Is proof dead in the computer-age school curriculum?

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During the last decades of the twentieth century, changes in school curricula have resulted in proof-free or proof-lite curricula. This article argues that proof is, and should be seen to be (and that is another issue), a central component in the school curriculum — from at least the middle of the primary years and upwards. It identifies proof with problem solving, logical argument, explanation, and meaningful learning. The article includes examples of proofs, and sub-proofs, such as that $\sqrt{2}$ is irrational.

Proof has always been central to research mathematicians. It is the engine that has driven historical developments in mathematics, that vast body of knowledge with roots in the folk traditions and earliest historical records of ancient civilisations. For many centuries proof—specifically the classic geometry proofs of Euclid—was a staple of school mathematics. Why? Traditionally, through the primary years, the broad aim of the school mathematics curriculum was to prepare for secondary mathematics study, and to help students learn to handle the everyday demands of everyday life. Then, through secondary school, the broad aim was to prepare students for university study of mathematics, and to use school mathematics in other university subjects and in specialist vocations. Proof, albeit often confined to the geometry curriculum, was then a natural part of secondary preparation for further mathematical study, even though it might have no simple use in everyday life, or in most forms of paid work. At least it was, outside of *clear thinking* lessons in English (or similar venues for rigorous argument), an important place that school students experienced a "nice knock-down argument" (such as appealed to Alice's acquaintance, Humpty Dumpty in Carroll, 1971/1960 edition chapter VI 'Humpty Dumpty').

Through the last quarter of the twentieth century, or slightly longer, proof in school mathematics has been an oddity, a fossil survivor (where it has survived) like the coelacanth and the horseshoe crab. With changes in the school curriculum, recent generations of school students and school teachers have experienced (mainly) proof-free mathematics.

Proof in the Victorian Curriculum

Consider, for example, the *Victorian Essential Learning Standards (VELS) Discipline-based learning strand: Mathematics*. Within the Dimension (curriculum sub-section) Working Mathematically, at Level 4 (to be achieved by the end of Year 6) standards, we find:

> Students develop and test conjectures. They understand that a few successful examples are not sufficient proof and recognise that a single counter-example is sufficient to invalidate a conjecture. For example, in:
>
> - number (all numbers can be shown as a rectangular array)
> - computations (multiplication leads to a larger number)
> - number patterns (the next number in the sequence 2, 4, 6… must be 8)
> - shape properties (all parallelograms are rectangles). (VCAA, 2008, p. 25)

Apart from this interesting example, where the emphasis is actually on disproving a conjecture, the word “proof” does not appear below Level 4, and has no substantial mention in Levels 5 or 6 (ending around Year 10, before the post-compulsory specialist secondary years): it is mentioned, but with no examples! Similarly, a search of the related Progression Points (an online curriculum, assessment and reporting resource used in Victoria) finds the words “proof” and “conjecture” being used, but with little illustration by example.

A related interesting example, for the key word “conjecture”, occurs in Level 6 (to be achieved by the end of Year 10) Working Mathematically:

> At Level 6, students formulate and test conjectures, generalisations and arguments in natural language and symbolic form (for example, ‘if m^2 is even then m is even, and if m^2 is odd then m is odd’). They follow formal mathematical arguments for the truth of propositions (for example, “the sum of three consecutive natural numbers is divisible by 3”). (VCAA, 2008, p. 37)

It is very interesting to see the example-conjecture ‘if m^2 is even then m is even, and if m^2 is odd then m is odd’, as this is a crucial Lemma in a proof of the irrationality of the square root of 2—a very important conceptual exemplar.

An exhaustive and creative search of the Victorian Progression Points can actually uncover a reasonably large curriculum for Proof. For example, as early as Years 1 and 2, working towards the completion of Level 2, we find in Working Mathematically [WM]:

> WM: 1.25 — Students select appropriate materials and diagrams to model and describe mathematical ideas and test simple conjectures. They use basic mathematical facts and symbols to describe their thinking when solving problems.
>
> WM: 1.5 — Students test simple conjectures by transferring known facts to unfamiliar situations using examples of objects, patterns, shapes and numbers. They use the calculator and describe how they use it to explore numbers and solve simple equations and problems.

WM: 1.75 — Students test simple conjectures by describing examples and counter-examples using materials, diagrams and models.

Ideas of "proof" (indicated by "describe their thinking", and "test simple conjectures") or a convincing argument are implicit here, and in other Dimensions, in the explanations of how a problem has been solved, the discovery of a counter-example, the recognition of a pattern, or the logical sequence of definition and concepts, notation and process used in some multi-step computation. The catch is that neither the teachers nor the students are likely to recognise any of this in terms of the word "proof". That is, a Proof sub-curriculum does exist within the larger Victorian mathematics curriculum, but it lacks explicit labelling that makes it coherent, and visible. For example, Working Mathematically has the Progression Point:

WM: 5.75 — Students follow a formal mathematical argument of several steps presented by the teacher, such as Pythagoras' theorem.

Here, even though a major theorem is named, are students to experience one or more formal proofs of the theorem, or just the algebraic-computational and measurement uses of it?

Proof in the New South Wales Curriculum

The New South Wales syllabus is an interesting comparison (Board of Studies, NSW, 2007). In the sub-syllabus for Space and Geometry, in Stage 4 (to be achieved by the end of Year 8), we find, for example:

SGS4.3 — Triangles: prove, using a parallel line construction, that the interior angle sum of a triangle is 180°;
SGS4.3 — Triangles: prove, using a parallel line construction, that any exterior angle of a triangle is equal to the sum of the two interior opposite angles.

Elsewhere in the NSW syllabus we find in the Background Information (to guide teachers implementing the syllabus): "Memorisation of proofs is not intended. Every statement or theorem presented to students to prove could be confirmed first by construction and measurement" (Board of Studies, 2003, p. 159). This is interesting; but it is unclear what can be proved by the specific example that results from a "parallel line construction": a construction is different from a schematic sketch. Similarly, it is unclear how physical constructions and measurement can aid with the conceptual generalisation that is the essence of a "proof". Further exploration of the 7–10 NSW syllabus finds an extensive sub-syllabus of Proof, in mid-secondary—but exclusively concerned with geometry, of the formal Euclidean kind. This has traditionally been the case in North America: much of the secondary curriculum is compartmentalised into subjects: Algebra, Trigonometry, and Geometry—only the last with any explicit attention to proof.

By contrast, even though it wears a disguise, the Victorian curriculum for Proof, albeit expressed not as "proof" but in terms of problem solving, reasoning, logic, making and testing conjectures (and so on), begins early, and ranges across all the topics in the curriculum, not just geometry. Hidden, or not, that Proof is so central in the current Victorian curriculum is a credit to the curriculum designers—but is that how Victorian teachers see it or are the richer challenges of Working Mathematically neglected, the better to concentrate on the more familiar teaching of Number, Measurement, and other seemingly stand-alone sub-curricula (referred to as Dimensions, in the Victorian system)? That Working Mathematically is supposed to apply to all the other Dimensions tends to lose emphasis, because although it is meant to be ubiquitous, it is in the background of other focal Dimensions.

Hence I suggest that, as seen in these cases of the Victorian and NSW curricula, proof, as a sub-curriculum, does exist; but it is either concealed, or appears only in Geometry, and only as an important topic in mid-secondary, and beyond. We can quantify the NSW focus on Proof. Across 11 years of primary and pre-Year 11 secondary, perhaps three of the years include Proof —only in Space and Geometry, one of five strands. Within Space and Geometry, Proof appears only in some of the topics, such as Deductive Geometry. Overall, out of the NSW compulsory school syllabus, rather less than 3/55 is identifiable as Proof—not more than one-twentieth of the syllabus! Is this proof-lite?

But should this be the case? What is proof? And why is it educationally problematic? We might think that, given centuries of school tradition, proof is fundamentally concerned with Euclid and formal geometry. Against this, I argue that a stronger, less limiting, more general view sees proof as what we do whenever we present a logical argument for something (or against something). We can even be proving, using logic, in contexts other than mathematics, such as in Law (e.g., du Sautoy 2008, p. 233), the sciences, or even in history.

The English word "proof" has several meanings. Here I will set aside the misleading proverbial alternative usage: "The proof of the pudding is in the eating". Here "proof" means only "test": we taste the pudding to test that it is edible. This is also equivalent to a "trial". This meaning of "proof" is equivalent to the cognate words "probe" and "probity". Legally, the conclusions of a court trial are based on acceptable evidence—physical experience and personal reporting—provided by witnesses and forensic science. By contrast in mathematics when we prove something we usually do not rely on physical evidence or witnesses' reporting. Instead we ensure that every step of the argument is logically sound. A mathematical proof is only partly like a court trial—it is partly a test of the logic. The mathematician's proof is more than just a test or trial of the logic used in the argument. It also tests the rightness of the assumptions, or clarity of definitions (as explored by Lakatos: 1976), that begin the argument. (The conclusion reached at the end of a proof is sometimes referred to as a "theorem". Basic assumptions made at the beginning of a proof may be referred to as "axioms". The words, or concepts announced at the beginning of a proof are usually called "definitions". I mention these technical terms for completeness.)

I must stress that mathematical proof is not simple, philosophically. I refer the reader to Jaquette's encyclopaedic exploration in *Philosophy of Mathematics* (2002) and to Lakatos' pioneering analysis (1976). Nor is mathematical proof simple, educationally. I refer the reader to *Proof: International Newsletter on the Teaching and Learning of Mathematical Proof*, and to Flener (2001) for an account of an educational experiment in the 1930s based on the explicit use of mathematical and other proof in a school curriculum, and its life-long impact on the students in the experiment.

In considering school mathematics, for simplicity and brevity and to emphasise what I believe is the general importance of proof (whatever this may be philosophically, and however it may be implemented educationally), I will adopt a generic sense of "proof" and "proving", and omit more rigorous and potentially controversial details. Hence, for example, we are proving when we ask, as MacNeal does, why does $2 \times 2 = 4$? MacNeal's discussion leads to his "Proposition 15: Until you work it out for yourself, two times two makes four only because the teacher says so. You have to *do* multiplication before you can understand what it means" (MacNeal, 1994, Chapter 11). Similarly we are proving when we read and make sense of Paulos's exploration of the connections and slippages between contextualised natural languages and formal mathematics, and especially symbolic logic (Paulos 1998, Chapter 3). Also we are proving when we consider Devlin's compelling arguments for a newly emerging "soft mathematics"—a form of mathematics which retains aspects of rigour while being embedded in natural language, and in a purposeful human context. A rigorous axiomatic approach runs the risk of breaking down, or becoming so highly technical that it is unusable: but if this occurs, common sense interpretations and informal arguments will fill the gaps (Devlin, 1997, p. 282). We can even be proving when we play a logic game (e.g., Layman E. Allen's neglected *Wff N' Proof or Queries N' Theories*, Allen, 1962).

Leaving aside these complications, we are proving whenever we start with (mathematical) definitions and a (mathematical) question, and pursue the logical consequences of this beginning with as much clarity as we can, so we reach conclusions that are objective and compelling—for ourselves and our audience. This is a reason-based (rather than experience-based or physical-evidence-based) logic-managed version of general curiosity and reasoning. Since it uses logic, this kind of curiosity does not apply to the creative emotive Arts or Humanities; but proof as part of mathematical thinking applies to all subject-areas that are, or aspire to be, objective and evidence-based, as well as logic-based. Proving is another example of mathematics being applicable across the curriculum!

In our ordinary mathematics classrooms we will be proving whenever we or our students ask the questions: "What does this mean?" "What does it not mean?" and "Why is this true?" School students should be asking these questions purposefully—to learn!

If proof is understood in my generic way as a convincing logical argument, then proof is virtually identical to reasoned problem solving. Ever since *An agenda for action: Recommendations for school mathematics of the 1980s* (NCTM, 1980) problem solving has been accepted as an identified crucial component of any school mathematics curriculum, at any year-level. In my generic sense, with a strict emphasis on logic, proof is equivalent to, or part of Polya's classic fourth stage of problem solving: *Check* (Polya, 1945). I have also argued that learning is essentially what we do when we are problem solving (Gough, 1989). This suggests that if we accept that problem solving is central to school mathematics, then so is proof, as a form of problem solving and mathematical argument—and learning!

John Mason also sees proof as an aspect of communication—convincing someone:

> The word *proof* is off-putting for many, perhaps because it has a sense of certainty or finality: once something is proved there seems to be no going back. I find it helpful to think in terms of convincing yourself, then convincing a friend, and then convincing a very sceptical 'enemy' who looks for every possible little gap that you might have overlooked. (1998, p. 4)

Incidentally, a proof need not be a verbal sequence of argument statements: it could be a diagram that shows the argument. For example, the limit of the infinite addition of fractional powers of 2, namely,

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

Why? We can "see" this is true because of the visually obvious way the unit square contains an infinite sequence of successive halvings (as shown in Figure 1). As the ancient Hindu mathematics textbook by Bhaskara (circa AD 1150) said about the classic diagrammatic proof of Pythagoras's theorem: "Behold!" (end of argument); see, for example, www.scribd.com/doc/40860/The-Pythagorean-Theorem-a-Wonder-for-all-Ages.

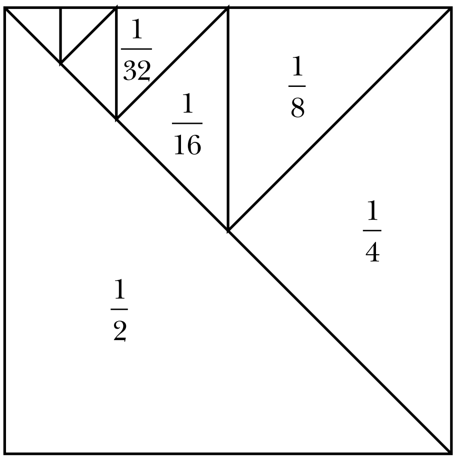

Figure 1. A traditional diagrammatic proof that $1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$

Unfortunately, no easy visual diagram can prove the following amazing series, that can be proved by integration (Pedoe, 1958, p. 133):

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} - \frac{1}{9} - \frac{1}{11} + \ldots$$

Despite its geometric connections, what diagram could possibly represent π?

Why is proof educationally and mathematically problematic?

In part, the removal of proof from the school curriculum is due to the overcrowded curriculum versus the information explosion. In part, it reflects the perceived intellectual difficulty of teaching and learning about proof. In part, it is also due to concern that proof was misunderstood by many students as a matter of merely memorising steps in a standardised formal argument, rather than as a fluid exploratory process. The more the curriculum (in all subject areas) emphasised the importance that students understand what they learn, the less value was placed on students memorising technical vocabulary, definitions, formulas, and mechanical computation processes. In the final stage of Year 12 students were allowed to attend examinations with a sheet of summary notes and formulas: previously this would have been forbidden and all examination answers would have to be based on memorised knowledge. Being able to repeat a formal proof (e.g., for Pythagoras' Theorem, or to differentiate from first principles) from memory under exam conditions was no longer required: being non-examinable, did students need to experience such proofs at all?

In 1984, the Fredkin Foundation, established by Edward Fredkin, an artificial-intelligence expert at MIT, announced a prize of $100 000 for the first computer program to make a mathematical discovery—a major new theorem based on mathematical ideas not implicit in the program that discovers it. (Gardner, 1996, p. 39). As far as I know, the prize for a computer-created new theorem has not yet been claimed. Computers have been used to assist humans construct massively complicated proofs: du Sautoy's fascinating discussion of symmetry (or group theory) repeatedly refers to the extensive use of calculators and computers to support human reasoning (2008; e.g., pp. 38–139, 144–146, 306, 312).

In my view, we should not be too worried about the increasing role of computers in supporting proofs. Some extreme writers claim this means we have reached the era of the "Death of Proof" (Horgan, 1993). Computers, for example, were essential experimental testing tools in the celebrated proof of the famous Four Colour Theorem, that only four different colours are needed on a flat map so that each country can be given a colour, without sharing any part of a border with another country which has the same colour. It is unlikely that humans could have completed all the tests of different colouring possibilities needed to prove, or verify that, this theorem holds, for all mathematically different configurations of countries. Whether the computer made a proof, as such, or only verified the truth of the theorem, is discussed in Jaquette (2002, pp. 167, 193–208, reprinting a 1978 discussion by Appel & Haken).

Obviously a human programmed the computer that tested these configurations. In my experience, constructing a computer program is essentially the same mathematical task as constructing a proof: manipulating the defined program commands and logical rules of syntax to obtain a demonstrable result, rather like a theorem. Until computers can program themselves, and create their own programming language, we can be assured that proof is alive and well as an essential part of at least some research mathematicians' and computer programmers' professional activities. Hence, it deserves active attention in school classrooms.

We currently have a mathematics curriculum that is largely proof-free (or light), and a professional body of active mathematics teachers who may themselves have studied little, if anything, about proof, at primary or secondary school levels. At the end of the first decade of the 21st century, any discussion of proof by, or for, school teachers must begin with a careful personal analysis of what each teacher already knows about proof, possibly teaches about proof, or recalls studying about proof during school and university mathematics learning: When was the last time you proved something? When you were a student at school, were you ever required to prove anything? Have you ever proved anything, mathematically speaking, in your life? I ask these questions because I believe proof is important. Hence, all mathematics teachers and students should be exposed to ideas of proof and proving. Others agree with this (e.g., Mason, 1998; Mason, Burton, & Stacey, 1982).

However, despite the upheavals of the New Math, de-emphasis of memorising, refreshed emphasis on problem solving, dilution or elimination of formal study of (Euclidean) geometry, introduction of more statistics and probability, and the widespread use of calculators and computers, the broad content of the existing school curriculum (which contains minimal or no proof) has not changed much since the 1960s, and earlier.

Consider examinations, and memorising: it is interesting that important explanatory terms in the constructivist theory of learning are "internalisation", "automation" and "encapsulation" (e.g., Gray & Tall, 1994). Having students memorise meaningfully is a constructivist's ally: by memorising students conceptualise—"internalise". Exam preparation also helps students form higher-level conceptions that encompass separate conceptions or skills—"encapsulation". Requiring students to perform fluently and rapidly within the time-limit of an exam is a stimulus for fast practice—"automation". Notoriously, memorising for an exam is widely believed to last only for the duration of the exam. Yet surprisingly often the strangest things memorised for exams stick in our minds. This is particularly so when they are made interesting and meaningful, a related diagram is visually striking, or a step in a calculation is breathtakingly sneaky (ingenious) or elegant (witty and/or aesthetic).

Consider fitting together two copies of the staircase made by an arithmetic sequence. To make a formula for the sum of *one* staircase, who would guess an easy way is to solve it for two (see Figure 2). Who would guess another easy way is to proceed by adding pairs of terms of the staircase from each end of the staircase—a pincer attack!

As a simple geometric context, for exploration and proving as reasoning and generalising, consider my concept of "Hole Numbers" (Gough, 1978). A hole number is whole number which can be represented by a rectangular array of dots, except that at least one dot is missing from inside the complete boundary of dots. For example, 12 is a hole number, because it can be represented by a 3×5 rectangular array of dots, with three dots omitted from within the array. (Draw the diagram!)

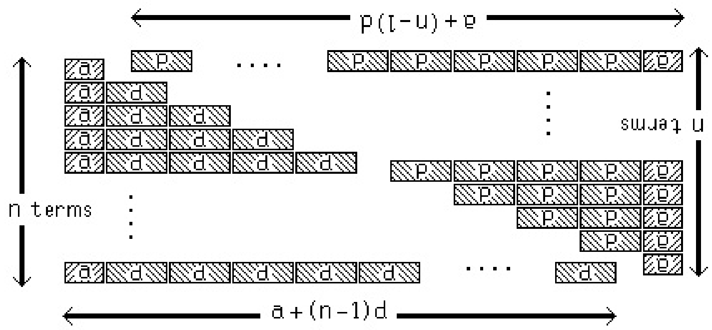

Figure 2. The rectangle made by fitting together two stair-case "triangles": the sum of n terms of an arithmetic sequence.

What is the smallest whole number you can find which is a hole number?
What are the first ten hole numbers?
How many hole numbers are there?
What is the largest whole number which is not a hole number?
Similarly, consider *plug-hole numbers*, where only one hole is allowed, and it must be at the geometric centre of the rectangle.
Consider *fence numbers*, the set of hole numbers which consist only of boundaries.

Shortly after students encounter square roots they may be ready to grapple with the formal proof that some square roots cannot be expressed exactly as fractions, finite decimals, or infinite recurring decimals. Unless students experience this proof, they face a version of MacNeal's Proposition 15: Until you see the proof for yourself, root-two is irrational only because the teacher says so. You have to *do* proof before you can understand what it means (adapted from MacNeal, 1994, p. 290.) The distinctions and relationships between fractions, finite decimals, and infinite recurring decimals is a rich topic for exploration.

The initial discovery that the square root of 2 cannot be exactly represented as a ratio of two whole numbers led to an intellectual crisis in mathematics in Ancient Greek times. The method of proof, involving *argument by way of contradiction*, is not too difficult for confident upper secondary students to grasp. However, we should expect that many students will not understand the complete idea the first time around. This proof is worth revisiting every year, being aesthetically lovely to behold, as is Baskhara's implicitly algebraic proof of Pythagoras's Theorem, which uses a diagram of a hypotenuse inner-square inside a larger square made with four other triangles: consider these two alternatives (see Figure 3).

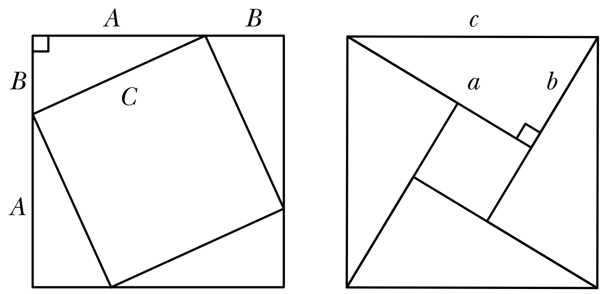

Figure 3. Two dissections of a square, demonstrating the theorem of Pythagoras.

Theorem: The square root of 2 is irrational — proof by contradiction

Without including all the algebraic steps, we begin by making the deliberate assumption that the square root of 2 can be expressed as a fraction, such as

$$\frac{m}{n}$$

with m and n positive whole numbers, with no common factors:

$$\sqrt{2} = \frac{m}{n}$$

Squaring both sides gives: $2 = \dfrac{m^2}{n^2}$ and hence $2\left(n^2\right) = m^2$

This means that m^2 is an even number.

We detour slightly to prove the lemma (a minor or subsidiary supporting theorem) that if m^2 is even, then m must be even. That is, there is a positive whole number P that satisfies the equation $m = 2P$. Hence $2(n^2) = 4(P^2)$. We divide both sides by 2, and find that: $n^2 = 2(P^2)$. This means that n^2 is an even number, and n must be even.

But that is a *contradiction*: n and m are both even, contrary to our initial assumption. The only way to avoid reaching this logical contradiction is to reject our initial assumption: namely, the square root of 2 can not be expressed as a fraction, such as

$$\frac{m}{n}$$

with m and n positive whole numbers, with no common factors. When this occurs we say that such a non-fraction-expressible number is irrational. QED: *quod erat demonstrandum* — Latin words, meaning, "which was to be proved", but often glossed, sarcastically, as "quite easily demonstrated"—sarcastic, because in the eyes of some students this argument has been far from easy. But just follow the ideas and the steps, logically: it is, aesthetically, a beautiful proof, with the possibly stunning result that some numbers are very different from whole numbers, or simple ratios of whole numbers.

Given a proof of any particular statement it is often hard to think of other statements that could be suggested, and either proved or disproved. It helps, then, to offer a range of statements, indicating possibilities for further questioning and conjectures. For example, now that we know we can prove that $\sqrt{2}$ is irrational, so what? Where could this lead us? Easily enough, perhaps, we could move to the next number, and ask whether $\sqrt{3}$ is irrational? (The irrationality of the cube-root of 3 is proved in Gough, 2009). Of course $\sqrt{4}$ is not irrational. Where next? What numbers can be proved to be irrational? Can we construct a general proof?

Similarly, what other numbers can be shown to be rational or irrational? We can, for instance, prove that $\log_{10}2$ is irrational. (I refer curious readers to Johnson & Rising, 1972, pp. 324–325). It is hard to identify other numbers whose irrationality might be tested. The fundamental number π can be shown to be irrational, but the proof would challenge most university mathematics graduates (Spivak, 1967, pp. 277–280). Spivak also includes a proof that the logarithmic constant *e* is transcendental (pp. 362–368).

This method of proof—by contradiction—is notoriously harder for students to grasp than direct proof. An easier example of proof by contradiction is demonstrating Euclid's proof that there are infinitely many prime numbers. Moreover, there are other methods of proof that cannot be considered here, such as proof by (mathematical) induction (not to be confused with scientific induction, as a method of developing an evidence-based generalisation).

When we solve a problem we do something (e.g., draw a diagram, manipulate a spreadsheet, zoom in on a graphed function, fiddle with some algebra, make numerical calculations) that convinces us (as Mason, 1998, says) that we have answered a question about which we were initially uncertain. But would anyone else believe you? You need a proof!

Centuries ago proof was a geometric demonstration based on Euclidean ideas. Nowadays a proof is often a string of algebraic statements, linked by logical inferences, beginning with scrupulously defined terms and rules, possibly based on axioms. Outside the realms of abstract senior secondary and undergraduate curricula, and technical research mathematics, a proof is a story that persuades other people and one’s self that some claim is mathematically correct.

Conclusions

Conceptual understanding depends on logic, albeit, possibly simple, or naturalistic, or naïve human logic. Consider the reasoning from first principles needed to answer this question:

> You have three coins. One is double-headed. One is double-tailed. One is an ordinary head–tail coin. All the coins are “fair” or unbiased in the way they fall. You choose one of the coins, randomly, and toss it. The coin lands heads uppermost. What is the probability that the other side of the coin is a tail? (Adapted from Donaldson, 1978.)

If you learn mathematics by understanding the reasons for what you know, and do, then you are experiencing proof. Where you do not have proof, you have rote learning—learning something without being able to meaningfully explain it or connect it sensibly with other existing knowledge (see Gough, 2004). If you have not taught a proof, you have taught by rote. It is as simple as that, and a central, necessary thread throughout the mathematics curriculum!

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