Parabolas: Connection between algebraic and geometrical representations

Atara Shriki Oranim Academic College of Education, Israel <atarashriki@gmail.com>

A parabola is an interesting curve. What makes it interesting at the secondary school level is the fact that this curve is presented in both its contexts: algebraic and geometric. According to the intended curriculum in mathematics, in 9th grade, students should learn about quadratic functions, including simplification techniques. In the 10th grade, they are expected to solve a wide range of quadratic equations, construct graphs of parabolas, and connect algebraic and graphical representations of quadratic functions.

Being one of Apollonius' conic sections, the parabola is basically a geometric entity. It is, however, typically known for its algebraic characteristics, in particular as the expression of a quadratic function. How do these two entities, the geometric and the algebraic, coincide with one another? In this paper we try to answer this question.

Geometric and algebraic definitions of the parabola

We start by discussing some definitions of curves, followed by an examination of the relations between them.

Consider the following four definitions of sets of curves (Shriki & David, 2001):

- 1. Set No. 1 (Λ_1) : λ_1 is an element of Set No. 1 if and only if given line *l* and point *F* that is not on line *l*, λ_1 is the locus of points on the plane that are equidistant from both line *l* and point *F*.
- 2. Set No. 2 (Λ_2) : λ_2 is an element of Set No. 2 if and only if λ_2 is a graph of a function of the form $y = ax^2 + bx + c$, where $a \neq 0$ and a, b, $c \in R$.
- 3. Set No. 3 (Λ_3) : λ_3 is an element of Set No. 3 if and only if λ_3 is the graph of an implicit function of the form $y^2 = 2px$, where $p \neq 0$ and $p \in R$.
- 4. Set No. 4 (Λ_4) : λ_4 is an element of Set No. 4 if and only if λ_4 is the graph of a function whose form is a product of two non-constant linear expressions.

Now, (i) for each of the above definitions, draw a schematic curve that corresponds with the definition and describe its characteristics, and (ii) draw

a Venn diagram that describes the logical relations between the four sets of curves.

Algebraic definitions versus a geometric definition

It is apparent that the first definition is of a geometric nature, while the other three are algebraic in nature. The first definition is actually Apollonius' definition of a parabola. But what about the three other definitions? Do they also describe parabolas?

We start by examining the curve of λ_2 . High school students recognise the curve described by the expression $y = ax^2 + bx + c$, where $a \neq 0$ and $a, b, c \in R$, as a parabola. However, as mentioned, a parabola is defined using geometric characteristics, and thus, in order to justify the use of the term "parabola" for both curves, we must show that curve λ_2 satisfies the requirements presented for λ_1 . In other words, we must prove that for λ_2 , there exists a point *F* (the "focus") and a line *l* (the "directrix") as described for Set No. 1 (David & Shriki, 2002).

Figure 1 presents the graph of the function $y = ax^2$. The vertex of this graph lies on (0,0). If this graph is a parabola, the distance of this vertex from the focus should be equal to its distance from the directrix. Therefore, a suitable candidate for the focus is any point on the *y*-axis, F(0,k), and a suitable candidate for the directrix is any line that runs parallel to the *x*-axis, k = -k.

In order to find the value of k, we use the equidistance constraint $(x-0)^2 + (ax^2-k)^2 = (ax^2+k)^2$ and obtain $x^2 + (ax)^2 - 2ax^2k + k^2 = (ax)^2 + 2ax^2k + k^2 \Rightarrow x^2 - 4ax^2k = 0 \Rightarrow x^2(1-4ak) = 0$

 $x^{2} + (ax)^{2} - 2ax^{2}k + k^{2} = (ax)^{2} + 2ax^{2}k + k^{2} \Rightarrow x^{2} - 4ax^{2}k = 0 \Rightarrow x^{2}(1 - 4ak) = 0$ Thus, the solution is $k = \frac{1}{4a}$, and $F\left(0, \frac{1}{4a}\right)$; $l : y = -\frac{1}{4a}$.

Since we succeeded in finding a focus and a directrix that do not depend on the selection of points on λ_2 , we have proved that the curve described by the function $y = ax^2$ is indeed a parabola.

It is easy to generalise the proof for any quadratic function by considering the canonic equation $y = a(x - m)^2 + n$, where (m, n) is its vertex. We show that by applying two translations to the graph of $y = ax^2$: (1). In Figure 2, the graph is translated *m* units in parallel to the *x*-axes, so that its vertex is (m,0). Obviously, the directrix remains the same, but the focus is now $F\left(m, \frac{1}{4a}\right)$; (2).



In Figure 3, the graph is translated *n* units in parallel to the *y*-axes, so that its vertex is (m, n). Its focus is therefore $F\left(m, n + \frac{1}{4a}\right)$. The directrix should simultaneously translate *n* units in parallel to the *y*-axes, so its equation is $l: y = n - \frac{1}{4a}$.

To conclude, we proved that λ_2 is a parabola with $F\left(m, n + \frac{1}{4a}\right)$ and $l: y = n - \frac{1}{4a}$.

Note that we have shown that all λ_2 are λ_1 , but not vice versa. In other words, $\Lambda_2 \subset \Lambda_1$.

The same process can be repeated for λ_3 . Using the above considerations, the focus and directrix are described in Figure 4. From $y^2 = 2px$ we obtain $y = \pm \sqrt{2px}$, and therefore we have to solve the equation:

$\underbrace{\left(x-k\right)^2+\left(\sqrt{2px}\right)^2}_{=}=$	$= \underbrace{\left(x+k\right)^2}_{k=1}$
The distance of	The distance of
a point from the	a point from the
focus	directrix

 $x^{2} - 2kx + k^{2} + 2px = x^{2} + 2kx + k^{2} \Rightarrow 2px - 4kx = 0 \Rightarrow 2x(p - 2k) = 0 \Rightarrow k = \frac{p}{2}$ Consequently, $F\left(\frac{p}{2}, 0\right)$ and $l: y = -\frac{p}{2}$.

As in the case of λ_2 , we managed to find a focus and a directrix that do not depend on the selection of points on λ_3 .



Figure 4

Thus, all of the λ_3 curves are λ_1 (but not vice versa). It is now left to show that curve λ_4 is a parabola. The expression of λ_4 is

 $h(x) = f(x) \times g(x) = (ax + b) \times (cx + d) = acx^2 + (ad + bc) x + bd.$ This expression is identical to the algebraic expression of curve λ_2 , which has already been proved to be a parabola.

The question now is: Are sets Λ_2 and Λ_4 identical? Examining the expression of h(x) reveals that its roots coincide with those of the linear functions f(x) and g(x), since the solution of equation h(x) = 0 is equivalent to that of $f(x) \times g(x) = 0$. Given that f(x) and g(x) are not constant functions, h(x) must

have at least one real root. Curves that belong to Λ_2 , however, might not have real roots. This means that not all of the λ_2 curves are λ_4 curves, but all of the λ_4 curves are λ_2 curves.

Let us now further examine the relations between λ_2 and λ_4 . What does it mean to have two real roots? One real root? No real roots?

We usually find the number of real roots of a quadratic function by examining its discriminant. However, observing the expression of λ_4 and realising that the roots of $h(x) = f(x) \times g(x) = (ax + b) \times (cx + d)$ are determined by the roots of f(x) and g(x), is it evident that a quadratic equation has no roots whenever it is impossible to factor its expression into a product of two linear functions. This is equivalent to $b^2 - 4ac < 0$. As for one real root: since the root of f(x) is $x = -\frac{b}{a}$ and the root of g(x) is $x = -\frac{d}{c}$, we obtain one root only if $-\frac{b}{a} = -\frac{d}{c}$. In other words, when c = ka and d = kb ($k \neq 0$), namely: $-\frac{b}{a} = -\frac{d}{c} = -\frac{kb}{ka}$. We therefore obtain

$$h(x) = f(x) \times g(x) = (ax + b) \times (kax + kb) = k(ax + b)^{2}$$

which is the known expression for a quadratic equation with a single real root. This is equivalent to $b^2 - 4ac = 0$. Finally, the quadratic equation has two real roots only if $-\frac{b}{a} \neq -\frac{d}{c}$, as the roots of f(x) and g(x) are different. This is equivalent to $b^2 - 4ac > 0$.

A Venn diagram describing the four sets of curves

Set Λ_1 is obviously the most comprehensive one. As mentioned, $\Lambda_4 \subset \Lambda_2$. For curves λ_2 and λ_4 , the directrix is parallel to the *x*-axis and for λ_3 it is parallel to the *y*-axis. Λ_2 and Λ_3 are, therefore, distinct sets.

The following Venn diagram represents the relations between the four sets (Figure 5):



Figure 5

Are there any other sets of parabolas that are sub-sets of Λ_1 ?

Since we differentiated between the parabolas in Λ_2 and Λ_3 according to the position of their directrix, other sub-sets of Λ_1 include parabolas with directrices that are not parallel to any one of the axes. Indeed, the geometric definition of curve λ_1 does not impose any restriction concerning the direction of the directrix.

It should be noted that the algebraic expression that refers to all possible parabolas is derived from the general, second-degree equation of the conic sections, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. When $h^2 - ab = 0$, the obtained curve is a parabola (for further details, see Wolfram MathWorld, at http://mathworld.wolfram.com/ConicSection.html).

Comments

Use the term "a graph of a quadratic function" with care

If we ask students to define a parabola, they will probably say that it is a graph of a quadratic function. It is, however, evident that a parabola is not necessarily a graph of a quadratic function (for example, curve λ_3 ,). Students should therefore be instructed to say, "The graph of a quadratic function is a parabola," but not vice versa.

Is the graph of $y = x^4$ a parabola?

Many students tend to identify graphs of the family $y = x^n$ (n > 2, even) as parabolas. It is easy to refute this using the geometric definition. Consider the function $y = x^4$. If the graph of the function $y = x^4$ is a parabola, then according to the definition of the parabola, the following should exist (see Figure 1):

$$x^{2} + (x^{4} - k)^{2} = (x^{4} + k)$$

$$\Rightarrow x^{2} + x^{8} - 2x^{4}k + k^{2} = x^{8} + 2x^{4}k + k^{2}$$

$$\Rightarrow x^{2} (1 - 4x^{2}k) = 0$$

$$\Rightarrow k = \frac{1}{4x^{2}}$$

Since the coordinates of point *F* depend on the choice of point (x,x^4) , we can safely conclude that the graph of $y = x^4$ is not a parabola. The same is obviously true for other functions of the form $y = x^n$ (n > 2, even).

References

- David, H. & Shriki, A. (2002). Connections within mathematics: What questions should be asked and what answers should be given? *Proceedings of the 2nd International Conference on the Teaching of Mathematics*, Hersonossos, Crete. Available from http://www.math.uoc.gr/~ictm2/
- Shriki, A. & David, H. (2001). Parabola and other conic sections. In *Promoting mathematical and didactical knowledge*. Israel National Center for Mathematics Education, Technion Israel Institute of Technology (in Hebrew).