A square becomes a regular octagon: An authentic experience in proof writing

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> The promotion of proof as a process through which mathematics knowledge and understanding have been constructed will not necessarily motivate students, though, unless they believe that they are participating in meaningful mathematical discovery (Vincent, 2005, p. 94).

In high school geometry courses, students are often given a prepackaged statement that they are asked to prove. In these situations, the process of writing proofs is being abridged, if not misrepresented. To provide my students with a more authentic experience in writing a proof, I provided them with a summative project for which they had the opportunity to explore a problem, make a conjecture, and then write a proof to justify that conjecture.

In particular, the assignment examined in this article, the octagon problem, was developed from a piece of children's literature. Although the intended audience of the book is the elementary grades, the content was extended to a high school honours geometry course. The octagon problem is a rich mathematical task requiring deductive reasoning that incorporates several geometric concepts including regular polygons, angle measures of a regular polygon, transformations (reflection, rotation, and dilation), scale factors, properties of right isosceles triangles, and properties of quadrilaterals.

After discussing how the project was conceived, I provide the details of the assignment and information about my students' prior experiences in geometry. For the remainder of the article, I share the students' work and the challenges encountered with writing the proof. I wanted to experience the exploration with my students; so I deliberately chose not to work out the problem for myself prior to giving the task to them. As a result, I describe a unique aspect of this particular proof, discovered when I was assessing the students' arguments and make some suggestion for teachers who find themselves in similar situations.

The origin of the project

Long and Crocker (2000) wrote an article about the use of Cindy Neuschwander's (1997) book, *Sir Cumference and the First Round Table: A Math Adventure*, in a fifth-grade mathematics class. The story revolves around the trials of Sir Cumference, his wife, Lady of Diameter, and his son, Radius, as they attempt to create a table that fits King Arthur and his knights comfortably. Originally, the table is a rectangle which is then sequentially transformed into a square, a parallelogram, an octagon, an oval, and then finally a circle.

In the article, there is a diagram demonstrating the transition from a rectangle to a square. Since I asked my students to make a parallelogram from a rectangle in my geometry classes in order to discover the formula for the area of a parallelogram, I was able to visualise the change from a square to a parallelogram as I read the article. Then, I came upon the following description: "The text shows how to cut off irregular quadrilaterals from the corners of the parallelogram, to form an octagon; parts of the quadrilaterals are then attached to the first octagon to make it a regular octagon" (Long & Crocker, 2000, p. 243). Since, without a diagram, I was not able to see this transformation, I bought the book. Even after seeing the diagram, shown in Figure 1, I was still curious: How does one know where to make the cuts so that the final octagon is, in fact, a regular octagon?

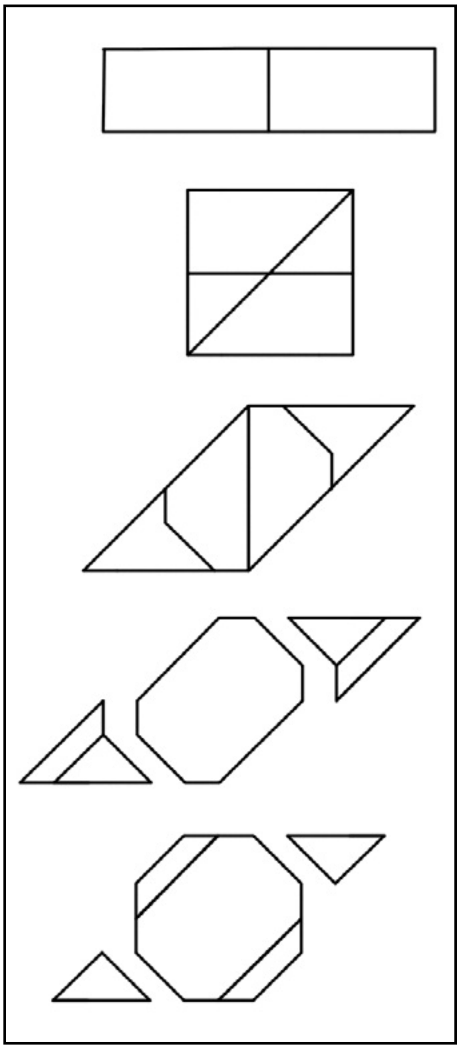

Figure 1. The transformation of the table from a rectangle to a square to a parallelogram to a regular octagon (Neuschwander, 1997).

The activity

I decided to use the regular octagon construction as a culminating project in my Honours Geometry class. To introduce the assignment, I read the book to the students. As the story progressed through the different shapes, the students used a pre-made template (Figure 2), which included the cuts, to imitate the changes to the table. The students were then presented with the directions:

1. Sketch [in The Geometer's Sketchpad (GSP)] the square with the cut lines that will form a regular octagon [Figure 2]. Make

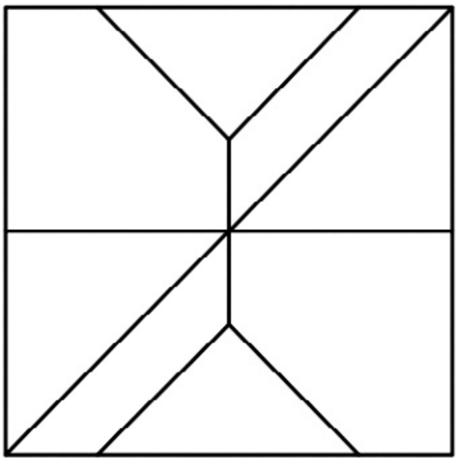

Figure 2. The square with the cuts that the students were asked to recreate using GSP.

sure that your diagram remains similar when you move any segments or vertices. Explain how you constructed the figure.

2. Write a proof showing that the octagon constructed from the square is a regular octagon. The proof can be written in any format.
3. Did drawing the diagram in Sketchpad help you write your proof? If yes, explain how. If no, how did you develop your proof?

I encourage the reader to take some time to understand and investigate the problem before continuing.

My students, ninth and tenth graders, had used GSP throughout the year so they were very familiar with the basic tools as well using them to construct figures such as square. The Honours Geometry course also had a strong emphasis on proof and the students were exposed to different formats for

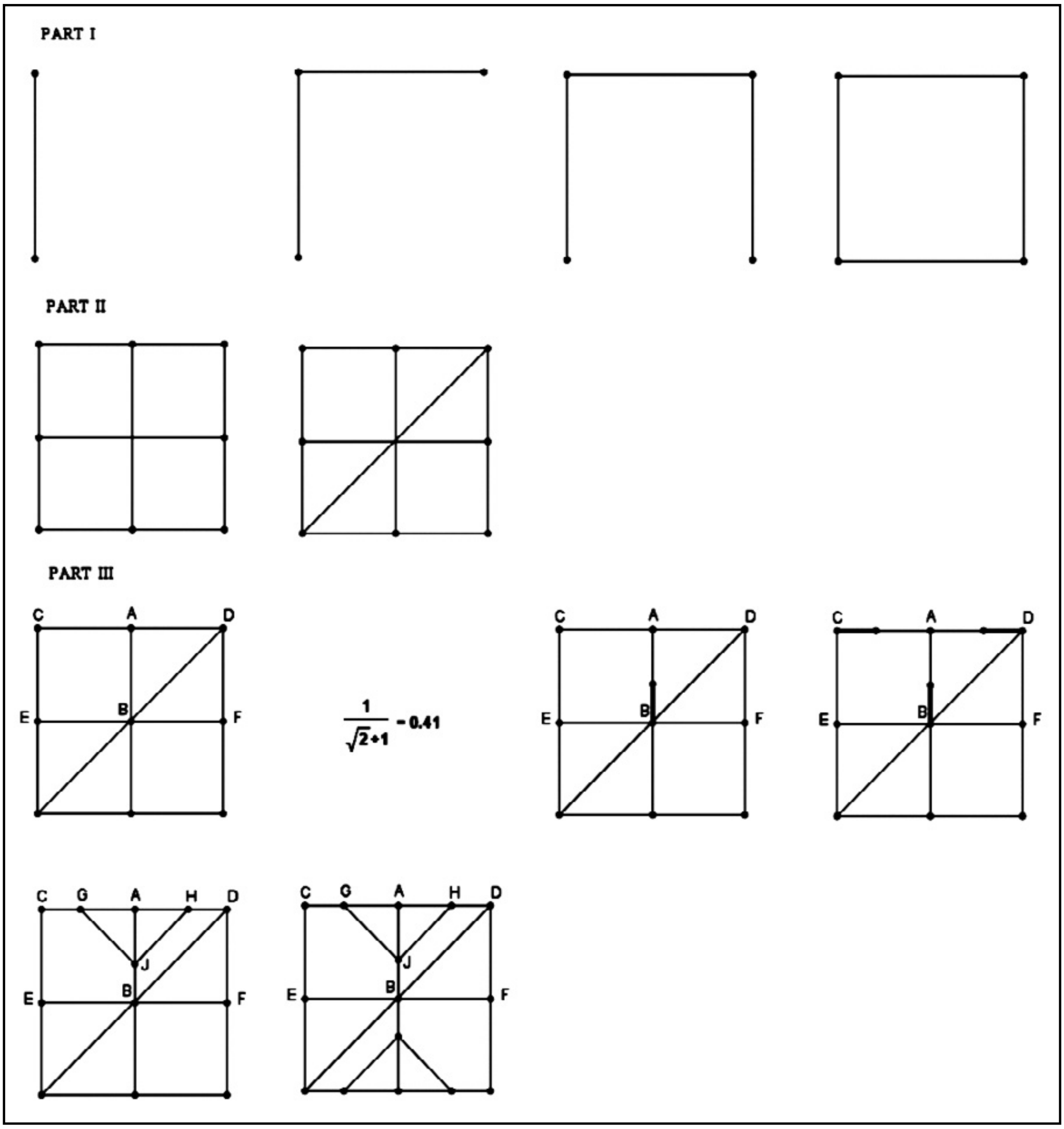

Figure 3. The student diagrams that correspond to the directions using the scale factor approach.

writing arguments such as paragraph, flow, and two-column. The students spent two 60-minute class periods in the computer lab creating their sketches (Question 1). Questions 2 and 3 were to be completed on the students' own time. However, several of the students stayed after school to work with me and their classmates.

The sketches

The ways in which the students constructed the square with the cuts can be classified into two different strategies. The first of these is referred to as the 'scale factor strategy' (see Figure 3) and the second the 'working backwards strategy'. The students who used the first strategy found the ratio of two segments (see Figure 3)

$$\frac{CG}{CA}$$

and used this as a scale factor for a dilation to determine the point (*G* in Figure 3) which would guarantee a regular octagon. For example, one of the students gave the following directions. Refer to Figure 3 for the diagrams that correspond to the steps. (The portions within brackets are added by the author for clarification purposes.)

PART I: Construct a square
1. Create a segment.
2. Mark one end as the centre (double click).
3. Select segment and other point (not the centre [of rotation]).
4. Click transform and rotate [90°].
5. Select new point and mark it as your new centre.
6. Repeat steps 3 to 5 until you have a square.

PART II: Setting up
1. Select segment and construct midpoint.
2. Repeat with opposite segment.
3. Connect the two midpoints.
4. Repeat for the [other pair of opposite] sides.
5. Create a diagonal from the upper right to the lower left of the square.

PART III: Construct the octagon
1. Mark *B* as the centre.
2. Select calculate and enter $\frac{1}{\sqrt{2}+1}$.

   [The calculation of this scale factor follows.]
3. Create segment $\overline{AB}$ and then select segment $\overline{AB}$.
4. Dilate the answer [segment] from calculating $\frac{1}{\sqrt{2}+1}$.
5. Create a segment on top of $\overline{CD}$ to create $\overline{CA}$ and another to create $\overline{AD}$.

6. Mark C as the centre and select $\overline{CA}$ and A.
7. Repeat [step number] 4.
8. Do the same for $\overline{AD}$ with D as the centre.
9. Connect [G to J and H to J].
10. Mark $\overline{EF}$ as your mirror and reflect the new points (G and H [and J])
11. [Repeat step number] 9.

Following the directions, the student justified the use of

$$\frac{1}{\sqrt{2}+1}$$

as the scale factor for making the dilation. She focused on the top right corner of the square with the cuts (see Figure 4) and let x represent one-half the length of one of the sides of the regular octagon ($\overline{CG}$). It then follows that $\overline{GJ}$, a side of the regular octagon, is $2x$. Since $m\angle BJG = 135$ (because it is an angle of a regular octagon) and $m\angle A = 90$, $m\angle AJG = m\angle AGJ = 45$. Therefore, ΔAJG is a right isosceles triangle and $\overline{AG} = \overline{AJ} = x\sqrt{2}$. Using this information, the scale factor is calculated:

$$\frac{CG}{CA} = \frac{x}{x\sqrt{2}+x} = \frac{1}{\sqrt{2}+1}$$

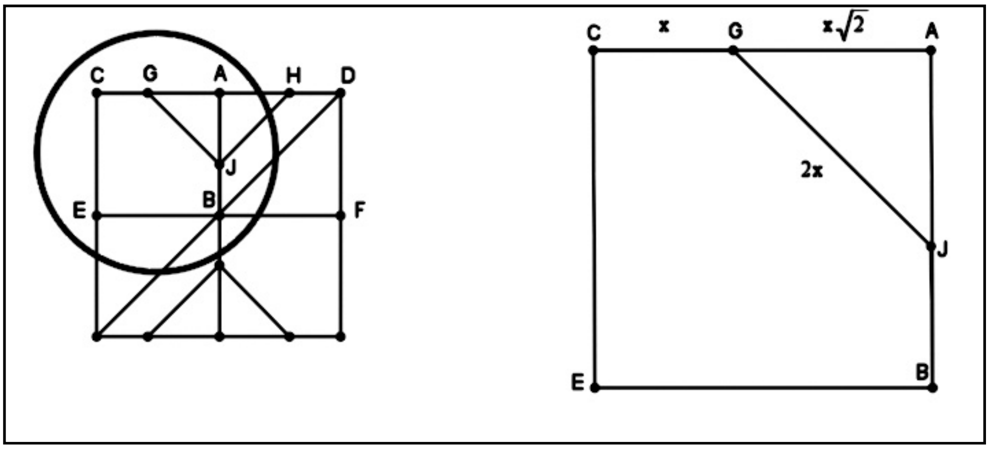

Figure 4. The student diagram used to explain the calculation of the scale factor $\frac{1}{\sqrt{2}+1}$.

The students who used the second strategy to construct the square with the cuts worked backwards. Since the square was transformed into the regular octagon, the students created a regular octagon first. The directions written by one of the students and the corresponding diagrams (Figure 5) follow.

1. Create octagon: a. create segment, b. rotate entire segment (including points) 135 degrees around the centre of rotation (change centre of rotation each time you rotate)
2. Create your octagon.
3. Pick two parallel segments and connect midpoints of these segments.
4. Hide the 5 segments (including parallel ones, above) to the left of the middle segment.
5. Connect the ends of the middle segment to the rest of the drawing.

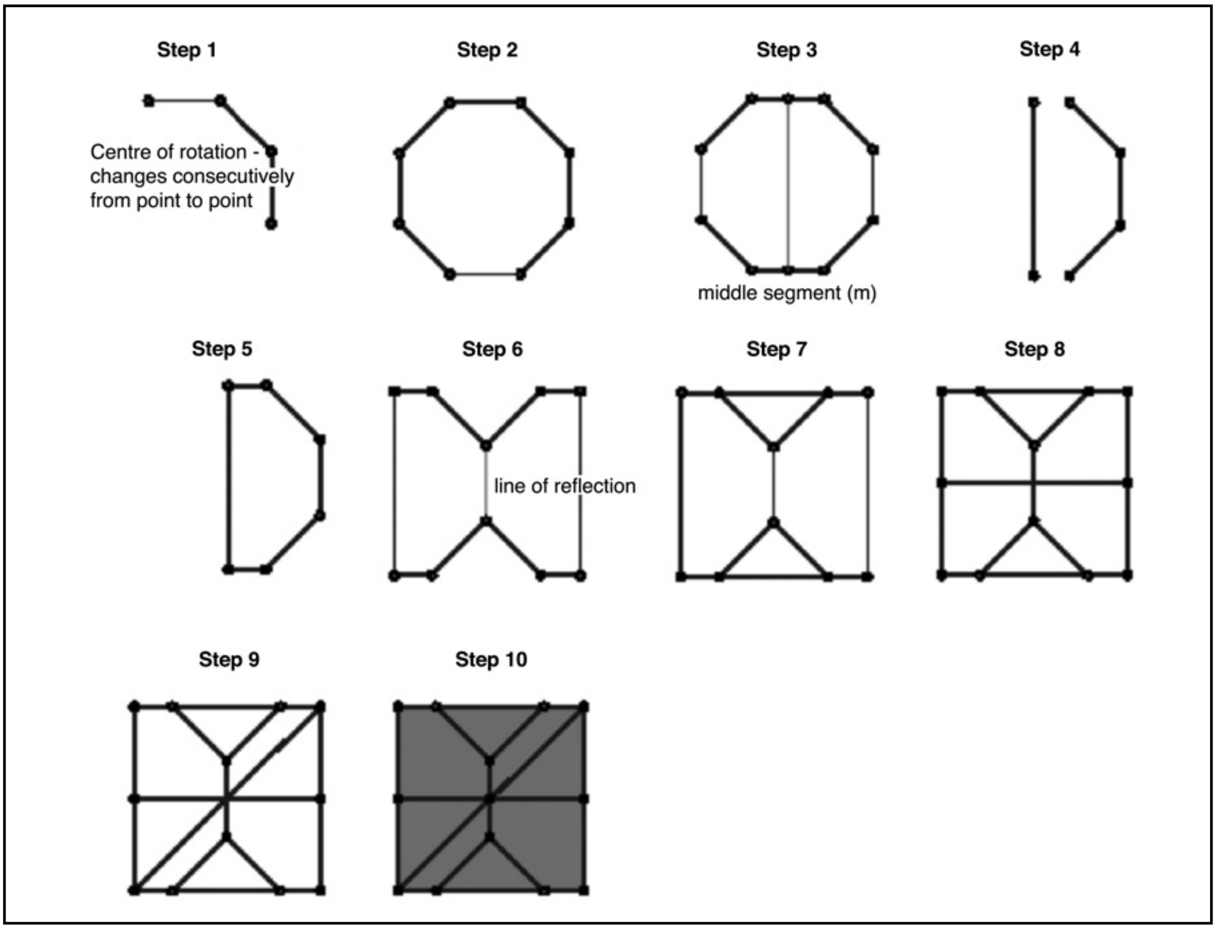

Figure 5. Student sketches for the working backwards strategy.

6. Reflect [half of the] octagon using the segment parallel to the middle segment as the line of reflection.
7. You have a concave polygon; connect the points to make a square.
8. Create the midpoints of the two sides of the square parallel to the middle segment.
9. Create diagonal of square.
10. Select 4 points of square and create quadrilateral interior.

Initially, the students were focused on the square. Once they saw the octagon within the square, they were able to work in the opposite direction.

The proofs

When it came to writing their proofs, the students encountered some road-blocks. Approximately half of the students wrote arguments that mirrored the scale factor strategy. Instead of explaining why the octagon was a regular octagon, the 'proof' read more like directions on how to construct the regular octagon. The basic premise is that the side length of the regular octagon can be determined in relationship to the side length of the square. With that ratio, the location of the cuts can be determined. However, instead

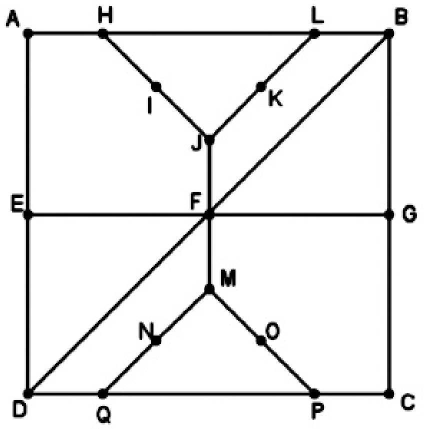

Figure 6. The diagram used by the student with the circular argument.

of proving that the octagon was a regular octagon, the students assumed the octagon was a regular and used this fact to determine the ratio of the side of the regular octagon to the side of the square.

Other students did attempt to prove that the octagon was a regular octagon by showing that all of its angles measured 135° and all of its side lengths were congruent. Many of the issues in their proofs involved circular arguments. For example, in explaining why all of the angles measured 135°, one student wrote (refer to Figure 6): "If angle *BDQ* is 45 degrees, then angle *MQP* is 45 degrees, because corresponding angles are congruent where parallel lines are present (parallel lines are present because of the square)." Two lines later, the student continued: "Now, a trapezoid *DQMF* exists (*DF* is parallel to *QM*, because it is already established that the angles *FDQ* and D*QM* are 45 degrees and 135 degrees respectively, and if the same side interior angles are supplementary, then the lines are parallel)." In the first statement, the student essentially assumes the segments are parallel to determine the angle measures. The student then uses those angle measures to explain why the two segments are parallel.

In discussing the flaws in the students' proofs with a colleague, we discovered the problem. There are infinite number of locations in which the cuts can be made to form an octagon (Figure 7). Yet, there is a unique cut that will transform the parallelogram into the regular octagon (bold cuts shown in Figure 7).

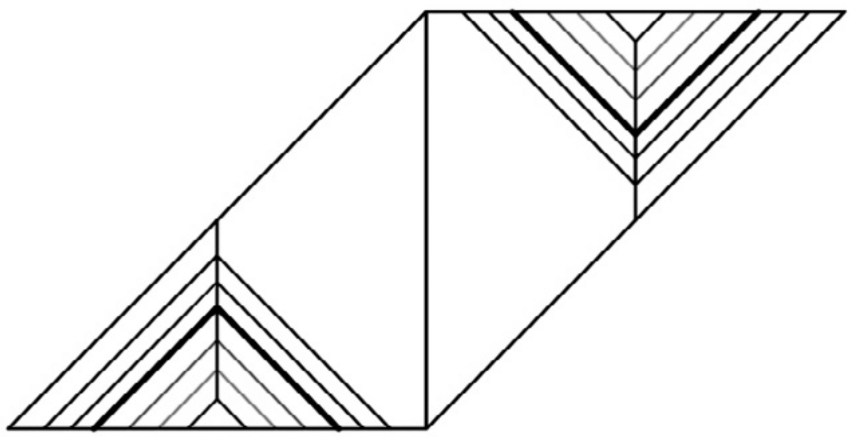

Figure 7. Examples of cuts that will produce an octagon, with the bold cuts producing a regular octagon.

Therefore, to write the proof you need to know and use the location of the point on the side of the parallelogram (or square) at which the cuts are made. Herein lies the challenge with this proof. To construct the argument, one cannot simply start at the beginning and progress to the desired end, proving the octagon is regular. Instead, one has to work backwards in order to figure out the given information. (For a similar problem, refer to Fermat's problem to Torricelli: "Determine a point P in a given triangle ABC such that the sum $PA + PB + PC$ is a minimum" (Honsberger, 1973, p. 24)). In essence, the students whose "proofs" were instructions actually discovered the statement of the problem, which can be written as follows (refer to Figure 8):

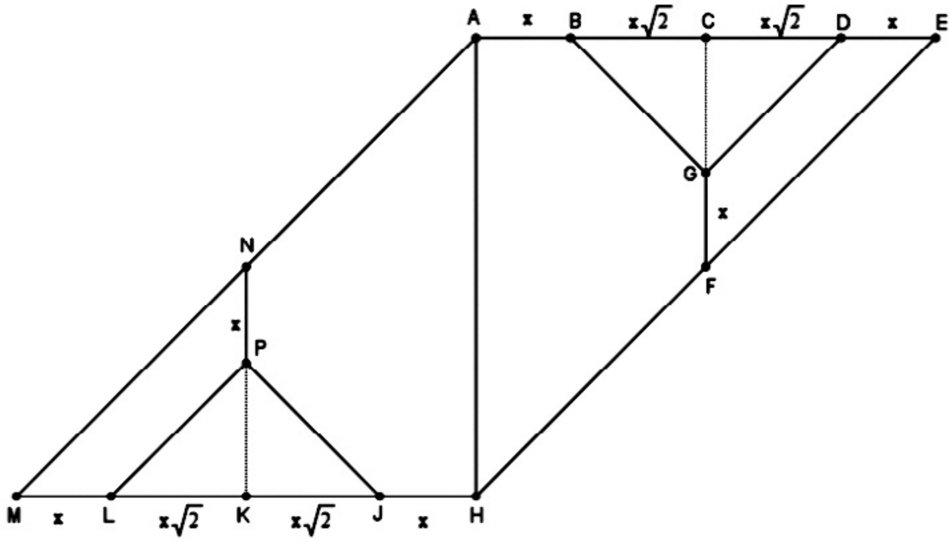

Figure 8. Parallelogram used to write the problem statement.

Suppose a square is cut on a diagonal and one of the triangles is translated to form a parallelogram.

- Mark B a distance $\frac{1}{\sqrt{2}+1}$ of one-half the length of the horizontal side of the parallelogram (or square) from A.
- Mark D the same distance from E.
- Mark G the same vertical distance from F.
- Connect B and G.
- Connect G and D.
- Cut along $\overline{BG}$, $\overline{GF}$ and $\overline{GD}$.
- Rotate trapezoid $DEFG$ 180° clockwise about F.
- Repeat for the opposite side of the parallelogram.

Prove that the resulting octagon is regular.

For an example of a possible proof, refer to the Appendix.

Conclusion

Initially, with the octagon task, it appeared as if the problem statement was given to the students: prove that the octagon constructed from the square is a regular octagon. Later, I realised that the problem statement was incomplete, and, in order to determine the complete statement, it was necessary for the students to explore, discover, and conjecture. In describing how she teaches proof, Muller (2010) further delineates the stages through which her students progress: "exploration, discovery, conjecture, further exploration, brainstorming, writing, assessment, and rewriting" (p. 438). The last two stages, assessment and rewriting, were unfortunately not a part of my implementation of the octagon project since it occurred in the last days of the school year. A class discussion about the student errors could have led the students to the discovery, made by me and my colleague, and enabled them to complete the proof. All of the stages outlined by Muller should be incorporated into this or any similar assignment so that students authentically experience the process of proof writing.

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Appendix

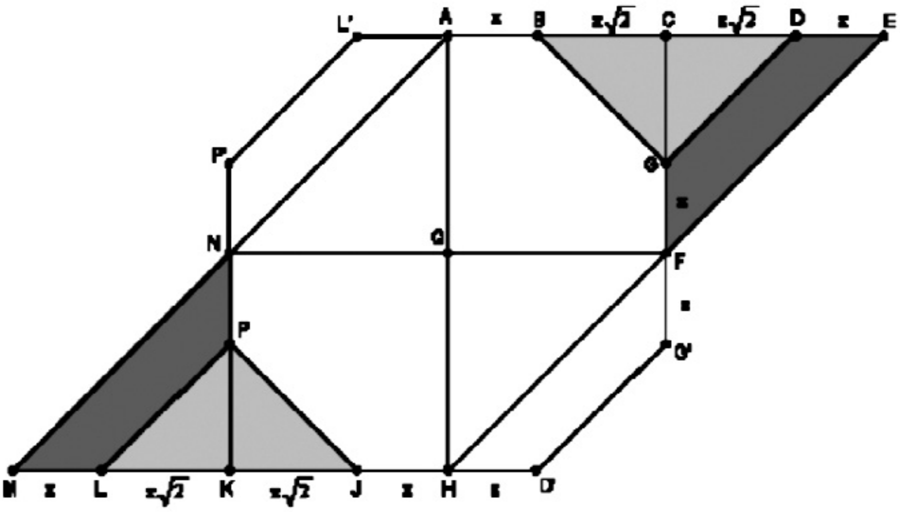

Figure A1. Diagram used for proof.

Suppose a square is cut on a diagonal and one of the triangles is translated to form a parallelogram.

- Mark B a distance
  $$\frac{1}{\sqrt{2}+1}$$
  of one-half the length of the side of the parallelogram (or square) from A.
- Mark D the same distance from E.
- Mark G the same vertical distance from F.
- Connect B and G.
- Connect G and D.
- Cut along $\overline{BG}$, $\overline{GF}$ and $\overline{GD}$.
- Rotate trapezoid $DEFG$ 180° clockwise about F.
- Repeat for the opposite side of the parallelogram.

Prove that the resulting octagon is regular.

To prove that octagon $P'L'BGG'D'JP$ is a regular octagon, we must show that all of its angles measure 135° and all of its sides are congruent.

Angles

First, to show that $\Delta Q'MF$ and ΔQAN are congruent, right isosceles triangles, refer to Figure A2. $m\angle Q'MF = m\angle QAN = 45°$ since the diagonal of a square bisects opposite angles. $\angle MQ'F$ and $\angle AQN$ are right angles since they were originally angles of a rectangle. Similarly, $Q'M = QA$ because they were originally opposite sides of a rectangle. Therefore, $\Delta Q'MF$ and ΔQAN are congruent, right isosceles triangles, F is the midpoint of $\overline{Q'Q}$ and $QF = AC$ (Figure A1). (It is given that C is the midpoint of $\overline{AE}$.)

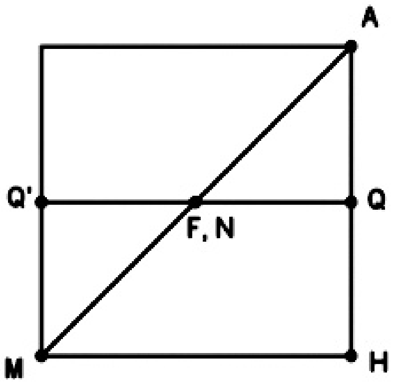

Figure A2. Diagram used to show that $\Delta Q'MF$ and ΔQAN are congruent right isosceles triangles.

Return to Figure A1. Since $\angle QAC$ and $\angle AQF$ are right angles (from the rectangle) and $QA = QF = AC$, quadrilateral $ACFQ$ is a square. Therefore, $\angle GCB$ is a right angle and $AC = FC$. Combining this information with the given information leads to the facts that $\overline{GC}$ is the perpendicular bisector of $\overline{BD}$ and GC = $x\sqrt{2}$. These two pieces of information indicate that ΔBCG and ΔDCG are congruent, isosceles right triangles, and $m\angle CBG = m\angle CDG = 45°$ and $m\angle BGD = 45° + 45° = 90°$. Since $\angle CBG$ and $\angle ABG$ form a linear pair, $\boldsymbol{m\angle ABG = 135°}$. Similarly, $m\angle GDE = \boldsymbol{m\angle G'D'H} = 135°$. Since $m\angle E = 45°$ (bisected angle of square), $\angle GDE$ and $\angle FED$ are supplementary. Combining this information with $DE = GF$ (given), $\overline{GD} \parallel \overline{FE}$ and quadrilateral $GDEF$ is an isosceles trapezoid. Therefore, $m\angle GDE = m\angle FGD = \boldsymbol{m\angle FG'D' = 135}$. Since $m\angle FGD = 135°$ and $m\angle BGD = 90°$, $\boldsymbol{m\angle BGF = 135°}$. Repeat this argument for the opposite side of the parallelogram to show that

$$\boldsymbol{m\angle HJP = m\angle JPN = m\angle NP'L' = m\angle P'L'A = 135°}.$$

Sides

In the process of arguing that all of the angles of the octagon measured 135°, it was shown that ΔBGD is an isosceles right triangle ($m\angle CBG = m\angle CDG = 45°$ and $m\angle BGD = 45° + 45° = 90°$). Since $BD = 2x\sqrt{2}$, $\boldsymbol{BG} = GD = \boldsymbol{G'D' = 2x}$. From the given information, $GF = FG' = x$ and $GF + FG' = \boldsymbol{GG' = 2x}$. Likewise, $D'H = HJ = x$ and $D'H + HJ = \boldsymbol{D'J = 2x}$. This argument can be repeated to show that $\boldsymbol{JP = PP' = P'L' = L'B = 2x}$.