of the rectangle and its area. I wondered if it made a difference how they were interpreting the model, as long as they were "seeing" some of the relevant connections to procedures they already knew.

Figure 1. Base-10 Area Model of 57 x 14
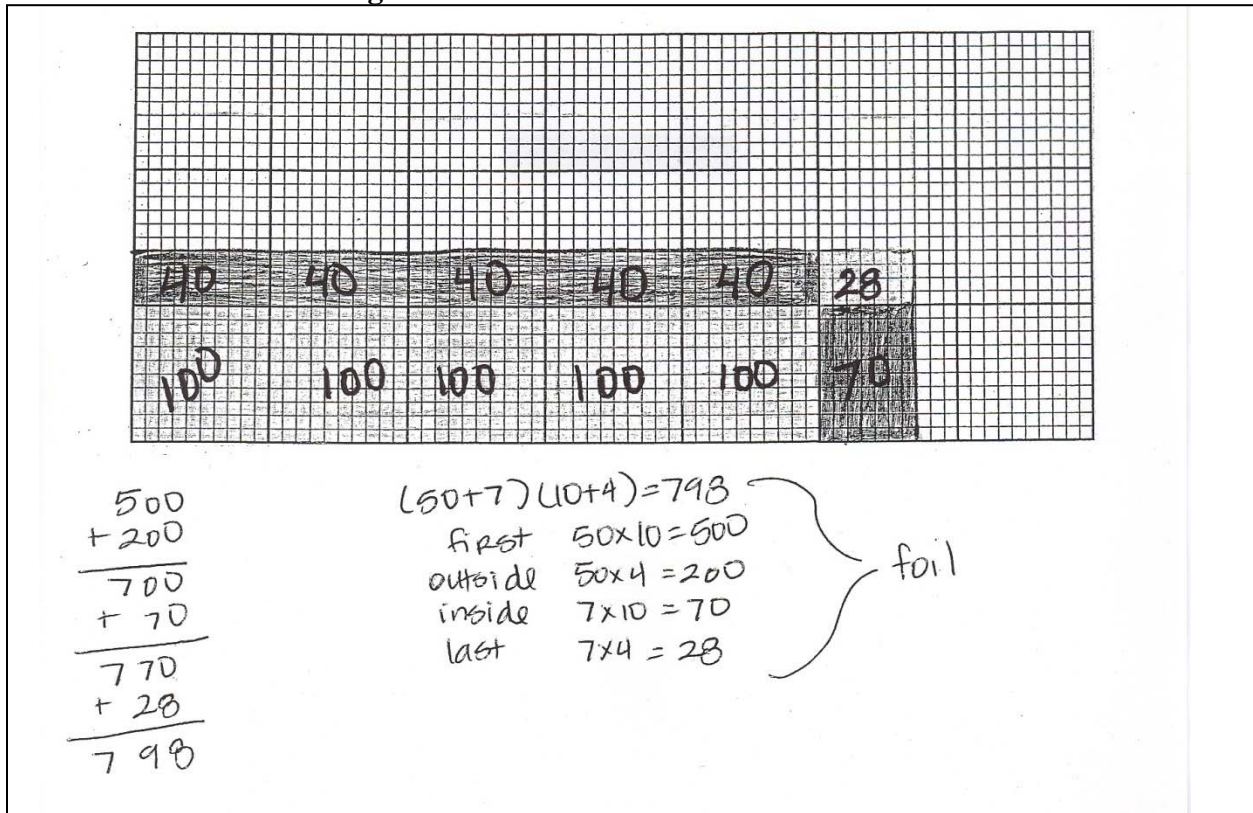
Decimal Fraction Multiplication Area Models

PSTs were then asked to make connections between models for whole number multiplication and related ones for decimal fraction multiplication. The goal was for PSTs to gain a tool to explain why multiplying tenths by tenths yields hundredths, why the decimal point is "moved" during multiplication, and how the partial products of the standard algorithm fit together. Indeed, the decimal multiplication problem 5.7 x 1.4 could be shown with the same base-10 diagram (see Figure 2) as the whole number multiplication 57 x 14, but with dimensions that are labeled as one-tenth the size of those in Figure 1.
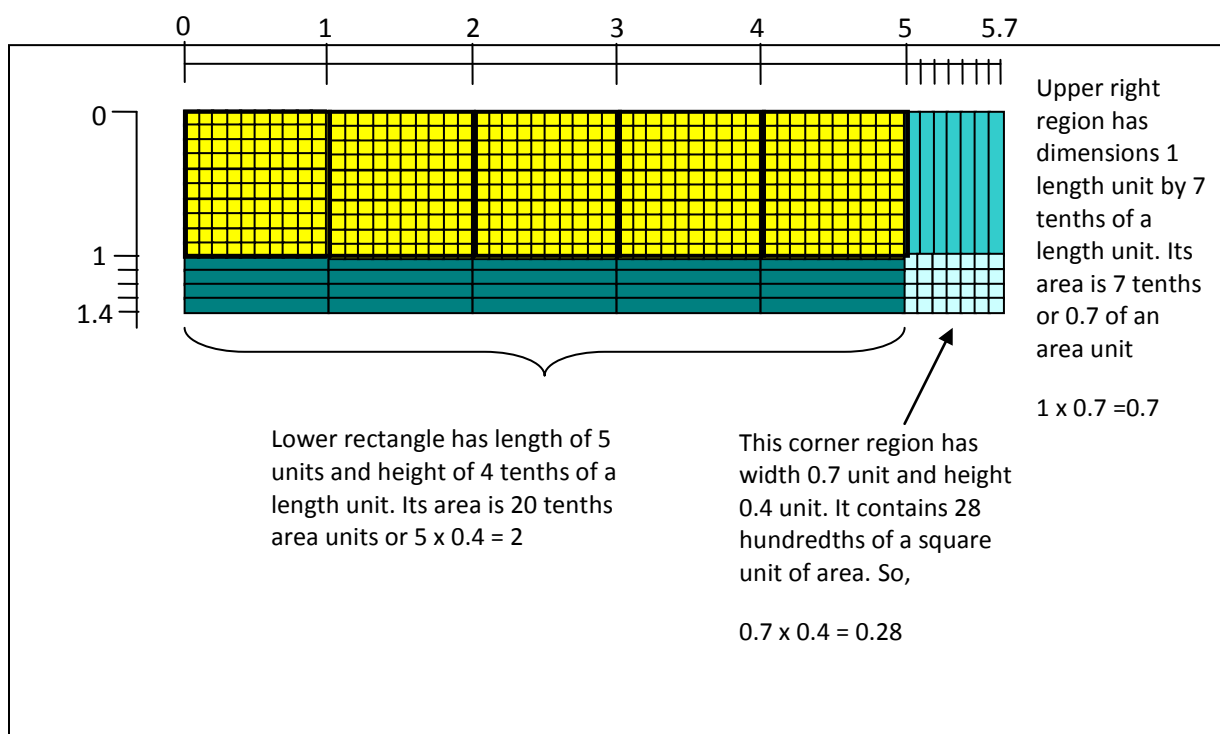
By constructing such models with decimal factors represented by the length and width of a rectangle and the product as the area, learners visualize the result of decimal fraction multiplication. However, in order to use representations to make connections among procedures they know and the concepts that underlie those procedures, learners must be able not only to construct the models, but also to interpret them. In particular, we would like PSTs

to note that the multipliers are the dimensions of the rectangle and the area shows the product. Further, in order to find the total product, one sums the areas of the smaller rectangles. In Figure 2, the smaller rectangles have the following dimensions: (1 x 5), (1 x .7), (.4 x 5), (.4 x .7) and respective areas of 5, 0.7, 2.0, and 0.28 square units, for a total of 7.98 square units. Finally, we want PSTs to make the connection that the smaller regions have the same dimensions and the same areas as the partial products produced using the distributive property as this calculation shows:

$5.7 \times 1.4 = (5 + .7) \times (1 + .4) = (5 \times 1) + (5 \times .4) + (.7 \times 1) + (.7 \times .4)$

$= 5 + 2.0 + 0.7 + 0.28 = 7.98.$

Figure 2. Base-10 Area Model of 5.7 x 1.4

Upper right region has dimensions 1 length unit by 7 tenths of a length unit. Its area is 7 tenths or 0.7 of an area unit

1 x 0.7 =0.7

Lower rectangle has length of 5 units and height of 4 tenths of a length unit. Its area is 20 tenths area units or 5 x 0.4 = 2

This corner region has width 0.7 unit and height 0.4 unit. It contains 28 hundredths of a square unit of area. So,

0.7 x 0.4 = 0.28

Selecting and Coordinating Attributes: Linear vs. Area Units

Several PSTs in this study successfully produced rectangular area models for multiplication, but failed to show the relevant relationships between the features of the rectangle (length, width, and area) and the numerical amounts in the calculation (factors and product). This lack of focus on two types of units became evident as some PSTs struggled to label the partial products appropriately. Examples shown in Figures 3-5 demonstrate how a lack of unit coordination may have led to an inappropriate interpretation of these representations.

In Figure 3, the PST used area units to indicate the "length" and "width" of the rectangle. Although the four partial products **circled** are those that group like place values, the **shading**

(see schematic in grey scale below student work) indicates a mis-interpretation of length and width as 2-dimensional area units.

Figure 3. Using Area Units to "Measure" Dimensions of a 4.3 by 2.6 Rectangle
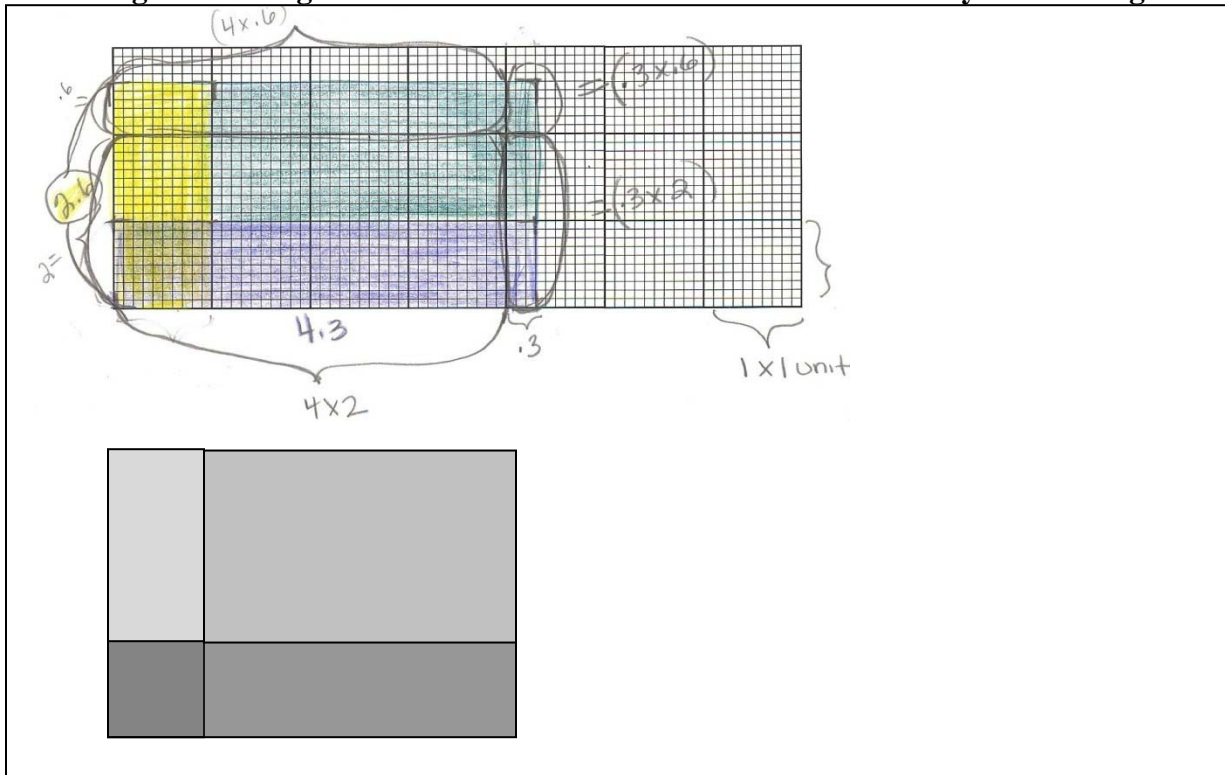
A similar mis-interpretation of length and width is seen in Figure 4, where the PST modeled the dimensions as 4.3 "strips" by 2.6 "strips." Although this PST found the correct product, the representation of one-dimensional length and width of the rectangle as "skinny area" units led to confusion for others in interpreting the value of the shaded area of the entire rectangle (see work in Figure 5).

The PST whose work is shown below in Figure 5, modeled 1.7 x 3.2, assigning a value of 1 unit to a "rod" (and, consistently, 0.1 unit to a "small box"). It seems that he used the **length** of these rods to measure the dimensions of the rectangle, 1.7 and 3.2, and then used the **area** of each rod to measure each partial product. Consistent with this interpretation, the three pink 1 x 1 regions are labeled "10 ones," the three dark green 1 x 0.7 regions are labeled "7 ones," and the light green region is labeled "20 tenths." Curiously, the black region is correctly (but inconsistently) labeled "14 hundredths." It appears that his mis-identification of length units as area units led to confusion about what values to give to the partial products, as noted by the crossed out labels.

Figure 4. Using "Longs" to "Measure" Length and Width

Figure 5. Confusion in Labeling Values of Partial Products

Analyzing PST Errors and Misconceptions

What can we learn about the thinking of pre-service elementary teachers from the types of errors we notice in this decimal multiplication task? How is their thinking related to that of children learning the same concepts? What are some conjectures as to the origin of the misconceptions? And how might these conjectures inform possible changes in professional development for elementary teachers?

The literature on children's learning of mathematics suggests that children struggle with measurement and geometry concepts (Clements, 2003; Lehrer, 2003). Children are not clear on attributes that can be measured or the units (e.g. inches vs. square inches) that should be used to measure those attributes. Procedural knowledge has been emphasized in mathematics classrooms to the extent that "Area" has a meaning for children and adults only in terms of the **computation** used to determine the area of a rectangle "L x W", not in terms of a measurable quantity (Izsak, 2004; Kamii & Kysh, 2006; Simon & Blume, 1994).

I conjecture that the adult student whose work is presented in Figure 3 saw one (large) area unit as "one unit" throughout the model (i.e. for both the factors and the product). This can happen when students are not comfortable measuring (or are not aware of the existence of) attributes from two different domains, one-dimensional lengths and two-dimensional areas, or interpreting units associated with each domain (Lehrer, 2003; Menon, 1998). An alternative explanation is that an array interpretation misled the PST to count large boxes as their units for rows and columns as is appropriate with whole number multiplication using arrays. In models with several possible "units," which of these units to attend to may be ambiguous (Izsak, 2005).

To understand the origin and problems with the length/area confusion, consider the transition a learner must make from discrete, array thinking to continuous area thinking. In doing so, they are forced to confront concepts of geometry (one vs. two dimensions) and measurement (attributes and units) that may have been latent with whole number multiplication. Note that with array models for whole numbers, whether a student is thinking about "one unit" as the side length of a small box or as the area of a small box, there is no confusion about what to call the product (10 rows of small boxes x 10 columns of small boxes = 100 small boxes). By contrast, with decimal multiplication, one small box has a side measure of one-tenth of a length unit, but an area measure of one-hundredth of an area (length^2) unit. Obviously, there could be some ambiguity if a student measures the "length" and "width" of the rectangular model by counting rows and columns of the small boxes. How many of the small boxes make one unit? Ten or one hundred? Examples of this type of confusion were noted previously by Ball, Lubienski, and Mewborn (2001) and are evident in the PST's work shown in Figure 5. His labels are all consistent with attempting to maintain the same size and type of unit (all area) for the product as for the factors. However, the 14 small boxes are inconsistently (but correctly) labeled "14 hundredths." As learners confront new ideas that are incompatible with their current knowledge, confusion and cognitive dissonance such as this is often observed.

Supporting PST Learning of Measurement Concepts

After examining the common types of errors modeling multiplication of decimal fractions, it became evident that adults re-learning mathematics for understanding encounter some of the same issues as children do when learning mathematical concepts for the first time (Graeber et al, 1989; Kouba and Franklin, 1993; Menon, 1998; Simon and Blume, 1994; Kamii & Kysh,

2006). To begin to address concerns, I designed and piloted activities that emphasized concepts of measurement of area and length. The course content for PSTs was re-sequenced so that these activities could occur early in their first mathematics course. Measurement concepts were revisited when the students attempted to use the models as thinking tools to understand decimal multiplication. An example of one of the activities can be seen in Figure 6.

One of the deliberate features of this activity sheet is that the base-10 grid lines are absent, requiring students to construct their own meaningful regions. Also, tools (metric ruler and base-10 tiles) are used to measure the attributes (length, width, and area). The order of tasks was considered as well: length measurement is emphasized before area measurement and the familiar formula for the area of a rectangle is used to calculate at the end of the activity. When discrepancies exist between the measured area and the calculated area, students confront those in small-group discussions. Throughout, students are required to explain their reasoning and to make connections between different representations.

Figure 6.

Length, Width, and Area Measurement and Justification

1. Measure (with a ruler) the length and width of the rectangle above using 1 decimeter as your unit. Briefly explain why the values make sense and show work on rectangle above (i.e. show what you measured).

2. Measure (with base-10 blocks to cover the interior) the area of the rectangle above using 1 sq. dm. (1 flat) as your unit of area. Briefly explain why the value makes sense (in terms of the sizes and quantities of the blocks). Show work on the rectangle above (i.e. show what you measured).

3. Calculate the area of the rectangle above using the formula $L \times W = A$. Use 1 dm as your unit for length and width and use 1 sq. dm as your unit of area.

Results from Measurement Task

Typical responses from piloting this activity confirmed that PSTs are comfortable with measurement concepts and units in a single domain (e.g. length only or area only), but struggle when they must coordinate two different types of units (i.e. length and area). For example, in response to the first question on the worksheet, one PST measured the length and width correctly and explained, "[the] length [is] 2.4 decimeters because 10 cm equal one decimeter. Width equals .3 decimeters because it is 3/10." Another wrote, "For every 10 cm it makes 1 dm. The width is 3 cm so that makes it 0.3 dm." However, when area was measured by covering the rectangle with base-10 tiles, these same prospective teachers reasoned the following, "Area of the rectangle is 7.2 sq decimeters…because there are six longs and 12 individual units. The 12 individual units turn into one long and 2 left over units, which is 7 longs and 2 individual units." They appear to be using a "long" to represent both 1 dm in length (it is, after all, one decimeter long) and, incorrectly, 1 square decimeter in area.

A third pre-service teacher seemed to rely on rules for moving the decimal point: "First I measured in cm and then converted to dm. 10 cm equals one dm so I converted by moving the decimal one place." To measure the area, "I used the blocks to fill the rectangle and then moved the decimal points 2 points to the left to make it squared." A picture indicates that she is counting 12 small cubes and 6 longs for a total of 72 (small cubes). She writes: "72 = .72 dm^2." When learners try to connect their procedural knowledge to physical models, often they "show" the answers they want without really explaining where the answers came from.

Only one of the twelve PSTs assessed in the pilot study was able to give an answer to the questions that showed complete understanding of the different units being used to measure length and area. For the first prompt, she wrote, "Unit in dm. 10 cm = 1 dm; .1 dm x 10 = 1 dm; The length is 1 dm plus 1 dm plus 4/10 dm. When added up it equals 2.4 dm. The height is only .3 dm, which is equal to 3 tenths of a dm. So the length is 2.4 dm while the height equals .3 dm."

For question 2, she appears to shift appropriately to discussing area: " 100 cm^2 = 1 dm^2; 10 rods = 1 sq dm; Flat = 1 sq. dm (10 x 10 sq. cm); rod = 1/10 sq dm; cm cube = 1/100 sq. dm. The rectangle is part of one flat. One flat is equal to 1 sq. dm. or 10 x 10 rods [ten groups of 10 rods as opposed to a region that is 10 rods long by 10 rods wide]. There are only seven rods and two small cubes which is less than a flat. Less than a flat is a fraction or percentage of the whole. Where 7 rods equals .7 of a flat and 2 squares equal .02 of a flat. 6 rods, 12 sm. cubes = 7 rods, 2 sm. cubes. .72 sq. dm."

PSTs' efforts on this measurement task support the conjecture that these learners can focus on one-dimensional length and width of the rectangle and seem to do so more easily in this case where there are no "grid lines" shown than in the case of the base-10 grid paper. However, measuring the area of the same rectangle proved more difficult even though the PSTs traced the base-10 block pieces with which they had covered the region, thus providing their own grid lines.

Lessons Learned from Studying PSTs' Mathematical Thinking

In creating curriculum for prospective teachers who are relearning mathematics for understanding, educators must be cognizant of the ideas these adult learners bring with them as well as which models to introduce. Assessing PST understandings prior to instruction, designing tasks to shore up conceptual knowledge, helping forge explicit connections to prior procedural knowledge, and refining the tasks based on PST interactions with the tasks, provides a useful cyclical model for improving course materials. In order for the process to operate effectively, however, student work must be analyzed thoroughly with an eye toward not only underlying conceptions, but also their accessibility to supportive representations (language, symbols, or diagrams). This analysis can then be used to consider the tasks and their sequencing.

In this study, area models were introduced to enrich learners' understanding of decimal number multiplication. Evidence of their difficulties with the model surfaced as frequent inability to track the type of unit (length or area) or its magnitude (does a small square correspond to a tenth or a hundredth?). The choice of several possible units to attend to on base-ten grid paper or base-10 blocks is a potential liability of these materials. In this unfamiliar territory of decimal numbers, PSTs confronted concepts of dimensionality, unit labeling, and unit coordination that were not at issue in the domain of whole numbers. Dufour-Janvier and colleagues (1987) claimed that "representations will be useful to the child only to the extent that they have been 'grasped' by him (p. 116)." The designers of the course discovered that this same idea appears to hold true for adult learners.

I have also found that adult learners (re-learners) often have deeper misconceptions than children. Instruction with these learners required "peeling off" these misconceptions and sometimes reordering the curriculum to allow students to confront their own misconceptions (Ball, 1988). For example, student work with the area models on base-10 paper indicated that we needed to back up to discover what is being measured and how each of these attributes is measured differently. Measurement activities such as the one shown in Figure 6 began to address these ideas, but clearly more support is needed.

Importantly, construction of mathematical ideas takes time and often involves a struggle. As we introduce models and meanings unfamiliar to our PSTs, these adult learners should be given the same opportunities to grapple with the mathematics as we expect them to give to their students in the future (Hiebert & Grouws, 2007). Also, when students are learning new material, course instructors might consider whether they are merely substituting a new procedure to replace an old one (see Resnick & Omanson, 1987; Thompson, 1992). In the case discussed in this paper, PSTs may "proceduralize" the creation of area models rather than try to make sense of the operation of multiplication by using the models as thinking tools that help them form ties to their prior algorithmic knowledge and build deeper understanding of concepts like the distributive property.

Finally, we should consider the expectations we have of students to infer the meanings and symbolism hidden in the models we introduce. Ultimately, students of all ages have to connect new ideas to their own meanings and models to make sense of the mathematics they use. Pressing learners to justify their reasoning and to explain the connections they find among representations can help to clarify and solidify the mathematical concepts behind familiar procedures.

Acknowledgements

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