Exploring Scaling: From Concept to Applications

Marina Milner-Bolotin
University of British Columbia, Vancouver, BC, Canada
mmilner@interchange.ubc.ca

Abstract

This paper discusses the concept of scaling and its biological and engineering applications. Scaling, in a scientific context, means proportional adjustment of the dimensions of an object so that the adjusted and original objects have similar shapes yet different dimensions. The paper provides an example of a hands-on, minds-on activity on scaling that can be adapted to a middle school, high school, or even undergraduate science curriculum. The student activity is preceded by an introduction and followed by a summary discussion with possible suggestions on how a teacher might guide student exploration.

A number of fundamental concepts in science fascinate students and teachers, yet the students require only basic algebra and very general science knowledge to understand them. As a result, these concepts can be studied at different levels and are well suited for middle or high school students, as well as college undergraduates. Moreover, the concepts often have fascinating applications connecting science to students' everyday lives. Biological scaling is a good example of such a concept as it provides a great opportunity to teach interesting physics and to see how it applies to biological systems. Scaling, in this context, means the proportional adjustment of the dimensions of an object such that the adjusted and original objects have similar shapes, yet different dimensions. In other words, an object is scaled when each one of its dimensions is changed (increased or decreased) by the same factor, referred to as a scaling factor (S.F.). The concept of scaling can be also successfully applied to engineering, architecture, the film industry, and other fields.

This paper presents a brief discussion of scaling and suggests a hands-on, minds-on activity that explores some of its interesting applications. A more in-depth discussion of scaling and its applications can be found elsewhere (Barnes, 1989; Fowlers, 1996; Goth, 2009; Haldane, 1970; Peterson, 2002; Thompson, 1992; Tretter, 2005; West & Brown, 2004). Having taught the topic of scaling to thousands of students over the years (from middle school to undergraduate non-science and science majors in college), I find it to be a topic that generates hot debates and raises students' interest and excitement about science.

Activity: Discovering Scaling

Materials

For each group: set of 27 or more small wooden or plastic cubes such as the ones used in elementary school mathematics classes, two or three metallic spheres of different sizes (wooden spheres do not sink in water and it is difficult to measure their volumes), a graduated cylinder large enough to fit the spheres and used to measure their volume, play dough, and a ruler.

Student Independent Investigation

Imagine a small cube with side 1 cm (Figure 1). The volume of such a cube is 1 cubic cm (1 cm^3), while its surface area is 6 square cm (6 cm^2) (a cube has six faces and each has an area of 1cm^2). Notice that, if you double every edge of the cube (i.e., enlarge it by a factor of 2), the volume of the cube increases by a factor of 8, while the surface area only increases by a factor of 4:

V_{small} = 1 cm^3; V_{large} = (1 cm x 2)^3 = (2 cm)^3 = 8 cm^3

A_{small} = 1 cm^2 x 6 = 6 cm^2; A_{large} = (1 cm x 2)^2 x 6 = 4 cm^2 x 6 = 24 cm^2

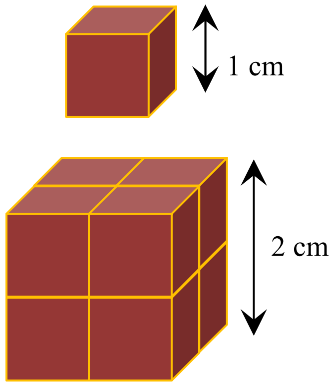

Figure 1. Two scaled cubes with a scaling factor of 2.

Stop and Think

Q1: What will happen to the surface area and the volume of the original cube if every edge of the original cube triples?

Definition of Scaling

Two objects are said to be *scaled* if one object can be obtained from the other by increasing its every dimension by the same factor, called the scaling factor (S.F.). In other words, two objects are scaled if one can be obtained from the other by proportional adjustment of all its dimensions. Notice that the scaling factor is a pure number (i.e., it has no unit). In the example above, the scaling factor is 2 (i.e., S.F. = 2).

Use the cubes provided to you to explore different scaling factors. Fill in your results in Table 1 below.

Table 1
Exploration of Scaling With Different Scaling Factors: Finding the Pattern in the Data

Length/m	Surface area/m^2	Volume/m^3	Ratio of Surface Area to Volume/m^{-1}	Scaling factor (compared with the smallest cube)
1				
2				
3				
10				
100				
1000				

Stop and Think

Examine Table 1 carefully and answer the following questions:

Q2: What interesting/surprising patterns have you observed in Table 1? Describe them.

Q3: When the scaling factor increases, the surface area and the volume of the object also increase. Do they increase at the same rate? Explain.

Q4: Any two cubes are always scaled. The same applies to any two spheres. Is it going to be true for any two rectangular prisms? Explain. (*Hint*: a cube is a rectangular prism, but is any rectangular prism a cube?)

Q5: In the SI system of measurement, 1 m = 10 dm = 100 cm = 1000 mm. What is the relationship between:

a) 1 m^2 and each of 1 dm^2, 1 cm^2, and 1 mm^2?
b) 1 m^3 and each of 1 dm^3, 1 cm^3, and 1 mm^3?
c) How are these relationships related to the concept of scaling?

Q6: For biological systems, surface area and volume play distinctively different roles: surface area (skin for example) is responsible for energy dissipation (or heat loss) while the volume is responsible for energy generation. How do you think the pattern you discovered in this activity might be relevant to biological systems?

Q7: A cross-sectional area of an object represents its strength (object's ability to withstand a load). For example, the larger the cross-sectional area of a bone is, the stronger is the bone. If the mass of an object is proportional to its volume, what can you say about the relative strengths of two scaled objects?

Q8: You are asked to help resolve an argument between three of your friends. David claims that when you enlarge every side of a cube n times, its volume also increases n times, Jane says that the volume of a cube increases 3n times, and Anne is convinced that the volume increases n^3 times. Who do you agree with and why?

Q9: Scaling is widely used in map-making. A map of a certain town is produced to a scale of 1:10 000. The town has a circular shape, and the map is 0.5 m across. What are the town's dimensions? What is the town's area? What is the town's area as represented on the map?

Q10: Rachel and Daniel have been assigned the task of peeling potatoes for the entire summer camp. Rachel is given 60 kg of potatoes that average 1 kg in mass, while Daniel is given 30 kg of potatoes that average 0.5 kg in mass (so Rachel's potatoes are on average twice heavier than Daniel's). Assuming that Rachel's and Daniel's peeling skills are equal, and if Rachel finishes her task in two hours, how long will it take Daniel to accomplish his task?

Q11: How do you think the scaling phenomenon might be relevant to other aspects of everyday life?

Activity Summary: Comments for the Teacher and Ideas for Class Discussion

At first glance, Table 2 does not hold any particular significance. But let us take a closer look at the ratio of the surface area of an object to its volume: the larger the scaling factor, the smaller is the ratio of the surface area to volume. For very large objects, the amount of surface area (or for that matter, cross-sectional area) compared to their volume becomes relatively small.

Galileo Galilei (1564-1642) noticed the phenomenon of scaling almost 400 years ago. In 1635, Galileo wrote in his *Dialogs Concerning Two New Sciences*:

> I am certain you both know that an oak two hundred cubits high would not be able to sustain its own branches if they were distributed as in a tree of ordinary size; and that nature cannot produce a horse as large as twenty ordinary horses or a giant ten times taller than an ordinary man unless by miracle or by greatly altering the proportions of his limbs and especially his bones, which would have to be considerably enlarged over the ordinary. (Galileo, 1635/2002, p. 402)

Table 2
Exploration of Scaling With Different Scaling Factors: Finding the Pattern in the Data. (The table shows that the ratio of the surface area to volume of scaled cubes decreases as the scaling factor increases.)

Length/m	Surface area/m^2	Volume/m^3	Ratio of Surface Area to Volume/m^{-1}	Scaling factor (compared with the smallest cube)
1	6	1	6	1
2	24	8	3	2
3	54	27	2	3
10	600	1000	0.6	10
100	60 000	1 000 000	0.06	100
1000	6 000 000	10 000 000 000	0.0006	1000

The reason for this trend in surface area to volume ratio is that the mass of an object is proportional to its volume (considering that the scaled objects have similar densities), while the cross-sectional area of a bone or a tree branch, which is responsible for an object's strength, is proportional to the square of the scaling factor. As a result, when one scales the object up, its mass increases more than does its surface and cross-sectional area (see solution to Q10 earlier). A cross-sectional area influences the strength of an animal's bones (large animals have disproportionally large legs to support their weight, unless they live in water!). On the other hand, the surface area for many animals (their skin) has many important bodily functions: one of them is to help warm-blooded animals keep their temperature via heat exchange with the environment. When it is too hot, the animals sweat or pant to lose heat. And what happens if a large animal does not have enough surface area?

Nature came up with many interesting solutions. For example, elephants have very large ears that provide additional surface area and help them to cool down by losing heat (the size of elephants' ears depends on the climate they live in). More interesting is that the laws of scaling tell us that one cannot scale up living organisms (humans, plants, and animals), without modifying their shape. There is no way of making a chicken 1 meter tall without changing its shape! Unfortunately it also applies to Hollywood famous giants, such as Mighty-Joe-Young or King-Kong. A 15-foot-tall gorilla cannot have the same shape as a 6-foot-tall gorilla.

Scaling down has similar limitations, as exemplified by bonsai trees. Although they look very much like a reduced replica of the larger trees, the looks can be deceiving. If one makes a careful comparison, the differences between the trees' structure will be apparent (Barnes, 1989):

> The physics of things that we can only imagine is often more interesting and exciting than the physics of things that are real. However, when entering the world of imagination one must be careful. Although physics is an experimental science, in the imaginary world, it is impossible to verify ones' theories. So we must not let our imaginations carry us too far. (p. 234.)

Following Barnes' observation, it is pedagogically valuable to remind the students about the value of experiment in testing scientific theories. A valid scientific theory must be able to generate predictions that can be verified (Etkina & Van Heuvelen, 2001; Etkina, Van Heuvelen, Brookes, & Mills, 2002; Kalman, 2008). The following two testing experiments can serve this purpose.

Testing Experiment 1

Measure the diameters of your spheres. Calculate the scaling factor. Predict the volume of the larger sphere based on the diameter of the smaller sphere and the scaling factor. Conduct an experiment to test your prediction. (The volume of the metal sphere can be measured by submerging it in water and measuring the volume of the displaced liquid.) Do your experimental results confirm your prediction?

Testing Experiment 2

Use play dough to build three scaled rectangular prisms (Figure 2): a small prism, a medium prism (S.F. = 2) and a large prism (S.F. = 3). Before building the larger prisms, hold the smallest prism by its base and make sure you can hold it horizontally (as a cantilever). If you cannot hold it (i.e., the unsupported end of the prism falls), make it a little shorter. Now build the other two prisms. Predict if it is going to be easier to hold the other two prisms horizontally by their bases and use them as cantilevers compared to the smallest prism. Test your predictions. How might what you found be relevant to architectural designs?

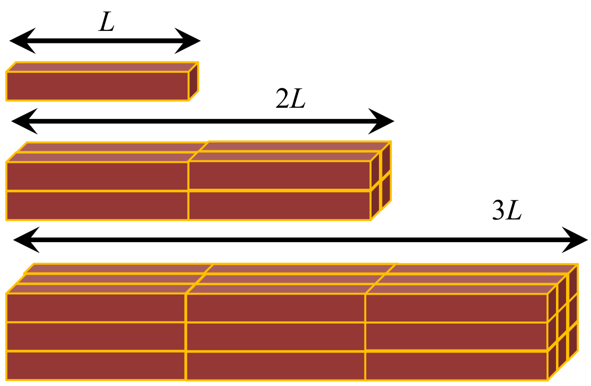

Figure 2. Three scaled rectangular prisms.

In addition to Barnes (1989), a very interesting explanation of scaling laws and their applications can be found in the following references: Fowlers (1996), Haldane (1970), Peterson (2002), and West and Brown (2004). Scaling plays a central role in our lives; in its biological applications (Ahlborn, 2004), as well as in engineering, architecture, geography (Wiegand, 2006), art, and design.

Answers to Some of the Stop and Think Questions From the Student Activity

Q3: When the scaling factor increases, the surface area and the volume of the object also increase. Do they increase at the same rate? Explain.

Answer. The volume increases faster than the surface area. This can be illustrated using small cubes to build bigger ones. While stacking small cubes together, some of the faces of the smaller cubes will become internal, decreasing the surface area. For example, if you stack 27 small black cubes together to create a larger cube and paint the surface area of the larger cube in red and then take the 27 small cubes apart, you will see that 1 of the 27 cubes will be completely black, 6 cubes will have one red face and five black faces, 12 cubes will have two red faces and four black faces, and 8 of the cubes will have three red faces and three black faces. Since the red faces represent the surface area of the larger cube, one can see that smaller cubes, when considered separately, have more surface area compared to when they are stacked together.

Q4: Any two cubes are always scaled. The same applies to any two spheres. Is it going to be true for any two rectangular prisms? Explain. (*Hint*: a cube is a rectangular prism, but is any rectangular prism a cube?)

Answer. By definition, all the dimensions of a cube (width, length, and height) must be equal. Therefore, if the ratio of two sides of any two cubes is found, the ratio between any two other sides of the two cubes must be the same. The same applies to any two spheres. However, when considering two arbitrary rectangular prisms, the ratios of their corresponding edges might be different, as shown in Figure 3. The ratio of the heights of these prisms is 3:2, yet the ratios of their lengths and widths are 2:1 and 1:1 respectively. So, when you enlarge or reduce different dimensions of an object by different factors, the original and enlarged/reduced objects are not scaled.

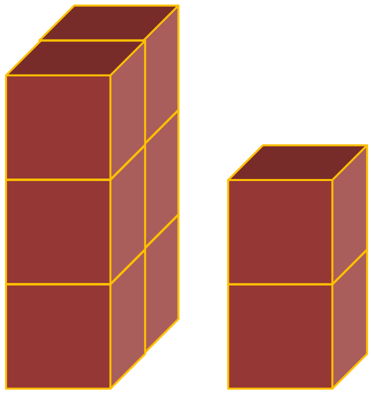

Figure 3. Two prisms that are not scaled.

Q5: In the SI system of measurement, 1 m = 10 dm = 100 cm = 1000 mm. What is the relationship between:

a) 1 m^2 and each of 1 dm^2, 1 cm^2, and 1 mm^2?
b) 1 m^3 and each of 1 dm^3, 1 cm^3, and 1 mm^3?
c) How are these relationships related to the concept of scaling?

Answer. a) $1\ m^2 = (10\ dm)^2 = 100\ dm^2 = 10^2\ dm^2$ (S.F. = 10)
$1\ m^2 = (100\ cm)^2 = 10\ 000\ cm^2 = 10^4\ cm^2$ (S.F. = 100 or 10^2)
$1\ m^2 = (1000\ mm)^2 = 1\ 000\ 000\ mm^2 = 10^6\ mm^2$ (S.F. = 1000 or 10^3)

b) $1\ m^3 = (10\ dm)^3 = 1000\ dm^3 = 10^3\ dm^3$ (S.F. = 10)
$1\ m^3 = (100\ cm)^3 = 1\ 000\ 000\ cm^3 = 10^6\ cm^2$ (S.F. = 100 or 10^2)
$1\ m^3 = (1000\ mm)^3 = 1\ 000\ 000\ 000\ mm^3 = 10^9\ mm^3$ (S.F. = 1000 or 10^3)

Q6: Answered in the text of the paper.

Q7: A cross-sectional area of an object represents its strength (object's ability to withstand a load). For example, the larger the cross-sectional area of a bone is, the stronger is the bone. If the mass of an object is proportional to its volume, what can you say about the relative strengths of two scaled objects?

Answer. If two objects are entirely scaled, a larger object is going to be weaker and will have less surface area per unit of mass than a smaller object. This is especially important in architecture and engineering science, while building models and testing the effects of wind, air ventilation, and load. If an engineer tested a small model of a bridge and found that the model of the bridge can support its weight, it does not mean that a real bridge will be able to support its weight!

Q8: You are asked to help resolve an argument between three of your friends. David claims that when you enlarge every side of a cube n times, its volume also increases n times, Jane says that the volume of a cube increases 3n times, and Anne is convinced that the volume increases n^3 times. Who do you agree with and why?

Answer. Anne is right. The reasoning is described earlier in the paper.

Q9: Scaling is widely used in map-making. A map of a certain town is produced to a scale of 1:10 000. The town has a circular shape, and the map is 0.5 m across. What is the town's real dimension? What is the town's area? What is the town's area as represented on the map?

Answer. The real dimension of the town is 0.5 m x 10 000 = 5000 m, or 5 km across. Therefore, the area of the town is $\pi D^2/4 = 3.14 \times 25\ km^2/4 \approx 20\ km^2$. The area of the town, as represented on the map, is $\pi D^2/4 = 3.14 \times 0.25\ m^2/4 \approx 0.2\ m^2$, which is also $20\ km^2/100\ 000\ 000$ or $20\ km^2/(S.F.)^2$.

Q10: Rachel and Daniel have been assigned the task of peeling potatoes for the entire summer camp. Rachel is given 60 kg of potatoes that average 1 kg in mass, while Daniel is given 30 kg of potatoes that average 0.5 kg in mass (so Rachel's potatoes are on average twice heavier than Daniel's). Assuming that Rachel's and Daniel's peeling skills are equal, and if Rachel finishes her task in 2 hours, how long will it take Daniel to accomplish his task?

Answer. Although Rachel and Daniel have, on average, the same number of potatoes (60) to peel, the surface areas of these potatoes (the area of potato skin) are not equal. Let us compare the surface areas (the area of the potato skin) of Rachel's and Daniel's potatoes. Since an average Rachel's potato has a mass of 1 kg and an average Daniel's potato has a mass of 0.5 kg, the volume of an average Rachel's potato must be twice the volume of an average Daniel's potato (assuming the potatoes have the same densities, ρ).

From the earlier discussion on scaling, we saw that if we assume that Rachel's (R) potatoes are a scaled version of Daniel's (D) potatoes, then the ratio of their volumes (V) is equal to the cube of the scaling factor (see Table 2). Therefore, the scaling factor can be found as follows:

$$V_{R\_potato}/V_{D\_potato} = 2 = (S.F.)^3 \qquad \text{Hence, S.F.} = \sqrt[3]{2}$$

On the other hand, we saw (see Table 2) that the ratio of the surface areas (A) of two scaled objects equals the square of the scaling factor. Therefore, the ratio of the area of the skin (surface areas) of Rachel's potato to the area of the skin of Daniel's potato can be calculated as the square of the scaling factor:

$$A_{R\_potato}/A_{D\_potato} = (S.F.)^2 = (\sqrt[3]{2})^2 \approx 1.59$$

Finally, since peeling time (t) will be proportional to surface area, and Rachel's peeling time is 2 hours, Daniel's peeling time can be calculated as follows:

$$t_R/t_D = 1.59 \qquad \text{So, if } t_R = 2 \text{ hours, } t_D = 2 \text{ hours}/1.59 = 1.26 \text{ hours}$$

Notice that, even though the mass of each of Rachel's potatoes is twice as much as the mass of each of Daniel's potatoes, it will take Rachel only 1.59 times longer than Daniel to peel her potatoes!

Q11: How do you think the scaling phenomenon might be relevant to other aspects of everyday life?

Answer. See earlier discussion and the references at the end of this paragraph. In addition, the topic of sound generation by musical instruments is another great application of the law of scaling. Larger instruments produce lowers sounds, but how is the ratio of produced tones related to the ratio of the sizes of these instruments? And what about the sounds produced by human vocal cords; how are they scaled? Think of the voices of kids versus the voices of adults, or males compared with females. These and many other interesting questions can be discussed qualitatively and quantitatively with more advanced students. A good start for the discussion of the applications of scaling to music can be found in Hoon and Tanner (1981) and Jeans (1968).

Conclusion

We hope that this paper will whet the appetite of readers for considering scaling phenomena in science classrooms. We have shown how the concept of scaling can be illustrated visually, as well as mathematically, and offered relevant hands-on and minds-on activities, as well as additional questions to think about. We have also shown that, when an object is scaled, its surface and cross-sectional areas change more slowly than its volume. Despite its straightforward formulation, scaling has profound effects on many aspects of our lives. The sources mentioned earlier, as well as Hewitt (1997), Levy and Salvadori (1994), and Salvadori (1980) will provide curious and creative science teachers and students with many additional scaling examples from the arts, science, engineering, and architecture. We hope that the discussion in this paper will help science teachers come up with exciting and unexpected activities for students of different ages and interests.

Acknowledgements

The author would like to express her gratitude to Peter Eastwell, Geetha Iyer, Anne Marie Janis, Gillian Kidman, and Noelene Wood for their helpful feedback and suggestions during the preparation of this manuscript.

References

Ahlborn, B. (2004). *Zoological physics: Quantitative models of body design, actions, and physical limitations of animals*. Berlin: Springer Verlag.

Barnes, G. (1989). Physics and size in biological systems. *The Physics Teacher, 27*(4), 234-253.

Etkina, E., & Van Heuvelen, A. (2001, July). *Investigative science learning environment: Using the process of science and cognitive strategies to learn physics.* Paper presented at the Physics Education Research Conference, Rochester, NY.

Etkina, E., Van Heuvelen, A., Brookes, D. T., & Mills, D. (2002). Role of experiments in physics instruction: A process approach. *The Physics Teacher, 40*(6), 351-355.

Fowlers, M. (1996). *Scaling: Why giants don't exist.* Retrieved June 30, 2009, from http://galileoandeinstein.physics.virginia.edu/lectures/scaling.html .

Galilei, G. (1635/2002). Dialogues concerning two sciences. In S. Hawking (Ed.), *On the shoulders of giants* (pp. 399-626). Philadelphia: Running Press.

Goth, G. (2009). *Physics and size in biological systems*. Retrieved June 30, 2009, from http://www.smccd.net/accounts/goth/MainPages/Scaling/Scaling.htm .

Haldane, J. B. S. (1970). On being the right size. In G. Holton, F. J. Rutherford, & F. G. Watson (Eds.), *Unit 1: The concepts of motion* (Vol. 1) (pp. 23-27). New-York: Holt, Reinhart and Winston.

Hewitt, P. G. (1997). *Conceptual physics* (Vol. 1) (8^{th} ed.). Reading, Massachusetts: Addison-Wesley Longman.

Hoon, S. R., & Tanner, B. K. (1981). The physics of music. *Physics Education, 16*, 300-311.

Jeans, S. J. (1968). *Science & music*. Toronto: General Publishing.

Kalman, C. S. (2008). *Successful science and engineering teaching: Theoretical and learning perspectives*. Secaucus, NJ: Springer.

Levy, M., & Salvadori, M. (1994). *Why buildings fall down: How structures fail*. New York: W. W. Norton.

Peterson, M. A. (2002). Galileo's discovery of scaling laws. *American Journal of Physics, 70*(6), 575-580.

Salvadori, M. (1980). *Why buildings stand up: The strength of architecture*. New York: Norton.

Thompson, D. A. W. (1992). *On growth and form*. New York: Dover.

Tretter, T. R. (2005). Godzilla versus scaling laws of physics. *The Physics Teacher, 43*, 530-532.

West, J. B., & Brown, J. H. (2004). Life's universal scaling laws. *Physics Today* (9), 36-42.

Wiegand, P. (2006). *Learning and teaching with maps*. London: Routlege.