An ordinary but surprisingly powerful theorem

Alan Sultan
Queens College, New York
<asultan956@aol.com>

Some years ago there was a young woman who sat in the front of my calculus class who said little. It was only when I gave a challenging problem that no one else could solve that she would say something. She consistently gave me not only correct solutions to the questions, but also ingenious solutions! I used to marvel at how ordinary she seemed on the outside, but how powerful the mathematical ideas she generated were. Being a mathematician, I started wondering, "Are there any theorems in mathematics that seem very ordinary on the outside, but when applied, have surprisingly far reaching consequences?" I thought about this for a while and came up with the following unlikely candidate which follows immediately from the definition of the area of a rectangle as length multiplied by width. I refer to this theorem as OT (Ordinary Theorem).

Theorem
The area of a right triangle is half the product of the perpendicular sides.

Proof
Every right triangle with perpendicular sides a and b is half of a rectangle whose sides are a and b. Hence, the area of a right triangle with perpendicular sides a and b is $\frac{1}{2}ab$. Simple!

You may wonder, "What is so wonderful about this?" Our plan here is to show you the answer. In fact, we will guide you through what promises to be an amazing journey into just how important this seemingly ordinary theorem really is. As we will show, hidden in this theorem is a treasure trove of almost all the familiar theorems we learn in high school, some of whose proofs are elusive.

The basic results

From OT, (rather than the other way around) we obtain the following:

Theorem
The area of a triangle with base b and height h is $\frac{1}{2}\,bh$.

Proof
We start with triangle ABC where the altitude to the base is inside the triangle, as shown in Figure 1.

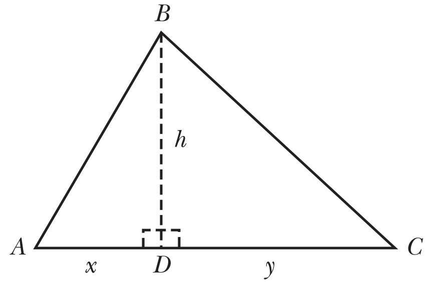

Figure 1. Altitude BD inside triangle ABC.

The area of ABD is $\frac{1}{2}xh$ by OT, and similarly, the area of triangle DBC is $\frac{1}{2}yh$. The area of triangle ABC is the sum of these areas. Thus, the area of triangle ABC is

$$\tfrac{1}{2}xh + \tfrac{1}{2}yh = \tfrac{1}{2}(x + y)\,h = \tfrac{1}{2}bh$$

If the altitude is outside the triangle, a similar proof is used, only this time we need to subtract areas. We leave this to the reader or to the reader's students! So from OT we find the area of any triangle is $\frac{1}{2}$ base × height.

An immediate consequence of this is:

Theorem
In a right triangle with perpendicular sides a and b and hypotenuse c, $a^2 + b^2 = c^2$.

Proof
The proof is the standard proof one often sees. We begin with four right triangles with sides a and b and hypotenuse c, and position them as shown in Figure 2.

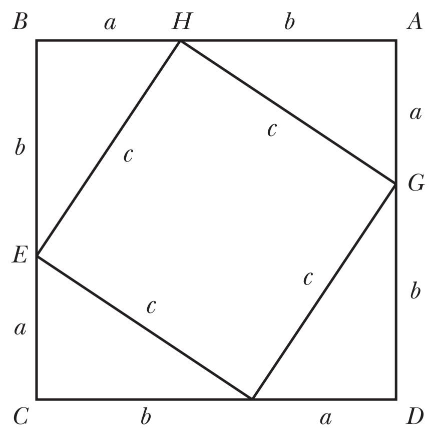

Figure 2.

Then the area of the square $ABCD$ is the area of the square in the middle, plus the sum of the areas of the four right triangles, that is,

$$(a+b)^2 = c^2 + 4\left(\frac{1}{2}ab\right)$$

Squaring the left side and then subtracting $2ab$ from both sides, we immediately obtain $a^2 + b^2 = c^2$. Voila! We have the Pythagorean Theorem, and we all know how useful and practical that theorem is!

Sine and cosine

When we define the cosine of an angle in standard position (where the initial side is the positive x-axis), we pick any point (x, y) on the terminal side of the angle, and define

$$\cos A = \frac{x}{r}$$

where r is the distance the point is from the origin. The first question that arises is, "If we are choosing *any* point on the terminal side, how do we know that we will get the same answer regardless of which point we use?" The usual answer is, "We use similar triangles," but our goal in this paper is ambitious. We want to show you that *all* the results on congruence and similarity can themselves be obtained from OT. So we cannot use similar triangles in our development to prove that the sine and cosine are well defined or we will be engaging in circular reasoning. We need to bypass similar triangles. How do we do that? We do that using none other than the formula for the area of a triangle, the first consequence of OT!

We need a lemma first.

Lemma

Given triangle ABC, suppose that a line segment is drawn from B to $\overline{AC}$, intersecting $\overline{AC}$ at D (see Figure 3). Then the ratio of the areas of triangle ABD to triangle CDB is the same as the ratio of AD to DC.

Proof

Figure 3 shows that both triangles ABD and DBC have the same height, h. Thus,

$$\frac{\text{Area } ABD}{\text{Area } DBC} = \frac{\frac{1}{2}AD \cdot h}{\frac{1}{2}DC \cdot h} = \frac{AD}{DC}$$

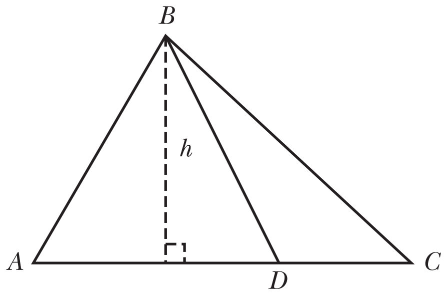

Figure 3. Two triangles with height, h.

Corollary

Given a right triangle ABC with right angle at C, draw a line segment $\overline{DE}$ parallel to $\overline{CB}$ where D is any point on $\overline{AC}$ and E is on $\overline{AB}$ (see Figure 4). Then

$$\frac{AC}{AB} = \frac{AD}{AE}$$

That is, $\cos A$ is the same whether we take the ratio of the adjacent side to hypotenuse in right triangle AED or right triangle ABC.

Proof

We begin with our diagram shown in Figure 4.

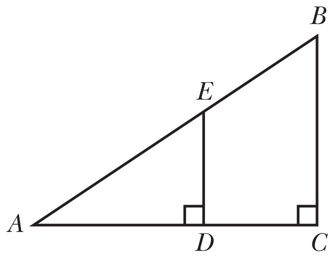

Figure 4.

First, draw $\overline{EC}$. The triangle ACE is then divided by $\overline{DE}$ into triangles I and II as shown in Figure 5.

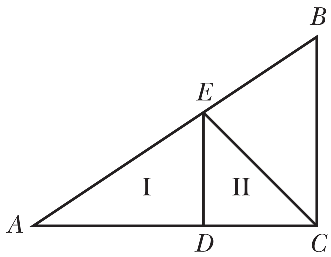

Figure 5. Dividing triangle ACE into triangles I and II.

Now from our lemma applied to triangle AEC, we have

$$\frac{\text{Area of I}}{\text{Area of II}} = \frac{AD}{DC} \tag{1}$$

Now draw $\overline{DB}$ in Figure 4, yielding the diagram as shown in Figure 6.

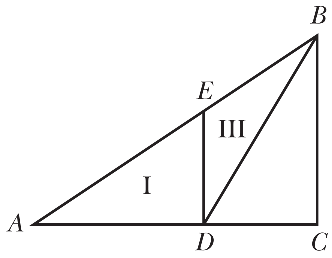

Figure 6

Again by the lemma, only this time applying it to triangle *ABD* we have

$$\frac{\text{Area of I}}{\text{Area of III}} = \frac{AE}{EB} \tag{2}$$

Finally, we note that both triangles II and III have base $\overline{ED}$ and height $\overline{DC}$ (since $\overline{ED}$ is parallel to $\overline{BC}$ and hence are the same distance from each other everywhere). Thus, the areas of II and III are the same. Replacing Area III by Area II in (2) we get that

$$\frac{\text{Area of I}}{\text{Area of II}} = \frac{AE}{EB} \tag{3}$$

Comparing (1) and (3) we see that

$$\frac{AD}{DC} = \frac{AE}{EB} \tag{4}$$

We are not quite there. Now, add the number 1 to both sides of (4) and combine. This gives

$$\frac{AD+DC}{DC} = \frac{AE+EB}{EB} \tag{5}$$

Dividing (5) by (4) and simplifying,

$$\frac{AD+DC}{AD} = \frac{AE+EB}{AE}$$

Since $AD + DC = AC$ and $AE + EB = AB$, this is just

$$\frac{AC}{AD} = \frac{AB}{AE} \tag{6}$$

From this proportion it follows that

$$\frac{AC}{AB} = \frac{AD}{AE} \tag{7}$$

and we are done.

The importance of this lemma cannot be underestimated. It says that if A is any angle, $\cos A$ is unique and well defined. We did not use similar triangles to prove it, rather *area*, which is surprising! There is such a nice interplay here between the algebra, geometry and trigonometry.

If we have two right triangles, *ABC* and *AED*, both containing an angle whose measure is that of *A*, then we can just overlay them to get Figure 4 and work from there.

Now that we know that the sine and cosine are well defined, we can accomplish our goal. We begin with the standard high school theorems.

The law of cosines

From OT, we obtained the Pythagorean theorem. Using the Pythagorean theorem, we easily derive the formula for the distance, D, between two points, (x_1, y_1) and (x_2, y_2), in the plane:

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Furthermore, using the Pythagorean theorem, we can derive, in the standard manner, the formula that $\sin^2\theta + \cos^2\theta = 1$. From this we immediately obtain:

Theorem (the law of cosines)
In any triangle ABC with sides a, b, c, $c^2 = a^2 + b^2 - 2ab\cos C$.

Proof
The proof we give here is for all triangles. We start with a triangle ABC and place the triangle on a coordinate plane as shown in Figure 7. Let the coordinates of A be (x, y). Using the definition of cosine and sine, we have

$$\cos C = \frac{x}{b} \quad \text{and} \quad \sin C = \frac{y}{b}$$

So $x = b\cos C$ and $y = b\sin C$.

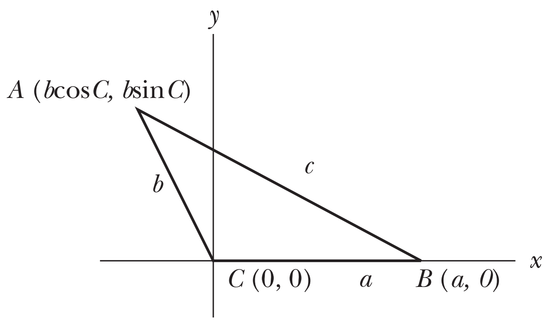

Figure 7. Triangle ABC on a coordinate plane.

Thus, the coordinates of A are $(x, y) = (b\cos C, b\sin C)$. Now the distance from A to B is c and can be computed by the distance formula.

$$c = \sqrt{(b\cos C - a)^2 + (b\sin C)^2} \tag{8}$$

Squaring both sides of (8) expanding and simplifying using the fact that $\sin^2\theta + \cos^2\theta = 1$ gives us

$$c^2 = b^2 - 2ab\cos C + a^2$$

Now we just rearrange our terms to obtain

$$c^2 = a^2 + b^2 - 2ab\cos C \tag{9}$$

What a nice interplay between algebra, Cartesian coordinates, geometry, and trigonometry!

Law of sines

We now know that the sine of an angle is well defined. If we draw an altitude to side $\overline{AC}$ of a triangle ABC, we divide the triangle into two right triangles (see Figure 1). This altitude's length, h, can be expressed in either of two ways: $h = a \sin C$ (from triangle BDC in Figure 1) or $h = c \sin A$ (from triangle ADB in Figure 1). Setting these two expressions equal to each other, we find that $a \sin C = c \sin A$, or that

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

(When the altitude is outside of the triangle, one will need to use the fact that $\sin(180° - A) = \sin A$ to prove the result, and this easily follows from the definition of the sine of an angle.) If we draw an altitude to side $\overline{AB}$ and proceed similarly, we obtain

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Combining our two proportions,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

This is the *law of sines*, which has great practical use. Now, let us turn to geometry.

The relationship to geometry

Congruence

We are taught in geometry that two triangles can be congruent under various conditions; for example, if SSS = SSS or if SAS = SAS or if ASA = ASA and so on, where S represents a side of the triangle and A an angle. Many books write one or more of these results as axioms and then derive the others. In fact, we can derive all of them from what we have done so far. Thus all the congruence results are (indirectly) a result of OT. Let us give the reader the flavour of how one of these is done by corroborating that if SSS = SSS for a pair of triangles, then the two triangles are congruent. Similar verifications can be done for SAS and ASA using the laws of sines and cosines. We encourage the reader to try some of these.

Theorem
If three sides of one triangle are congruent to three sides of another triangle then the triangles are congruent.

Proof
In two triangles ABC with side lengths a, b, c and DEF with side lenghts d, e, f, let us assume that $\overline{BC}$ is congruent to $\overline{EF}$, $\overline{AC}$ is congruent to $\overline{DF}$, and $\overline{AB}$ is congruent to $\overline{DE}$, as shown in Figure 8.

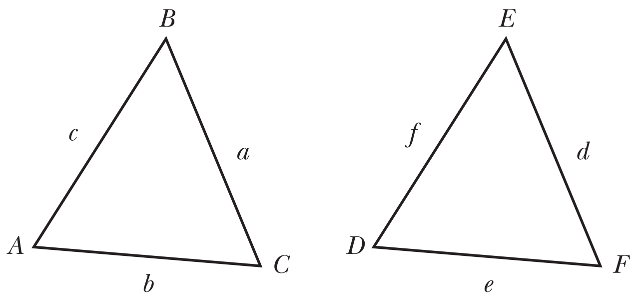

Figure 8. A pair of congruent triangles.

So $a = d$, $b = e$ and $c = f$. From the law of cosines, (9), we have that

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \tag{10}$$

and using the same law in triangle DEF with the corresponding sides, we have

$$\cos F = \frac{d^2 + e^2 - f^2}{2de} \tag{11}$$

Since $a = d$, $b = e$ and $c = f$, we can substitute them in (10) to get

$$\cos C = \frac{d^2 + e^2 - f^2}{2de} \tag{12}$$

and we see from (11) and (12) that $\cos C = \cos F$.

It follows that $\angle C \cong \angle F$ since $\angle C$ and $\angle F$ are less than 180 degrees. In a similar manner using the other versions of the law of cosines, we find that $\angle A \cong \angle D$, and that $\angle B \cong \angle E$. Thus, if three sides of one triangle are congruent to three sides of another triangle, then the corresponding angles are congruent, and so the triangles are congruent. Congruence theorems are the basis of most theorems one proves in geometry.

Another important concept used in obtaining practical geometric results is similarity.

Similarity of triangles

We now show that OT gives us all the similarity results we learn in high school as well! The main result in similarity is the following:

Theorem

If two angles of one triangle are congruent to two angles of another triangle, then the lengths of the corresponding sides of the two triangles are in proportion.

Proof

The third angles of the triangles are also congruent since the sum of the measures of the angles in a triangle is 180 degrees. We will use Figure 8 to give the proof. We will assume that $\angle A \cong \angle D$, $\angle B \cong \angle E$ and $\angle C \cong \angle F$. Then by the law of sines, applied to triangle ABC we have

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

which can be written as

$$\frac{a}{b} = \frac{\sin A}{\sin B} \tag{13}$$

Using the law of sines in triangle DEF we have in a similar manner that

$$\frac{d}{e} = \frac{\sin D}{\sin E} \tag{14}$$

However, since $\angle A \cong \angle D$ and $\angle B \cong \angle E$, we can substitute A and B for D and E in (14) respectively, and we get

$$\frac{d}{e} = \frac{\sin A}{\sin B} \tag{15}$$

Since the right hand sides of (13) and (15) are the same, the left hand sides are also, so we have that

$$\frac{a}{b} = \frac{d}{e}$$

Rewriting this as

$$\frac{a}{d} = \frac{b}{e}$$

we have part of our result. In a similar manner you can show that

$$\frac{a}{d} = \frac{c}{f}$$

So, with your work and ours, we get

$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$$

which says the lengths of the corresponding sides of the two triangles are in proportion.

Let us give another illustration:

Theorem
If the lengths of two sides of one triangle are proportional to the lengths of two sides of another triangle, and the angle between the proportional sides of these triangles are congruent, then the lengths of the third sides are in the same proportion as the lengths of the other corresponding sides.

Proof

Again, we use Figure 8. We may suppose that the two sides of triangle ABC we are talking about are $\overline{AC}$ and $\overline{AB}$ and that the lengths of these sides are proportional respectively to the lengths of $\overline{DF}$ and $\overline{DE}$. We also assume that $\angle A \cong \angle D$. Saying that these side lengths are proportional in our diagram means that

$$\frac{b}{e} = \frac{c}{f} = k \tag{16}$$

for some real number k. To prove our result, we will show that the lengths of the third sides are also in the same proportion. That is, we will show $\frac{a}{d}$ is also k. Now, using the law of cosines in triangle ABC we have that

$$a^2 = b^2 + c^2 - 2bc\cos A \tag{17}$$

Using the law of cosines in triangle DEF we have

$$d^2 = e^2 + f^2 - 2ef\cos D \tag{18}$$

From (16) we have $b = ke$ and $c = kf$, and we were given that $\angle \mathrm{A} \cong \angle \mathrm{D}$. Replacing b by ke, c by kf and A by D in (17) we get

$$\begin{aligned} a^2 &= (ke)^2 + (kf)^2 - 2(ke)(kf)\cos D \\ &= k^2(e^2 + f^2 - 2ef\cos D) \\ &= k^2d^2 \qquad \text{[Using (18)]} \end{aligned}$$

This string of equalities shows that $a^2 = k^2d^2$; hence, $a = kd$. It follows that $\frac{a}{d}$ is also k and using (16) we see that

$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f} = k$$

So, we have shown that the lengths of all three corresponding sides are in proportion which is what we wanted to prove. The approach taken to these proofs can help the teacher better understand how mathematical ideas interconnect and build on one another.

The circle

As a final illustration of the power of our ordinary theorem, let us now turn to the circle, only this time we examine its circumference and area, topics that are ordinarily discussed in calculus.

Before we begin, let us observe some things. If we let P_n represent a regular polygon of n sides inscribed in a circle of radius r, and p_n represent its perimeter and a_n the length of its apothem. (An apothem is a perpendicular segment from the centre of a regular polygon to one of its sides.) Then it is clear that as the number n of sides of the polygon increases, the apothems get closer and closer to the radius of the circle and the perimeters of the polygons get close to the circumference of the circle. In symbols,

$$\lim_{n\to\infty} a_n = r \tag{19}$$

and

$$\lim_{n\to\infty} p_n = C \tag{20}$$

We are now ready to proceed. The next theorem uses the following consequence of OT which follows from our results on similarity: the perimeters of two similar polygons are in the same ratio as their side lengths. A proof of this is rarely seen by high school teachers.

Theorem

The ratio of the circumference of a circle to its diameter is constant for all circles.

Proof

Begin with two circles, C_1 and C_2 where the radius of C_1 is smaller than the radius of C_2. Place circle C_1 inside C_2 circle so that their centres coincide. Inscribe a regular polygon P_{1n} with n sides inside circle 1 and draw "spokes" from the centre of the circle C_1 through the vertices of P_{1n} until they intersect C_2 forming a polygon P_{2n} with n sides. We get the figure shown in Figure 9.

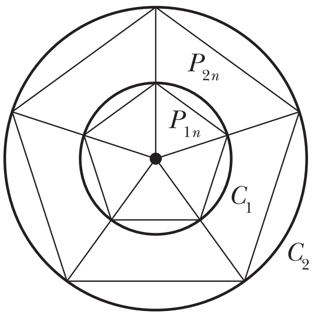

Figure 9.

Now P_{1n} and P_{2n} are similar, so the ratio of their perimeters is in the same ratio as the lengths of their corresponding sides, which in turn are in the same ratios as the lengths of their apothems, a_1 and a_2; that is

$$\frac{p_{1n}}{p_{2n}} = \frac{a_1}{a_2} \tag{21}$$

Now let the number of sides of P_{1n} and P_{2n} increase and approach infinity. Then a_1 approaches r_1 the radius of C_1, and a_2 approaches r_2 the radius of C_2. If the circumferences of C_1 and C_2 are c_1 and c_2, the left side of (21) approaches

$$\frac{C_1}{C_2}$$

while the right side approaches

$$\frac{r_1}{r_2}$$

(Here is where we are using limits.)

It follows that

$$\frac{c_1}{c_2} = \frac{a_1}{a_2}$$

This can be rewritten as

$$\frac{c_1}{r_2} = \frac{c_1}{r_2}$$

Dividing both sides by 2 we get

$$\frac{c_1}{2r_2} = \frac{c_1}{2r_2}$$

Since $2r_1 = d_1$, the diameter of circle 1, and $2r_2 = d_2$, the diameter of circle 2, we have that

$$\frac{c_1}{d_2} = \frac{c_1}{d_2}$$

and we are done! [Editor: This is clearly equal to the well-known constant π.]

We can go further with this. We know that from the area of a triangle, (OT's first corollary,) we can find the area of a regular polygon, and that it is equal to $\frac{1}{2}\,ap$ where a is the length of the apothem and p is the perimeter. It is also intuitively clear that we can find the area of a circle by expressing it as the limit of the areas of inscribed regular polygons of larger and larger number of sides. Thus we have as a corollary:

Corollary
The area of a circle is πr^2.

Proof
If we denote the areas of the typical inscribed regular polygon with n sides A_n and the area of the circle A of circumference C, then using (19) and (20) we have

$$\begin{aligned}
A &= \lim_{n\to\infty} A_n \\
&= \lim_{n\to\infty} \frac{1}{2} a_n p_n \\
&= \frac{1}{2} \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} p_n \\
&= \frac{1}{2} \cdot r \cdot C \\
&= \frac{1}{2} \cdot r \cdot 2\pi r \\
&= \pi r^2
\end{aligned}$$

and we are done. [Editor: This, of course, assumes you have derived $C = 2\pi r$ from the result of the previous theorem.]

Thus, from yet another consequence of OT follows the result that the area of a circle is πr^2. Our ordinary theorem has shown us a side we have never expected to see and makes us wonder, "Should the area of a right triangle be taught as a fundamental concept in high school?" We leave this for you to ponder.