DEVELOPING MATHEMATICS CONTENT KNOWLEDGE ALONG NEW PATHS: SHIFTING FROM STUDENT TO TEACHER

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Today's classrooms pose many challenges for new mathematics teachers joining the teaching force. As they enter the teaching field, they bring a wide range of mathematical experiences that are often focused on calculations and memorization of concepts rather than problem solving and representation of ideas. Such experiences generally minimize what teachers are capable of practicing in the classroom. Ball and Cohen (1999) suggest that teachers need to become serious learners of practice rather than learners of strategies and activities. This involves teachers taking steps to understand the mathematics they will teach and what it means to reason and understand mathematically. This process also entails teachers moving from their own teaching experiences as learners into a more deliberate perspective of what it means for students to learn.

The Conference Board of the Mathematical Sciences [CBMS] (2001) has published *The Mathematical Education of Teachers*. In this document, several recommendations are established. Two such recommendations focus on experiences in prospective teachers' mathematics courses. They include:

- Courses on fundamental ideas of school mathematics should focus on a thorough development of basic mathematical ideas. All courses designed for prospective teachers should develop careful reasoning and mathematical "common sense" in analyzing conceptual relationships and in solving problems.
- Along with building mathematical knowledge, mathematics courses for prospective teachers should develop the habits of mind of a mathematical thinker and demonstrate flexible, interactive styles of teaching (CBMS, p. 8).

In addition, *Mathematics Teaching Today* (National Council of Teachers of Mathematics [NCTM], 2007) promotes a vision of what teachers at all levels should know and be able to do. This image embraces:

- Knowing about mathematics, general pedagogy, and students' mathematical learning;
- Ways to stimulate engagement and exploration;
- Implementing teaching and learning activities, including selecting worthwhile mathematical tasks; and
- Creating a challenging and nurturing classroom environment and facilitating meaningful discourse (NCTM, pp. 9-10).

In order to achieve such a vision, new teachers need exposure to learning opportunities that incorporate the following notions:

- equity, in the form of high expectations and strong support for all learners;
- curriculum that is coherent and comprehensive;
- teaching that is competent and caring;
- learning that is focused on understanding and using mathematics;
- assessment that monitors, enhances, and evaluates learning to guide instruction; and
- technology use that enhances mathematical understanding (NCTM, p. 122).

The recommendations and ideas promoted in these documents set the stage for this article. We believe that providing prospective teachers with "rich" investigative learning experiences in their mathematics content courses may lead these future teachers to reflect on how such experiences can potentially play out in their future classrooms as well as impact their own students' life-long learning of mathematics.

An Exemplar: Why Study Similarity?

A number of mathematics concepts is addressed in the elementary grades up through high school, but teachers often have limited exposure and experiences to gain insight into recognizing potential connections. Possessing the "know-how" of a mathematical idea – that is, studying an idea as it begins with a simplified form in elementary school and develops into

related components through middle and high school – is of great value to teachers' development of a deep understanding of content. Providing teachers with such experiences gives them the power to weave mathematical ideas across various grade levels or content areas into their instruction.

We have evidence that what teachers know about mathematics is an important factor for what they do in the classroom (Fennema & Franke, 1992). Prospective teachers' knowledge of similarity is an aspect of mathematics where inconsistencies in their understandings surface. Because similarity is a concept which is best developed across the grade bands in school mathematics, we provide opportunities for all our prospective teachers to study this concept. In content courses for prospective elementary school teachers, the focus of this effort is intended to highlight aspects of early algebra understandings (Carraher, Schliemann, & Schwartz, 2007) and a sense of geo-spatial relationships. In comparable courses required of prospective middle school mathematics teachers, the study of similarity is driven by aspects of the concept associated with understandings of proportional reasoning (Lamon, 2007).

Furthermore, for prospective secondary school mathematics teachers, similarity and its connections to trigonometric functions are presented by use of the hand-held Texas Instrument-Nspire device. We now elaborate on the path taken to empower prospective teachers in the development of their content knowledge as it relates to the theme of similarity. It is our hope that this excursion is one that continues beyond the content courses and into future classrooms when attempts are made to adopt comparable instructional practices with pre K-12 students.

Similar Models: Moving from Geometric to Algebraic Thinking

In a geometry course for elementary prospective teachers, similar figures are an area of study that typically follows congruent figures. Embracing the previous recommendations and experiences, an investigation of similar models is presented to the class. The investigation's framework, generated from the Grades K-6 Geometry and Spatial Sense Addenda series

(NCTM, 1993), consists of three parts: (1) building geometric models, (2) expressing geometric arrangements algebraically (or numerically), and (3) extending the initial problem situation. A snapshot of the investigation follows: Teachers are instructed to use the pattern blocks to complete the following geometric models and answer related questions:

1. Using the orange square pattern block, begin with one square.
2. Build models for "larger" squares that are similar to the first model (What does similar mean?). Maintaining the properties/look of a square, build the model for a second square, third square, fourth, etc. Draw a picture of what your models look like.
3. Now look at the number of pattern blocks required for each model. Record the number. Now predict how many pattern blocks would be needed for the 10th figure. Build it. Make a generalization for any size square.
4. Repeat the same process for the green triangle pattern block and the white rhombus pattern block. What do you notice about the models and the number of blocks required?
5. Analyze the pairs of figures (for example, the first square and the second square you built). Identify the relationship of side lengths for the squares as a ratio. Using this ratio, try to find a relationship of the areas between the first square and the second square. The first square and the third square, etc. How does this ratio apply to the triangles and rhombi?
6. How can this relationship extend to three dimensions? Begin with one cube. Build a second cube that is similar in appearance but twice as high, three times as high, etc. What do you notice?

Prospective teachers often enter the course with very narrow viewpoints of what similarity means. They often do not see relationships that exist beyond "same shape, but not same size." In addition, they rarely have opportunities to see the parallels between the mathematics they study in their coursework and how it relates to school mathematics. This investigation attempts to reveal such connections. Encouraging prospective teachers to progress from building models to generating

algebraic expressions forces them to think more deeply about relationships that exist among lengths, areas, and volumes of similar shapes and solids. This experience provides them with means to consider how they could take steps in introducing such investigations – or components of them – into their future elementary and/or middle grades classrooms.

Similarity and Scale: Traveling with Gulliver

In an example highlighting proportional reasoning, the context for the prospective teacher's mathematical thinking is drawn from Jonathon Swift's (1931) *Gulliver's Travels*. By extending an activity designed for middle grades students (*Gulliver's Worlds*, 2005), teachers gather and interpret data about measurable aspects of the context.

The reader may recall Swift's Gulliver, his travels to Lilliput, and a passage where Gulliver describes the people he meets as six inches tall. After accessing a digital image depicting Gulliver being restrained by the Lilliputians where the average person is measured to be 0.8 cm tall, teachers should:

1. Use this scale factor to determine how large the other objects in the digital image would be in Lilliput and in 'Our World'.
2. Find at least two other images of Gulliver and his time in Lilliput.
3. Establish the scale(s) of the new image.
4. Determine measurements of the depicted objects in both Liilput and 'Our World'.
5. Use multiple representations to describe the collected and computed measurements.

The effort to organize these data might lead some teachers to use Microsoft Excel to keep track of their findings. A sample of how one might use this tool is provided in Table 1.

If teachers chose Microsoft Excel (or some other spreadsheet software) to build tables of values, they could then use the software's (XY) scatter chart to construct graphs with varied axes. For example, a graph on which the size of a real object is represented on the horizontal axis and its size in

Lilliput along the vertical axis could be created. Likewise, a teacher could generate a graph such that the size in Lilliput is represented on the horizontal axis and the size in a digital image along the vertical axis. Regardless, it is important for prospective teachers to invest the time interpreting the different graphed representations attending explicitly to the connections between the scale factors, the numbering of the axes, the perceived steepness of the line, and the direct variation inherent to similarity. Given the connected nature of the topics taken up while "Traveling with Gulliver," it is our assertion this experience supports understandings of similarity. If middle school students are provided analogous materials and time to investigate and organize information with a spreadsheet, they too will extend their understandings of similarity.

Table 1
One Way Data May Be Organized

Objects Gulliver Might Encounter	Measurement in 'Our World'	Measurement in Lilliput	Measurement on the Digital Image
		From provided image	
People	6 feet	6 inches	0.8 cm
Ladder			1.6 cm
Lengths of Rope			
		From student-chosen image	
House		18 inches	
Grain Silo		4 feet	
Palace		9 feet	
		From student-chosen image	
Table (length)		1 foot	3.75 cm
Loaf of Bread	15 inches		

Similarity and Trigonometric Functions: Taking a Technology Path

Prospective teachers at the secondary level experience similarity—more specifically relationships between similar triangles and trigonometric functions. Many teachers have limited understandings of these relationships. In the following exploration (Johnson, 2010), teachers use hand-held TI-Nspire devices to explore connections between these topics.

This investigation requires teachers to conjecture and test for triangle similarity. Upon representing the corresponding sides as proportional, (i.e., , $\frac{BE}{AE} = \frac{CD}{AD}$: see Figure 1), the focus is then directed to investigating the position of point E relative to point D. The intent is for the teachers to determine $\sin(m\angle A) = \frac{BE}{AE}$. As point D is moved, point C moves along $\overrightarrow{AB}$, so that the right triangle properties are preserved. This exercise provides teachers with an environment to connect the trigonometric functions generated by the side lengths of the right triangles formed (from the initial Δ ABE) to the similarity of the newly-constructed triangles.

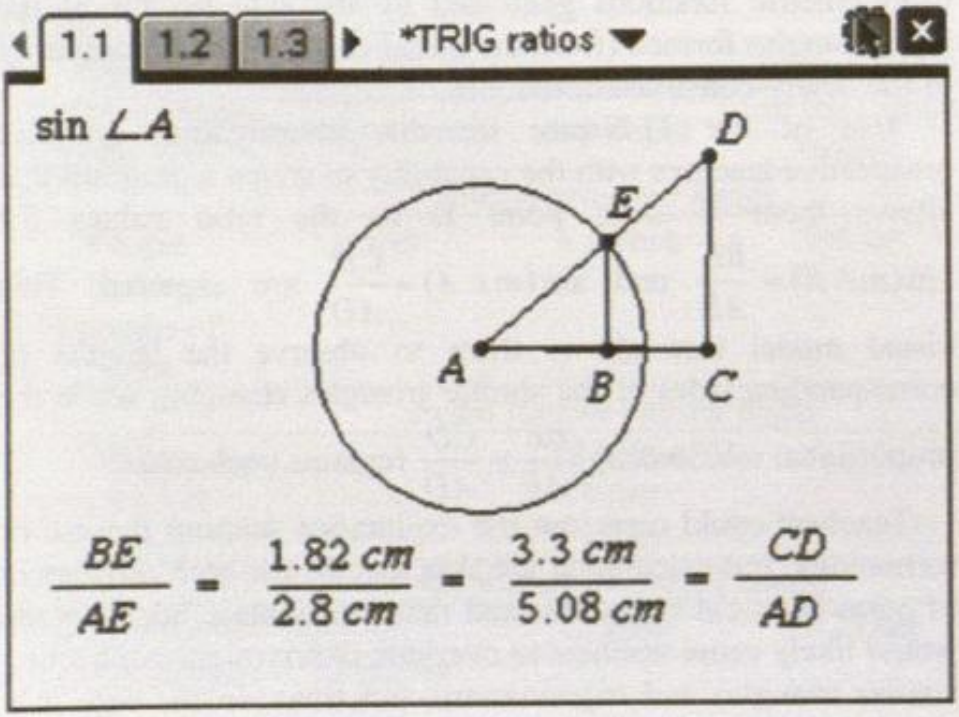
Use of the TI-Nspire for this investigation provides prospective teachers with the capability to create a program that allows them to *drag* point D as the ratio values for $\sin(m\angle A) = \frac{BE}{AE}$ and $\sin(m\angle A) = \frac{CD}{AD}$ are captured. This visual model also allows them to observe the lengths of corresponding sides of the similar triangles changing while the proportional relationship $\frac{BE}{AE} = \frac{CD}{AD}$ remains unchanged.

Teachers could carry out the exploration without the use of technology, but calculating the side lengths for each movement of point D would be tedious and time-consuming. Such a route would likely cause teachers to overlook observations connecting similar triangles and trigonometric functions. In the end, it is our hope that these secondary prospective teachers will provide their future students opportunities to examine connections between other related topics.

An On-Going Journey

A teacher preparation program should empower prospective mathematics teachers for the on-going journey of cultivating their content knowledge as they grow into experienced teachers. Thus, it is critical for prospective teachers to have opportunities exploring mathematical connections in their coursework related to pedagogy as well as content. NCTM (2007) articulates, "if teachers are to change the way they teach, they need to learn significant mathematics in situations in which good teaching is modeled" (p. 186). Through the featured exemplar, as well as additional open-ended tasks, new teachers can gain an innovative vision into the power of implementing high-quality investigations. This on-going journey provides them with the realization and insight to use mathematical connections where appropriate, in guiding all students to learn mathematics (Boaler, 2006; Gutierrez, 2009; Gutstein, 2006).

Figure 1. TI-Nspire hand-held screen capture for sine function.



Resources for Developing Rich Mathematical Tasks

For new teachers, we provide here a sample list of valuable resources to help develop good mathematical tasks that are rich in content: Make these resources a part of your library to enhance not only your mathematics content knowledge, but that of your students as well. Here's to an exciting journey!!!

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National Council of Teachers of Mathematics. (2007). *Mathematics teaching today* (2nd Ed.). Reston, VA: Author.

Swift, Jonathan, (1936, c1912). *Gulliver's Travels*, illustrations by Milo Winter. Windermere series: Rand McNally & Co., New York, Chicago, retrieved from http://www.jaffebros.com/lee/gulliver/winter/p1.jpeg, February 26, 2010.