The Mathematics Educator
2011, Vol. 21, No. 1, 11–21

Using Technology to Unify Geometric Theorems About the Power of a Point

José N. Contreras

In this article, I describe a classroom investigation in which a group of prospective secondary mathematics teachers discovered theorems related to the power of a point using *The Geometer's Sketchpad* (GSP). The power of a point is defines as follows: Let P be a fixed point coplanar with a circle. If line $\overleftrightarrow{PA}$ is a secant line that intersects the circle at points A and B, then $PA \cdot PB$ is a constant called the power of P with respect to the circle. In the investigation, the students discovered and unified the four theorems associated with the power of a point: the secant-secant theorem, the secant-tangent theorem, the tangent-tangent theorem, and the chord-chord theorem. In our journey the students and I also discovered two kinds of proofs that can be adapted to prove each of the four theorems. As teacher educators, we need to design learning tasks for future teachers that deepen their understanding of the content they are likely to teach. Having a profound understanding of a mathematical idea involves seeing the connectedness of mathematical ideas. By discovering and unifying the power-of-a-point theorems and proofs, these future teachers experienced what it means to understand a mathematical theorem deeply. GSP was an instrumental pedagogical tool that facilitated and supported the investigation in three main ways: as a management tool, motivational tool, and cognitive tool.

The judicious use of technology enhances the teaching and learning of mathematics. Technology frees the user from performing repetitive and computational tasks, and thus, it allows more time for action and reflection. As a consequence, when students use technology as a cognitive tool, they develop a deeper understanding of mathematical concepts, patterns, and relationships (Battista, 2007; Clements, Sarama, Yelland, & Glass, 2008; Hollebrands, 2007; Hollebrands, Conner, & Smith, 2010; Hollebrands, Laborde, & Sträβer, 2008; Hoyles & Healy, 1999; Hoyles & Jones, 1998; Koedinger, 1998; Laborde, 1998; Laborde, Kynigos, Hollebrands, & Sträβer, 2006).

For example, Battista (2007) describes how two fifth graders constructed meaning for a spatial property of rectangles--each of the four angles of a rectangle measures 90°--within the Shape Makers environment (Battista, 1998), a GSP microworld for investigating geometric shapes. In their review of research on learning and teaching geometry within interactive geometry software (IGS) environments, Clements, Sarama, Yelland, and Glass (2008) concluded that IGS "can be beneficial to students in their development of understandings of geometric shapes and figures" (p. 131). Similarly, research reviewed by Hollebrands, Conner, and Smith (2010) suggests that IGS environments "enable students to abstract general properties and relationships among geometric figures" (p. 325).

IGS such as The Geometer's Sketchpad (GSP) (Jackiw, 2001) and Cabri Geometry II (Laborde & Bellemain, 1994) are powerful instructional technology tools. IGS allows the user to construct dynamic figures that can be manipulated or moved without altering the mathematical nature of the geometric figure. This feature allows the user to quickly generate many examples of a geometric diagram. This feature is in marked contrast to the static nature of textbook and paper-and-pencil illustrations. A diagram that can be resized by dragging flexible points also motivates the user to investigate invariant geometric relationships. As a result of motivation, action, and reflection, students construct a more powerful abstraction of mathematical concepts (Battista, 1999).

This article describes a classroom activity in which a group of 13 prospective secondary mathematics teachers (hereafter referred to as students) investigated the power of a point with GSP. My objective was to guide my students to discover and unify several geometric theorems related to the power of a point. The power of a point is defined as follows: Let P be a fixed point coplanar with a circle. If $\overleftrightarrow{PA}$ is a secant line that intersects the circle at points A and B, then

Dr. José N. Contreras, jncontrerasf@bsu.edu, teaches mathematics and mathematics education courses at Ball State University. He is particularly interested in integrating problem posing, problem solving, technology, history, and realistic mathematics education in teaching and teacher education.

$PA \cdot PB$ is a constant called the power of with respect to the circle.

The Classroom Setting

The students were enrolled in my college geometry class for secondary mathematics teachers. The textbook I used was *Geometry: A Problem-Solving Approach with Applications* (Musser & Trimpe, 1994). All of my students had completed the calculus sequence, discrete mathematics, and linear algebra. In addition, by this point in the course, my students were proficient using GSP, as they had employed it to complete several tasks involving constructing geometric figures (e.g., centroid of a triangle, squares, etc.), detecting patterns, and making conjectures. We conducted our power of a point investigation in the computer lab where each student had access to a computer with GSP. To facilitate and manage the investigation more efficiently and accurately, I provided students with geometric files relevant to the investigation. I had my laptop computer connected to an LCD projector.

Starting the Investigation: Discovering the Power of a Point

We began our investigation with the problem shown in Figure 1.

Find the value of PD in the configuration below where $PA = 1.60$ cm, $PC = 1.50$ cm, $PB = 3.30$ cm. Justify your method.

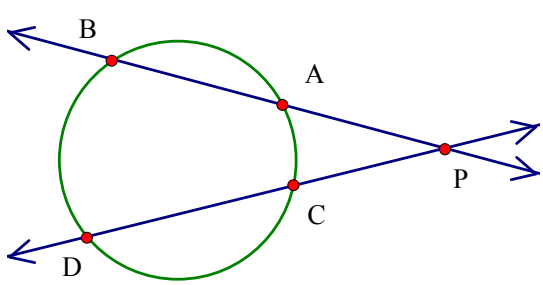

Figure 1. The initial problem.

Some students used the proportion $\frac{3.30}{1.60} = \frac{PD}{1.50}$ or one of its equivalent forms, others said that they did not remember how to do this type of problem, while a third group claimed that they had never seen a problem like that before. I then asked students to open the "power of a point" file to investigate this problem using GSP. I had hoped for students to attempt to discover the general relationship. A few students quickly used the measurement capabilities of GSP to find or verify their solution. When they realized that their solution was incorrect, they concluded that their proposed relationship $\frac{3.30}{1.60} = \frac{PD}{1.50}$ did not hold. Another student reached this conclusion by noticing that dragging point B changed PA, PB, and $\frac{PB}{PA}$, but did not influence PC and PD. Therefore, the proportion $\frac{PB}{PA} = \frac{PC}{PD}$ did not hold. The measurement and dragging capabilities of GSP allowed students to disconfirm their initial conjectures.

After confirming that dragging point B changed PA and PB, I told them that a hidden quantity involving only PA and PB remained constant and challenged them to find it. Some students tried $PA+PB$ and $PB–PA$. One of the first students who discovered that $PA \cdot PB$ remains constant said, "I can't believe it. $PA \cdot PB$ remains the same no matter where points A and B are." Other students verified this hypothesis by dragging point B and calculating $PA \cdot PB$ (see Figure 2). One student was puzzled because she noticed that PB increases in some instances but the product remained the same. Another student said, "Yes, but PA decreases. When one number increases the other decreases. So they balance each other." At this time I mentioned that the constant $PA \cdot PB$ is called the power of point P, $\mathscr{P}(P)$, with respect to the circle. In this case, the computational and dynamic capacities of GSP allowed some students to discover that $PA \cdot PB$ remains invariant regardless of where points A and B are located in the circle.

Continuing the Investigation: An Unanticipated Discovery

As we did with other investigations involving GSP, we systematically tested our conjecture for different circles and points. To test our power-of-a point conjecture for a given circle, we dragged point P and then point B to verify that $PA \cdot PB$ is constant. Students also noticed that for a given circle, the farther point P was from it, the greater its power. A couple of students also dragged the point controlling the radius of the circle and noticed that the radius influenced the power of a point as well. I had originally planned to just test our conjecture for different points and different circles, but our systematic testing led us to investigate an unexpected conjecture related to how both the length of the radius (r) of the circle, and the distance from P to the center (O) of the circle impacted its power.

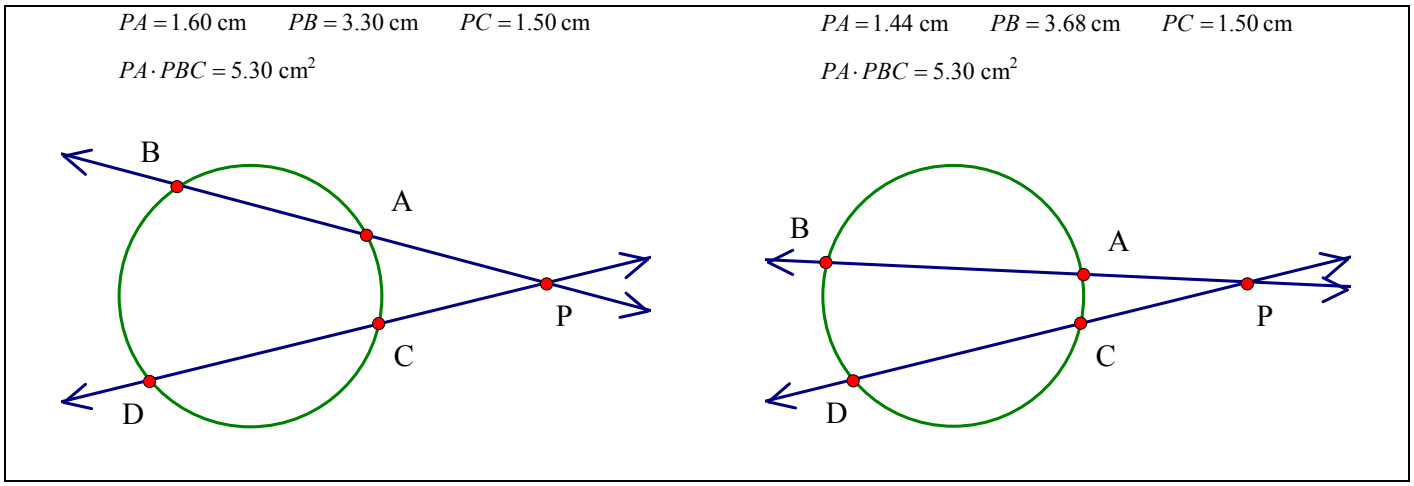

Figure 2. PA·PB seems to be constant for a given point P and circle.

I hid the product *PA·PB* on my GSP sketch and asked students to predict the behavior of the power of point P as I increased the radius of the circle from 0 with both its center *O* and point *P* fixed. A student claimed that the power of the point would remain constant because *PB* increases and *PA* decreases. Another student refuted this explanation saying that the power would decrease because *PB* increases but *PA* approaches zero and becomes zero when the circle goes through *P*. The second student added that the power would increase as the radius of the circle increased "beyond *P*". Students confirmed this conjecture on their GSP sketches. At this time, it occurred to me to ask students for the maximum value of the power of the point when the point is still in the exterior of the circle (i.e., the radius of the circle is less than *PO*). Some students provided a numerical value while others argued that the maximum value did not exist because *PA*, *PB*, and *PB·PA* disappear when the circle becomes a point. One student said that we could still consider a point as a circle of radius zero, and another student mentioned that a point could be considered as the limiting case of a circle when the radius approaches zero. However, most students in the class agreed that a point is not a circle because the radius has to be greater than zero. I then asked students to consider what conception would be more helpful or convenient to describe the behavior of *PA·PB*. We then formulated the following conjecture:

> Let *P* be a fixed point and *C* a circle with fixed center *O* but variable radius r. As the radius of the circle increases from zero, the power of the point with respect to *C*
>
> a) decreases from a maximum, the square of the distance from the point to the center of the circle (when the radius of the circle is zero), to zero (when the circle contains *P*) as the radius increases from 0 to *OP*.
>
> b) increases from zero without limit as the radius increases without limit from *OP* (*P* is an interior point).

At this point, I wanted to investigate the relationship between the power of a point and the radius of a circle. Since I knew my students were not familiar with the graphing capabilities of GSP, I asked them to use pencil and paper to sketch a graph of the power of a point as a function of the radius. While they did this, I constructed the graph in GSP using the trace feature. I asked students how we could conveniently position a circle in the coordinate plane to simplify the computations. One student suggested putting the center of the circle at the origin and points *P*, *A*, and *B* on the *x*-axis. This student provided the table shown (see Figure 3) for the point *P* whose coordinates were (2, 0). Other students constructed similar tables using the same or different coordinates for point *P*.

	$\mathcal{P}(P)$
0	2(2) = 4
1	3(1) = 3
2	4(0) = 0
3	5(1) = 5
4	6(2) = 12
5	7(3) = 21

Figure 3. Student-constructed table examining the relationship between radius of a circle and power of

point.

All students agreed with the GSP graph (see Figure 4) since it looked like their sketches, and that the first piece of the graph seemed to be a parabolic arc. To better visualize the nature of the second piece of the graph, I changed the scale of the *y*-axis. Notice that the circle is not shown on the second graph. We conjectured that the graph appeared to be two pieces of parabolic arcs.

As we tried to make sense of the table in Figure 3 and the graphs in Figure 4, we generalized the pattern depending on whether P is an exterior or an interior point as:

$$\mathcal{P}(P) = (2 + r)(2 - r) = 4 - r^2$$

or

$$(r + 2)(r - 2) = r^2 - 4.$$

I then asked students for the geometric interpretation of the number 2 in this formula. After some reflection and discussion, students realized that 2 was the distance from the point P to the origin, which is the center of the circle O. Therefore we could rewrite our equations as:

$$\mathcal{P}(P) = PA \cdot PB = (OP - r)(OP + r) = OP^2 - r^2$$

and

$$\mathcal{P}(P) = r^2 - OP^2 .$$

when *P* is exterior to the circle and when *P* is interior to the circle, respectively. Since my objective for this activity was to unify theorems related to the power of a point, I asked the students, "How can these two graphs be unified? How we can have one parabolic arc instead of two pieces?" In a previous activity we had unified the theorems related to the measures of angles formed by secant lines when the vertex of an angle is an is an exterior point and when the vertex is an interior point by considering directed arcs, so it was natural for a student to suggest using directed distances. Another student said that using directed distances could "flip" the second piece across the *x*-axis. The first student inferred from the graph that we could unify the two formulas by considering the power of an exterior point to be positive and the power of an interior point to be negative. In order to do this, we needed to consider *PA*. and *PB* as directed distances, similar to directed arcs. As a result, we obtained the graph displayed in Figure 5. The equation of this graph is $\mathcal{P}(P) = OP^2 - r^2$.

I was particularly delighted that we had also discovered a formula for the power of a point in terms of its distance to the center of the circle and the radius. The interactive, graphing, and dynamic capabilities of GSP motivated us to follow our intuitions and test the resulting conjectures. It minimized the managerial and logistic difficulties of performing this part of the investigation with paper and pencil.

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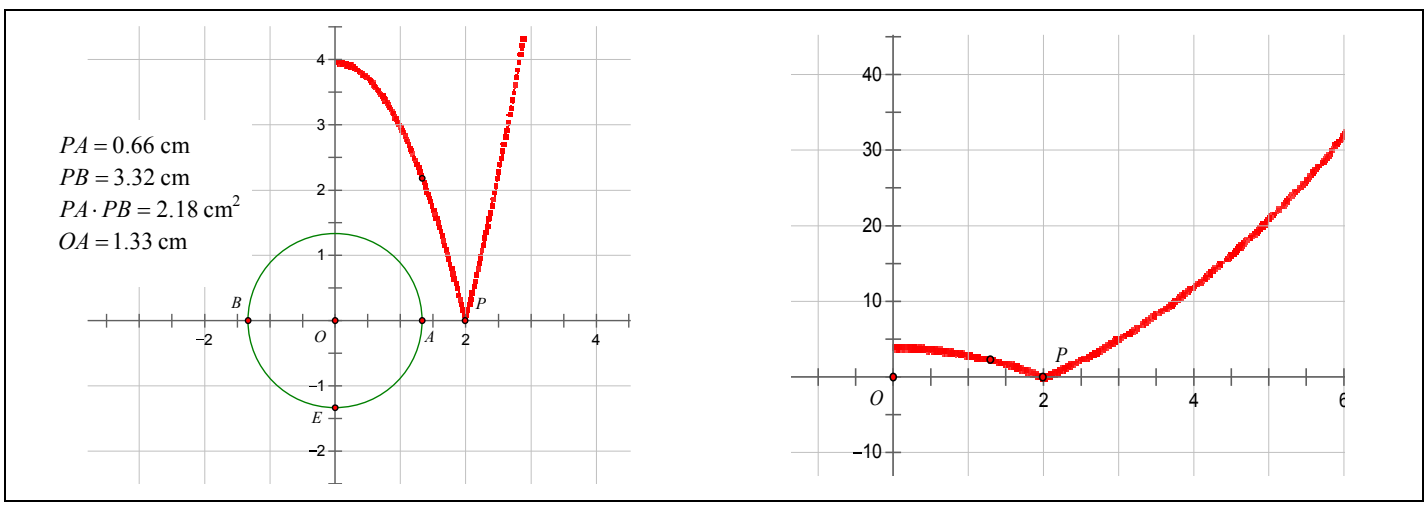

Figure 4. The power of a point as a function of the radius of the circle.

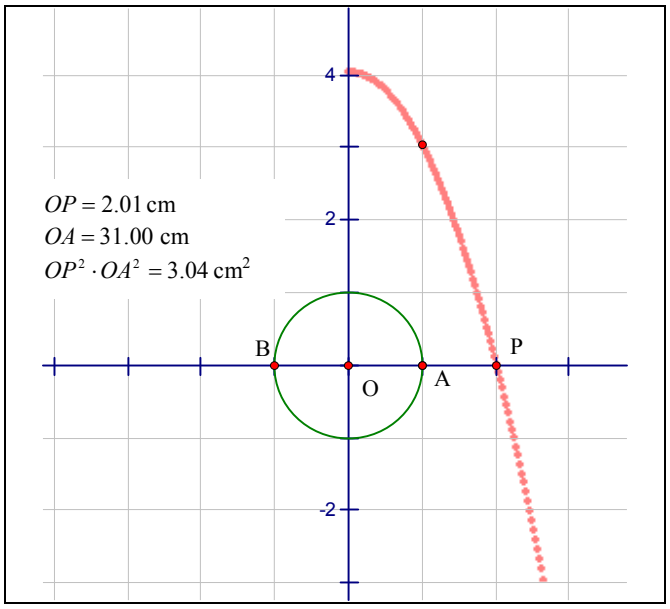

Figure 5. The unified graph of the power of a point as a function of the radius of the circle.

Continuing the Investigation: Establishing the Secant-Secant Theorem

After these unexpected but productive digressions, we came back to our original problem. Two students admitted that they did not know how to use $PA{\cdot}PB$ to find PD. After I dragged point B around the circle hoping that these students could see the connection that $PA{\cdot}PB = PC{\cdot}PD$ because $PA{\cdot}PB$ is a constant, only one student still failed to see the connection. A classmate provided the following explanation: "PA times PB is a constant no matter where points A and B are. So if $A = C$ and $B = D$ we have that $PA{\cdot}PB = PC{\cdot}PD$." The student computed the product $PC{\cdot}PD$ to see the pattern. After we established the relationship $PA{\cdot}PB = PC{\cdot}PD$, I asked the class how we could prove it. Since nobody provided any hint or suggestion about how to prove the relationship, I suggested rewriting $PA{\cdot}PB = PC{\cdot}PD$ in another way. Some students suggested rewriting $PA{\cdot}PB = PC{\cdot}PD$ as $\frac{PA}{PC} = \frac{PD}{PB}$. This prompted one student to suggest using similar triangles. Several students immediately proved the equality by using the AA similarity theorem to prove $\Delta APD \sim \Delta CPB$ (see Figure 6), and one student shared his proof with the rest of the class.

By proving that $PA{\cdot}PB = PC{\cdot}PD$ for arbitrary B and D on the circle, we established that $PA{\cdot}PB$ is a constant for a particular exterior point of a given circle. We then formulated the corresponding theorems in the following terms:

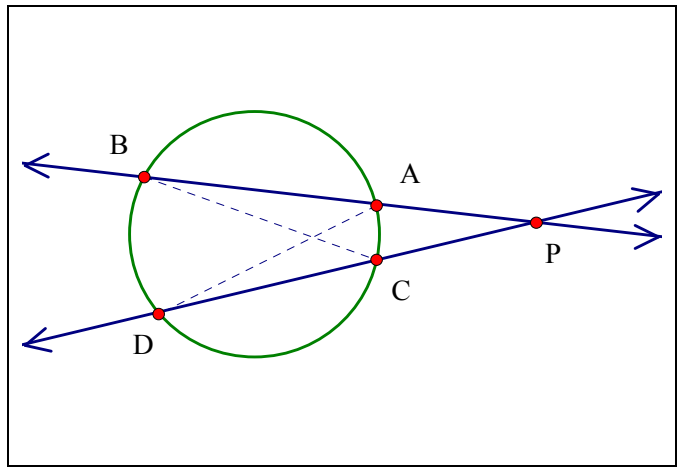

Figure 6. $\Delta APD \sim \Delta CPB$.

(i) The secant-secant theorem: Let P be an exterior point of a circle. If two secants $\overleftrightarrow{PA}$ and $\overleftrightarrow{PC}$ intersect the circle at points A, B, C, and D, respectively (see Figure 6), then $PA{\cdot}PB = PC{\cdot}PD$.

(ii) P is an exterior point and $\overleftrightarrow{PA}$ is a secant of a circle. If the secant $\overleftrightarrow{PA}$ to the circle intersects the circle at points A and B, then $PA{\cdot}PB$ is a constant. This constant is called the power of P with respect to the circle.

GSP allowed students to dynamically manipulate and interact with the power of a point, an abstract object, in a "hands-on" manner. By moving points along the circle, they gained experience with one of the representations of the power of a point.

Modifying the Secant-Secant Theorem: The Tangent-Secant Theorem

Since my goal was to formulate theorems related to the secant-secant theorem, I asked students what other theorems could be generated from this theorem. The class listed the following possible cases to consider:

1. P is on the exterior
2. One secant and one tangent
3. Two tangents
4. P is on the circle
5. P is in the interior of the circle

We then proceeded to investigate the case when P is an exterior point of a circle, one line is a secant, and the other is a tangent. With my computer, I illustrated the situation as D approaches C (see Figure 7a) and

asked students to predict the relationship $PA{\cdot}PB = PC{\cdot}PD$ when line $\overleftrightarrow{PC}$ (or $\overleftrightarrow{PD}$) is a tangent line to the circle. Most students predicted that $PA{\cdot}PB = PC^2$ (or PD^2). To further test their conjecture, I had my students open a file containing a pre-constructed configuration to illustrate the "secant-tangent" situation (see Figure 7b). After testing our conjecture for several cases by dragging point P and varying the size of the circle (see Figure 7c), students were confident that the conjecture was true and, therefore, that we could prove it.

Since ΔAPD approaches ΔAPC (see Figures 7a and 7b), I was expecting students would use the similarity of ΔAPC and ΔCPB to prove the tangent-secant conjecture. However, only two students thought of using the fact that $\Delta APC \sim \Delta CPB$ (see Figure 8a) to prove our conjecture. Since I wanted to unify the two theorems (the secant-secant theorem and the tangent-secant theorem), I illustrated on my computer how, as line $\overleftrightarrow{PC}$ approaches a tangent line, ΔAPD approaches ΔAPC (see Figure 8b). All students were able to justify that $\Delta APC \sim \Delta CPB$ by the AA similarity theorem and derived the tangent-secant relationship. Initially two students measured angles $\angle ACP$ and $\angle CBP$ to convince themselves that those angles are congruent. Eventually both of them "saw" why they are congruent: By the inscribed angle theorem $m(\angle CBP) = \frac{1}{2}\overset{\frown}{AC}$ and, by the semi-inscribed angle theorem, $m(\angle ACP) = \frac{1}{2}\overset{\frown}{AC}$. We formulated our theorems as follows:

(iii) The tangent-secant theorem: Let P be an exterior point of a circle. If a secant $\overleftrightarrow{PA}$ and a tangent $\overleftrightarrow{PC}$ intersect the circle at points A, B, and C, respectively, then $PA{\cdot}PB = PC^2$.

(iv) If P is an exterior point and $\overleftrightarrow{PA}$ is a tangent line of a circle with point of tangency A, then the power of the point is $= PA^2$.

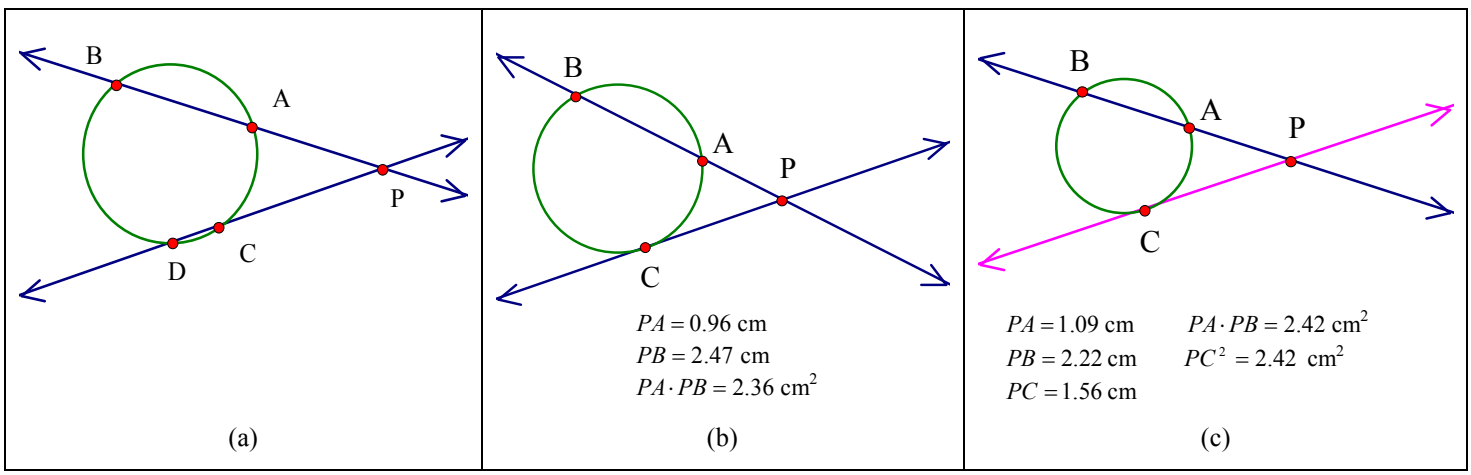

Figure. 7: Discovering the tangent-secant theorem.

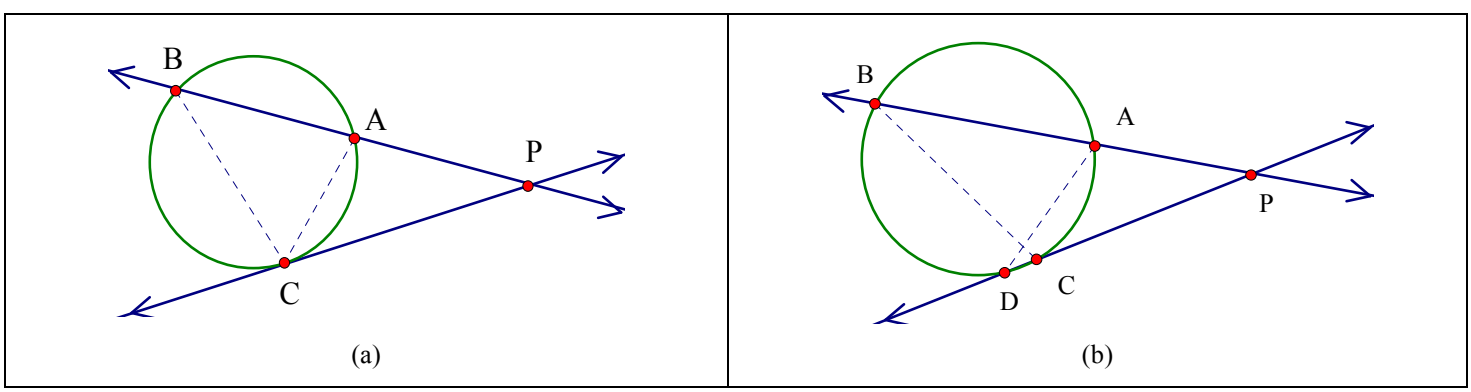

Figure 8. ΔAPD approaches to ΔAPC as C and D get closer.

The dynamic geometry environment facilitated our examination of what varied and what remained invariant as one secant line approached and eventually became a tangent line. Students gained experience with a second representation of the power of a point. They were also able to see similarities and differences between the new proof and the proof for the secant-secant theorem.

Modifying the Secant-Secant Theorem: The Tangent-Tangent Theorem

Our next task was to investigate the case when both lines are tangent (see Figure 9a). I asked students to conjecture a new relationship by applying our knowledge of the power of a point to Figure 9a. One student said that $PA = PC$ but he was unable to explain the connection between this relationship and the tangent-secant theorem. He could only say that the figure suggests such a relationship. As a hint, I used the tangent-secant configuration, dragging point B until it got close to point A (see Figure 9b), and asked students what would happen when $\overleftrightarrow{PA}$ becomes a tangent. After some reflection, two students were able to deduce that $PA = PC$. One of the arguments was as follows: By the secant-tangent theorem, $\mathscr{P}(P) = PA^2$ and $\mathscr{P}(P) = PA^2$, so $PA^2 = PC^2$. After taking the square root of both expressions, we got $PA = PC$. We formulated our theorem as follows:

(v) Let P be an exterior point of a circle. If $\overleftrightarrow{PA}$ and $\overleftrightarrow{PC}$ are tangent lines to the circle, with tangency points A and C, then $PA = PC$ (see Figure 9a).

To illustrate the interconnectedness of these mathematical theorems, I challenged my students to find as many additional proofs as they could that $PA = PC$. As a group, students provided two more proofs, which refer to the diagram in Figure 10.

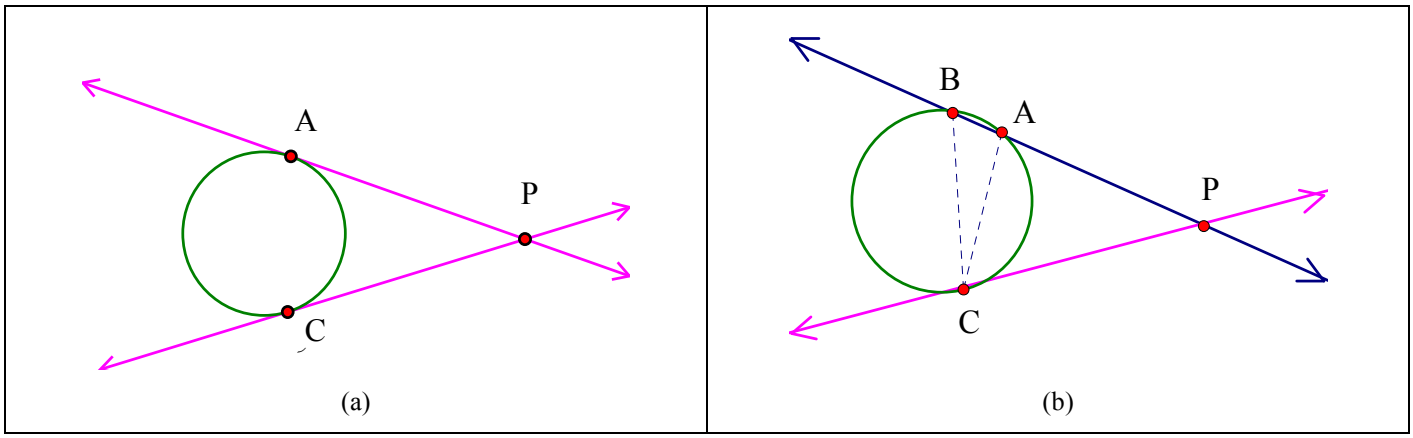

Figure. 9: Discovering the tangent-tangent theorem.

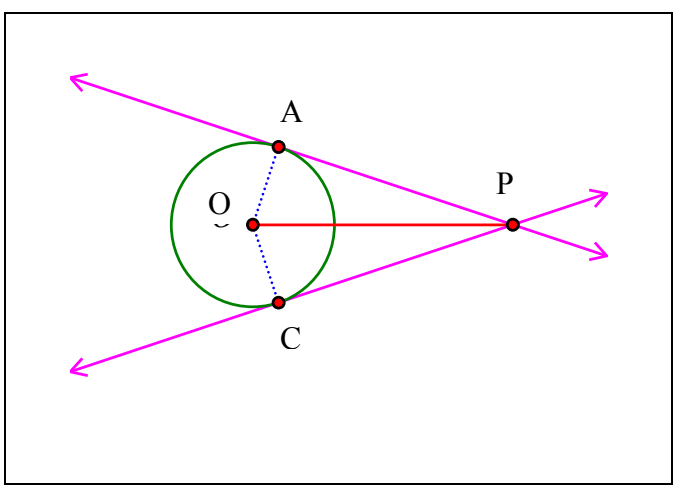

Figure 10. Diagram students used to prove $PA = PC$

Sketch of proof 1. Since lines $\overleftrightarrow{PA}$ and $\overleftrightarrow{PBC}$ are tangent lines, they are perpendicular to the radii that go through their points of tangency. Therefore, triangles ΔAOP and ΔCOP are right triangles. Since $AO = CO$ (by definition of a circle), $\Delta AOP \cong \Delta COP$ by the Hypotenuse-Leg congruence criterion. As a consequence, $AP = CP$.

Sketch of proof 2: As in proof 1, $\angle OAP$ and $\angle OCP$ are right angles. In addition $AO = CO$. Since O is equidistant from the sides of $\angle APC$, it belongs to its angle bisector. Therefore, $\overleftrightarrow{PCO}$ is the angle bisector of $\angle APC$, which means that $\angle APO \cong \angle CPO$. We conclude that $\Delta AOP \cong \Delta COP$ by the AAS congruence criterion. By definition of congruent triangles, $AP = CP$.

Since one of my objectives was to unify the theorems related to the power of a point, I asked students to prove that $PA = PC$ by modifying the proof for the tangent-secant theorem. Since $\Delta APC \sim \Delta CPB$ and points A and B collapse into one point, all of the students were able to see that $\Delta APC \sim \Delta CPA$. Some students established that $PA^2 = PC^2$ using the proportion $\frac{AP}{CP} = \frac{PC}{PA}$, another established directly that $PA = PC$ using the proportion $\frac{AP}{CP} = \frac{AC}{CA} = 1$, and others used the fact that $\Delta APC \cong \Delta CPA$ by the ASA congruence criterion. Finally, following my suggestion, the class proved that $PA = PC$ using the converse of the isosceles triangle theorem since $\angle PAC \cong \angle PCA$.

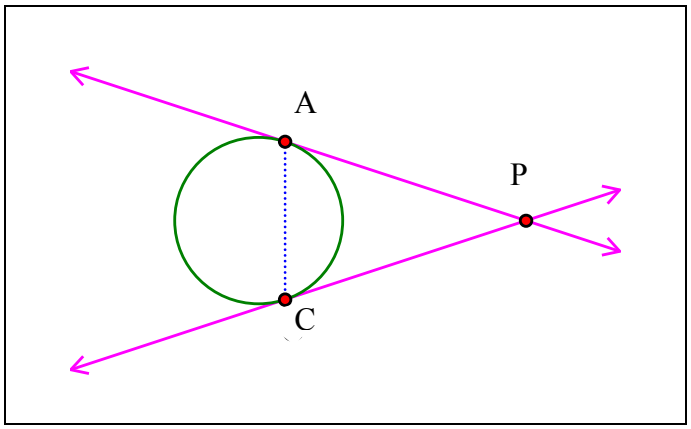

Figure 11. The tangent diagram.

GSP was a powerful pedagogical tool because it allowed students to adapt the proof of the tangent-secant theorem to develop another proof of the tangent-tangent theorem. They were able to dynamically see how the two original triangles were continuously transformed into a single triangle.

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The Secant-Secant Theorem Again: The Chord Theorem

As we continued working towards the unification of all the theorems related to the power of a point, I had my students consider the case when P is an interior point of the circle and both lines are secant to the given circle (see Figure 12a). The theorem states:

(vi) If $\overline{AB}$ and $\overline{CD}$ are two chords of the same circle that intersect at P, then $PA \cdot PB = PC \cdot PD$.

By now, all of my students were able to predict that $PA \cdot PB = PC \cdot PD$. As I expected, all but two students proved this relationship by using the fact that $\Delta APD \sim \Delta CPB$ (see Figure 12b).

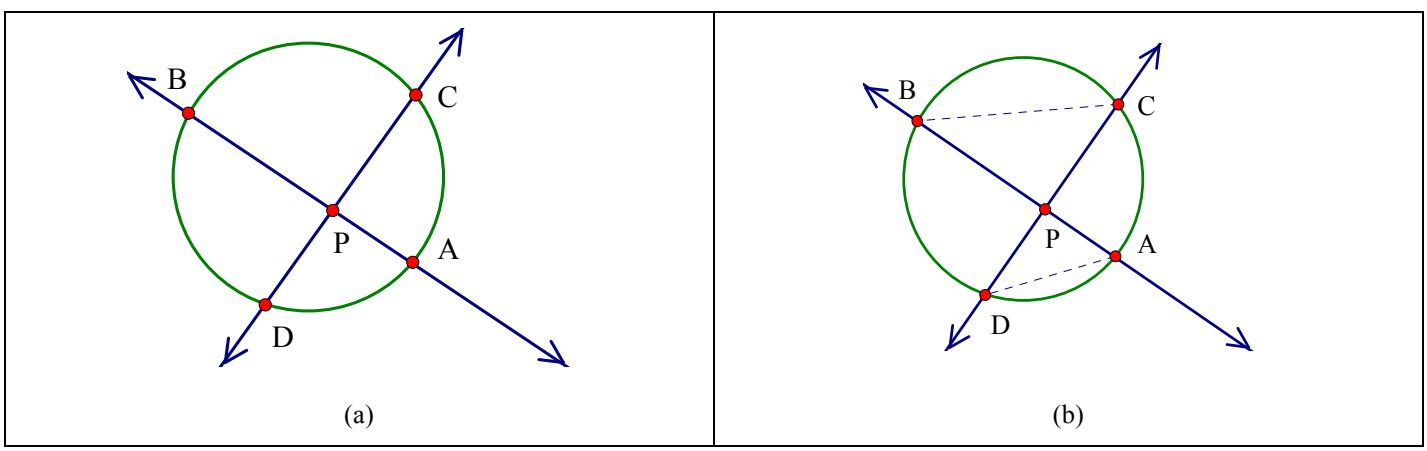

Figure 12. Proving that $PA \cdot PB = PC \cdot PD$ using $\Delta APD \sim \Delta CPB$.

The Investigation Concludes: The Unification and Another Discovery

At this point, the investigation took another unexpected turn: Two students proved the power-of-a-point relationship using triangles ΔACP and ΔDBP (see Figure 13a). At that time, it occurred to me that this proof could be extended to the other cases, so I challenged the class to adapt the proof to the other situations. While there were no changes for the tangent-secant theorem and the tangent-tangent theorem, all of my students were challenged by the secant-secant theorem (see Figure 13b).

Some students argued that the proof could not be adapted to the secant-secant theorem because triangles ΔACP and ΔDBP did not look similar. I myself was not sure whether triangles ΔACP and ΔDBP were similar. Based on visual clues, one student thought that $\Delta ACP \sim \Delta BDP$, but another student refuted her proposition because lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are not necessarily parallel. To investigate whether triangles ΔACP and ΔDBP were similar, we measured their angles and, to our surprise, we found that $\angle ACP \cong \angle DBP$ and $\angle CAP \cong \angle BDP$. Our next task was to explain these congruencies. After some reflection and discussion, and without my guidance, a student concluded that $\angle CAP \cong \angle BDP$ if and only if $m(\angle BDP) + m(\angle CAB) = 180°$. Since we had not proved that angles $\angle BDP$ and $\angle CAB$ are supplementary, I challenged the class to prove their claim. Some students were able to prove the claim using the inscribed angle theorem as follows:

$$m(\angle BDP) + m(\angle CAB) = \tfrac{1}{2}m(\overset{\frown}{CAB}) + \tfrac{1}{2}m(\overset{\frown}{BDC})$$
$$= \tfrac{360°}{2} = 180°$$

We stated our theorem as follows:

(vii) The opposite angles of a cyclic quadrilateral are supplementary.

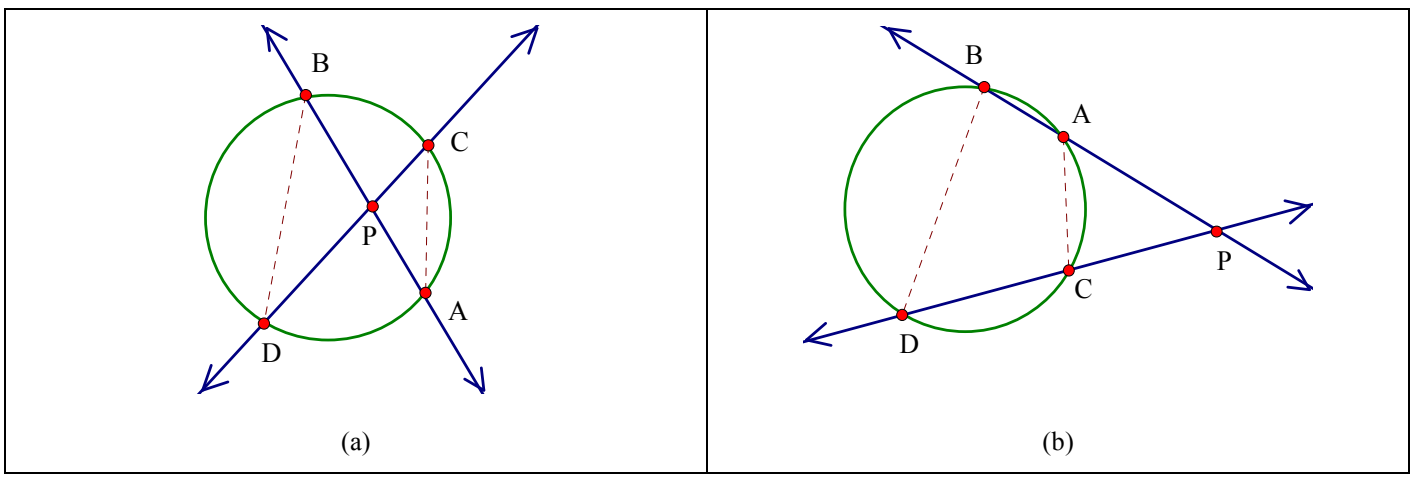

Figure 13. Triangles ΔACP and ΔDBP support our theorems.

We concluded our investigation of the power of the point by combining our theorems into one theorem that we called the power-of-the-point theorem:

(viii) Let C be a circle and P be any point not on the circle. If two different lines $\overleftrightarrow{PA}$ and $\overleftrightarrow{PC}$ intersect the circle at points A and B, and C and D, respectively, then $PA \cdot PB = PC \cdot PD$.

In addition, we came back to our formula for the power of a point in terms of its distance to the center of the circle and the radius of the circle:

(ix) The power of a point with respect to a circle with center O and radius r is $OP^2 - r^2$.

GSP was instrumental in investigating the possibility of developing a second proof for the secant-secant theorem based on two triangles that did not look similar to us at first sight. GSP motivated us to question our initial impression that the triangles are non-similar and to go beyond empirical evidence to justify mathematically why those two triangles are similar.

We then discussed why textbooks presented the four theorems (secant-secant, secant-tangent, tangent-tangent, and chord-chord) separately if they could be stated as a single theorem. My goal was to help my students recognize that our knowledge of a mathematical theorem deepens as we discover or come to know the new relationships or patterns that emerge

in special cases of a theorem. If we do not make explicit that the four theorems can be unified, we tend to learn each one as a separate, compartmentalized theorem. As a consequence, we may fail to remember one case (e.g., the tangent-secant case) even when we know another case (e.g., the secant-secant case).

Discussion and Conclusion

In the power of the point investigation, we used the power of the dynamic, dragging, computational, graphing, and measurement features of GSP to discover and unify several theorems related to the power of a point. We *all* discovered some theorems. my students, under my guidance, discovered the main theorems related to the power of a point and the supplementary property of the opposite angles of a cyclic quadrilateral, and I discovered alongside my students the formula of the power of a point in terms of both the distance from the point to the center of the circle and the length of the radius of the circle. In addition, we unified the five main power-of-a point theorems. As I have shown, GSP was an essential pedagogical tool that was instrumental in our investigation.

I used GSP as a pedagogical tool in three main ways: as a management tool, a motivational tool, and a cognitive tool (Peressini & Knuth, 2005). As a management tool, GSP allowed us to perform the investigation more efficiently and accurately avoiding computational errors and imprecise drawings and measurements associated with lengthy paper and pencil constructions needed to examine multiple examples. As a motivational tool, GSP enhanced our dispositions to perform the investigation. The dynamic and interactive capabilities of GSP allowed us to follow our intuitions, question our predispositions, and test the resulting conjectures easily and accurately. As a cognitive tool, GSP provided an environment in which all of us were active in the process of learning the concepts and procedures at hand. We were able to actively represent and manipulate this abstract geometric object in a hands-on mode. As we experienced first hand the meaning of the power of a point, we reflected on the factors that influenced its behavior. As a result of our actions and reflections, we constructed a more powerful abstraction of this concept, and, thus, we developed a deeper understanding of it.

Understanding the unification of the four theorems is important from both pedagogical and mathematical perspectives. From a pedagogical point of view, understanding the relationships among different representations of mathematical theorems and concepts helps us to generate the special cases, to remember the different forms that a theorem can take, to reduce the amount of information that must be remembered, to facilitate transfer to new problem situations, and to believe that mathematics is a cohesive body of knowledge (Hiebert & Carpenter, 1992). From a mathematical point of view, doing mathematics involves discovering special relationships as well as unifying known theorems. Even concepts that are apparently different can be unified when examined from another viewpoint. For example, from the perspective of inversion theory, lines and circles are the same type of geometric objects. Yet, from a Euclidean point of view, the circles and lines are absolutely different geometric entities. In our case, the power of a point P with respect to a circle with center O and radius r is the product of two directed distances from P to any two points A and B of the circle with which it is collinear. By allowing $A = B$, the theorem is transformed into useful instances from which we derive special and useful corollaries. By considering the case when points P, A, B and O are collinear, we obtain another useful instance of the theorem (i.e., $\mathscr{P}(P) = OP^2 - r^2$).

In this mathematical investigation, students experienced learning mathematical concepts with a specific piece of technology. They were engaged in the process of constructing mathematical knowledge by discovering and justifying their conjectures and making sense of classmates' explanations. They justified their conjectures not only with the technological tool (i.e., testing a conjecture for several instances), but also with mathematical theory (i.e., justifying why a conjecture is plausible and proving that a theorem is true). By learning mathematical concepts within technology environments, these future teachers further developed not only specific content knowledge but also their conceptions about the nature of mathematical activity and their pedagogical ideas about learning mathematics with technology. They deepened their knowledge of the connections among the various special cases of the secant-secant theorem. They experienced that *doing mathematics* involves formulating and testing conjectures and generalizations, as well as discovering and proving theorems. From a pedagogical point of view, these future teachers experienced what it means to teach and learn mathematics within IGS environments. The students take a more active role in their own learning under the guidance of the teacher whose main responsibility becomes facilitating. Making connections among mathematical ideas is a powerful

tool for prospective teachers' learning that they can transfer to their own teaching practice.

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