What you see is what you get: Investigations with a view tube

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This paper presents an investigation by pre-service secondary school teachers in a geometry class of the relationship between the perpendicular distance from the eyeball to the wall (x) and the viewable vertical distance on the wall (y) using a view tube of constant length and diameter. In undertaking the investigation, students used tabular and graphical representations to determine the relationship. They also used a TI-84 calculator to investigate the relationship, and also modelled the scenario with the aid of *Geometer's Sketchpad* software.

An investigation is defined as "a systematic or formal inquiry to discover and examine the facts (of an incident, allegation, etc.) so as to establish the truth" (Oxford University Press, 2002, p. 701). For mathematical investigations, students are engaged in inquiry through exploration to develop problem-solving skills. The following are reasons to use investigations.

Firstly, investigation can promote mathematical problem solving. It is very important that students interpret mathematical statements and be able to establish patterns, relationships, and reason mathematically. Many people use the terms "mathematics investigation" and "problem solving" interchangeably, although there are some differences.

> A mathematical investigation stresses mathematical processes such as searching regularities, formulating, testing, justifying and proving conjectures, reflecting, and generalizing. When one starts working on an investigation, the question and the conditions are usually not completely clear, and making them more precise is the first part of the work. That is, investigating involves an essential phase of problem posing by the pupil—something that is problem solving is usually done by the teacher. However, investigations go much beyond simple problem posing and involves testing conjectures, proving, and generalizing (Ponte, 2001, p. 54–55).

Secondly, investigation can motivate students through active learning (Gadanidis, Sedig & Loamg, 2004). The National Council of Teachers of Mathematics (NCTM) *Principles and Standards for School Mathematics* (2000) values mathematics investigation and argues that students learn best when they are actively engaged in constructing their own understanding of mathematical ideas. The inquiry-based method motivates and encourages students to think creatively through active learning in order to discover multiple solution paths and critique solutions, as well as justify why they work (Chapin, 1998; Lesh & Doerr, 2000; National Council of Teachers of Mathematics, 2000).

Thirdly, investigation promotes conceptual understanding. Investigation tasks usually take a period of time to accomplish. They provide an opportunity for students to engage in inquiry-based tasks from different perspectives. This type of environment enables students to be engaged in open questioning, debate, explanation, clarification and modification of their mathematical thinking (Baroody & Coslick, 1998; Bowers, Cobb & McClain, 1999; Mokros, 2003).

Fourthly, investigations may show relevance in the "real" world. Modelling real-world tasks gives students the opportunity to explore real-life situations (Ponte, Ferreira, Brunheira, Oliveira & Varanda, 1998). To help students benefit from this experience, teachers need to provide students with "worthwhile mathematical tasks … [and] proactively and consistently support students' cognitive activity without reducing the complexity and cognitive demands of the task" (Henningsen & Stein, 1997, p. 546).

The investigation

This article presents the first of three investigations posed to a group of pre-service teachers in a geometry class for pre-service secondary school teachers in Texas, USA. In the activity, pre-service teachers investigated the relationship between the perpendicular distance from the eyeball to the wall (x), and the viewable vertical distance on the wall (y) using a view tube of constant length and diameter. In the two follow-up activities, not addressed in this paper, my students:

- investigated the relationship between the length of the view tubes (x) and the viewable vertical distance on the wall, while keeping constant the perpendicular distance from the eyeball to the wall and the diameter of the view tube (Obara, in press);
- investigated the relationship between the diameter of the view tubes (x) of constant lengths and the viewable vertical distance (y) on the wall while keeping constant the perpendicular distance from the eyeball to the wall (Obara, 2009).

One should note that similar investigations have been suggested by Day, Kelly, Libby, Lott and Hirstein (2001) and Wilson and Shealy (1995). The main goal of this activity was to allow students to examine the situation and be able to use various techniques investigating, conjecturing, and modelling in the learning process to enable students understand and communicate mathematically.

Data collection and tabular representation

Before the investigation, I divided the class into three groups and provided each group with a metre ruler and PVC tube 26.67 centimetres long and 4.064 centimetres in diameter (i.e., 10.5 and 1.6 inches respectively). Tubes are easily accessible, for example, toilet roll tubes or tubes from wrapping paper. However, as there was construction on campus, I was able to collect pieces of scrap PVC tubes of the desired length and diameters. Each group was supposed to have at least three people: one to look through the tube (viewer), the next to measure, and the third to record data. The goal of the investigation was to determine how much they could see using the view tube. Before the students started working on the investigation, I asked each group to predict what kind of relationship they expected to find (linear, hyperbolic or exponential).

In this investigation, students worked collaboratively and changed roles as the activity proceeded. I asked them to measure the distance from the eyeball to the wall (parallel to the floor, x), and the vertical distance (y), as shown in Figure 1.

Before they started the investigation, the students discussed among themselves what the relationship might be when plotting y versus x. One student said it should be exponential and another said that it was linear. The rest of the group was not sure what the relationship would be. They started to discuss the best way to carry out the experiment and came up with the following recommendations:

- students decided to use centimetres and metres as a unit of measure and to record their data in a table;
- they resolved that it would be easier and more accurate to measure distance x along the floor from the foot to the wall;
- since the floor was made of tiles and each tile was one foot in length, they decided to use tiles to measure x initially and then convert to metric units;
- they advised each other to make sure to hold the tube parallel to the floor when viewing the wall.

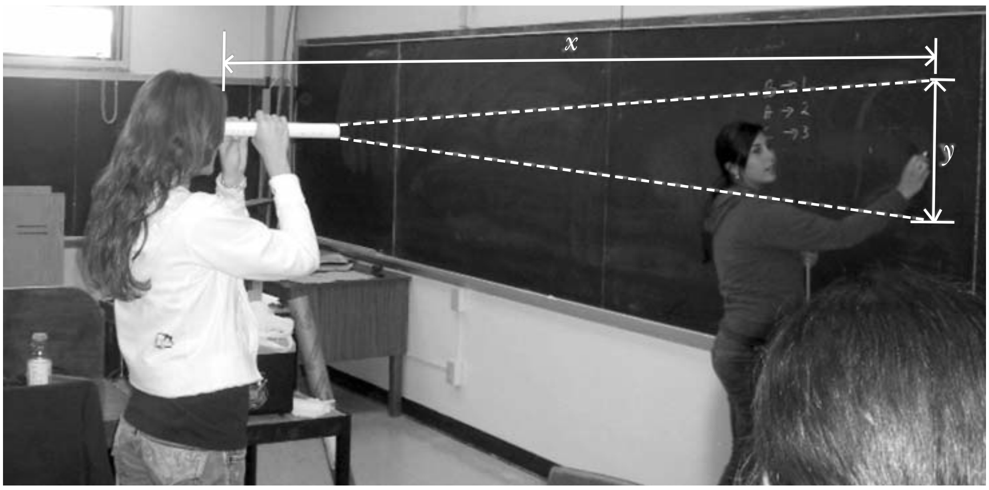


Figure 1. Students viewing through the view tube and measuring lengths.

The students started the investigation. They moved to the end of the room and marked a point on the floor to be a point where to place the end foot. They counted the number of tiles to measure x_1 from this point to the wall with a blackboard, and used chalk to mark the lower and upper points of the view area to determine the vertical distance y (Figure 1). Using x_1, all students in the group used the view tube to determine how much they could see on the wall, took the average, and recorded that figure as the first data point (y_1). When asked why they took the average, they said that it was a way of minimising error in the measurement process. Then they moved one step towards the wall and measured (x_2). Then using x_2 as the horizontal distance, they repeated what they did for x_1, took the average of the vertical distances, and recorded it as the second data point (y_2). They continued this process and generated the data shown in Table 1.

Table1 Distance from eye to wall (x) and the viewable vertical distance (y)

x (metres)	y (centimetres)
5.18	86.36
4.87	81.28
4.57	73.66
4.27	68.58
3.96	66.04
3.66	55.88
3.35	50.8
3.05	45.72
2.74	43.18
2.43	38.1
2.13	33.02
1.83	27.94
1.52	25.4
1.22	20.32
0.914	15.24
0.61	10.16
0.30	7.62

After collecting the data, the students started a discussion on what kind of a relationship might exist. Some students could not describe the relationship by looking at the table (see Table1). I asked them to look for a pattern in the way the x values changed and y values changed in the tabular representation, and I asked them to build a conjecture. Using the table, most of my students were unable to see any existing relationship, but some did conjecturing. They started by noting that x values were almost double the values of y, and that when the values of x changed by some value, y changed by about two times that value, which led the students to talk about slope. They talked about the slope being 2, and then conjectured that the relationship was linear. Asked to talk about their experience collecting data, one student noted:

I would advocate having one person as the viewer the entire experiment rather than trading jobs to get each other's reading and take the average. This is because one member of my group may use glasses or have deeper-set eyes, which might make his reading considerably different from others'. I like the concept of trading jobs because that way everyone gets a chance to do the experiment. If we need accuracy, then this suggestion might be considered.

Graphical representation

Students started plotting the data on a graph (as shown in Figure 2) to make sense of the relationship and test their conjecture. They joined the data points with simple curves and also drew an approximate line of best fit that they thought best fits the data. They all agreed it was a linear relationship. Asked why they joined points with simple curves, the students pointed out that they learned it in algebra [class], where after finding values of *x* and *y*, they simply joined the points. The students also discussed why it was difficult to see a relationship using the data in Table 1.

Teacher: Why was it difficult to see the relationship in the data presented in Table 1?

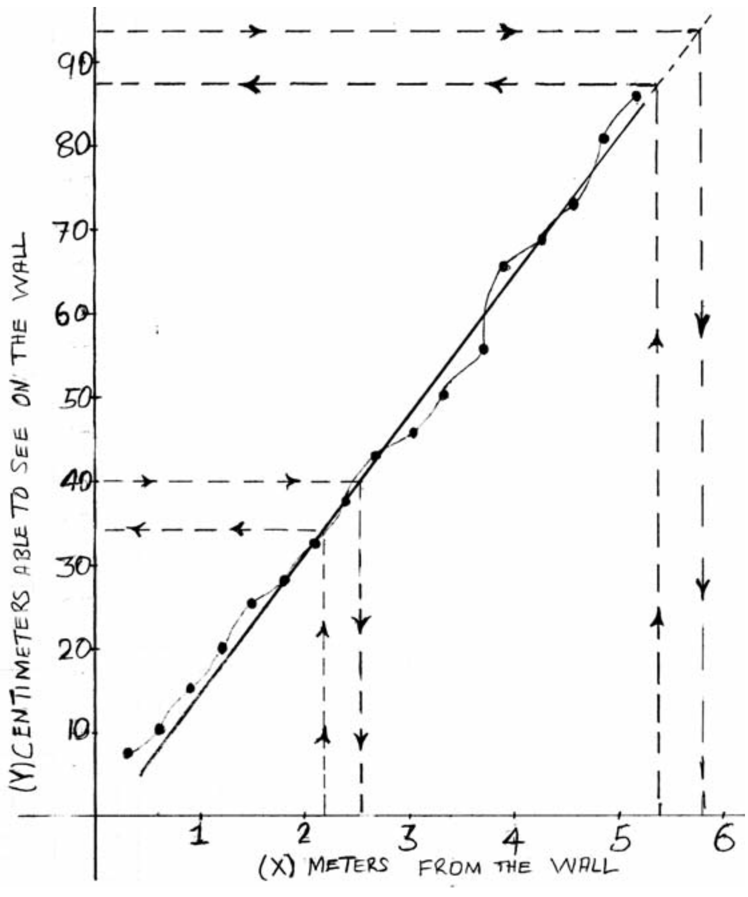


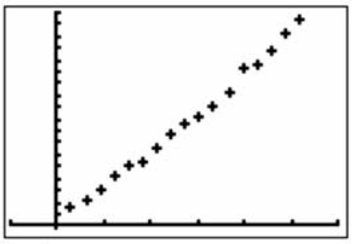
Figure 2. Centimetres visible on the wall plotted against distance from the wall.

Student: I focused on the values of x and y separately and not as a pair. Yes, I noted that y was changing, but also observed that x was changing. That confused me, but now with the graph, I can see the pair of points plotted and can see the relationship much more easily.

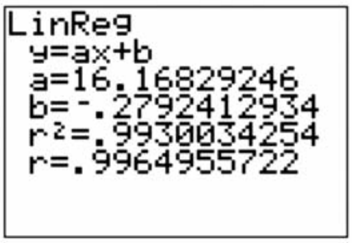
The students pointed out that using graphical representation enabled them to interpolate data by approximating a line of best fit. For instance, from the line they found that one could see 30.4 centimetres if positioned 2.2 metres from the wall. Conversely, how far away from the wall can one be in order to see 40 centimetres on the wall? The answer is 2.6 metres (Figure 2). Since the data of this experiment for x ranges from 0.3 to 5.18 and for y from 7.62 to 86.28, students also talked about extrapolation. They considered how much one could see, using the line of best fit, when 5.4 metres away from the wall. Conversely, how far away from the wall can one be to see 93 centimetres on the wall (Figure 2)?

Algebraic representation

Since the students simply approximated the best-fit line (as shown in Figure 2), they wanted a calculated regression model. Some students wanted to find the slope of the line and then determine the equation but others wanted to use the TI-84 graphing calculator. They noted that the TI-84 would calculate a line of best fit; thus, they would not have to find it from their by-hand plot of the data (Figure 2). Since all agreed that it was a linear relationship, they were eager to know what linear model might fit the data. They entered the data into the lists of the TI-84 in order to further investigate this relationship (Figure 3) through their use of the linear regression feature of the graphing calculator.



(a)



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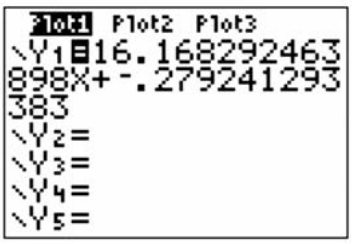

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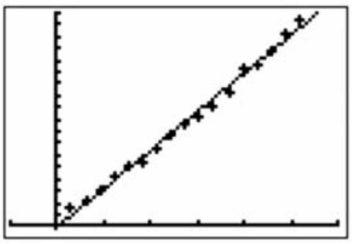

(d)

Figure 3. Plot of the data, linear regression analysis output, and function graph of regression function with plot

Each group selected a group leader in discussing their investigation results with the rest of the class. Jennifer, one of the group leaders, presented the following:

> We first used the graph paper and noticed that the data appeared to be consistent with a linear relationship. We then decided to enter data into the TI-84 and plotted the scatter plot [shown in Figure 3], which appeared to be consistent with a linear relationship. We calculated the linear regression $y = ax + b$ (a line of best fit) and determined the values $a = 16.168\ldots$, $b = -0.279\ldots$ So, the linear regression is $y = 16.168x - 0.279$ [Figure 3c] with $r = 0.996$ [Figure 3b]. Since r is very high, the linear regression model is the best fit [Figure 3d]. Note that r is a measure of the strength of a linear relationship.

The class agreed that the line of best fit can be used for interpolation and extrapolation of data. Students noted that the line of best fit ($y = ax + b$) has slope a and y intercept b. The students started to make sense of the line of best fit ($y = 16.168x - 0.279$) and guided the discussion.

Teacher: What will happen when $x = 0$, and what does it mean when $x = 0$? What about when $y = 0$?

Student: When $x = 0$, $y = -0.279$, and to me this does not make any sense. How can y be negative, and how can one see negative distance?

This discussion generated a lot of questions but did not lead to useful interpretations about what certain values in the regression line meant. I reminded the students that the line of best fit was just the best model and cannot be seen as a real life scenario. As the students were not able to relate the coefficient of the regression line in relation to the experiment, I introduced geometrical analysis to clarify some ambiguities as discussed below.

Geometric representation

I asked students to model the experiment using *Geometer's Sketchpad* (GSP) software (Jackiw, 2001). Some students were very creative: one student came up with the model shown in Figure 4 where $CDEF$ is the view tube and HI is the viewable portion of the wall.

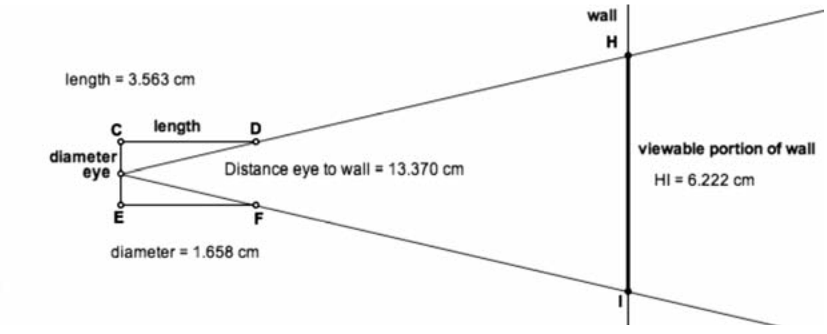

Figure 4 Modelling of view tube length and viewable vertical distance.

In this model (Figure 4), while students were tabulating the data to do an analysis similar to what was done with the paper-and-pencil drawing and graphing calculator, they concentrated on the dynamic and functional relationship between the perpendicular distance from the eyeball to the wall and the viewable vertical distance on the wall. They clicked and dragged the point representing the eye to simulate moving to and from the wall in order to see how the varying distance from the eye to the wall changes the viewable portion of the wall. In this process, the students shared their models among themselves, made some adjustments.

Discussion of model

As noted, $DCC'D'$ is the view tube. E and I are midpoints of $\overline{DD'}$ and $\overline{CC'}$. A is a point on the origin and B is the point on the wall. The point (EB, $m\overline{FF'}$) traced gives a line. The vertical distance $\overline{FF'}$ is a linear function of independent variable $\overline{\text{EB}}$. What is the equation of the traced line? What is its slope and y-intercept? In Figure 5, note triangle ECC' and EFF'. By the angle–angle property of similarity of triangle, the two triangles are similar. With that we can state that

$$\frac{FF'}{CC'} = \frac{EB}{EI}$$
$$\Rightarrow \frac{y}{d} = \frac{\overline{EI} + \overline{IB}}{EI}$$
$$\Rightarrow \frac{y}{d} = \frac{x}{l}$$
$$\Rightarrow y = \frac{dx}{l}, \quad x \geq l$$

where l represents the length of the tube, x represents the distance from the end of the tube to the wall, d represents the diameter of the tube, and y represents the viewable vertical distance.

Note that d and l are constants because the diameter and length of the view tube are kept constant.

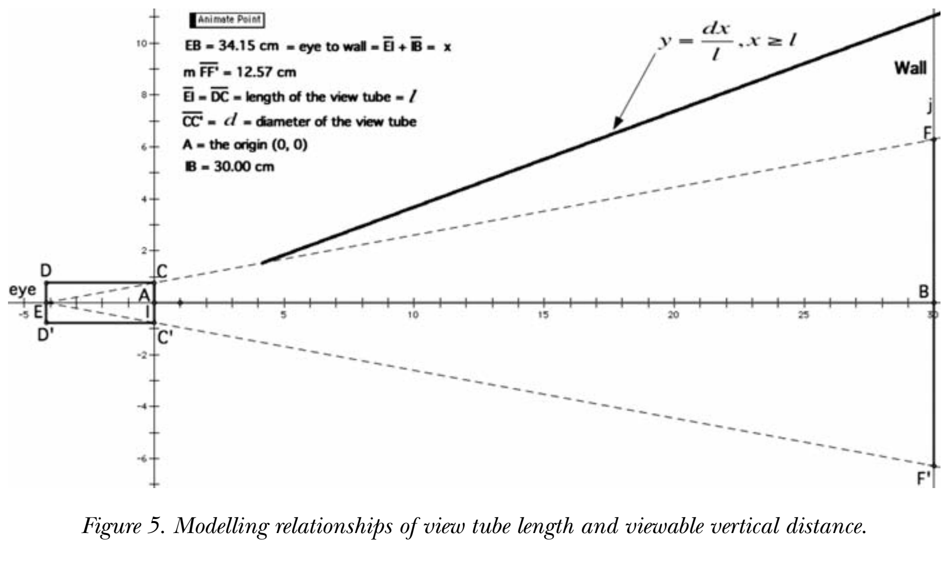

Figure 5. Modelling relationships of view tube length and viewable vertical distance.

The linear relationship

$$y = \frac{dx}{l}$$

stimulated a lot of discussions among students. One student asked, "What will happen when $x \rightarrow l$?" The class responded that $y \rightarrow d$; which means that one can only see a distance the size of the diameter of the view tube when $x \rightarrow l$. When x is l, it means that the end of the view tube is touching the wall, and one can only see the vertical distance of the size of the diameter of the view tube. For the case of $y = 0$, $x = 0$ did not make any sense since $x \geq l$. Students also noted that such a scenario could not exist since the minimal vertical distance viewable on the wall is d (diameter of the view tube). This made sense to students as opposed to what they discussed with the linear regression equation.

The students noted that for $y = \frac{dx}{l}$, the y-intercept is 0 and $\frac{d}{l}$ the slope of the linear relationship equation.

One thing the students liked about this equation was that it was expressed in terms of the diameter of the view tube (d) and the length of the tube l. The students noted that l and d cannot be 0, otherwise there will be no view tube. Also, the smaller that l is, the steeper the slope is and the bigger d is; likewise, the steeper the slope is and *vice versa*. They noted that this model could be used with a view tube of different dimensions, and found the desired distance from the wall or viewable vertical distance. Based on this discussion, the students were given a graph of two curves as homework in order to discuss how the dimensions of the two tubes may be different (Figure 6).

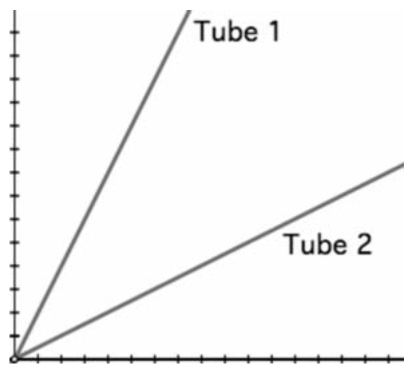

Figure 6. Distance from eye to wall (x) versus viewable vertical distance (y).

The students had a brief discussion relating to the statistical best line to the GSP model. Since the statistical model is $y = 16.168x - 0.279$, and the GSP model is

$$y = \frac{dx}{l}$$

they asked what would happen if they substituted the values of $l = 26.67$ centimetres and $d = 4.064$ centimetres in the GSP model (note that $l = 26.67$ centimetres and $d = 4.064$ centimetres are the dimensions of the view tube used in the investigation). Thus, would they get the same equation as the statistical model? The students pointed out that for the statistical model to be compared to the GSP model, the equation

$$y = \frac{dx}{l}$$

becomes

$$y = \frac{100dx}{l}$$

since the units in y are in centimetres and in x are in metres in the statistical model (Figure 2). Therefore using the dimension of the view tube, the GSP model

$$y = \frac{100dx}{l}$$

becomes

$$y = \frac{100 \times 4.064x}{26.67} = 15.238x$$

They compared the two models and noted that the statistical model ($y = 16.168x - 0.279$) and the GSP model ($y = 15.238x$) were very close but not the same. The statistical model had a slope of 16.168 and y-intercept of –0.279, whereas the GSP had a slope of 15.238 and y-intercept of 0. The students noted that this difference might be due to an error resulting from measuring the distance from the eyeball to the wall (x) and the viewable vertical distance on the wall (y), both of which were used to find the statistical model.

Conclusion

This view tube activity demonstrates why investigation is very important. It was evident that the activity helped students to develop their problem solving skills. The task gave the students a chance to collect data, and from the data to determine what kind of relationship exists. They solved the problem by conjecturing what the relationship might be and testing their conjecture. The students employed different strategies, first by graphing the data on a graph paper, and then using it to test the conjecture. To refine their solutions, the students also used a TI-84 graphing calculator and *Geometer's Sketchpad* (GSP) to investigate more about the relationship in which they learned that "there is more than one way to approach or solve each problem" (Greenes, 1996, p. 38).

Secondly, the activity gave the students the opportunity to work together cooperatively. They shared roles and they indicated that they enjoyed working together and motivated each other in the process. As one student noted, "It was helpful to have others helping you and to help each other throughout the whole experiment. It is better to view and compare results from others."

Thirdly, the activity promotes conceptual understanding. The investigation of relationships between the variables by using the TI-84 graphing calculator and the GSP provided the students with the opportunity to understand mathematical concepts (slope, intercepts, interpolation, and extrapolation) and relationships. Using GSP allowed students to not only see the relations between the dependent variable (y) and independent variable (x), but also expressed y in terms of the length of the tube (l) and its diameter (d):

$$y = \frac{100dx}{l}$$

Fourthly, the activity provided students with the opportunity to relate to real world experiences. As one student noted:

> I like "hands on" activities and being involved in the learning process, making conjectures/predictions, and testing them. This activity was a real world experience that I don't get in many of my classes. The experiment makes sense and showed me that geometry is everywhere. I'm sure that I will adopt this experiment when I start teaching my own students.

The modelling of the activity using GSP provided real world experience. By building on the concept of similar figures, students could use the activity to find the height of objects (buildings, tress, poles, etc.). Mathematical modelling provides the opportunity for students to relate to real world experiences. It should be noted that the relationship

$$\frac{FF'}{CC'} = \frac{EB}{EI}$$

established from Figure 5 brings forth the issue of proportional reasoning and application to real world situations. Using this relationship, the application of the view tube can be used to find the height of a building, person or any object. Teaching investigation should be emphasised at all grade levels, more so in pre-service mathematics courses.

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