Sequences of rational numbers converging to surds

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Consider the sequence $\frac{1}{1}, \frac{7}{5}, \frac{41}{29}, \frac{239}{169}, \ldots$

Thomas notes that the sequence converges to $\sqrt{2}$ (1965, pp. 840–841). By observation, the sequence of numbers in the numerator of the above sequence, have a pattern of generation which is the same as that in the denominator. That is, the next term is found by multiplying the previous term by six and then subtracting the term before the previous one. This sequence can be expressed as the second order difference equation $t_{n+2} = 6t_{n+1} - t_n$.

In order to find the value, that the above sequence of rational numbers converges to, both the difference equation generating the numerators and that generating the denominators has to be solved. This is necessary in order to obtain an explicit expression for the elements of the sequence in terms of the term number n. Then the effect of allowing $n \to \infty$ can be determined.

The solution of a second order difference-equation with constant coefficients is carried out as indicated in the web document (Polyanin, 2004).

The general case is usually expressed as $t_{n+2} = at_{n+1} + bt_n = 0$ and this leads to what is called the characteristic equation $\lambda^2 + a\lambda + b = 0$. In this case, $a = -6$ and $b = 1$, giving $\lambda^2 - 6\lambda + 1 = 0$, the solutions of which are $\lambda = 3 \pm 2\sqrt{2}$.

If the roots of the characteristic equation are real and distinct, as they are here, namely $\lambda_1 = 3 + 2\sqrt{2}$ and $\lambda_2 = 3 - 2\sqrt{2}$, this leads to a particular solution of the difference equation, which is $t_n = c_1(\lambda_1)^n + c_2(\lambda_2)^n$.

Applying the particular solution to the sequence of numbers in the numerator gives the following pair of simultaneous equations:

$$t_1 = c_1\left(3 + 2\sqrt{2}\right) + c_2\left(3 - 2\sqrt{2}\right) = 41 \qquad (1)$$

$$t_2 = c_1\left(3 + 2\sqrt{2}\right)^2 + c_2\left(3 - 2\sqrt{2}\right)^2 = 239 \qquad (2)$$

Solving these manually or with a CAS enabled calculator gives:

$$c_1 = \frac{7 + 5\sqrt{2}}{2} \text{ and } c_2 = \frac{7 - 5\sqrt{2}}{2}$$

Applying the particular solution to the sequence of numbers in the denominator gives the following pair of simultaneous equations.

$$t_1 = k_1\left(3+2\sqrt{2}\right)+k_2\left(3-2\sqrt{2}\right)=29 \qquad (3)$$

$$t_2 = k_1\left(3+2\sqrt{2}\right)^2+k_2\left(3-2\sqrt{2}\right)^2=169 \qquad (4)$$

Solving these gives $k_1 = \dfrac{10+7\sqrt{2}}{4}$ and $k_2 = \dfrac{10-7\sqrt{2}}{4}$.

The initial sequence of fractions can now be expressed explicitly as a function of n, the term number as follows

$$t_n = \frac{c_1\left(3+2\sqrt{2}\right)^n + c_2\left(3-2\sqrt{2}\right)^n}{k_1\left(3+2\sqrt{2}\right)^n + k_2\left(3-2\sqrt{2}\right)^n}$$

Now because $|\,3-2\sqrt{2}\,| < 1$, as $n \to \infty$ the value of $\left(3-2\sqrt{2}\right)^n \to 0$.

Hence $t_\infty = \dfrac{c_1}{k_1} = \dfrac{\frac{7+5\sqrt{2}}{2}}{\frac{10+7\sqrt{2}}{4}}$ which simplifies to $\sqrt{2}$.

I now wondered if other rational sequences exist which converge to a surd value. What about $\sqrt{3}$?

The number to which the rational sequence converges, as shown above, is determined by the ratio

$$\frac{c_1}{k_1}$$

and both c_1 and k_1 are solutions emanating from two simultaneous equations with coefficients derived from solutions to the characteristic equation. Hence, if the rational sequence is to converge to $\sqrt{3}$ then the discriminant of the characteristic equation must be an element of

$$X = \{3, 12, 27, 48, \ldots\} \text{ that is, } X = \{x\colon x = 3n^2,\ n \in z\}.$$

The characteristic equation $\lambda^2 + \lambda a + b = 0$ has solutions

$$\lambda_1 = \frac{-a+\sqrt{a^2-4b}}{2} \text{ and } \lambda_2 = \frac{-a-\sqrt{a^2-4b}}{2}$$

Consider $b = 1$ as previously: the discriminant is $a^2 - 4$; if the roots are to be real and distinct then $a^2 - 4 > 0 \Rightarrow |\,a\,| > 2$ and because a is negative, $a < -2$.

Try: $a = -3$: $a^2 - 4 = 5$ which is not an element of X;
$a = -4$: $a^2 - 4$ which is an element of X.

So a difference equation needed to generate the numerators and denominators of a rational sequence which converges to $\sqrt{3}$ is $t_{n+2} = 4t_{n+1} - t_n$.

The next step is to find the element in the sequence after $\frac{1}{1}$.

This is necessary in order to fully define the terms occurring in both the numerator and denominator using a second order difference equation. To assist in this trial and error search, a spreadsheet was used, as shown in Figure 1.

a	b	numerator	denominator	fraction	sqrt(3) =	1.7320508
-4	1	1	1	1		
		5	3	1.66666667		
		19	11	1.72727273		
		71	41	1.73170732		
		265	153	1.73202614		
		989	571	1.73204904		
		3691	2131	1.73205068		
		13775	7953	1.7320508		
		51409	29681	1.73205081		

Figure 1. Searching for term two of the sequence.
Note: This spreadsheet can be downloaded from www.aamt.edu.au/Professional-learning/Journals/files

Once the spreadsheet is established with the appropriate cell formula to generate the numerators, denominators and is filled down, the following is carried out. First the values of $a = -4$ and $b = 1$ are entered. Then the numerator = 1 and denominator = 1 of the first fraction are entered. Finally, trial values for the numerator and denominator of the second fraction are entered and systematically altered until the fractions converge to the sought after surd as they have in Figure 1.

This method was repeated in order to find rational sequences which converge to each of the surds involving prime numbers less than twenty. In each case, calculations as shown above were carried out in order to prove the result indicated by the spreadsheet. A summary of the results is shown below.

***Summary of converging sequences found with b* = 1**

$\frac{1}{1}, \frac{7}{5}, \frac{41}{29}, \frac{239}{169}, \ldots \qquad \sqrt{2} \qquad t_{n+2} = 6t_{n+1} - t_n$

$\frac{1}{1}, \frac{5}{3}, \frac{19}{11}, \frac{71}{41}, \ldots \qquad \sqrt{3} \qquad t_{n+2} = 4t_{n+1} - t_n$

$\frac{1}{1}, \frac{4}{2}, \frac{11}{5}, \frac{29}{13}, \ldots \qquad \sqrt{5} \qquad t_{n+2} = 3t_{n+1} - t_n$

$\frac{1}{1}, \frac{11}{5}, \frac{76}{34}, \frac{521}{233}, \ldots \qquad \sqrt{5} \qquad t_{n+2} = 7t_{n+1} - t_n$

$\frac{1}{1}, \frac{29}{11}, \frac{463}{175}, \frac{7379}{2789}, \ldots \qquad \sqrt{7} \qquad t_{n+2} = 16t_{n+1} - t_n$

$\frac{1}{1}, \frac{43}{13}, \frac{859}{259}, \frac{17137}{5167}, \ldots \qquad \sqrt{11} \qquad t_{n+2} = 20t_{n+1} - t_n$

$\frac{1}{1}, \frac{25}{7}, \frac{274}{76}, \frac{2989}{829}, \ldots \qquad \sqrt{13} \qquad t_{n+2} = 11t_{n+1} - t_n$

$\frac{1}{1}, \frac{169}{41}, \frac{11153}{2705}, \frac{735929}{178489}, \ldots \qquad \sqrt{17} \qquad t_{n+2} = 66t_{n+1} - t_n$

$\frac{1}{1}, \frac{911}{209}, \frac{309739}{71059}, \frac{105310349}{24159851}, \ldots \qquad \sqrt{19} \qquad t_{n+2} = 340t_{n+1} - t_n$

The effect of changing the value of "b" in the characteristic equation is now examined. In order to find the restrictions placed on the value of "b" it must be remembered that one of the solutions of the characteristic equation of the difference equation must have a magnitude less than 1 to get convergence as $n \to \infty$ (as discussed earlier). If a is always negative, the root which can have a magnitude less than 1 will be

$$\lambda_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}$$

and hence
$$-1 < \frac{-a - \sqrt{a^2 - 4b}}{2} < 1$$
$$-2 < \left(-a - \sqrt{a^2 - 4b}\right) < 2$$
$$(a-2) < -\sqrt{a^2 - 4b} < (a+2)$$
$$\left(a^2 - 4a + 4\right) > a^2 - 4b > \left(a^2 + 4a + 4\right)$$
$$(-4a + 4) > -4b > (4a + 4)$$
$$(a-1) < b < -(a+1)$$

It must also be remembered that b cannot be zero because the characteristic equation would not have two distinct roots. The restrictions on "a" and "b" can be summarised as follows: $a \in Z$ and $a < 0$, $b \in Z$ and $(a-1) < b < -(a+1)$ and $b \neq 0$ and if $b > 0$ then $b < \frac{a^2}{4}$.

For $a = -4$ the following values of b satisfy the above criteria: –4, –3, –2, –1, 1, 2.

$a = -4$ b	Surd part of the characteristic equation	Sequence and convergence value	Second order difference equation
–4	$\sqrt{32} = 4\sqrt{2}$	$\frac{1}{1}, \frac{6}{4}, \frac{28}{20}, \ldots, \sqrt{2}$	$t_{n+2} = 4t_{n+1} + 4t_n$
–3	$\sqrt{28} = 2\sqrt{7}$	$\frac{1}{1}, \frac{9}{3}, \frac{39}{15}, \ldots, \sqrt{7}$	$t_{n+2} = 4t_{n+1} + 3t_n$
–2	$\sqrt{24} = 2\sqrt{6}$	$\frac{1}{1}, \frac{8}{3}, \frac{34}{14}, \ldots, \sqrt{6}$	$t_{n+2} = 4t_{n+1} + 2t_n$
–1	$\sqrt{20} = 2\sqrt{5}$	$\frac{1}{1}, \frac{7}{3}, \frac{29}{13}, \ldots, \sqrt{5}$	$t_{n+2} = 4t_{n+1} + t_n$
1	$\sqrt{12} = 2\sqrt{3}$	$\frac{1}{1}, \frac{5}{3}, \frac{19}{11}, \ldots, \sqrt{3}$	$t_{n+2} = 4t_{n+1} - t_n$
2	$\sqrt{8} = 2\sqrt{2}$	$\frac{1}{1}, \frac{4}{3}, \frac{14}{10}, \ldots, \sqrt{2}$	$t_{n+2} = 4t_{n+1} - 2t_n$

Obviously converging sequences can be found when b values other than 1 are used. Points of note were:

1. The sequence terms had common factors in the numerator and denominator.
2. When b is negative the sequence was oscillatory convergent.

Sequences which converge to surdic expressions

The second term in the sequence, the one after $\frac{1}{1}$, determines the value to which the sequence converges. We know that for $a = 6$ and $b = 1$ and a second term of $\frac{7}{5}$, the sequence converges to $\sqrt{2}$. If the second term is altered to, say, $\frac{8}{5}$, to what will the sequence now converge? The new sequence is:

$$\frac{1}{1}, \frac{8}{5}, \frac{47}{29}, \frac{274}{169}, \ldots$$

with extra terms being generated by $t_{n+2} = 6t_{n+1} - t_n$.

Using the method shown previously it can be shown that this sequence converges to

$$\frac{3\sqrt{2}-1}{2}$$

Hence, sequences of rational numbers generated by the same second order difference equation in the numerator and denominator, converge to surds and surdic expressions. This leads to finding a rational sequence which converges to

$$\phi = \frac{\sqrt{5}+1}{2}$$

The surdic expression contains $\sqrt{5}$ so the values of $a = -3$ and $b = 1$ were entered into the spreadsheet and, perhaps not surprisingly, when the second rational number of $\frac{3}{2}$ was entered, the sequence converged to ϕ.

$$\frac{1}{1}, \frac{3}{2}, \frac{8}{5}, \frac{21}{13}, \ldots, \phi = \frac{\sqrt{5}+1}{2} \qquad t_{n+2} = 3t_{n+1} - t_n$$

It is interesting to see the Fibonacci numbers generated in this way.

Because $\sqrt{5}$ is also inherent in a characteristic equation with $a = -7$ and $b = 1$, another sequence was sought and found which converges to ϕ.

$$\frac{1}{1}, \frac{8}{5}, \frac{55}{34}, \frac{377}{233}, \ldots, \phi \qquad t_{n+2} = 7t_{n+1} - t_n$$

This sequence has the even numbered terms of the previous sequence removed and hence converges towards ϕ more quickly.

There are other ways to get $\sqrt{5}$ in the surdic expression for the characteristic equation; another is to use $a = -1$ and $b = -1$ this leads to

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \ldots, \phi \qquad t_{n+2} = t_{n+1} + t_n$$

This of course is the well known sequence of the ratio of successive values of the Fibonacci sequence which is oscillatory convergent to ϕ.

Two other sequences which converge to ϕ are shown below.

$$\frac{1}{1}, \frac{144}{89}, \frac{17711}{10946}, \ldots, \phi \qquad t_{n+2} = 123t_{n+1} - t_n$$

$$\frac{1}{1}, \frac{377}{233}, \frac{121393}{75025}, \ldots, \phi \qquad t_{n+2} = 322t_{n+1} - t_n$$

Note that in the last sequence the third term has converged to ϕ correct to ten significant figures.

With regard to student coursework, this investigation seems to be ideally suited to student project work. The only mathematical content that is not included in the curriculum of upper secondary school mathematics, is the solution of second-order constant-coefficient linear difference equations. However, students will be aware of first order difference equations and their recursive nature so the step up will be minimal.

As a forerunner or introduction to a project, students could be given the first page and a bit of this article as an example of a problem solving report which proves that the given sequence converges to $\sqrt{2}$. Each student could then be given the task of determining a sequence which converges to their own given surd. Those students who cannot make progress could then be given page two of this article to assist them in their investigation. A CAS-enabled calculator would be of assistance to students for solving the simultaneous linear-quadratic equations and for the spreadsheet work if computers were not available.

References

Thomas, G. B. Jr (1965). *Calculus and analytic geometry* (3rd ed.). Reading, MA: Addison Wesley.

Polyanin, A. D. (2004). *EqWorld: Second-order constant-coefficient linear homogeneous difference equation.* Retrieved 17 March 2009 from http://eqworld.ipmnet.ru/en/solutions/fe/fe1203.pdf