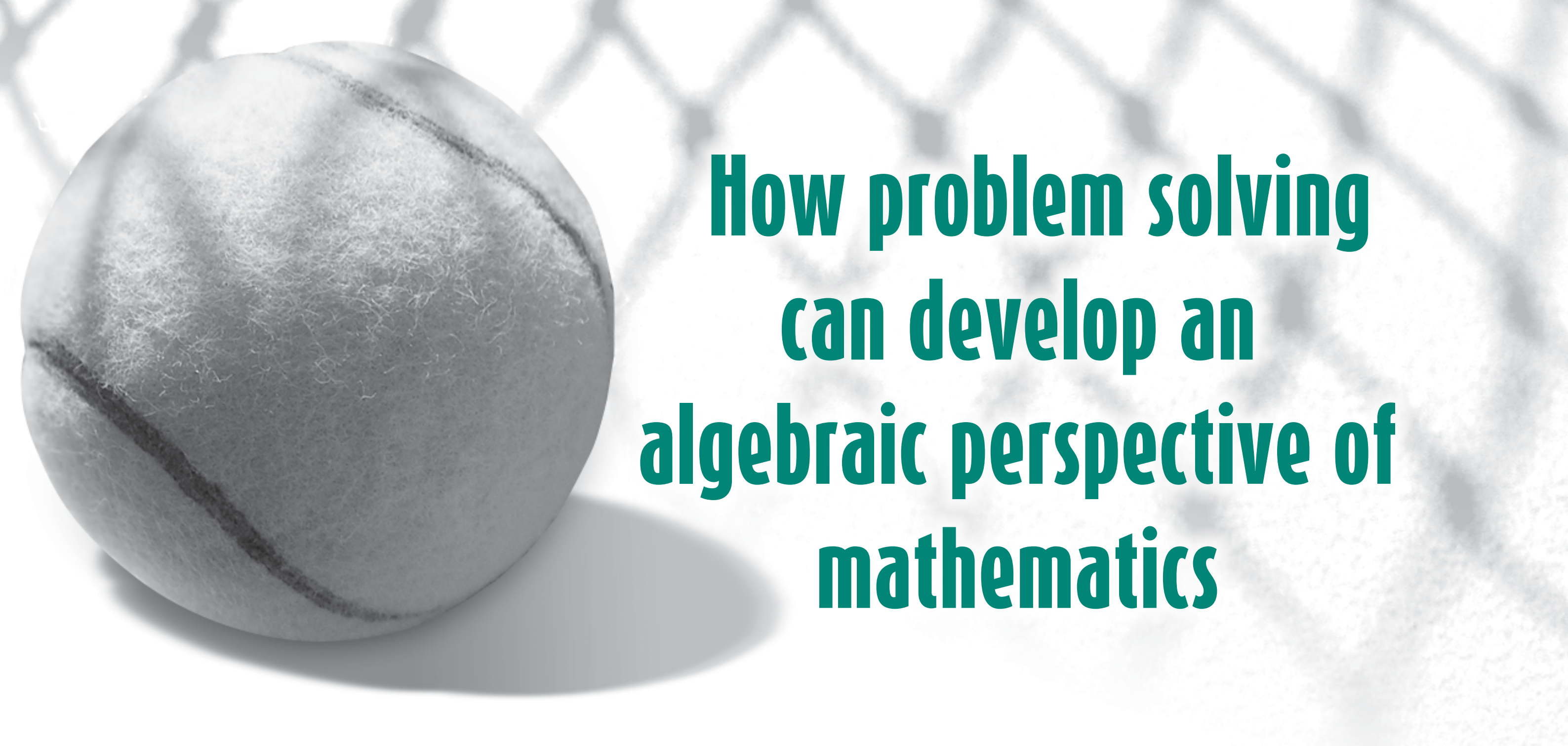

How problem solving can develop an algebraic perspective of mathematics

WILL WINDSOR provides a brief overview of problem solving and links it to algebraic thinking. Readers are encouraged to try the tasks with their own students and then re-read the article.

Introduction

Problem solving has a long and successful history in mathematics education and is valued by many teachers as a way to engage and facilitate learning within their classrooms. The potential benefit for using problem solving in the development of algebraic thinking is that "it may broaden and develop students' mathematical thinking beyond the routine acquisition of isolated techniques and procedures often associated with secondary school algebra" (Booker, 2007; Kaput, 2008; Carraher & Schliemann, 2007; Lins, Rojano, Bell & Sutherland, 2001). Furthermore, the thinking required to solve problems can be extended from methods tied to concrete situations—the backbone of primary school mathematics—to experiences that develop an ability to solve problems using abstractions based on the relationships within the problems. By establishing an algebraic perspective of problem solving it acknowledges that "students can adapt their ways of thinking, they can express mathematical generalisations and it can provide an entry to algebraic symbolism that is meaningful" (Carraher & Schliemann, 2007).

Using mathematical problems to develop algebraic reasoning

This article describes a lesson that was undertaken in a Year 7 class. Six groups, each with four students, were given structurally similar problems. The approaches taken by two of the groups are presented, to demonstrate how algebraic reasoning can build from their experiences, discussions and interpretation of mathematical problems. Their voices explain the strategies and thinking they used to solve two structurally similar problems.

Brush Up

At an art store, brushes have one price and pencils have another. Eight brushes and three pens cost $7.10, but six brushes and three pens cost $5.70. How much does one pen cost?

A Good Sport

At the local sports store, all tennis balls are sold at one price and netballs are sold at another price. If three netballs and two tennis balls are sold for $47.00, while two netballs and three tennis balls are sold for $38.00, what is the cost of a single tennis ball?

Solving problems using relational thinking

Many primary school teachers might regard solving this type of problem as a difficult task, only accessible by those students with an aptitude in mathematics. Conversely, students are simply shown an algebraic procedure with little understanding of the meaning behind the symbols. By analysing the processes required for using two relationships simultaneously, there is a greater likelihood of understanding the algebraic sequences encountered in secondary school algebra. The two examples describe the thinking used by students to isolate the objects and show what the students did in order for the problem to be less complex. As Lins, Rojano, Bell and Sutherland (2001) note, “no matter how suggestively algebraic a problem seems to be, it is not until the solver actually engages in its solution that the nature of the thinking comes to life.”

The two problems are sometimes classified as *algebra problems* because they can be solved using a system of simultaneous equations (Lenchner, 2008). However, this interpretation limits and possibly predetermines a procedural approach that simply mimics the processes of the teacher or text book. It should be noted, that sometimes the classifications of problem solving strategies are used by experts when giving “after-the-fact explanations of their own or others problem solving behaviours” (Lesh & Zawojewski, 2007). The two examples described in this paper demonstrate how students can solve problems from an algebraic perspective when mathematical objects are considered relationally and not simply as specific numbers. This change in student thinking is necessary as students develop the ability to think algebraically (van Amerom, 2002)

Isolating one unknown to find a solution

In solving the problem Brush Up, Dougal, a Year 7 student, identified the relationship between the brushes and pencils using blue and yellow counters and two calculators. As shown in Figure 1, the yellow counters represented the paintbrushes and the blue counters represented the pens, and, interestingly, he recorded the cost of each statement on the two calculators. He then removed six yellow counters and three blue counters from the top statement and subtracted $5.70 from $7.10. Once the counters were removed, Dougal explained (pointing at the 1.4 on the display), “Divide by two is 70, so we know that one brush equals 70 cents.” Once he had found the cost of one the brushes, he logically found the cost of the pens. His use of the two calculators helped him to reduce his cognitive load and freed his thinking to concentrate on the mathematical objects rather than computational processes. Furthermore, the materials helped his group

Figure 1. Dougal using counters and calculators to solve the problem Brush Up.

to see the problem clearly and they developed their mathematical argument in a strategic and organised manner.

Using factors to isolate one unknown

At the beginning of the lesson, Holly and Amelia had solved a structurally similar problem to Dougal's group and throughout the prior lessons both girls, especially Amelia, had demonstrated the capacity to think about a variety of problems from a generalised and relational perspective. Amelia wrote down the statement:

3 netballs and 2 tennis balls = \$47
2 netballs and 3 tennis balls = \$76

Unlike the problem *Brush Up*, both girls found the problem *A Good Sport* demanding and difficult to understand because it required two mathematical statements to be manipulated simultaneously. Their teacher, having seen Amelia's statement, asked them to reflect on their prior understanding and to consider how they solved addition or subtraction problems involving "unlike common fractions." Holly, who was proficient at arithmetic, proceeded to explain to Amelia that when both common fractions were "unlike fractions" (e.g., $\frac{3}{4}$ and $\frac{1}{3}$), she renamed each fraction as a "like fraction" or an equivalent fraction using a factorisation method. Interestingly, during this short episode, the teacher did not set about explaining a solution but asked them to reflect on the mathematics they already knew and allowed the girls to develop their own solution. The teacher's knowledge of the students and her pedagogical understanding was crucial in guiding Holly and Amelia's mathematical thinking as well as developing an algebraic perspective that acknowledges the inter-connectedness of mathematics.

After Holly explained how she solved common fraction addition problems, Amelia pointed at the statement she had written about the netballs and tennis balls. She suggested that the relationship between the balls could be maintained if each statement was renamed by a factor of two and three. Figure 2 demonstrates Holly and Amelia's thinking with the coloured text highlighting what the two girls actually recorded. The table has been extended to show the girls thinking in more detail and how as each statement is renamed by a factor of 2, 3, 4 or 5, the relationship between the netballs and tennis balls can be maintained.

Relationship	x2	x3	x4	x5
3 Netballs 2 Tennis Balls \$47	6 Netballs 4 Tennis Balls \$94	9 Netballs 6 Tennis Balls \$141	12 Netballs 8 Tennis Balls \$188	15 Netballs 10 Tennis Balls \$235
2 Netballs 3 Tennis Balls \$38.00	4 Netballs 6 Tennis Balls \$76	6 Netballs 9 Tennis Balls \$114	8 Netballs 12 Tennis Balls \$152	10 Netballs 15 tennis balls \$190

Figure 2. A summary of Holly and Amelia's thinking.

Once they established a "like part" or common coefficient, in this case the six netballs, the two girls could logically solve the problem. The two girls reasoned that if they took six netballs and four tennis balls (\$94) away from the statement "six netballs and nine tennis balls" (\$114) then only five tennis balls would remain (\$114 – \$94 = \$20), thus isolating one of the variables. Knowing that five tennis balls cost \$20, the two girls simply ascertained that the cost of one tennis ball was \$4. The girls presented their thinking to their classmates, as shown in Figure 3; their explanation was enthusiastically applauded by everyone in their class, but, importantly, other students used the girls' ideas that they had to build their own understanding,

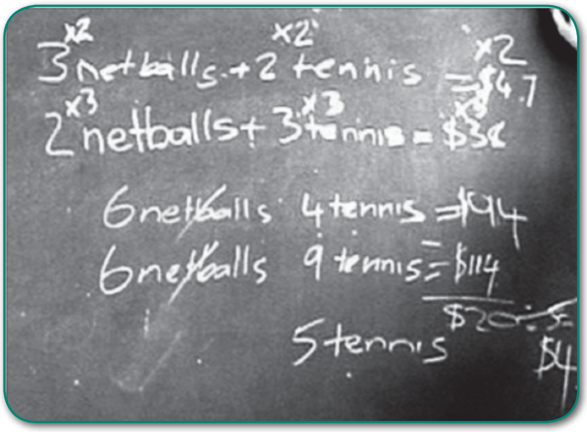

Figure 3. The solution Holly and Amelia presented to the class.

and this was reflected in the work of other students following the girls' explanation. On reflection, the applause and enthusiasm for the both Dougal's and Amelia and Holly's explanations highlights the value that students place on "mathematically significant" events when they are actively involved and engaged in teaching and learning.

Using materials

As Dougal, Holly and Amelia demonstrated, once an object or value has been isolated, students can logically solve for the other unknown. However, difficulties arise in developing the thinking required to isolate one of the objects, especially when the coefficients are not equal. Building on Dougal's example, counters can be helpful for showing the thinking that Holly and Amelia used for renaming their mathematical statements and the thinking required for eliminating and isolating the mathematical objects. Using coloured counters may help to illustrate this process and allow students to 'see' the relationships by reducing their cognitive load and allowing them the opportunity to compare, identify and manipulate the amounts of each statement more readily and with greater understanding.

In the problem A Good Sport, what is known can be presented using two different coloured counters and the price recorded beside each statement. Similarly to Dougal's example (see Figure 1), a calculator can be used to record the price beside each statement. As shown in Figure 4, black and white counters can show how the original statement, "Three netballs and two tennis balls cost $47 and two netballs and three tennis balls cost $38" can be renamed. Using the Amelia and Holly's thinking the "three netballs and two tennis balls" can be renamed by a factor of two, thus: six netballs and four tennis balls will cost $94. The "two netballs and three tennis balls" can be renamed by a factor of three, thus: six netballs and nine tennis balls will cost $114. Now both statements have a "like amount" or common coefficient, in this case the six netballs. Figure 4 illustrates how the like terms from both statements are removed, thus leaving only five tennis balls (five white counters) at a cost of $4 per tennis ball.

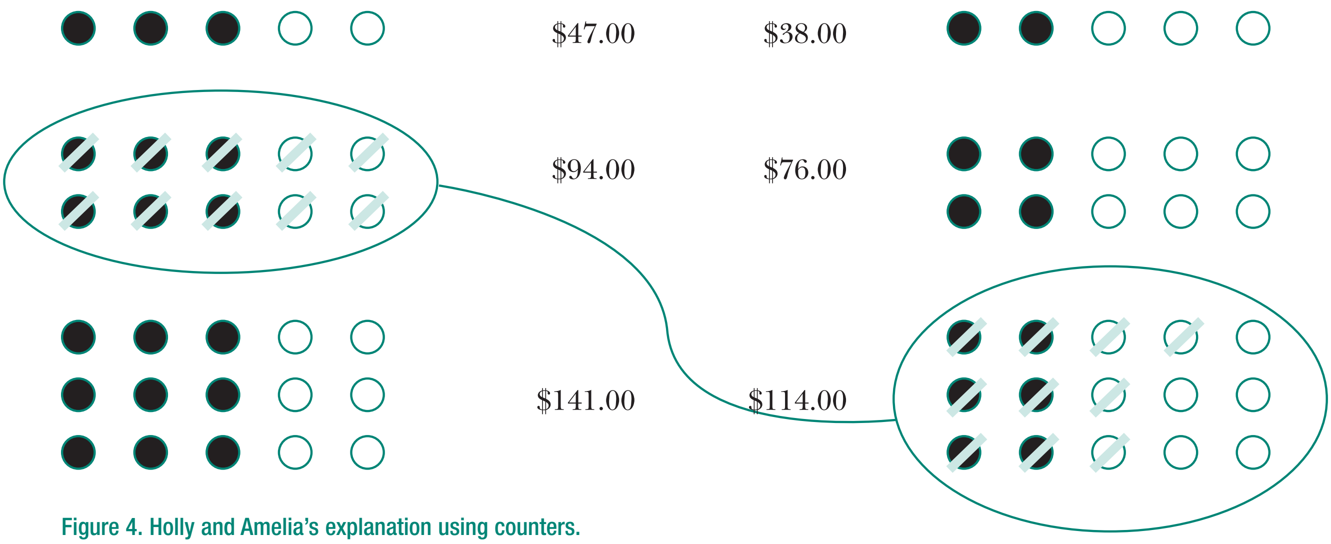

Figure 4. Holly and Amelia's explanation using counters.

Conclusion

Developing algebraic thinking initially relies on students' ability to consider the structures of problems from a generalised and relational perspective. From this position, students can develop increasingly sophisticated ways of thinking about the problems and their solutions, as well as organising and manipulating their thinking in order to solve the problems. As demonstrated by Dougal, Holly and Amelia, students are capable of exploring sophisticated problem solving approaches in a coherent and organised way. The two student examples present their own unique approaches yet their thinking has elements often associated with the algebra of secondary school. One example shows how materials and calculators can complement students thinking as they grapple with new ideas. The other highlights how simple mathematical ideas such as renaming and equivalence can be used by students to understand concepts often associated with algebra. The problems provide a context in which to introduce the ideas of algebra but more importantly facilitate a generalised way of thinking that can transfer to other situations beyond mathematics. Can you use the thinking and strategies that Dougal, Holly and Amelia used to solve these two problems?

Flourish & Botts

At the Flourish & Botts bookstore, the first wizard book and the second wizard book together cost $45. Two copies of the first wizard book and three copies of the second wizard book costs a total of $125. At this book store how much is the first wizard book?

Pencils and Pens

In a stationary store, pencils have one price and pens have another. Two pencils and three pens cost 78 cents, but three pencils and two pens cost 72 cents. How much does one pen cost?

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