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Reasoning with Paper and Pencil: The Role of Inscriptions in Student Learning of Geometric Series

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The purpose of this article is to analyse how students use inscriptions as tools for thinking and learning in mathematical problem-solving activities. The empirical context is that of learning about geometric series in a small group setting. What has been analysed is how students made use of inscriptions, self-made as well as those provided by text books and teachers, and the role these inscriptions played in the coordination of students' learning/communication. Through the use of inscriptions (made on the chalkboard and with paper and pencil), the students externalised their thinking while engaging in mathematical reasoning on the topic of geometric series. The inscriptions were significant as anchor points for arguments in the ongoing discussions. Three main issues are highlighted: (a) how the inscriptions used contribute to the process of appropriation, (b) how the students use inscriptions to externalise and clarify their ideas and attempts at meaning-making, and (c) how the inscriptions are conducive to closing the gap between the original problem as given in the text book and the mathematisation necessary. It is argued that inscriptions, through their material nature, play a decisive role in learning mathematical reasoning.

This study is an attempt to document and analyse how students are using inscriptions in reasoning and learning about geometric series. Following a sociocultural perspective on learning and development (Rogoff, 1990; Säljö, 2005; Wertsch, 1998), it is important to study students' actions and use of artifacts in problem-solving settings to gain insight into how they appropriate mathematical concepts and modes of reasoning. The focus of the present study is on how students rely on and use the specific type of cultural tools that we refer to as *inscriptions* when engaging in mathematical problem solving. Inscriptions are understood as artifacts such as graphs, drawings, and mathematical symbols used for cognitive, communicative, mathematical, and problem-solving purposes in interactional settings[1]. The inscriptions may appear in text books, they may be made by the teacher for instructional purposes, or they may be self-made by learners while engaging in problem solving.

The terms *tool* and *artifact* are in this study understood as 'two sides of the same coin', following the position of Trouche (2004): "When speaking of a tool before considering its users and its uses, I will speak of an *artifact*" (p. 282, emphasis in original). Our use of the term tool here is thus different from Rabardel's (2002) use; Rabardel defines a tool as an artifact enriched with a cognitive scheme. Our view of tool is also different from the term instrument as coined by Vérillon and Rabardel (1995). They use the term instrument as a psychological construct that is intermediary between the subject and the object.

In the literature, different terms have been used about tools people use to communicate and externalise thinking, for instance *visualisation* and *representation*. The notion of visualisation, as described and defined by Arcavi (2003), is used to denote both the process and product of creation, and includes both internal and external pictures, images, and diagrams. The term visualisation is thus used differently from our term inscription. Goldin and Kaput (1996), in a study that can be viewed as a joint summary of their previous individual research about representations, distinguish between internal mental representations and external physically embodied representations. The term representation is also used generally in everyday language, and the term implies that people must extract the relevant information that the representation seeks to communicate. According to Yerushalmy (2005), this is exactly the case when students interact with what she calls *interactive visual representations*. The term representation is also ambiguous since it does not explicitly pinpoint *what* it is a representation of. Thus, we agree with Roth and McGinn (1998) that the notion of representation is too general for the purpose of this study. To avoid the term representation and its connotations, we prefer to use the term inscription; a term that has some similarities with the term external representation as identified by Goldin and Kaput (1996). Inscription is in this study the notion used to label drawings, signs, diagrams, and graphs that are made explicit and inscribed on paper or at the chalkboard. Such inscriptions are used by the students *in situ* to solve mathematical problems and to appropriate mathematical concepts. Thus, inscriptions are accessible signs whose meaning are contextual and situated and may be shared by several participants. This definition relates to the semiotic classification of a sign as icon, index, or symbol. In accordance with Cobb (2002), we see inscriptions as intimately related to the notion of symbolising in mathematics. Meaning-making and ways of symbolising are co-evolving and mutually constitutive processes. This study will show that the inscriptions used serve as indexes and symbols more than icons.

In educational situations, particularly in problem-solving contexts in small groups, self-made inscriptions (which will be in focus here) function as mediating devices. These inscriptions make public and visible steps in students' individual and collective thinking, and they provide foci for the successive coordination of reasoning during a problem-solving process (Säljö, 2005). It is important to emphasise that students' use of inscriptions, such as graphs, tables, and drawings in reasoning, is a components of the appropriation process. Inscriptions are not mere additions or illustrations. Rather, they are materially externalised instances of student collective and individual thinking, and in this sense they do real work in the students' process of appropriating the concept of geometric series.

Theoretical Framework: Inscriptions as Tools for Reasoning

According to a sociocultural perspective, a central feature of learning among humans is that we can take advantage of and use cultural tools in social interaction to communicate and solve problems (Rogoff, 1990; Säljö, 2005; Vygotsky, 1979). Human thinking and learning are intimately intertwined with the use of external and symbolic tools that are simultaneously intellectual and material (Sfard & McClain, 2002). These tools have emerged in the culture in which we have been socialised, and as individuals we appropriate how to use them in informal and formal learning situations. We learn to read, write, count, and use many symbolic systems and material artifacts for various purposes. Leont'ev (1981) emphasises the use of artifacts as mediational means. In this view, artifacts and tools connect "humans not only with the world of objects but also with other people" (p. 56). Inscriptions are mediating tools since they are used purposively to support verbal arguments (Roth & McGinn, 1998). Thus, artifacts are co-constitutive of human interaction in social practices. The overview presented here concentrates on studies and theoretical perspectives that scrutinise the relationships between appropriation and use of inscriptions in mathematics learning (Cobb, 2002; Forman & Ansell, 2002; Roth & McGinn, 1998; Säljö, 2005).

Inscriptions in scientific reasoning. According to Latour (1987), inscriptions are *intermediaries* in human exchanges; this definition indicates the mediating role of inscriptions. Latour studied how scientists make use of inscriptions, such as diagrams, maps, and diaries, to transform a diffuse and ambiguous reality into something distinct and concrete. Through these processes of objectification it becomes possible for scientists to name and describe abstract processes and substances. For instance, in a laboratory study of the effects of endorphin and naloxone on the guinea pig gut, inscriptions such as graphs served as tools and visual displays to communicate results and convince readers of articles in which the research findings ultimately were communicated. In this account of scientific reasoning and communication of findings, the central role of inscriptions as resources for meaning-making is very clear. What is seen in the laboratory ultimately ends up as an inscription in a scientific paper. However, the direction is also the reverse: What can be seen in the laboratory relies on interpretive frameworks provided by inscriptions in books and articles (cf. Latour, 1987).

Inscriptions as tools for communicating in contexts of learning. When students make use of inscriptions in a problem-solving process, the tool mediates meaning and contributes as an intermediary 'actant', as Latour (1987) puts it, between the original problem and the mathematisation. For inscriptions to fulfil the function of mediating and communicating ideas and thoughts, an active, cognising person, who has the ability to retrieve the intended meaning of the inscription, is needed. When encountering inscriptions, people thus have to transform information to locally relevant knowing (Säljö, 2005).

An additional point that is important in this context is that when students produce inscriptions as parts of their activities, these resources can be seen as externalisations of students' thinking. Inscriptions are hence tools in which intellectual and material aspects merge and coexist. As will be seen in the empirical study, students make drawings in order to clarify to themselves and their fellow students various features of the nature of geometric series. This practice implies that students think by means of their drawings. Lehrer, Schauble, and Petrosino (2001), when studying elementary school students' development of scientific reasoning, showed how students engaged in the enterprise of modelling by using inscriptions such as maps and graphs to support arguments and interpretations. Moreover, Lehrer et al. (2001) argue that student production of inscriptions of this kind during experiments and observations is of vital importance for the development of skills in scientific reasoning. In addition, they show how the inscriptions themselves sometimes were made into objects of talk and reflection, and thus how they triggered new ideas and questions in relation to the phenomena studied. This demonstration illustrates that inscriptions are not dead objects; they are actants (cf. Latour, 1987).

The production of inscriptions, and their capacity to contribute to the transformation of human thinking and reasoning, has been studied by Roth and McGinn (1998). For instance in studying biotic and abiotic aspects of soil compositions, students were producing and using inscriptions, including drawn models and graphical representations such as tables and graphs, to be able to compare different soil samples, to support verbal arguments, and as illustrations in their written reports. Again, research showed how inscriptions were made into topics of discussion, and how they were used to support communication (Roth & McGinn, 1998; Sandoval & Millwood, 2005). Roth and McGinn (1998) also emphasise the explanatory functions of inscriptions. These functions depend on whether the inscriptions are recognised by the collaborators as legitimate representations of the particular phenomenon under investigation. The inscriptions rely on communities and collective communicative practices in which their meaning potentials are recovered by active subjects with relevant cultural experience (Säljö, 1998, 2005).

Inscriptions as ways of thinking in processes of appropriation. The use of inscriptions in problem-solving settings is a goal-directed activity and may also be seen as instrumental in the promotion of cognitive activities from participants. Inscriptions, when integrated into the flow of activities, thus contribute to structuring the ways users will reason about a problem (Sandoval & Millwood, 2005).

Inscriptions, Säljö (2005) argues, are tools for thinking, communicating, and acting. They also serve as important means for reifying experiences, as Wenger (1998) discusses in detail. To reify human experiences means committing them to some kind of permanent or semipermanent form such as text. In mathematical small-group problem solving, the use of inscriptions is a substantial part of the problem-solving process. However, inscriptions do not contain any inherent meaning *per se*, rather the meaning "arises in the context of other inscriptions and sign forms" (Roth & McGinn, 1998, p. 38). In our study this feature is exemplified by the contextual dependency of the inscriptions used in problem-solving sessions. Use of inscriptions facilitates joint activity and the achievement of shared foci of attention. Thus, it can be assumed that reasoning with inscriptions is central in processes of appropriation when people "get to know, attempt to and successively increase their familiarity with the use of knowledge and tools in social actions" (Säljö, 2005, p. 51, my translation; cf. Forman & Ansell, 2002; Moschkovich, 2004; Rogoff, 1990).

Inscriptions as mediating devices in the learning of mathematics. To learn mathematics is to learn to produce and handle inscriptions for multiple purposes. Doing mathematics simply would be impossible without mastering intellectual technologies for making inscriptions; they are so much part of the mathematical practice (Duval, 2006). In learning addition, for instance, a child has to learn the number symbols as well as to use the symbols while learning how to add numbers. Thus, inscriptions appear at different levels of activity.

The central role that inscriptions play in student interaction when mathematising is discussed by Lehrer, Schauble, Carpenter, and Penner (2000). They studied science learning and issues in elementary mechanics (such as factors that affect the speed of objects that are dropped from fixed heights). The researchers claim that inscriptions such as tables, drawings, diagrams, and graphs allow for the visualisation and externalisation of thinking, the communication of plans and of ideas, and the tracking of arguments and conjectures. Lehrer et al. (2000) claim that the students' use of inscriptions can be seen as governing the evolution of their problem-solving process and thinking, through the progressive mathematising of ideas and arguments made by means of inscriptions. Mathematical concepts, attempted representations of those concepts through inscriptions, and individual and collective thinking are often intimately connected. Moreover, Lehrer et al. (2000) argue about the necessity of students producing inscriptions themselves to meaningfully solve problems. Hence, the use of inscriptions may be important in bridging 'the gap' between a problem situation and the mathematisation needed. Analogically, Wyndhamn and Säljö (1997) show how students use inscriptions to close 'the gap' between two textual realities when solving mathematical word problems. One of these textual realities is the everyday semantics of language in the original problem situation, and the other one is the formal mathematical reasoning used to solve such problems.

Cobb (2002) studied students' reasoning with inscriptions (written notations and graphical displays of the lifetime of batteries and tables of T-cell counts). These inscriptions were used by the students at various steps of their reasoning in relation to data, warrants, and backings for argumentation. Cobb's study exemplifies the mediating function of inscriptions in problem-solving processes. Reasoning with inscriptions and symbols are integral parts of the process of mathematising. (For other examples and discussion of the role of inscriptions, see Roth & McGinn, 1998, and Sandoval & Millwood, 2005).

The Problem

An important question in this context then is of course how students fill 'the gap' between their own thinking and the inscription, or, rather, how they integrate inscriptions into their reasoning. The topic of study here is student learning of geometric series, and it is interesting to study how students use inscriptions to deal with this concept, what the artifacts present in the activities do, and where the artifacts lead. The following question will be addressed: How do students engaged in small-group work use inscriptions in their reasoning and learning of geometric series?

Methods and Participants

Video data have been analysed to scrutinise how students at upper secondary school make use of inscriptions when reasoning during mathematical problem solving. Following a qualitative and naturalistic case study design (Bassey, 1999; Bryman, 2001; Lincoln & Guba, 1985; Stake, 1995), a small group of six students has been followed for 12 weeks. The students participated in a theoretically demanding mathematics course preparing, among others, for university studies in mathematics. The whole class consisted of 15 students, each of whom was assigned by the mathematics teacher to one of three small groups, located in separate rooms with chalkboards. The mathematics teaching, including whole-class expositions and small-group sessions, was organised by the teacher. The class met for 5 hours of mathematics teaching each week. Usually 3 out of these 5 hours were devoted to allowing students to meet for collaborative problem-solving sessions in the small groups. These sessions were video recorded and transcribed[2]. Typically, the teacher made a whole-class exposition of a mathematical theme, and then the students met in the small groups to solve problems and to practise what had been discussed in class. The three groups were composed according to the students' previous marks in mathematics. The students in each group were thus fairly equal with respect to previous achievements in mathematics. The six students selected for this study were chosen due to their background as relatively high-achievers in mathematics. Three excerpts have been selected to illustrate the role that inscriptions play in student interaction and meaning-making when learning about geometric series. These excerpts were also selected because it was on these occasions the students made use of inscriptions in the appropriation process to such an explicit degree. Consequently, an evolution of these inscriptions across the twelve week period was not noticeable. Nevertheless, the excerpts are linked by the mathematical tool, geometric series, that was to be appropriated; and how the students made use of different inscriptions in this process to support communication, externalise thinking, and to close the gap between the problem situation and the mathematisation necessary.

The same six students, 17 and 18 years of age, participated in all three dialogues. Participants were three females named Aud, Eli, and Pia, and three males named Are, Jan, and Pål; names are pseudonyms. In the analyses, quotes from the dialogues are indicated by smaller fonts.

Additionally, certain informative aspects of the multimodal elements of communication will be incorporated into the transcripts (under the heading *accompanying activities*). When interpreting inscriptions and arguing about them, students often use strategies such as pointing, nodding, and looking at inscriptions in their books but also at the chalkboard.

In analysing the students' interaction, a dialogical approach has been used. This approach is an analytical tool derived from dialogism (Bakhtin, 1981, 1986; Linell, 2006; Markovà, 1990a, 1990b). The epistemological framework of dialogism is particularly interested in how people discuss, communicate, and negotiate in collective settings. Dialogism is hence, according to Wertsch (1990), compatible with a sociocultural perspective on learning and development. The analyses made also draw on such dialogical principles as *sequentiality*, *joint construction*, and *act-activity interdependence* (Linell, 1998). Following this analytical framework, one has to use the analytical principle of double dialogicality when analysing students' dialogues (Linell, 2006). Both spoken and written utterances have to be analysed with regard to their interrelationships both with prior and subsequent utterances. With respect to double dialogicality, the relationships between the dialogues' embeddedness in a global institutionalised setting and their embeddedness in a local small-group context have to be considered.

Results

Excerpt 1 Use of an Inscription as an Algorithm

In the following dialogue from the ninth group session, the students' work with the task 1.37a from their text book (Erstad, Heir, Bjørnsgård, Borgan, Pålsgård, & Skrede, 2002a). From a mathematical point of view, the students are working with a task that relates to the use of the concept of geometric series within economics.

> **1.37**
> A sum is put on an account 10 years in a row. Estimate the amount of money one year after the last insertion when
> **a** the annual amount is 7000 NOK and the interest is 5.5 %
> **b** the annual amount is 12 000 NOK and the interest is 6 %
> **c** the annual amount is 12 000 NOK and the interest is 7 %

Aud is absent on this occasion, and Pål does not contribute to the discussion. When solving this task, the students made an inscription (Figure 1). The inscriptions used had been presented by the mathematics teacher and to some extent the inscriptions were also present in the students' text book (Erstad et al., 2002a).

		Verbal activity	Accompanying activities
44	Eli	Can't you just write AHA? You do know what it will be. Why do we have to draw it? We do know	Eli looks down in her books while reasoning. She addresses her question to Jan by looking at him.
45	Jan	It becomes a geometric series, but e::	Jan directs his response to Eli by looking at her.
46	Eli	AHA, a geometric series	Eli looks down and points in her note book to reinforce her verbal claim.
47	Jan	Then we have to draw. It's important to draw	Jan looks down in his book while reasoning.
48	Eli	( )	Eli points in Pål's text book to focus their attention.
49	Are	A sum is put on an account::	Are reads aloud in Pål's text book.
50	Jan	Ten years	Everybody is making a drawing in their own note book.
51	Pia	Is it [ten xs- no nine]?	Pia is questioning through her books.
52	Eli	[One two three] four five six seven eight nine ten	Eli counts with her pencil in her note book to make her inscription represent the

			information given in the task.
53	Jan	That's why you have to draw, 'cause then you can just count, right?	Jan is reasoning through his books.
54	Pia	Four five six	Pia is simultaneously counting and drawing.
55	Eli	Why is it *x*'es now? It is seven thousands now	Eli addresses her question to Jan by looking at him.
56	Pia	(Have we done that)? Seven thousand (with) five point five percent intere::st. Then a_1 is, it is seven thousand. *k*, it is one point fifty-five	Pia looks down in her note book, and she is simultaneously reasoning and writing.

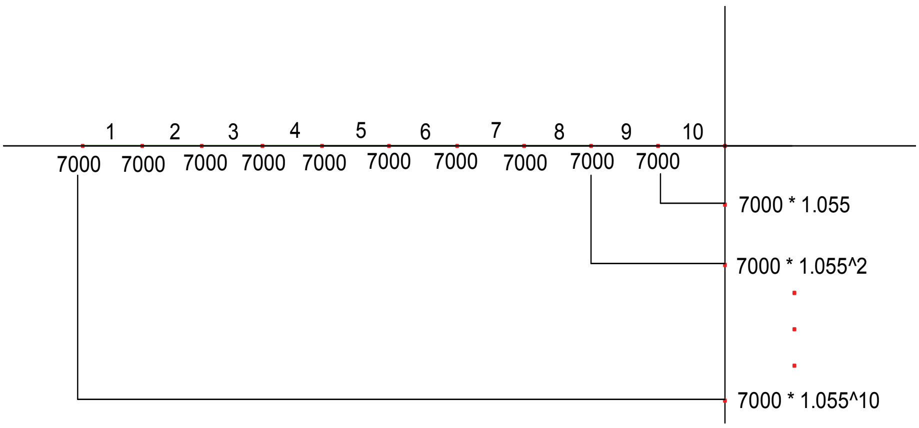


Figure 1. A copy of the inscription made by Eli, Pia, and Jan.

Eli (44) initiates this dialogue by asking a series of questions. Her use of the word AHA deserves an explanation. The mathematics teacher frequently used this expression to signal recognition of a mathematical situation and understanding of how to solve a problem. The expression in this sense is thus closely related to sense-making. Eli continues by accounting for her use of the word AHA: She claims to know what the task is all about, and she does not realise why they have to draw it. Jan (45) follows her reasoning and determines that this is a case where the concept of geometric series is applicable. The word It in (45) is interpreted as referring to the amount of money asked for in the task text, which can be estimated as the sum of a geometric series. However, Jan is still hesitant in spite of having mathematised the situation. Eli (46) responds to Jan's argument by repeating AHA. She thus underscores her agreement with Jan's mathematisation of the problem.

Jan (47) continues the dialogue following the path on which he and Eli have agreed, and he claims that they have to make a drawing. This claim is based on his argument that it is important to make drawings. Hence he argues against the initial claim by Eli (44) that it is not necessary to make a drawing. Further arguments are not given at this moment. The mathematics teacher has frequently expressed the view that it is important to make drawings, so this might be seen as a case in which the voice of the teacher intervenes into the reasoning of the students. Are (49) starts to read the text of the task aloud, and Jan's (50) short contribution ten years is a continuation of Are's comment. In reading the text aloud a shared focus of attention is established.

Pia (51) then initiates a small shift in the dialogue, starting to concretely address issues related to the resolving of the problem. In mathematical terms, her question concerns the estimation of the value of *n* in the formula for the sum of a geometric series, $S_n = \dfrac{a_1(k^n - 1)}{k-1}$. She is in doubt whether she has to write nine or ten *x*s, but she seems to believe more in nine than in ten. At the same time Eli (52) starts counting how many times she has written 7000 in her book. She ends up with ten and thus implicitly argues against the position of Pia (the correct value is ten). Jan's (53) argument supports his previous utterance (47) about the importance and advantages of making inscriptions. His argument now is that they can just count the number of *x*s or seven thousands if they have a drawing. Jan thus argues that the inscription can serve as an algorithmic device in their problem-solving process: That's why you have to draw, 'cause then you can just count. His point seems to be that the inscription illustrates and mathematises all the significant elements and steps needed to resolve the problem. The drawing keeps score of what to sum (a_1), the ratio (*k*), and how many terms have to be summed (*n*). Hence, the inscription bypasses the formula and is a tool for producing an answer.

Pia (54) then counts her number of *x*s, and she is thus responding to the argument by Jan. Eli (55) reacts to Pia's use of *x*s in (51), and she claims that they now should use seven thousands. This utterance signals a pragmatic strategy. If the amount of money is not known, the students previously used *x* to symbolise that amount. In this case, however, they do know the amount, and hence Eli wants them to use it[3]. Eli's initiative is responded to by Pia (56) in a very interesting utterance. Here, Pia reformulates the question asking Have we done that? Without getting an answer, Pia continues her reasoning and uses the information given within the frame of a geometric series. She concludes that a_1 is 7000 and *k* is 1.55. From a mathematical point of view, her estimations of a_1 and *k* are mistaken, since the first term should be $7000 \cdot 1.055$ since *k* equals 1.055. This is also made explicit in the inscription (cf. Figure 1).

This dialogue illustrates how the students attend to and discuss the advantages of using inscriptions in this context for solving the problem. They have experiences of such problems, and thus they realise that the tool of geometric series is applicable here. It only remains to establish the parameter values included in the sum $S_n = \frac{a_1(k^n - 1)}{k - 1}$ ($S_{10} = \frac{7000 \cdot 1.055 \cdot (1.055^{10} - 1)}{1.055 - 1}$, $n = 10$, $k = 1.055$, and $a_1 = 7000 \cdot 1.055$). In spite of this Jan argues that it is advantageous to use an inscription as a tool.

In this sequence, the intricate relationship between reasoning, learning, and use of inscriptions is illustrated. The students move between the problem and the mathematising by means of the drawing they have made to fit the problem. Jan (53) is also aware of the fact that this specific tool can serve as an algorithm for solving this kind of problem. In this sense the inscription may be looked upon as having semiotic iconic aspects. All of the activities undertaken can be understood as elements of a process of appropriating the concept of geometric series. This process includes features such as (a) familiarising oneself with how such a series can be used to mathematise a particular kind of problem, (b) analysing how a drawing can be made to represent such a series, and (c) establishing how one can perform mathematical operations by means of this drawing.

Also evident here is that the students literally are reasoning through the inscriptions present in the different books they use. The students are mostly looking down into their books while talking to each other. The gestures by Eli and Pia when counting with their pencils support their reasoning. The inscriptions thus elicit physical actions that fulfil communicative functions. Additionally, what is evident from this dialogue is that solving this problem without taking advantage of the inscription had been impossible for the students; a fact of which the students seem to be aware, since they are constantly consulting the inscription. The inscription is used both as a tool for communicating with the others and as a tool for individual thinking. However, also important to observe is that in spite of having an explicit inscription available, the students have to engage in cognitive work and make use of earlier experiences to know what to do, where to start, and how to coordinate the information given with the tool. Inscriptions never contain all information about how they are to be used, nor do they mean anything outside the purposive use by people (Goodwin, 1997). The students have to add relevant knowing *in situ* and engage in 'gap-closing' (Lave, 1988).

Excerpt 2 Reasoning with Inscriptions

In the 14th group session, the students are faced with a problem in which the heights of a bouncing ball are supposed to constitute a geometric sequence. In Excerpt 2 and Excerpt 3, the students work with the tasks 164a[4] and 164b respectively, from their task book (Erstad, Heir, Bjørnsgård, Borgan, Pålsgård, & Skrede, 2002b).

164 ΔΔ

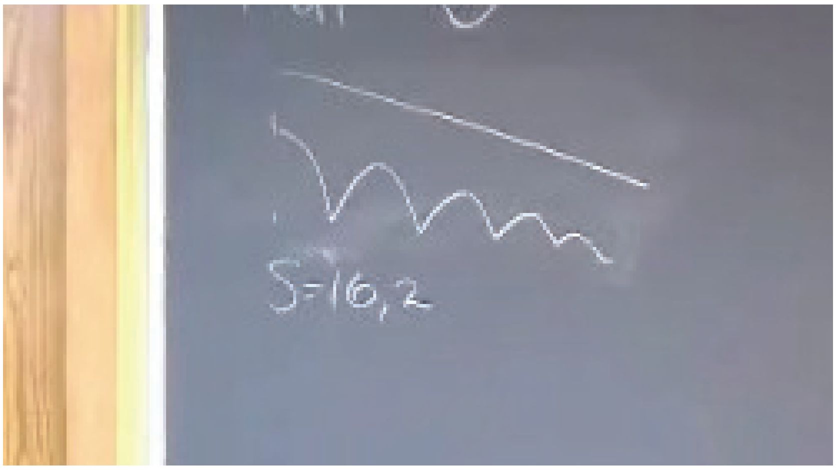
a We again deal with the bouncing ball from task 136. What distance will the ball cover all together from the first time it bounces till it comes to rest?

b A ball is dropped from a height of 1.6 m down to a floor and bounces up and down. Each time the height becomes 40 % of the previous height. What distance does the ball cover all together before it comes to rest?

136 ΔΔ (First part)

We drop a bouncing ball to a floor. Firstly, it bounces 1.80 m over the floor. Secondly it bounces up 1.60 m. We assume that the heights of the ball make a geometric sequence.

		Verbal activity	Accompanying activities
77	Eli	What? Can you think so that all are able to hear?	Eli looks down, but addresses her question to the other group members.
78	Pia	It travels do::wn	Pia looks down in her book while reasoning.
79	Jan	Yes I'll show you. First you just calculate the sum of that convergent series, right?	Jan responds to Eli's question and addresses his explanation to Eli by looking at her.
80	Aud	Actually we have a series that just, or that series But that ball, we have to figure out the distance it travels, it travels like, or does it become up? Then it goes up down up down up down up, right? So it travels much longer than just like that. It isn't thrown. So the answer we get when we calculate the sum, it wa::s 16.2, and so it actually is the answer we are going to=	Aud stands up, goes to the chalkboard, makes a drawing and writes S = 16.2. Everybody looks at Aud and the board (See below).



81	Eli	Yes we just get that series	Eli confirms Aud's reasoning.

82	Aud	= find in the end. In the Answer section it said 34 point- e:: two point, yes 32 point four, twice as much	
83	Pål	Ye:s	
84	Jan	32.4 But that is because it travels up and down	Jan discusses with Aud about the sum of the heights of the bouncing ball by looking at her.
85	Aud	Mm. I have just taken, what you said that it starts from above, and in the end it doesn't travel upwards again. There it just lands.	Aud points at the start and at the end of her graph respectively, to focus the group's attention and to reinforce her argument.
86	Eli	Yes is it right to multiply it by two then?	Eli addresses her question to Aud since Aud has taken a teacher's role.
87	Jan	That is actually what	Jan initiates an explanation meant for Eli and looks at her.
88	Aud	At least that's what we discussed. I was just going to get you into the way of thinking	Aud directs her explanation to Eli by looking at her, responding to Eli's question in (86).
89	Eli	Ok	

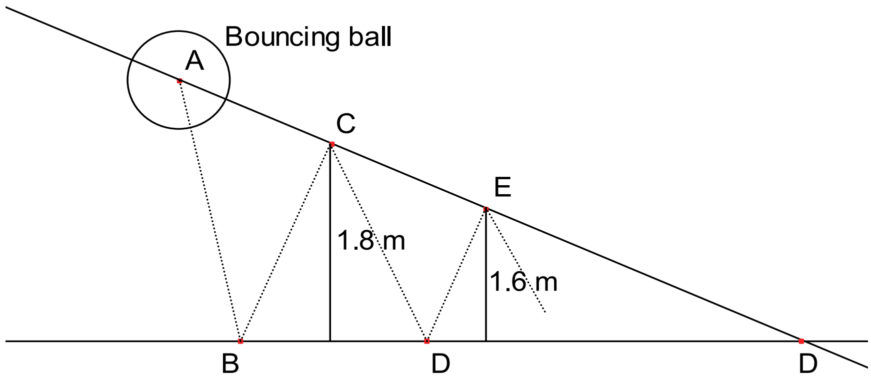


Figure 2. A stylised version of Aud's inscription (cf. Aud (82)).

Eli (77) initiates this dialogue by asking the other students to think aloud and share their ideas with the rest of the group. Pia (78) starts reasoning about the path of the bouncing ball. The prolonged sound of the word do::wn is an indication of her reasoning about the particulars of the movements of the ball. Jan (79) responds and continues the dialogue by telling Eli what one should do in this situation. When saying you just calculate the sum of that convergent series, Jan most likely has the following geometric series in mind: $1.8 + 1.8 \cdot \frac{8}{9} + 1.8 \cdot \left(\frac{8}{9}\right)^2 + \ldots$ This geometric series is the sum of the subsequent heights of the bouncing ball (cf. the task text of 136). Jan uses the word convergent, which is an additional emphasis of the relevant mathematical principle in this context. The word just indicates that Jan makes a claim of knowing how to solve the problem. Aud (80) continues the discussion and begins to make drawings of the path of the bouncing ball at the chalkboard (cf. Figure 2). By drawing the path of the ball in this manner, Aud explicitly uses the inscription as a resource to make clear to the others the nature of the problem. The distance they are asked to calculate is the length of the subsequent heights and not just the length of the line segment AD. Evidence for this interpretation is found in Aud's claims (80) that the ball is not thrown and thus does not follow the path of a straight line. Additionally, a point of interest is that Aud's inscriptions on the chalkboard do not describe the path of a bouncing ball that is dropped. However, to see what actually happens, Aud has to describe the ball's bouncing by adding a dimension. Actually, the inscription is rather misleading since it mathematises the problem incorrectly. Aud's inscription leads the students to calculate the lengths of the parabolas representing each bounce. However, none of the students react to this misleading nature of the inscription. Rather, Aud's attempt to communicate her interpretation of the nature of the problem is ratified by Eli (81) who says that they get that series Jan was talking about before.

Aud (80) mentions the sum of the series that Jan was talking about, 16.2, which is the sum of the convergent geometric series with $a_1 = 1.8$ and $k = \frac{8}{9}$. This reference seems to indicate that Aud has appropriated important features of the concept of a convergent geometric series. Aud (82) continues the discussion by referring to the answer they got, 16.2, and compares it with the answer presented in the Answer section of the text book [5]. After some negotiation and repetition of the movements of the ball (82-84), the students agree that the answer section gives the result 32.4. Aud (82) notices that this is twice the amount they got. Jan (84) explains this result. The total distance is twice the sum they got, since the ball is going up and down.

This exchange (82-84) exemplifies the students' highly coordinated co-thinking at this stage of the problem solving. However, they seem to have some difficulty understanding the result. Evidence of this is found in Aud's comments (85) regarding the issue of the first and the last movements of the ball. The students cannot multiply the series by two since the path does not follow the expected pattern either at the beginning or at the end. The students' difficulties are obvious in bridging the gap between the empirical illustration of a bouncing ball and the mathematisation necessary. They have contextualised the bouncing ball's path and the distance it should travel within the mathematical frame of a geometric series. Still the problem is how to coordinate these two realities; the physical and the mathematical. Eli (86) explicitly addresses this problem when she asks the other students if multiplying the sum of the series by two is correct.

Jan (87) seems to confirm Eli's suggestion that they should multiply the sum of the series by two. This interpretation is further supported by Aud (88), who, while interrupting Jan, says that the problem of multiplying the sum by two is what they have been discussing all along. Aud continues with a metacomment (88) that confirms the analysis presented above that her drawing is intended to be an illustration of her verbal argumentation. Eli's contribution (89) ends the sequence and confirms that she has followed Aud's reasoning.

As can be seen, throughout this dialogue Aud's inscription is an anchor in the discussion. The inscription is an externalisation of her thinking and it is a tool for sharing meaning among the students. The inscription illustrates the path of the bouncing ball and it communicates Aud's ideas and explanations. The inscription serves both as a semiotic index and symbol and bridges the gap between the problem situation and the mathematics through representing the ball's path in an approximate manner (Forman & Ansell, 2002; Lehrer et al., 2000, Lehrer et al., 2001). The inscription supports the verbal argumentation and enriches the communication since it adds another mode that is permanent and to which the interlocutors can refer continuously. It also distributes a specific manner of understanding the problem among the group members. Again, this dialogue indicates how students in a stepwise fashion, and by using inscriptions, appropriate features of how the concept of geometric series can be productively used in this specific setting. Moreover, it is interesting to note that in spite of the students' relative expertise in this area, they have to struggle with the particulars of how to coordinate the inscription with the specific information. This difficulty may have been caused by the misleading nature of the inscription, even though we claim that the inscription served as a tool in the students' problem-solving process.

Excerpt 3 Using Inscriptions in Explaining

Later on in the 14th session, the group is working with task 164b (see above). The dialogue is primarily a conversation between Jan and Pia, where Jan is explaining to Pia the mathematical reasoning of the group by using an inscription (Figure 3). Once again the inscription is invented *in situ*.

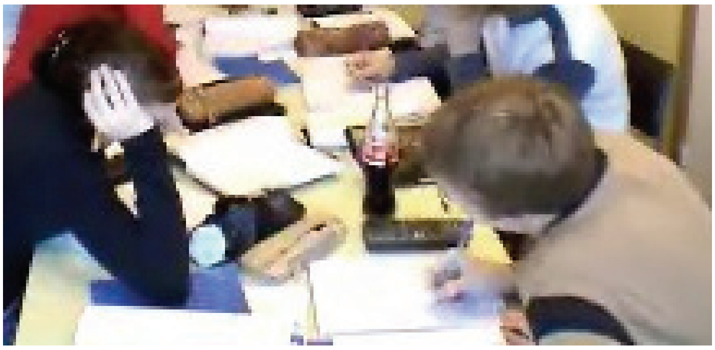
		Verbal activity	Accompanying activities
199	Jan	And then that is like *S*: of, equals e:: zero point sixty-four over one minus zero point four	Jan writes and explains.
200	Aud	Yes	Aud looks into her book.
201	Pia	Actually how zero point sixty-four?	Pia refers to what is written in Jan's book.
202	Jan	'Cause that i::s e::: that is a_2, and a_2 is Yes	Jan directly refers to the inscription.

203	Are	One point six times zero point forty	Are comments on procedure of calculating a_2 .
204	Pia	But do you use a_2 instead? Why don't you take a_1?	Pia asks Jan for a clarification.

Jan (199) initiates this sequence by telling Pia about the sum of a convergent geometric series. From the numbers it is possible to see that Jan is talking about calculating the sum of a convergent geometric series given the parameters a_1 equal to 0.64 and *k* equal to 0.4. Obviously, Jan uses the formula for the sum of a convergent geometric series, $S = \frac{a_1}{1-k}$. Aud (200) agrees that Jan has applied the correct parameter values for a_1 and *k*. This agreement indicates that Aud and Jan have a shared understanding of what *S* is in this case. Pia (201) wonders how and why Jan and Aud have used the number 0.64. The reason for Pia's question is related to the information given in the task, where it is said that the ball is dropped from 1.6 metres. If Pia is thinking about the formula mentioned above, the parameter a_1 is supposed to be used. This parameter refers to the first term of a geometric series. From the task text this first term seems to be 1.6 metres and not 0.64 metres.

Jan (202) initiates another explanation by saying that it is a_2 , the second height, which is 0.64. Are (203) tells how they have arrived at this number; it is the product of 1.6 and 0.4. These numbers come from the text, where 1.6 is the initial height and 0.4 the decimal notation of 40 %. Pia (204) is not satisfied with this explanation of the number 0.64. Pia wonders why they have used a_2 and not a_1 in this particular case, thus abandoning one of the premises for using the sum formula. At this point all four students participating in the dialogue seem to have reached a similar understanding of the symbol S and what it represents, but they do not as yet have a shared focus of attention with respect to how they should proceed. Pia's utterances are about calculating the distance from beginning to end. However, the three other students leave the first height and deal with the sum of the remaining heights, a strategy elaborated in the following dialogue:

		Verbal activity	Accompanying activities
205	Jan	E:: I'm gonna show you. Okey e:: what we do Okey like e:: that is (the ball), then it is like it is starting here, then it falls down like that, then it bounces up like that like that like that like that And then we are going to find out how long that=	
206	Pia	Yes	
207	Jan	=is So what we do is that we say that we might go up here, okay there we may=	While talking Jan simultaneously makes a drawing of the bouncing ball which looks like the following: 
208	Pia	Mm	Pia still follows the explanation by looking at the inscription (see below).



209	Jan	=(drop down) how much it <u>falls</u>, right? We have to find <u>that</u>, <u>that</u>, and that, that is *S* if like you add them. Then we say all of them and then we double them. =	Jan reinforces his argument by pointing at various points in the graph.
210	Pia	Yes	Pia still looks at the graph.
211	Jan	= That is, we are going to find that separately and then take, you multiply by two. Because then we get both up and down	Jan actively refers to the inscription by sweeping his hand over the graph.
212	Pia	Oh yes in that way	Pia looks at Jan while reasoning.
213	Jan	Then we got both that and at that and then we add this, 'cause that's a_1	Jan points at the first line of his graph to focus their attention.
214	Pia	Yes Oh yes	Pia refers to the inscription in her reasoning.
215	Jan	'cause we, that is like we first calculate (the decimal numbers) ( ) 'cause in the previous task it was like it was going like <u>that</u>. 'cause ( ) just take the sum, that is of the <u>whole</u> thing and multiplied by two	Jan actively applies the inscription while reasoning and explaining.
216	Pia	Yes	Pia looks at Jan.

217	Jan	But now we have to take like, add [that first]	Jan reinforces his explanation by referring to the inscription.
218	Pia	[You have] to, the part first yes. Okay, mm	Pia argues using the inscription.

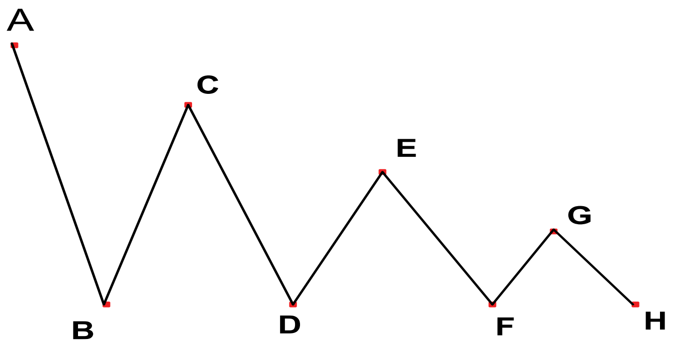


Figure 3. Jan's inscription in his notebook (the letters A-H have been added by the present author to facilitate understanding of the dialogue).

Jan (205) starts to explain to Pia what has been done. While doing this, Jan makes an inscription (Figure 3) to complement his verbal explanation. This inscription serves as a mediating device through which Jan's thinking is externalised in an explicit form. One could even argue that this inscription *is* a significant part of his thinking. Jan draws the path and starts pointing with his pencil at various places to support his argumentation. When uttering the emphasised word here (205), he points to point A. The word that refers to point B and the four repetitions of like that refer to the points C, D, F, and H, respectively. By using this inscription, Jan and Pia through joint activity achieve a shared interpretation of the path of the ball. This is confirmed by Pia (206), who through an overlapping Yes confirms that she follows Jan's reasoning. Jan and Pia (207-208) continue with their project of understanding how to calculate the length of the path. The emphasised word here (207) and Jan's simultaneous pointing refers to point A in the inscription. Jan puts emphasis on the distances the ball falls by uttering the word that three times. The first that refers to the segment CD, the next to EF, and the last to GH. The sum of these heights is what Jan calls *S* (209).

In order to interpret the inscription, some interpretations have to be made, since in Jan's inscription the heights AB, CD, and EF are not represented as verticals as would be expected when the ball bounces up and down. However, this misleading nature of the inscription is not commented upon by Pia. Instead she agrees with Jan's reasoning (210). An essential point in Jan's explanation (211) concerns doubling of the heights. Jan says that they have to calculate the sum of the distances the ball falls first, and then multiply by two to get the distances the ball bounces up and down. Pia (212) claims to follows Jan's reasoning. She is convinced by Jan's argument, which might be represented mathematically as follows:

$$S = 1.6 + 2\sum_{n=1}^{\infty} 1.6 \cdot (0.4)^n$$

Jan (213) continues his reasoning. The first that refers to CD and the second refers to EF, while this refers to the distance AB, the first term in the total distance sum. Pia (214) again claims to follow Jan's reasoning. Jan (215) continues by explaining their calculations of the decimal numbers, that is the consecutive heights. Furthermore, Jan (215) refers to the similar previous task on which they had worked, 164a. He recapitulates how they dealt with that problem. Again Jan actively uses his inscription and when expressing the word that, he moves his pencil in the order B-C-D-E-F, starting at the floor level (actually Jan makes a new drawing here, but it is similar to that shown in Figure 3). The students calculate the consecutive heights starting with CD, and multiplied it by two to include the distances BC, DE, et cetera. Pia (216) responds with a confirmation, before Jan (217) explains the difference between the two tasks by saying that now they have to add AB to the calculated sum. Through her utterance (218), Pia expresses the mathematical consensus of the whole dialogue. The students had to calculate the length of the path from B first, multiply it by two, and then add the distance AB.

Jan's explanation in this dialogue emerges in the context of the inscription (Figure 3). In this interaction the use of the inscription served a range of purposes, including explaining to Pia the nature of the mathematical reasoning that Jan considered relevant. By using this tool Jan made his reasoning public in such a manner that Pia could participate in it, as is seen through her confirmations. The inscription is a tool for articulating Jan's thinking and for taking it forward in a stepwise fashion. Most likely, Pia's potential to bridge the gap between the problem situation and the mathematics was facilitated. Thus, the sequence might very well have supported Pia's appropriation of some aspects of the concept of geometric series.

Discussion

Inscriptions are important to make information and understandings explicit; they serve as prosthetic devices for thinking. Mathematical reasoning is an extremely literate activity and relies on the successful use of such external resources. In our opinion, the significance of the multimodality of learning mathematics is not given enough emphasis in research. In the excerpts we have seen how students actively rely on and produce inscriptions to coordinate their interpretations of problems with the mathematisation necessary. Their activities unfold largely on the basis of the use of inscriptions that have a range of functions "intra-mentally" as well as "inter-mentally" (Wertsch, 1998) to use Vygotskian parlance. Although the evolution of inscriptions is not traced across the data collection period, we still claim that the inscriptions evolve as objects in the students' discourse along with coevolution of thinking about these objects (Latour, 1986, 1987; Lehrer et al., 2000; Lehrer et al., 2001).

The students experienced the advantages of applying inscriptions for problem-solving purposes, and received support for their reasoning in using them. The inscriptions constituted a substantial part of their reasoning. Even though the inscriptions are claimed to serve as anchor points and significant tools in the students' problem solving, the students may seem to be working against the leading of the inscriptions in excerpt 2 and excerpt 3. The inscriptions utilised are misleading because they indicate the distances to be calculated not as vertical heights. The misleading nature of the inscriptions may thus have caused parts of the students' difficulties in solving the mathematical problems. Nevertheless, the inscriptions supported and were intermediaries in the students' reasoning, but did not substitute thinking. The students still needed to add significant information and earlier experiences to take full advantage of the inscriptions and to overcome the obstacles (Goodwin, 1997). It is impossible to externalise all aspects of human understanding and make everything explicit in a cultural tool. Nevertheless, the students were learning through interacting with the inscriptions, and these discursive tools supported processes of appropriation both for those who produced them and for those who participated in the interaction in which they were used (Lehrer et al., 2000; Lehrer et al., 2001).

In Excerpt 1, it is interesting to observe that the student recognises that the inscription can be used as an algorithm. That's why you have to draw, 'cause then you can just count, right? as Jan puts it. In a very concrete sense the inscription takes the role of a mediating device used as a discursive tool for communicative purposes (Latour, 1987; Roth & McGinn, 1998; Sandoval & Millwood, 2005). However, in spite of this successful introduction of the tool into the flow of activities, it is important to note that there is still room for mistakes when applying the tool to the concrete case. Even the use of algorithms is never algorithmic. In spite of the correct nature of the drawing, there are still problems in coordinating the information given with the structure of the inscription. For instance, one of the students makes a mistake when identifying the first term. In our opinion, this is an important observation that testifies to the role of practising how to use inscriptions. Even though one is clear about the conceptual nature of the inscription, quite some time may be needed before one learns to see how the various elements are to be integrated. Thus, students of mathematics not only have to learn how to generalise, they also have to learn how to particularise (Billig, 1996), that is, to understand how a particular intellectual tool (the concept of geometric series) can be coordinated with a concrete problem.

In Excerpt 2, the students' problem is whether and why they can multiply their answer by two since their result was half of the amount provided in the answer section[5]. They discuss at length how they can arrive at this answer. This problem is the concrete background when Aud, *in situ*, externalises her thinking by introducing the inscription on the chalkboard (Figure 2). Even though the inscription is misleading, it indicates that the students have to calculate the length of the parabolas, and the students actively reason through using this inscription in the continued discussion. The inscription does, in spite of its misleading nature, show that the ball bounces up and down, and that is why doubling their result is correct. The inscription hence fulfilled a communicative purpose, in that it made explicit Aud's reasoning (cf. Lehrer et al., 2000). However, again we see that the coordination between the inscription, the verbal discussion, the information provided in the answer section of the book, and the formulation of the problem poses a considerable challenge for the students.

This case illustrates that inscriptions are essential for thinking and communicating, but they do not mean anything on their own.The cognising subject has to realise how the inscription can be productively invoked and how it can be made to fit a concrete problem.

In Excerpt 3, Jan utilises an inscription when explaining to Pia how to mathematise the problem. Jan's use of an inscription that is a mere copy of the one used previously by Aud indicates that Jan has appropriated the meaning of the inscription and hence uses it as a tool to support his reasoning. Thereby, the inscription seems to support Pia in her closing of the gap between the problem situation and the needed mathematisation.

The three excerpts are thus linked together on a macro level. They all serve as significant tools in the students' appropriation process, they support communication and problem solving, and they serve as tools in closing the gap between the presented task and the mathematisation necessary.

At a more principled level the results illustrate the fundamental manner in which human reasoning and learning are tool-mediated activities (Cobb, 2002; Lehrer et al., 2000; Lehrer et al., 2001; Säljö, 2005). As humans we rely on material artifacts as repositories of, and resources for, thinking. We constantly move between thinking and external symbolic tools in our meaning-making practices; indeed our thinking can be very much described as embodied in artifacts. This realisation should serve as an antidote against the heavily rationalist understanding that has tended to dominate research on mathematics learning. Conceptual mastery often has been exclusively seen as an internal, cognitive process. However, as Latour (1986) points out, we think as much with our hands and eyes as with our brains.

This multimodal and dynamic nature of human thinking and learning is richly illustrated in our data. The artifacts the students use serve a wide range of purposes: They externalise modes of reasoning, they clarify to the speaker and the listener what is being said, they add concretisation to complex talk, and they serve as anchor points for talking and gesturing. Suggestions can be made, arguments can be challenged, and claims can be tested by referring to the material artifacts (Latour, 1987; Roth & McGinn, 1998; Säljö, 2005). Collaboration and cothinking of the kind we have seen would be impossible without the presence of such devices. Moreover, the inscriptions also serve as tools the students utilise to close the gap between the presented problem and the mathematisation needed. As seen in the students' discussions, they are totally dependent on these inscriptions in solving the problems. The analysis thus shows the role of inscriptions in the students' problem solving leading to mathematisation. Thinking is embodied in material objects and all three inscriptions serve as intermediaries and mediating tools in the joint activity in which the students appropriate significant aspects of mathematics (Leont'ev, 1981; Moschkovich, 2004; Rogoff, 1990; Säljö, 2005; Wertsch, 1998). The inscriptions are hence significant tools in the individual student's appropriation process of mathematical concepts. Particularly, this fact is illustrated by Pia's response (212) to Jan's explanation in Excerpt 3: Oh yes in that way. Shared meanings are achieved in the students' joint activity of the inscription and their mathematisation of the problem.

The issue of generalisation of research results ought to be addressed in our study, since the number of students involved is small. One must be aware that the outcomes might be caused by the researched students being 'special cases'. The students are considered as being high-achievers. Objections might thus be raised that it is because of this that the students are able to externalise their thinking through the use of inscriptions as communicative and problem-solving tools to achieve shared meanings. Nonetheless, we argue that since it happened in this case there is potential for it to occur in other cases (cf. Bassey, 1999; Niss, 2004).

A pedagogical consequence of this suggestion is that teaching should encourage students to engage in reasoning by means of artifacts. Knowing when and how to rely on artifacts, whether self-made or borrowed from others, is an important kind of metaknowledge in the learning of mathematics. Learning to represent in physical form what is to be mathematised is a productive task in the process of appropriating abstract concepts and operations. Students will learn that concepts and inscriptions of the kind studied here can be expressed in a variety of ways. Inquiring into how these various types of representations correspond with each other will allow a richer mode of understanding to emerge.

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Notes

[1] The definition of inscriptions is inspired by Lehrer et al. (2001, p. 259) and Latour (1987).

[2] Transcription codes: = continued utterances, [ ] overlapping utterances, (( )) Non-verbal activity /comments, :: prolonged sound or letter, ( ) inaudible fragments, (guess) best guess, Under emphasised words, CAPS loud utterance.

[3] Later on in the group's problem-solving process Jan explains his choice of using x instead of the actual amount; to economise and make one inscription that

simultaneously mathematises all three cases (1.37 a, b, and c).

[4] The task is considered to be a difficult one by the task book authors due to the marking with two out of three possible triangles.

[5] According to the Norwegian tradition in making text books in mathematics, generally the correct answers to the tasks would be presented at the end of the book. In Norwegian it is called "Fasit" which translated to English becomes "the Answer section".

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