DISCOVERY

with Neville de Mestre

Patterns from division

All common fractions can be written in decimal form. In this Discovery article I suggest that you ask your students to calculate the decimals by actually doing the divisions themselves, and later on they can use a calculator to check their answers. The article is based on the research of Bolt (1982).

Start with the first five digits as denominators respectively (excluding unity) and only consider proper fractions (p/q with $p < q$), not those improper or vulgar ones where the numerator is greater than the denominator. Thus your students should produce:

$$\frac{1}{2} = 0.5$$

$$\frac{1}{3} = 0.\overline{3}, \quad \frac{2}{3} = 0.\overline{6}$$

$$\frac{1}{4} = 0.25, \quad \frac{2}{4} = 0.5, \quad \frac{3}{4} = 0.75$$

$$\frac{1}{5} = 0.2, \quad \frac{2}{5} = 0.4, \quad \frac{3}{5} = 0.6, \quad \frac{4}{5} = 0.8$$

$$\frac{1}{6} = 0.1\overline{6}, \quad \frac{2}{6} = 0.\overline{3}, \quad \frac{3}{6} = 0.5, \quad \frac{4}{6} = 0.\overline{6}, \quad \frac{5}{6} = 0.8\overline{3}$$

The bar above a single digit indicates that it is to be repeated forever. Note that some of these decimals terminate but others go on forever and are called recurring decimals.

Next ask your students to write $\frac{1}{7}$ to $\frac{6}{7}$ in decimal form and comment on the results. They should obtain

$$\frac{1}{7} = 0.\overline{142857}$$

$$\frac{2}{7} = 0.\overline{285714}$$

$$\frac{3}{7} = 0.\overline{428571}$$

$$\frac{4}{7} = 0.\overline{571428}$$

$$\frac{5}{7} = 0.\overline{714285}$$

$$\frac{6}{7} = 0.\overline{857142}$$

Here the whole sequence covered by the bar is to be repeated forever. Note the permutation of the six digits in each repeated decimal.

Your students can continue to investigate $\frac{1}{8}$ to $\frac{7}{8}$ (terminating), $\frac{1}{9}$ to $\frac{8}{9}$ (recurring), and $\frac{1}{10}$ to $\frac{9}{10}$ (terminating). So far they should have discovered that the decimal expression of the fraction in fundamental form will terminate when the denominator is 2, 4, 5, 8 or 10. This could lead them to conjecture that decimal fractions will terminate if their denominator is any combination of powers of 2 and powers of 5.

To draw meaningful conclusions about recurring decimals, further investigation is needed. Now

$$\frac{1}{11} = 0.\overline{09}, \quad \frac{2}{11} = 0.\overline{18}, \quad \frac{3}{11} = 0.\overline{27},$$

$$\frac{4}{11} = 0.\overline{36}, \quad \frac{5}{11} = 0.\overline{45}, \quad \frac{6}{11} = 0.\overline{54},$$

$$\frac{7}{11} = 0.\overline{63}, \quad \frac{8}{11} = 0.\overline{72}, \quad \frac{9}{11} = 0.\overline{81}, \quad \frac{10}{11} = 0.\overline{90}$$

Students should next consider $\frac{1}{12}$ to $\frac{11}{12}$.

For larger prime number denominators your students could now use their calculators. With 13 as the denominator some fascinating results emerge.

$$\frac{1}{13} = 0.\overline{076923}, \quad \frac{3}{13} = 0.\overline{230769}, \quad \frac{4}{13} = 0.\overline{307692}$$

$$\frac{9}{13} = 0.\overline{692307}, \quad \frac{10}{13} = 0.\overline{769230}, \quad \frac{12}{13} = 0.\overline{923076}$$

$$\frac{2}{13} = 0.\overline{153846}, \quad \frac{5}{13} = 0.\overline{384615}, \quad \frac{6}{13} = 0.\overline{461538}$$

$$\frac{7}{13} = 0.\overline{538461}, \quad \frac{8}{13} = 0.\overline{615384}, \quad \frac{11}{13} = 0.\overline{846153}$$

Here there are permutations of two separate recurring sequences. One would certainly not be able to predict this strange property.

To investigate the fractions with 14 as the denominator there is no need to use the calculator except as a check on students' division ability. Some of your students should observe that seven of the required results have already been obtained (which ones?) and a further three of the required results can be obtained by dividing some earlier answers by 2. This just leaves three to be calculated by long division or by dividing $\frac{9}{7}$, $\frac{11}{7}$ and $\frac{13}{7}$ respectively by 2. The results are somewhat surprising. Similar short cuts can be used for fractions with 15 and 16, with 15 adding further insight into the overall results.

Now fractions with the prime number 17 pose a problem. The average calculator cannot accurately produce the required number of decimal places in the recurring decimal sequence, and it appears that the sequences seem to terminate. This is because there are actually sixteen digits in the recurring sequence and basic calculators usually only give the first eleven significant figures and round off the twelfth. See if your students can now discover for themselves a way of finding the sixteen digits using their calculators.

This is one method illustrated by tackling $\frac{6}{17}$. The calculator gives

$$\frac{6}{17} = 0.352941176471$$

and the decimal appears to terminate. Next divide 6 000 000 by 17 and the calculator shows 352941.176471 as expected. Now multiply the number in front of the decimal point by 17 yielding

$$352\,941 \times 17 = 5\,999\,997$$

and subtracting this from 6 000 000 leaves a remainder of 3. Now

$$\frac{3}{17} = 0.176470588235\ldots$$

Thus the extension of

$$\frac{6}{17} = 0.352941176470588235\ldots$$

and the sixteen digit sequence has been obtained, ending with ...882. Some of this can be checked using Excel, but students will have to widen the cell width to reveal the first 14 digits of the sequence obtained by increasing the number of decimal digits in the Excel menu bar. Note that every digit appears at least once in this sixteen-digit sequence.

Composite denominators 18, 20, 21 and 22 add further information to the data collected, while the primes 19 and 23 produce only one sequence each respectively. The next most interesting denominator will be 2 × 13 = 26, because of the strange property shown by 13. What about 13 × 13 = 169? That would require a computer with double digit precision or a special mathematical program such as Maple or Mathematica.

So what should the investigations suggest so far? Simply that proper fractions with N as denominator, where N has factors other than 2 or 5, will produce repeating sequences of length $(N - 1)$ or less. Those with repeating sequences of length $(N - 1)$ will have N prime, but the converse is not true (see 2, 3, 5, 11 and 13).

The reverse operation of going from a decimal fraction to a proper fraction is well known for terminating decimals but not so for recurring decimals. For terminating decimals one just needs to write the terminating sequence as the numerator with one followed by the same number of zeroes as the length of the sequence in the denominator. Thus

$$0.125 = \frac{125}{1000} = \frac{5\times5\times5}{2\times2\times2\times5\times5\times5} = \frac{1}{2\times2\times2} = \frac{1}{8}$$

using prime factors to simplify to an equivalent fraction.

For recurring decimals, the conversion is a bit more complicated. Four examples will illustrate the method.

$$0.\bar{6} = \frac{(6-0)}{9} = \frac{6}{9} = \frac{2}{3}$$

$$0.\overline{36} = \frac{(36-0)}{99} = \frac{36}{99} = \frac{4}{11}$$

$$0.1\bar{6} = \frac{(16-1)}{90} = \frac{15}{90} = \frac{1}{6}$$

$$0.91\bar{6} = \frac{(916-91)}{900} = \frac{825}{900} = \frac{11}{12}$$

The rule for the numerator is to write down the sequence of numbers after the decimal point including the recurring sequence only once, and then subtract the part of the sequence which does not recur. The denominator consists of nines for the number of recurring decimal digits followed by zeroes for the number of non-recurring decimal digits. Thus

$$0.5\overline{63} = \frac{(563-5)}{990} = \frac{558}{990} = \frac{31}{55}$$

$$0.10\overline{51} = \frac{(1051-10)}{9900} = \frac{1041}{9900} = \frac{347}{3300}$$

The rule is based on the sum to infinity of a geometric progression.

Happy discoveries!

Reference

Bolt, B. (1982). *Mathematical activities* (pp. 104–106). Cambridge University Press.