Cognitive development of applying the chain rule through three worlds of mathematics

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The derivative of a composite function, taken with the chain rule is one of the important notions in calculus. For a function of a single variable, the chain rule is

$$\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx}$$

where $z = f(y)$ and $y = g(x)$. In the function notation, the chain rule indicates: $h'(x) = f'(g(x)).g'(x)$. In Turkey where the study described in this paper was conducted, the chain rule was given with the formula in function notation and/or the Leibniz notation without relating these formulas to life-related problem situations in the mathematics teaching program. As a result of changes in the teaching program in Turkey, the instructional way of presenting the chain rule changed focus to encourage students to obtain the chain rule with some life-related problem situations. Applications of the chain rule to life-related situations are also suggested as a learning experience in the Queensland Curriculum (QSA, 2009). On the other hand, verifying the chain rule by using either or both graphing software/ graphics calculator and an algebraic approach is considered for developing teaching and learning strategies of the chain rule in the mathematics teaching program of South Australia (SACE Board of South Australia, 2009).

Theoretical background

The mathematics education literature includes several studies about the chain rule. Some of them are about teaching of the chain rule (Lutzer, 2003; Mathews, 1989; Thoo 1995; Uygur & Özdas, 2007) while others are on understanding the rule. Webster (1978) investigated the effects of emphasising composition and decomposition of various types of composite functions on the attainment of chain rule application skills. Clark et al. (1997) investigated understanding of the chain rule in the APOS (Action-Process-Object-Schema) theoretical framework (Asiala et al.,1996). According to the genetic decomposition that they developed, the chain rule schema improves through the stages of the triad: *Intra*, *Inter*, and *Trans*. At the Intra stage, a student has a collection of rules for finding derivatives of special composite functions, but cannot recognise the relationships among them, although the general formula may be known. The student at the Inter stage begins to collect different cases and recognise that in some way they are related. To reach the Trans level, a student must link function composition and decomposition to differentiation, and recognise various instantiations of the chain rule, which follow from the same general rule through function composition. Cottrill's (1999) findings in his dissertation are consistent with the genetic decomposition for the chain rule developed by Clark et al. Additionally, Hassani (1998) and Capistran (2005) studied understanding of the chain rule. Although students' notational and structural difficulties were highlighted in some studies, I did not find a study investigating students' understanding and applying the chain rule in terms of symbolic and structural sense. Novotna and Hoch (2008) posited to focus on developing students` structure sense. Thus, it was decided to analyse students' applying the chain rule within Tall's (2007) framework, which considers symbolic development, in order to address this apparent gap in the literature.

The framework of "three worlds of mathematics", developed by David Tall, describes three levels of understanding (Tall, 2007). The first world is the *conceptual-embodied world*, in brief; it is known as the embodied world. This world includes thoughts about mathematical things after perceiving and sensing them. By reflecting on and using the language of mathematics, an image of a mathematical thing is produced. The second world is the *proceptual-symbolic world*. This world is the world of symbols that is used in calculation and manipulation, for instance in calculus. By using symbols, processes to do mathematics are switched effortlessly to concepts to think about. The last world is the *axiomatic-formal world*. This is the world of mathematical sense and knowledge. The axiomatic-formal world includes formal definitions and proofs of theorems. Tall labelled these three worlds as "embodied," "symbolic," and "formal", and he considered combinations of them as presented in Figure 1.

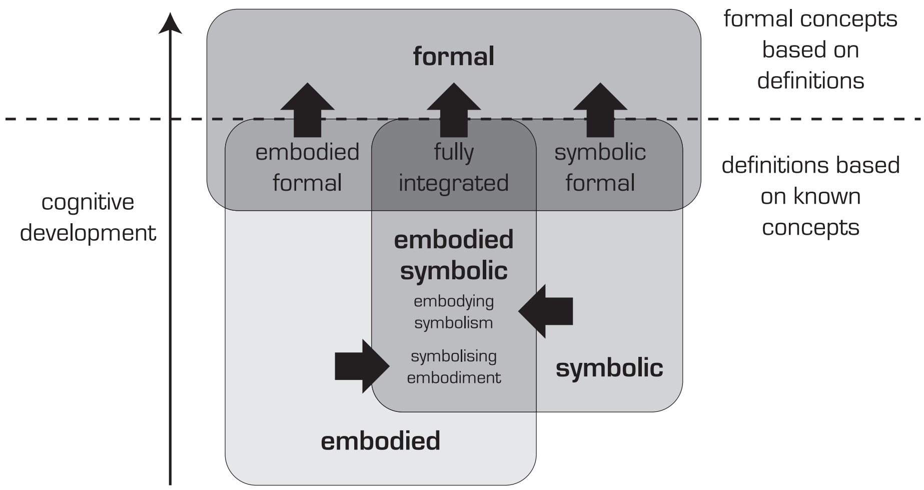

Figure 1. Cognitive development through three worlds of mathematics (Tall, 2008, p. 8).

As well as lack of studies on structural development of the chain rule, there is a need for a study on students' applying the chain rule to second order derivatives and to two-variable composite functions. Although students' difficulties intensify when taking partial derivatives of two-variable composite functions or second order derivatives of composite functions, the studies about the chain rule take into account only single variable functions and first order derivatives.

Based on the framework of "three worlds of mathematics", an attempt was made to address these gaps in the literature, by answering the following research questions:

- How does a student develop applying the chain rule?
- How does a student develop applying the chain rule for taking the second order derivative?
- How are the embodied route, symbolic route and combinations of these evident in the cognitive development of the chain rule?

Context, methodology and instruments

The present study was conducted in the mathematics education program in an education faculty of a public university in Turkey. Twenty-seven students whose ages ranged from 18 to 21 years participated in the study. Only three of the participants were male, so the number of females in the mathematics education program was high, biasing the sample in terms of gender. These pre-service mathematics teachers were taking a two variable analysis course (Analysis III). The students took Analysis I and Analysis II courses, which include differentiation and integration of single variable-functions respectively, in the previous year.

For the study, an arrow diagram was used as a mnemonic device while teaching the chain rule in the course. Various formats of the arrow diagram are used in most calculus textbooks (e.g., Barcellos, & Stein, 1992); and some studies about the chain rule (e.g., Thoo, 1995). For instance; for $w = f(x,y)$, where $x = x(t)$, the arrow diagram is as in Figure 2.

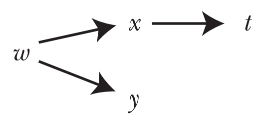

Figure 2. Arrow diagram.

In the study, after teaching the chain rule, a test was prepared and used to select students for interview. The test included four items, as follows:

1. Let $w = \cos(u + v)$, $u = \tan x$ and $v = \ln x$. Find the derivative of w with respect to x.
2. If $y = f(u)$ and $u = g(x)$, evaluate the second order derivative of y with respect to x.
3. Given $w = \sin\ (\tan x + e^x)$, find the derivative of w with respect to x.
4. If $w = f(x,y)$ and $x = x(t)$, evaluate the second order derivative of w with respect to t.

As appropriate to purposive sampling (Fraenkel & Allen, 1996), the students were grouped according to the results of the test and then students to interview were chosen.

Purposive sampling

While most students could take derivatives of special functions, some of these students had errors due to carelessness or confusion about memorised rules. We ignored procedural errors. Most students were successful taking first order (partial) derivatives of abstract composite functions also; however four students failed these items. Furthermore, some students had notational or structural difficulties that intensified in taking second order derivatives. To explain the students' difficulties with second order derivatives, we give the expected response to the fourth item. Accordingly, from $w = f(x,y)$ and $x = x(t)$, a student can produce Figure 2 and by using this figure, obtain:

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \times \frac{dx}{dt} \tag{1}$$

It should be highlighted that a second order derivative of w with respect to t is required in the fourth test item.

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial x} \times \frac{dx}{dt}\right) \tag{2}$$

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial t \partial x} \times \frac{dx}{dt} + \frac{d^2 x}{dt} \times \frac{\partial w}{\partial x} \tag{3}$$

The equality $$\frac{\partial^2 w}{\partial t \partial x} = \frac{\partial}{\partial t}(\frac{\partial w}{\partial x}) = \frac{\partial^2 w}{\partial x^2} \times \frac{dx}{dt} \tag{4}$$

is obtained by using the arrow diagram of the partial derivative function.

As mentioned above, most students could obtain the first order partial derivative successfully as in (1). The students having difficulties with the second order derivatives could do the calculation up to the equalities of either (2), or (3) or (4). On the basis of students' difficulties, five groups were obtained. Since the students' skills and difficulties taking second order derivatives were similar in the second and fourth items, grouping students was not ambiguous. To select the interview students, students' general performances in the test were used. One student, whose general performance could represent the whole group characteristics, was selected from each group to be interviewed. Selected students were called P1, P2, P3, P4, P5 for the groups 1 to 5, respectively.

First group (P1)

The four students in this group had structural difficulties in the second order derivative process and they were not able to use the diagram correctly, even if they were able to apply the product derivative rule. Therefore, they were not able to obtain (3), then naturally (4).

Second group (P2)
The seven students in this group were able to take second order derivatives completely as well as first order derivatives. They might have some minor notational errors, which did not influence the results.

Third group (P3)
This group comprised six students who were able to obtain (3), but they were not aware of other second order partial derivatives, which should be calculated in (3).

Fourth group (P4)
This group of four students did not attempt to calculate the second order partial derivative.

Fifth group (P5)
There were six students, who were able to express the second order partial derivative as (2). However, they were not able to carry out the calculation any further. We thought that these students were at least aware of which derivative they should take with respect to which variable. On the other hand, the students in the previous group missed the items in which second order derivatives were required.

After selecting the participants, the interview was prepared, piloted with a student by the researcher, and revised as follows:

Compute the first and second order derivatives of y with respect to x.
1. $y = x^4 + 3x + 1$
2. $y = (3x + 2)^2$
3. $y = (g(x))^2$
4. $y = f(u),\ u = g(x)$

Compute the first and second order derivatives of z with respect to x for the given equations in the following three items.

5. $z = \sin\ (x^3 + 2y^2 + xy)$
6. $z = f(u),\ u = g(x,y)$
7. $z = u^2 + v^2.u,\ u = x + 2y,\ v = \cos\ (x + y)$

Since the interview tasks are related to students' procedural knowledge when applying the chain rule, the following questions were asked as sub-questions, when needed, in order to assess the student's conceptual understanding.

SQ1: What is the chain rule?
SQ2: Could you write down a mathematical general formula for the chain rule?
SQ3: Could you compute the composition of these functions?
SQ4: Could you select the composite functions in the questions?
SQ5: Could you decompose the functions you selected as composite functions in the questions?

SQ6: What is this? (A notation like $\frac{dy}{dx}$ or $\frac{\partial y}{\partial x}$ was given.)

SQ7: What does this notation mean to you? (A notation like $\frac{dy}{dx}$ or $\frac{\partial y}{\partial x}$ was given.)

Clinical interviews (Clement, 2000), which lasted for approximately one to two hours, were recorded using a camera. The interview questions were written on a worksheet one by one by the researcher and the students were required to write the answers on it. Transcriptions of recordings were coded for key events by the researcher and worksheets were also used especially for the codes related to notations.

Data analysis and findings

The selected students were successful taking the first order derivatives of specific single or two-variable composite functions that are given by a single equation, like $y = f(x)$ or $z = f(x,y)$ respectively. All five students, differentiated the specific power functions successfully, considering the chain rule as "take the power down, subtract one, and take the derivative of inside" in a routine way. It appeared no student was aware which rule was used at the beginning. So, this was interpreted as even if the student had no awareness of applying the chain rule, he or she could take the derivative of the specific composite functions, which had become routine by practising such exercises. Therefore, such a student may begin with derivatives of specific composite functions in the embodied world. By the end of the interview, the differences between the subjects' cognitive developments emerged as they encountered abstract composite functions as in the third and fourth items as shown below. P2 and P3 realised the relation between the functions of $y = (g(x))^2$ and $y = (3x + 2)^2$, which is one of the indications that they were able to relate the derivatives of specific and abstract cases.

I: Okay. Now, let $y = (g(x))^2$. Could you compute the derivative of y with respect to x?
P2: Actually we will apply the previous rule. Now, we will write as the derivative of $g(x)$.

P2's comments after the fifth item were interesting:

P2: I take the derivative of inside automatically for such function. Are we using the chain rule, actually?
I: What is your opinion?
P2: For me, yes. Because, I have called by u just now, then $z = \sin u$.
I: A while ago, what did you say about the functions, for which the chain rule is used?
P2: It is used to composite functions.
I: Okay, how is this function?

P2: It is a composite function. I have realised that I have been applying the chain rule when I wrote u. We are computing automatically.

P1, P4 and P5, in applying memorised derivative rules to the specific composite functions in an unrelated manner, called these memorised rules by special names as this example shows.

P1: But, since the function of u depends on x and y, I thought x as a variable and y as a real number, then I am taking the derivative like that. But, I did these calculations with this rule.

I: What is this rule?

P1: Can it be the derivative rule of trigonometric functions?

As can be seen in the above quotation, the student could take the partial derivatives of two-variable specific composite functions easily, since such students knew the concept of partial derivatives.

For abstract composite functions, all students could take the first order derivatives of abstract functions in the third and fourth items using "derivative of function times derivative of inside" as in the derivatives of special composite functions. All tended to use function notation while applying the chain rule to special composite functions and single variable abstract composite functions for the first order derivative. In the interview it was revealed that some of the subjects (P1, P4, P5) were not aware of the relationship between the cases in which he or she used function notation and Leibniz notation. Furthermore, P1, P4 and P5 had symbolic and structural difficulties especially with applying the chain rule in Leibniz notation, since they did not use the Leibniz notation meaningfully. As an example of not perceiving the relation between applying the chain rule with function notation and the Leibniz notation, responses from the interview with P5 are:

I: Okay, what is the name of the rule that you applied by writing $\frac{dz}{dx} = \frac{dz}{du} \times \frac{du}{dx}$?

P5: The chain rule.

I: Did you apply this rule when you wrote $z' = \cos u.u'$?

P5: I did not apply the chain rule

I: Okay, when is the chain rule applied?

P5: The chain rule is applied when the function depends on its independent variable… How can I say?… If we inverted the function to another form, for instance, here, I connected z to u and the derivative with respect to x was required. We must pass from u to reach x. I apply the chain rule in such case.

According to these findings, I thought that a student might begin with the formula of the chain rule in function notation in the symbolic world. It appeared that the student could take the first order derivatives of simple single variable abstract composite functions in the function notation, when he or she related the formula of the chain rule in the function notation and derivatives of specific composite functions by embodying the formula and symbolising the specific derivatives. I inferred that in order to take the first order derivative of any single variable abstract composite function, the student should have the Leibniz notation and the formula of the chain rule in this notation in the symbolic world. If such a student has the partial derivative notion in the embodied world also, he or she can take the first order partial derivative of an abstract two-variable composite function. Moreover, I observed that students could take the first order derivatives of single or two-variable abstract composite functions by using the arrow diagram when needed, even if they did not embody the Leibniz notation or the formula in this notation. For instance, P4 using the Leibniz notation meaninglessly was able to obtain the first order derivatives in all items by routinely using the arrow diagram when needed. To progress in cognitive development of the chain rule, the formulas of the chain rule in the function and the Leibniz notations should be related in the symbolic world and this relation should be embodied. It was noted that the prerequisite knowledge of composite function is another significant notion for applying the chain rule by raising awareness of the relation among various cases. P2 and P3, for instance, had the composition of function notion as prerequisite knowledge and symbolising embodiment of this prerequisite knowledge. Moreover, they indicated that they could relate this symbolising embodiment with embodying the relation between the formulas of the chain rule in function and Leibniz notations. As a result of these skills, they had the general statement of the chain rule in the embodied-symbolic world. This is shown in the following comments of P2:

I: Okay, what was the chain rule?

P2: $\frac{dz}{dx} = \frac{dz}{du} \times \frac{du}{dx}$

I: Which functions did you apply the chain rule to?

P2: … [She was thinking]

I: For instance, when you had seen this case, you had said this was the Chain rule

P2: For the composite functions?

All subjects could obtain the composite function in the interviews. Whereas, two of five subjects, P1 and P4 could not decompose the composite functions given in the items. P4 related the concepts of the chain rule and the composite function only with the arrow diagram. When questioned about her knowledge of the composite function concept, she said:

I: Okay, is the function of $y = \cos\,(3x + 1)$ composite?

P4: In my opinion, it is not.

I: Okay, how did you take the derivative?

P4: [Thinking]… There is already a variable, there is another variable depending on the previous, and there is nothing to make a chain.

I: If I had given the function of $y = \cos u$, $u = 3x + 1$ would it be composite?
P4: In this case, the function would be composite.

P1 was able to perceive when prompted that the functions, for which she applied the chain rule, had common properties of composition. However, it was then realised that she did not have the prerequisite knowledge of composite functions:

I: You said you used the rule that you named "changing variable" in some items.
P1: Yes, I used here, here, I did not use for the first item, I took derivative directly.
I: Is there a common property of the functions in the items which you said you had used the rule?
P1: Because there is power here, and there is power here also.
I: Look, there is power also in the first item.
P1: Yes, but there is the power of whole function in the second and third items, so I used the rule. There is not such a function in the fourth, but I again used the rule because there is a function, which is one within the other.
I: Okay, here you said there was the property of one within the other, here there was power… [P1 interrupted]
P1: When the function is composite, I used the rule.
I: Okay, could you decompose this composite function? (the function in the second item).
P1: f… [thinking]… The inverse function… What does $f(x)$ equal to? … How were we doing?

P5 was not aware of the relationship between her applications in function and Leibniz notations; even if she had the prerequisite knowledge of composite function. Moreover, she was not aware of the variable according to which the derivative was taken. After P5 had the second-order derivative as $y'' = f(g(x))''.g(x)'.g'(x) + g''(x).f(g(x))'$ in the fourth item, the following exchange was revealed:

I: You said this [indicating $f(u)'$] with respect to u, and this [indicating $f(g(x))''$] with respect to x. Let's see again, please. Which variable was the second order derivative of $f(g(x))''$ taken with respect to?
P5: It depends on x, I wrote x instead of u; actually here it does depend on x.

Furthermore, we also noticed that P1 and P4 thought that they had taken the derivative with respect to the variable, according to which they obtained the result of the derivative. When they were reminded that the derivative with respect to another variable was required, they said they would obtain the required derivative when they wrote the required variable in the result. So, we inferred that variable notion is another significant prerequisite knowledge in the embodied world of the cognitive development of the chain rule.

Students' difficulties with the second order derivatives of composite functions were based on symbolic and structural knowledge, as is seen in the next exchange.

P1: $\dfrac{dy}{dx^2}$, can I take the derivative from here?
[She was indicating the first order derivative function of $18x + 12$, which she computed by a memorised rule, while laughing]
But, I think I can find the result in this way. [Indicating the formula in the Leibniz notation]

I: How can you find?

P1: $\dfrac{dy}{du}$, here du over…
I should take the second order derivative in order to obtain this
[she was indicating $\dfrac{dy}{dx^2}$]
after simplified. Derivative of this [indicating u^2] $2u$, the second order derivative of this [indicating $3x$] is 0. Then, the result is 0 in this way.

I: Did you again use a different way?

P1: I'd like to find this [$\dfrac{dy}{dx^2}$].
I wrote $2u$, the second order derivative of $3x$ is 0.

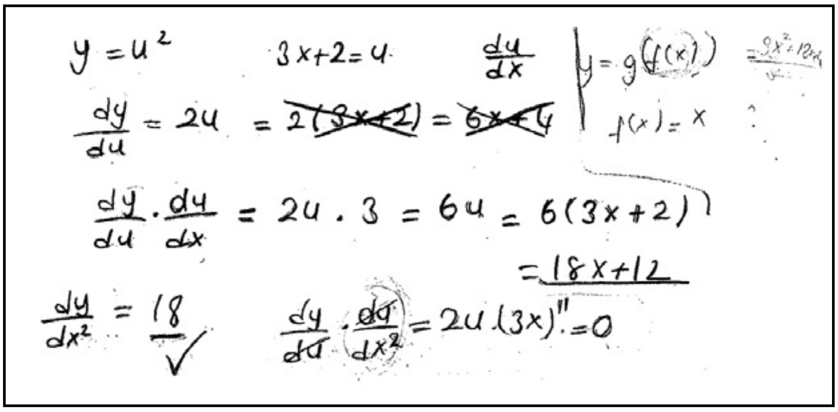

Figure 3. Second order derivative solution in the worksheet of P1.

After we saw the symbolic and structural difficulties of P1 (Figure 3), we realised the importance of structural knowledge in applying the chain rule in the cognitive development of mathematical concepts as Novotna and Hoch (2008) indicated previously. Therefore, we considered that a student might begin with the Leibniz notation of second order derivative in the symbolic world. In the embodied world, we saw having the knowledge that the result of taking the derivative of a function is again a function is prerequisite knowledge in applying the chain rule for the second order derivative. P2 and P3's successful computing of the second-order derivatives even for abstract functions demonstrated that they had this knowledge. Another student, P5, despite having the notion of derivative function, could not take the second-order derivatives of the abstract functions in the sixth and seventh items, due to difficulties with notations and variables. On the other hand, P1 and P4 were not able to obtain the second-order derivatives of the abstract functions despite prompts from the interviewer as they did not have this prerequisite knowledge. It was thought that a student having this prerequisite knowledge in the embodied world and the second order Leibniz notation in the symbolic world must also symbolise the derivative function notion and embody the Leibniz notation in order to identify the derivative function of an abstract function as a product. The findings of the test and the interviews indicated that students had difficulties with this identification. After applying the product rule, students can complete the second order derivative easily; if they do not have difficulty with embodying the second order Leibniz notation and symbolising the derivative function notion.

As a result of analysis of the findings, we obtained a model for the cognitive development of applying the chain rule as presented in Figure 4. The stages of the cognitive development of the chain rule developed by Clark et al. (1997) are indicated in the model also.

Discussion and conclusion

Data analysis has indicated that the students' difficulties with the chain rule increased in applying the chain rule to the abstract composite functions especially while taking the second order derivatives. Some of the students' difficulties could be attributed to their difficulties with prerequisite knowledge of composite function or the derivative function notion; however, it appears to us that most difficulties are related to symbolic or structural difficulties. This finding is consistent with Capistran's (2005) result in which he stated that most students do not like or understand the Leibniz notation. Obviously, the difficulties could not be attributed to only having the structural or conceptual prerequisite knowledge, but also relating them. For instance; a student should perceive the Leibniz notation as a function apart from as an operator. This result already foreshadows the framework of three worlds of mathematics in which

$$\frac{dy}{dx}$$

is regarded as a process and a concept, and therefore is a procept.

Moreover, we can explain a student's cognitive development of the chain rule with possible difficulties based on the three worlds of mathematics. After a student having the derivative rules of specific functions is taught the chain rule, he or she can compute the first or second order derivative of a specific composite function considering "the derivative of outside times derivative of inside". Such a student having the formula of the chain rule can either associate the derivatives of specific composite functions with the formula or not. A student, who cannot make the relation as mentioned above, takes specific composite function derivatives as unrelated by using memorised rules. This student can be considered as at the *Intra* stage, which is the first step of the chain rule schema developed by Clark et al. (1997). If this student has the

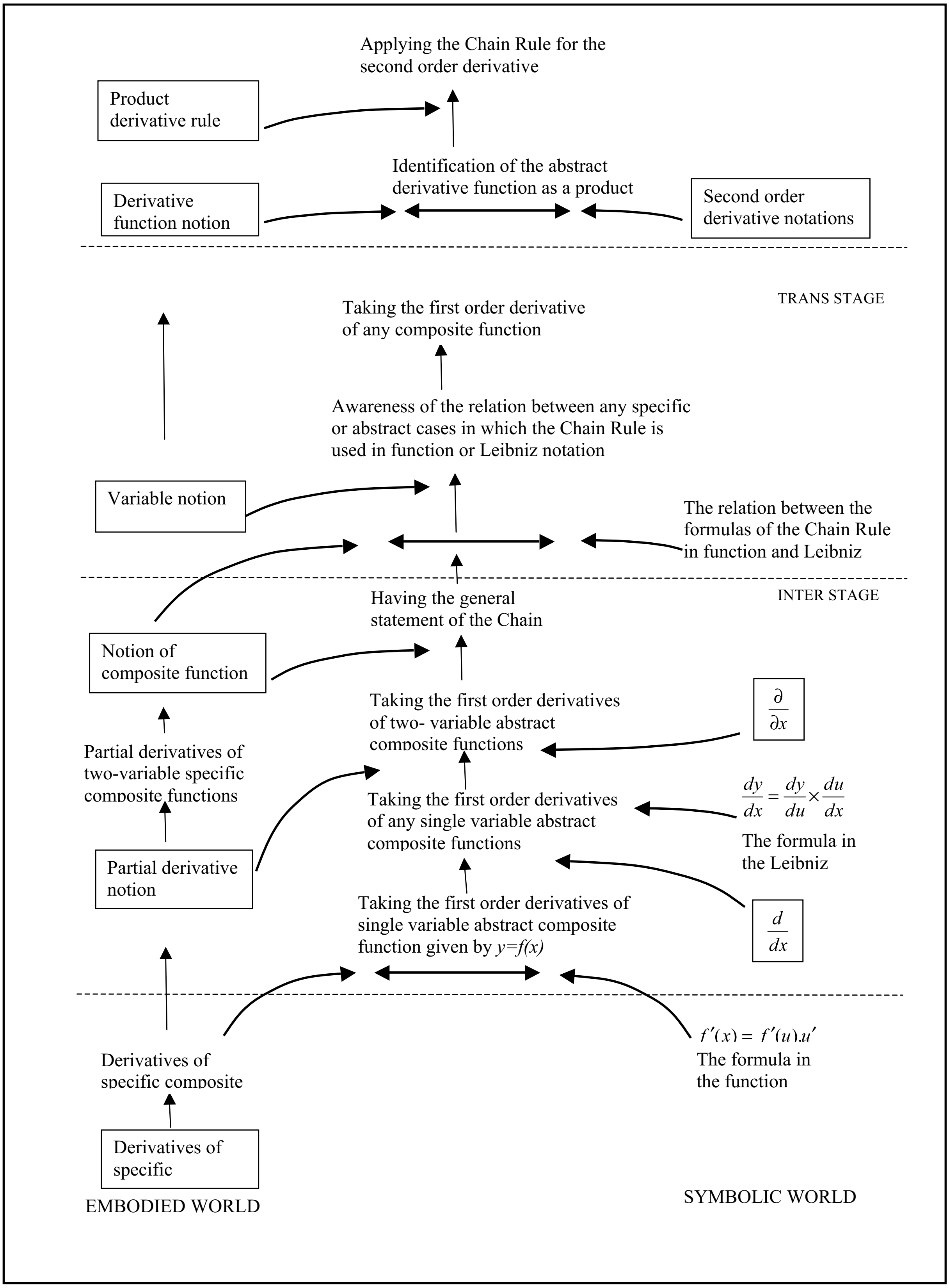

Figure 4. Cognitive delopment of the chain rule.

relation between derivatives of specific composite functions; and the formula of the chain rule in function notation, he or she can take the first order derivatives of single variable abstract composite functions given by a single equation such as $y = f(x)$. When such a student has the Leibniz notation and the formula of the chain rule in this notation in the symbolic world, and embodies them, he or she can take the first order derivative of any single variable abstract composite function. If the student at this level of the cognitive development of the chain rule knows the notion of partial derivative, he or she can take the first order derivative of an abstract two-variable composite function also. If the student does not have the composite function notion exactly, he or she takes the derivatives routinely by a memorised rule. Meanwhile, such a student not having the composite function notion exactly is able to take partial derivatives of an abstract two-variable composite function by using a diagram merely, as a mnemonic device.

When the student at this level of cognitive development of the chain rule has the notion of composite function exactly in the embodied world and symbolises this notion, he or she has the general statement of the chain rule. Such a student can be considered as at the *Inter* level of the chain rule schema developed by Clark et al. (1977). According to these authors, a student at the Inter level can collect the various derivative rules in a group and might provide the general statement of the chain rule, but he or she has not yet constructed the underlying structure of the relationships. According to the development model based on the three worlds of mathematics, when the student having the general statement of the chain rule has the relationship between the formulas of the chain rule in the function and the Leibniz notations in the symbolic world and relates this relationship with the composite function and variable notions, he or she has awareness of the relations among any specific or abstract cases in function or the Leibniz notations. Thus, he or she completes the cognitive development of applying the chain rule for the first order derivative, which is the first step in the development model. Meanwhile, such a student also completes the cognitive development of the chain rule, as developed by Clark et al. According to Clark et al., a student can relate function composition and decomposition with the chain rule, and recognise various instantiations of the chain rule to apply the same rule.

For second order derivatives of composite functions, the student having completed the first step of the development model must compute the second order derivatives of not only single variable but also two-variable specific composite functions. In the cognitive development of applying the chain rule for the second order derivative, such a student might begin with the notion of the derivative function as prerequisite knowledge in the embodied world. Thus, the student must have the notion that the result of any derivative is again a function, whose dependence on variables is the same as the original function' dependence. On the other hand, the student must have the second order Leibniz notations in the symbolic world and relate these notations with the notion of derivative function by embodying the notations and symbolising the notion of derivative function. This means that the student can see the Leibniz notation not only as a procedure, but also as an object (concept). If such a student symbolises the derivative function notion and embodies the formula of the chain rule in the Leibniz notation as much as recognising the derivative function to apply the product derivative rule, he or she could take the second order derivative of any function.

When the development model of the chain rule is considered, it is apparent that some prerequisite concepts such as composite function or derivative function notions in the embodied world and the Leibniz notation in the symbolic world are key aspects. Regarding teaching the chain rule, a revision of these key aspects is recommended at the appropriate time in the teaching process of the chain rule in order to ensure that students comprehend these concepts. Similarly, Hassani (1998) concluded that improving algebraic skills, understanding of functions and the composition of functions while studying the chain rule, had positive effects on understanding and applying the chain rule.

Beyond comprehending the prerequisite concepts, it is also concluded that the relations between them are essential in cognitive development of the chain rule. So, the teacher should design teaching activities that can help students relate prerequisite concepts with various chain rule applications. For instance, a mnemonic device like the arrow diagram can be used in order to help students grasp the relation between the notion of composite function and the chain rule. For this purpose, Uygur and Özdaş (2007) suggested using the arrow diagram should begin with the instruction of composition of functions. Meanwhile, even if students have a tendency to use only function notation in applying the chain rule to single variable composite functions, teachers should prompt them to use the chain rule in both notations in various problem situations. Moreover, the tasks in which the chain rule is used to specify composite functions with the Leibniz notation by using the arrow diagram after decomposing the function should be posed to students also. Otherwise, students take the derivatives of specific composite functions with the function notation separately from the chain rule applications with the Leibniz notation, in which they use the arrow diagram meaninglessly routinely. For the second order derivative of composite functions, the teacher should ensure students have the notion of derivative function firstly. Beyond emphasising that the Leibniz notation indicates both a derivative procedure and a function, the arrow diagram can be used to highlight that dependence of the derivative function on variables is the same as dependence of the original function. Teaching the chain rule in this way can facilitate understanding the extension of this rule to two variable composite functions. Particularly, emphasising the derivative function notion can facilitate students' taking second order partial derivatives of two-variable composite functions by using the arrow diagram. To sum up, it is emphasised again that such instruction of the chain rule should be exemplified with a variety of problem situations.

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