Concretising factorisation of quadratic expressions

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Many secondary mathematics teachers find it a challenge to teach factorisation of quadratic expressions to students who have weak foundations in algebra. This paper is a sharing of a project we embarked on to help students who have been generally unsuccessful in the topic to gain greater proficiency in the concepts and skills involved with quadratic factorisation.

Background of project

The way quadratic factorisation was usually taught to students in Bukit View Secondary (as well as other secondary schools in Singapore) was through the familiar "cross-method". An illustration of this method in the example of $x^2 + 3x + 2$ is shown in Figure 1.

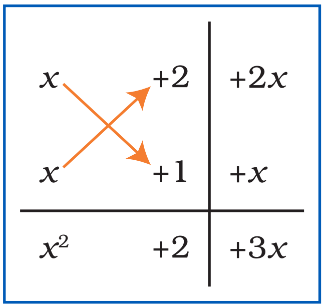

Figure 1. Factorisation of $x^2 + 3x + 2$ using the "cross-method".

However, some teachers felt that a significant number of students could not use the method effectively even after careful demonstration through repeated examples. As such, a *Lesson Study*[1] team was formed comprising the authors in this paper with the aims of identifying students' difficulties and devising ways to help them improve in their proficiency. We targeted the secondary Normal (Academic)[2] students in the school. A pre-test was administered to find out the extent of students' difficulties. The results confirmed the hunch that a majority of students was unable to factorise quadratic expressions correctly. This finding strengthened the resolve to devise another approach to teaching the topic.

After discussion, the team identified possible problems that students might experience regarding quadratic factorisation. These included:

1. Lesson Study originates in Japan and is now widely used as a school-based teacher development model in different parts of the world. For more information about Lesson Study practices, the reader is advised to check the reference list.
2. In Singapore, based on the results of their Primary School Leaving Examination, secondary students in mainstream schools are channeled into three ability streams: Express, Normal (Academic), and Normal (Technical).

- students' common conception of algebra as being "abstract" and thus beyond them;
- students' lack of pre-requisite algebraic skills (such as simplification of like terms);
- students' perception of the "cross method" as arbitrary, and the subsequent failure to make sense; and
- factorisation seen as an isolated skill to be learned without a broader context.

The team agreed that a reworking of another approach must address the concerns listed above. In addition, the team was convinced that for a teaching method to work practically in the classroom, it had to take into consideration realistic constraints of the classroom such as time constraints, large class sizes, and usefulness in actual test situations. In other words, the team was not working towards ideal pedagogies that could theoretically bring about radical changes in students' abilities; rather, we were targeting a realistic pedagogy that was useable in actual day-to-day lessons that could bring about improvements in students' learning of the topic. In response to these concerns and constraints, we decided that a new approach to teaching quadratic factorisation should satisfy the following criteria; that it would:

- appear concrete to the students;
- require minimum prerequisite algebraic skills;
- be perceived as sensible and non-arbitrary to the students;
- connect to the broader context of factorisation as reverse expansion;
- be useable directly by students as a method for test situations.

Exploring possibilities: Algebra discs and algebra tiles

We first explored Algebra Discs developed by the Ministry of Education, followed by Algebra Tiles[3]. For the reader who may not be familiar with these manipulatives, an example of how the Discs and the Tiles present the factorisation of $x^2 + 3x + 2$ is shown in Figure 2 and Figure 3 respectively. For more details on the methods, the reader is advised to check the listed references.

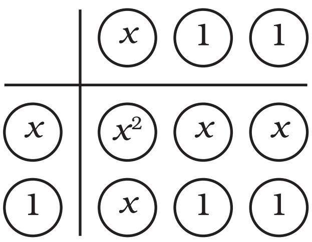

Figure 2. Factorisation of $x^2 + 3x + 2$ using the Algebra Discs.

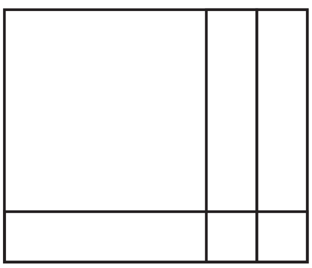
Figure 3. Factorisation of $x^2 + 3x + 2$ using the Algebra Tiles.

Looking at both Figures 2 and 3, the most immediate differences are their surface appearance: Discs are round and have "x^2", "x", and "1" labels while the Tiles are squares/rectangles and they do not have labels on them. Beyond these superficial differences, we analysed these presentations of quadratic factorisations in the light of the criteria listed above. We agreed that both these methods clearly satisfy Criteria 1.

As for Criteria 2, the methods require pre-requisite algebraic skills in subtly different ways: for Discs, the requirement is somewhat greater. For example, when one places the "1" disc on the top row, one needs to multiply against the left column for checking purposes; in the case of the tiles, visual checks of alignment of sides of the squares and rectangles suffice but there is still a need consciously to read off $(x + 1)$ and $(x + 2)$ as breadth and length of the rectangle at the end of the process.

3. Algebra Tiles is well-known and literature about its use and potential abounds (e.g., Howden, 1985; Norton 2007). As such, a detailed review about Algebra Tiles is not provided in this article; rather, we focus on how we modified the Tiles to suit the goals and constraints of our classroom innovation project.

With regards to Criteria 3, the Tiles appear to bring out the concept of "factorisation as forming rectangle and finding length/breadth given area" using the underlying idea of area conservation more conspicuously, thus strengthening its sense-making potential. This in turn links to Criteria 4. In the case of the Tiles, factorisation reverses the premise/conclusion of expansion, which is "given length and breadth of a rectangle, find the area". Similarly, the Discs can easily present the reversal by first placing the top row and left column followed by obtaining the discs in the internal array as a way of demonstrating expansion.

Looking at Criteria 5, the Discs have greater potential as the method extends to cases of negative coefficients quite naturally (see Figure 4). This cannot be said of the Tiles as "negative area" is not immediately intuitive to students[4]. Nevertheless, we deemed that both of these methods by themselves are unable to satisfy Criteria 5: these manipulatives are not directly transferable to conventional paper-and-pencil test situations; and, they are not suitable in dealing with larger coefficients.

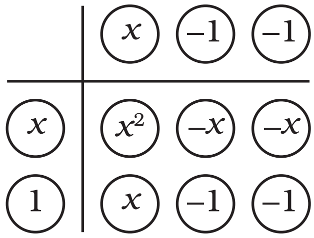

Figure 4. Factorisation of $x^2 - x - 2$ using Algebra Discs.

4. This does not mean that proponents of Tiles have not made attempts to represent negative areas. However, the problem we struggled with was that the concept of 'negative area' may not be easily acceptable to students as it does not correspond to their experience in the real word.

Making modifications: Fusing Algebra Discs and Algebra Tiles

We preferred the Tiles' presentation of factorisation as forming rectangles and finding length/breadth given area instead of the multiplicative approach of the Discs. We think that this would help students who require a more concrete and visual representation of factorisation. Moreover, this allows students to make geometric sense alongside the algebraic manipulation of factorisation. On the other hand, we appreciated the explicit labeling of the Discs with "x^2", "x", and "1" as a way to heighten students' awareness of the link to symbolic algebra. In other words, we wanted to fuse the geometric qualities of the Tiles (the squares and rectangles) and the algebraic features of the Discs (the "x^2", "x", and "1" labels).

We attempted the fusion by using laminated cardboard squares and rectangles with "x^2", "x", and "1" labels. Enough sets were made for all students. The teachers' sets were bigger and with magnetic backing so that they could be demonstrated on the whiteboard. Figure 5 shows an illustration of these manipulatives in the factorisation of $x^2 + 3x + 2$. The school has since given a name to this adaptation: AlgeCards.

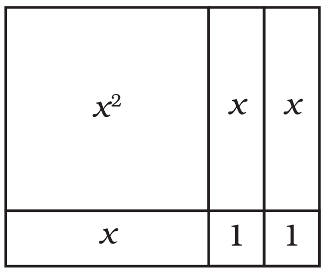

Figure 5. Factorisation of $x^2 + 3x + 2$ using AlgeCards.

With this modification, we were sufficiently satisfied that we have fulfilled Criteria 1–4. However, we still needed to contend with Criteria 5. How can we develop from the AlgeCards to a pictorial form that students can use easily in a paper-and-pencil test situation? Moreover, how do we manage the transition smoothly from AlgeCards to the final form?

From AlgeCards to Rectangle Diagram

The AlgeCards in its original form, even when drawn, is too cumbersome to deal with quadratic expressions with larger coefficients. As such, it has to

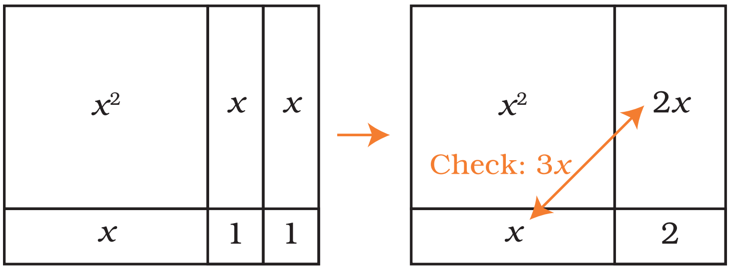

Figure 6. From AlgeCards to Rectangle Diagram.

be simplified into a form that is easily drawn and yet retain some links to the AlgeCards, especially the concept of finding length/breadth given area. We discussed the possibility of transiting from AlgeCards to the 'cross-method', since the latter is commonly known. But, even cursory visual comparison of the two (see Figure 5 and Figure 1) would reveal that the links are not easily established. We finally decided on the Rectangle Diagram, which initially is a kind of visual simplification of the AlgeCards. Figure 6 shows the link between the two representations in the case of factorising $x^2 + 3x + 2$. In using the Rectangle Diagram, we emphasised the need to check that the sum of the two x-terms in the diagram tallied with the x-term in the expression (a step similar to the checking step in the "cross-method").

Carrying out the lessons

First lesson

The Secondary 2 Normal (Academic) students who did not do well in the pre-test were selected to attend the lessons we prepared. Two lessons of 1 hour each were devoted to quadratic factorisation. In the first lesson, we focused on introducing factorisation as finding length/breadth given area. We introduced AlgeCards as a means to help students reinforce this concept for themselves through numerous examples. At the same time, we displayed the Rectangle Diagram right from the start alongside the AlgeCards. The message intended was that AlgeCards help to concretise the algebra; Rectangle Diagram is a simplification of the AlgeCards representation. In the worksheet that we give to the students, the two modes of representation were placed side-by-side to highlight the links between them. Figure 7 is an extract of the first example in the worksheet. All the examples in this worksheet contain quadratic expressions with positive coefficients.

Students proceeded with subsequent examples in the worksheet. The examples were crafted such that they were in increasing order of

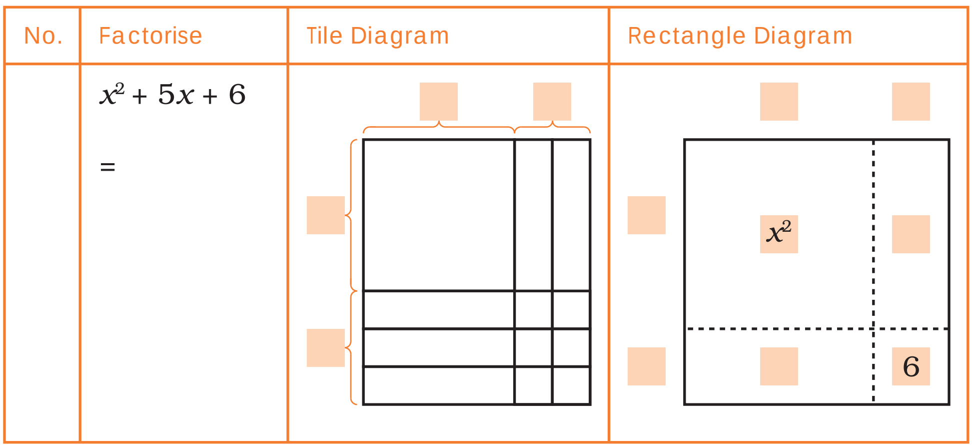

Figure 7. An extract of the worksheet used in the first lesson on factorisation.

complexity: for example, the constant term in the quadratic expression possesses more factors in subsequent examples. Initially, the students were encouraged to form the AlgeCard rectangles through experimentation. Most of the students at that stage tried to fit the pieces in random fashion without a clear strategy but were eventually successful. This initial period of letting students "play around" with the AlgeCards without a fixed strategy is intended to ease them into familiarity with the new manipulatives. Subsequently, they were taught a more efficient and methodical approach of observing the factors of the constant term as a way to form the rectangle. The steps advocated were:

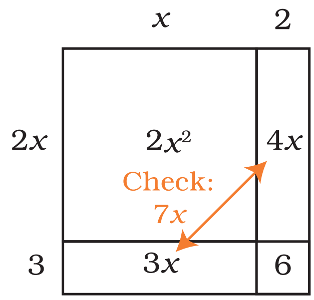

Figure 8. Rectangle Diagram method used to factorise $2x^2 + 7x + 6$.

- place the x^2-term;
- place the "ones" in a rectangle array at the bottom right corner;
- fill in the "x" cards and check if they add up to the x-term in the quadratic expression.

Students transferred these steps onto the Rectangle Diagram as well. The intention was that with increasing familiarity, students would on their own accord operate with the Rectangle Diagram with these steps in mind after receiving sufficient confidence through the scaffold of the AlgeCards.

Interestingly, some students were observed to move to the Rectangle Diagram very quickly without prompting (i.e., they stopped using the AlgeCards after the first few examples); some others relied on the AlgeCards throughout the entire first lesson. We were glad that the worksheet allowed the pace of transition from AlgeCards to Rectangle Diagram to be determined by the student. At the end of the first lesson, some students were able to attempt more complex factorisations like the one shown in Figure 8.

Second lesson

In the second lesson, we wanted to shift the emphasis away from the AlgeCards to the Rectangle Diagram (in line with Criteria 5). This shift is also important for another reason: to handle quadratic expressions with negative coefficients. AlgeCards, as discussed earlier, is based on the area concept and as such, it does not lend itself intuitively to "negative areas". The transition involves a gradual downplaying of the geometric significance (the idea of area) and the increasing emphasis on algebraic manipulation (checking for products and simplification of like x-terms).

We were concerned that some students became too "comfortable" with the AlgeCards that they would resist the transition to the Rectangle Diagram. For this reason, in the worksheet for the second lesson, we removed the "Tile Diagram" column (see Figure 7) as a way to urge students to operate in the visual territory of the Rectangle Diagram. Nevertheless, we continued to make available the AlgeCards as an option should they need to still make reference to these tools as an intermediate measure. As a further push for them to move on to Rectangle Diagram, the subsequent examples in the worksheet included expressions with larger coefficients (such as $3x^2 + 14x + 8$) as a way to help students realise the inadequacies of the AlgeCards in these situations.

The second part of the worksheet was on quadratic expressions with negative coefficients. We hoped that at that stage, students would be able to use the Rectangle Diagram as a sort of "template" to perform the computations and checks. We were initially apprehensive that that part of the lesson would be too challenging (and hence discouraging) for the students.

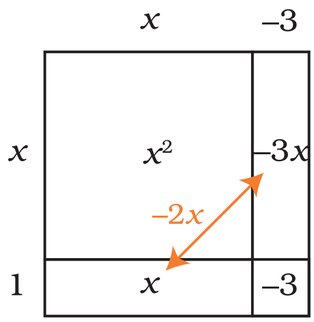

Figure 9. Factorisation of quadratic expression with negative coefficients using Rectangle Diagram.

Surprisingly, a significant number of students proceeded without teacher supervision steadily throughout the worksheet and even attempted successfully those examples involving negative coefficients. Figure 9 shows one such successful attempt at factorising $x^2 - 2x - 3$.

Reflections

With regards to the five criteria that were stated earlier, we were reasonably satisfied that the approach chosen of starting with AlgeCards and then moving on gradually to the Rectangle diagram fulfilled the intended parameters of the teaching innovation. In fact, on hindsight, we found that it also possibly addressed other criteria that we did not make so explicit from the beginning. One of these was the avoidance of introducing "negative areas" in the case of quadratic expressions with negative coefficients. While we focussed on area in the use of AlgeCards in the first lesson, there was a deliberate effort to present the Rectangle Diagram more as a "template" to fill in and check the expressions in the second lesson, thereby side-stepping the issue of dealing with negative areas. Students were quite happy with the transition, with none being particularly disturbed about filling in terms with negative coefficients in the rectangular spaces of the Rectangle Diagram. The other criterion would be that of coping with larger coefficients. With the Rectangle Diagram, students can quite easily handle larger numbers that would be a significant challenge if we stayed merely with the use of AlgeCards.

After the two lessons, we conducted a post-test that was very similar to the pretest. We also conducted interviews with selected students and held Lesson Study meetings within the team to discuss our observations of students' in-class work. We were encouraged by what we saw and read. The posttest showed significant improvements and evidences of the use of Rectangle Diagram successfully in quadratic factorisation. We noticed, in particular, that a few students, who used to perform poorly and were uninterested in algebra, became more engaged and successful through the project lessons. We were glad that even though the Lesson Study process was indeed time-consuming, there was observable positive effect on some students, which made the effort worthwhile.

On a casual note, the mathematics department of Bukit View Secondary is now so fired up with the Lesson Study experience that they intend to implement this method in all the other relevant classes in the school. They have since engaged an outside vendor to produce in larger quantities the AlgeCards. If you are keen to find out more about the AlgeCards, please feel free to contact the school via www.bukitviewsec.moe.edu.sg.

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