Graphical solution of polynomial equations

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Graphing functions

Graphing utilities, such as the ubiquitous graphing calculator, are often used in finding the approximate real roots of polynomial equations. The polynomial function whose zeros are to be found is simply graphed in the *graphing window* in the prescribed way; the points where the graph of the function crosses the x-axis are then the *real zeros* (or real roots) of the polynomial equation. The location of the roots can be found very accurately using the *zoom-in* capabilities of the device.

To illustrate, let us consider the "irreducible" quintic (5th degree) equation: $2x^5 - 10x + 5 = 0$. This is in one of several "reduced" forms of a quintic equation; the fact that the of the coefficients are real, however, is fortuitous and cannot be guaranteed by a "reduction" (Burnside & Panton, 2005, p. 181). The first step is to graph the *function*

$$f(x) = 2x^5 - 10x + 5 \tag{1}$$

The graphing utility produces the graph shown in Figure 1, which indicates three real roots (i.e., when $y = 0$) at approximately $x \in \{-1.6, 0.5, 1.3\}$. Note, that we find all five roots, to greater accuracy, later in this paper.

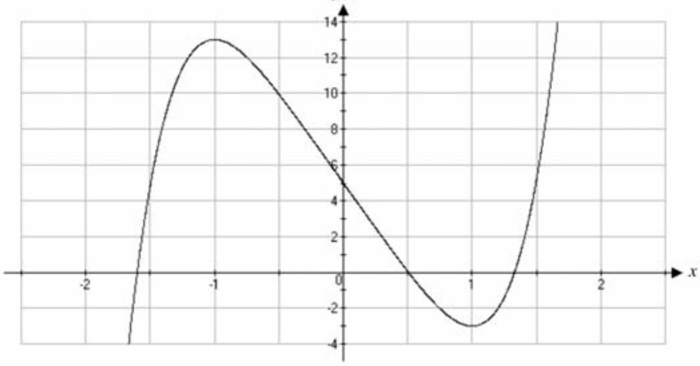

Figure 1. Graph of $f(x) = 2x^5 - 10x + 5$.

Depending on the type of the graphing utility used, a high degree of accuracy can be obtained for each root (by zooming in) if desired. To solve the quintic equation there are now only two more roots to identify; these—we know in advance—are complex conjugates of each other, as the coefficients of the original equation are all real, and we are well on our way to finding all five roots of the polynomial equation.

In this graphing procedure, two difficulties immediately suggest themselves.

1. If the quintic equation had only one real root—one that we had perhaps determined graphically as above—to find the remaining roots we would then be faced with the task of finding the four roots of a quartic, none of them real.
2. If any of the coefficients of the original equation had been complex, no part of this graphing technique could have been used.

In this paper we offer a very simple graphing technique that allows one to find all solutions of a polynomial equation of any degree:

- all roots, real and complex, are found;
- all coefficients, real and complex, are allowed.

The graphing tool, however, must have the ability to graph relations directly, without being forced to resort to the strenuous expedient of solving the equation for one variable as a function (or a set of functions) of others. (Two commercially available graphing utilities, capable of graphing relations directly, are *Advanced Grapher*—used exclusively in this paper, see www.alentum.com/agrapher/—and *Graphing Calculator*—not used in this paper, see www.pacifict.com/Products.html.) Thus, for example, the graphing tool must be able to graph the equation $x^2 + y^2 = 1$ directly, without having to express y in terms of two functions of x:

$$f_1(x) = \sqrt{1-x^2} \quad \text{and} \quad f_2(x) = -\sqrt{1-x^2}$$

Several commercially available graphing tools have the ability to do this.

In the next sections, we explain the technique, illustrating by solving a quadratic equation with complex coefficients and with no real roots. Finally, we illustrate the technique by solving a variety of higher-degree polynomial equations, with real and complex coefficients, and with real and complex roots.

Graphing relations

For purposes of the present study, we introduce a simple notational device: all polynomial equations are henceforth to be written in terms of the variable z. It is understood that z is a complex number, and that it can be separated into real and imaginary parts: $z = x + iy$, where x and y are both real. Thus, the general quadratic equation in standard form from now on will be rendered as $az^2 + bz + c = 0$ where the coefficients a, b and c are complex in the general case.

As our first example, let the equation to be solved be

$$z^2 + (-1 + i)z - 5i = 0 \tag{2}$$

Of course, this cannot be graphed by the method discussed previously in this paper. We know that both solutions will be complex, and that the solutions will not be complex conjugates of each other (because the coefficients are not all real).

We begin by substituting $z = x + iy$ into $z^2 + (-1 + i)z - 5i = 0$:

$$(x + iy)^2 + (-1 + i)(x + iy) - 5i = 0 \tag{3}$$

which yields

$$x^2 + 2ixy - y^2 - x - iy + ix - y - 5i = 0 \tag{4}$$

Now, separating real and imaginary parts (remembering that x and y are both real):

$$x^2 - y^2 - x - y + i(2xy - y + x - 5) = 0 \tag{5}$$

Equating the real and imaginary parts separately to 0, we obtain two equations:

$$x^2 - y^2 - x - y = 0 \qquad \text{[real part]} \tag{6}$$
$$2xy - y + x - 5 = 0 \qquad \text{[imaginary part]} \tag{7}$$

Graphs of these two equations are shown in Figure 2 and Figure 3.

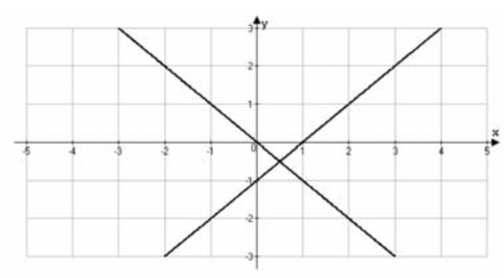

Figure 2. Graph of equation $x^2 - y^2 - x - y = 0$.

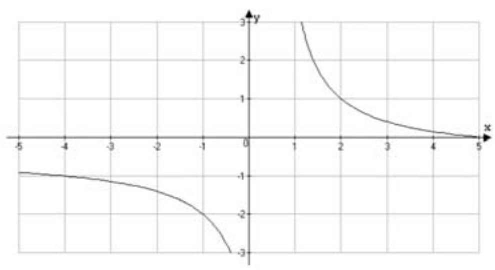

Figure 3. Graph of equation $2xy - y + x - 5 = 0$.

Note how the graphing tool automatically graphs both branches of the relations $x^2 - y^2 - x - y = 0$ and $2xy - y + x - 5 = 0$. To make sense of the graph of $x^2 - y^2 - x - y = 0$, it might be helpful to realise that this can factorised into $(x + y)(x - y - 1)$. Viewing the graphs shown in Figure 2 and Figure 3 simultaneously in the same viewing window gives the graphs shown in Figure 4.

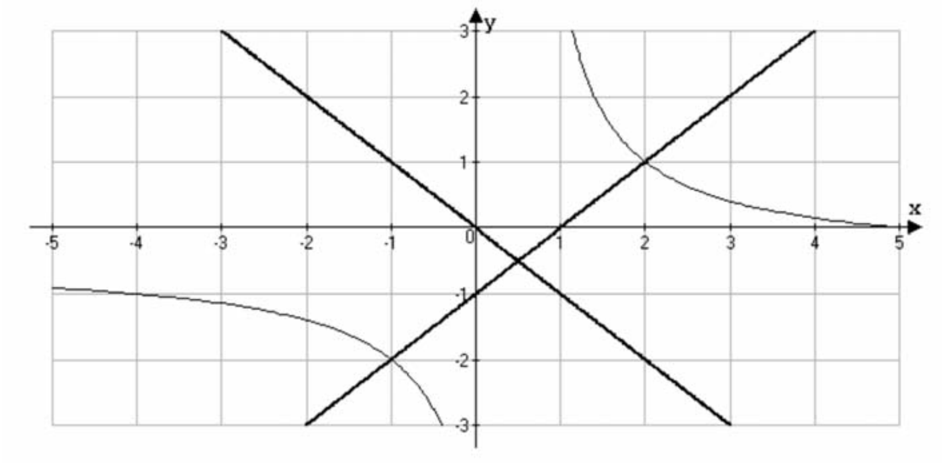

Figure 4. Graphs of the equations [6] and [7].

As in all graphs in this paper, the heavier (thicker) line represents the graph of the relation issuing from the real part of the equation; the lighter (thinner) line represents the imaginary part.

The two roots of $z^2 + (-1 + i)z - 5i = 0$ [2] are $z \in \{2 + i, -1 - 2i\}$. They occur *at the intersections* of the curves representing the two relations $x^2 - y^2 - x - y = 0$ [6] and $2xy - y + x - 5 = 0$ [7]; that is, they occur at the intersection of the graphs of the real part of the relation with the graph of the imaginary part of the relation.

We propose to call Figure 4 the *Argand image* of equation (2): $z^2 + (-1 + i)z - 5i = 0$. A two-dimensional graph that plots the real part of a complex number on the x-axis and the imaginary part on the y-axis is commonly called an *Argand diagram.*

Argand image of selected polynomial equations

Case 1: A cubic equation with no real roots

$$z^3 - iz^2 - 2iz - 2 = 0 \tag{8}$$

Replacing z with $x + iy$ gives

$$(x + iy)^3 - i(x + iy)^2 - 2i(x + iy) - 2 = 0 \tag{9}$$

Performing the elementary algebra, and equating real and imaginary parts separately to zero, we obtain two relations between x and y:

$$x^3 - 3xy^2 + 2xy + 2y - 2 = 0 \tag{10}$$
$$3x^2y - y^3 - x^2 + y^2 - 2x = 0 \tag{11}$$

Graphs of these are shown in Figure 5.

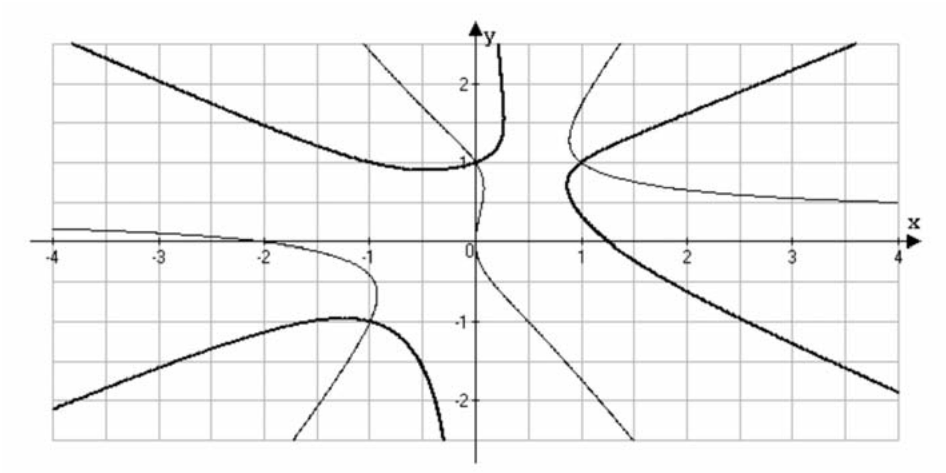

Figure 5. Graphs of the equations $x^3 - 3xy^2 + 2xy + 2y - 2 = 0$ [10]
and $3x^2y - y^3 - x^2 + y^2 - 2x = 0$ [11].

The three roots at the intersection points of the relations $x^3 - 3xy^2 + 2xy + 2y - 2$ (10) and $3x^2y - y^3 - x^2 + y^2 - 2x = 0$ (11) are $z \in \{-1 - i, i, 1 + i\}$.

Case 2: A quartic equation with two real roots and one imaginary root of multiplicity 2

$$z^4 - 2iz^3 - 2z^2 + 2iz + 1 = 0 \tag{12}$$

Note that $z^4 - 2iz^3 - 2z^2 + 2iz + 1 = 0$ can be factorised into $(z-1)(z+1)(z-i)^2 = 0$. Replacing z with $x + iy$ leads, as above, to two relations to be graphed:

$$x^4 - 6x^2y^2 + y^4 + 6x^2y - 2y^3 - 2x^2 + 2y^2 - 2y + 1 = 0 \tag{13}$$
$$4x^3y - 4xy^3 - 2x^3 + 6xy^2 - 4xy + 2x = 0 \tag{14}$$

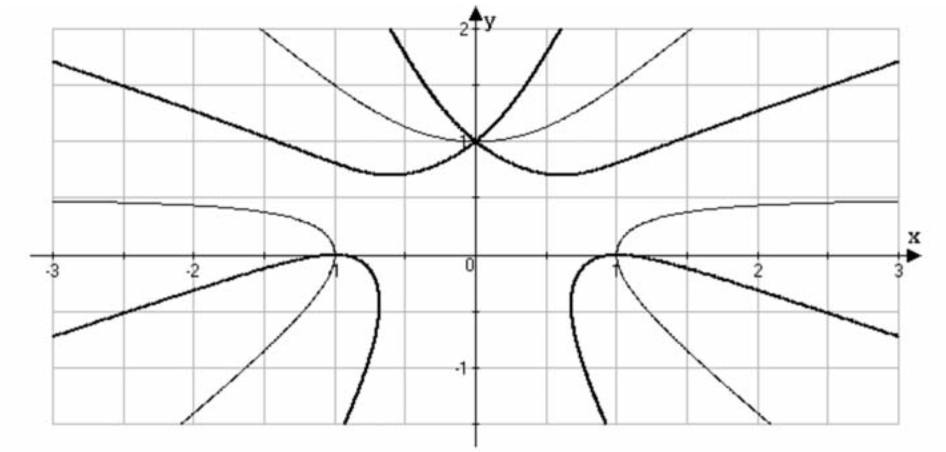

Figure 6. Graphs of $x^4 - 6x^2y^2 + y^4 + 6x^2y - 2y^3 - 2x^2 + 2y^2 - 2y + 1 = 0$ [13]
and $4x^3y - 4xy^3 - 2x^3 + 6xy^2 - 4xy + 2x = 0$ [14].

The roots of $z^4 - 2iz^3 - 2z^2 + 2iz + 1 = 0$ (12) are $z \in \{-1, i, i, 1\}$. Note that the double root occurs at the confluence of four curves (one is a straight line along the y-axis, and is not very discernible in the given graph, Figure 6).

Case 3: A quintic equation with three real roots —the Argand image of equation $2x^5 - 10x + 5 = 0$

Here we return to reconsider our original function.

$$2z^5 - 10z + 5 = 0 \qquad (15)$$

Replacing z with $x + iy$ produces two relations:

$$2x^5 - 20x^3y^2 + 10xy^4 - 10x + 5 = 0 \qquad (16)$$
$$10x^4y - 10x^2y^3 + y^5 - 10y = 0 \qquad (17)$$

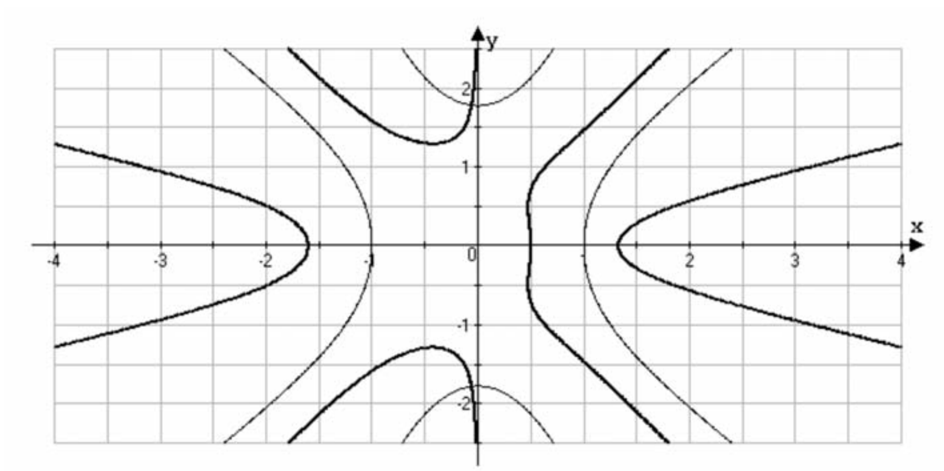

Figure 7. Graphs of $2x^5 - 20x^3y^2 + 10xy^4 - 10x + 5 = 0$ [16] *and* $10x^4y - 10x^2y^3 + y^5 - 10y = 0$ [17].

The approximate roots of $2z^5 - 10z + 5 = 0$ (15) are (to 2 decimal places): $z \in \{-1.60, -0.06 + 1.78i, -0.06 - 1.78i, 0.51, 1.33\}$.

At least five-decimal place accuracy can be found by zooming in on the intersection points of the graphed relations; here we supply this accuracy for the two complex roots: $z \approx -0.05508 \pm 1.78255i$.

Note that, of course, *all real roots are on the x-axis*, that is, the line $y = 0$. This is a branch of the relation (17). Pictorially, this curve is being masked here by the x-axis.

Case 4: Quintic equation with one real root

Here is an equation that has been analysed online at http://documents.wolfram.com/v4/MainBook/3.9.5.html:

$$z^5 + 7z + 1 = 0 \tag{18}$$

Replacing z with $x + iy$, and (as before) equating real and imaginary parts separately to 0, leads to

$$x^5 - 10x^3y^2 + 5xy^4 + 7x + 1 = 0 \tag{19}$$
$$5x^4 - 10x^2y^3 + y^5 + 7y = 0 \tag{20}$$

Graphs of these relations are shown in Figure 8.

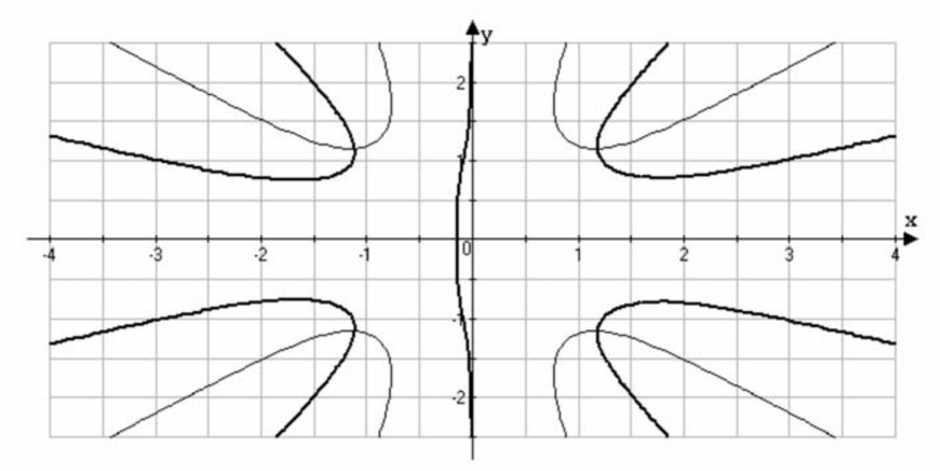

Figure 8. Graphs of $x^5 - 10x^3y^2 + 5xy^4 + 7x + 1 = 0$ (19) *and* $5x^4 - 10x^2y^3 + y^5 + 7y = 0$ (20).

Hence, the five roots of $z^5 + 7z + 1 = 0$ are at approximately $z \in \{1.1 + 1.1i,\ 1.1 - 1.1i,\ -0.14,\ 1.2 + 1.2i,\ 1.2 - 1.2i\}$.

We provide a five decimal place accuracy to the single real root of $z^5 + 7z + 1 = 0$ [18]: $z \approx -0.14285$.

Conclusion

In this paper we offer a simple technique for solving graphically any given polynomial equation

- of arbitrary degree;
- with real or complex coefficients;
- possessing both real and complex roots.

The technique uses a graphing tool endowed with the ability to graph relations directly, without resorting to the decomposition of these relations into sets of functions. The graph of a polynomial equation consists of curves in the complex plane, producing an Argand Image of the equation. Solutions of equations occur always at the intersections of curves representing the relations.

The method is simple enough to be introduced to mathematics students in secondary schools. It seems likely that the possession of an *Argand image* of an equation may help in the conceptual understanding of what a solution of a polynomial equation really is, why there is more than one solution, and indeed why there are exactly n solutions in any nth-degree equation. It seems possible that the Argand image may point the way to a more surveyable (or visualisable) form of a very abstract theory: the Galois Theory (see, for example, Littlewood, 1965, p. 202) of algebraic equations. Moreover, the "unceremonious" use here of the Argand diagram may well pave the way, in future, to a more informal approach to the mystery and mysticism of "imaginary" and/or complex numbers.

References

Burnside, W. S. & Panton, A. W. (2005). *The theory of equations.* Mineola, NY: Dover Publications.

Littlewood, D. E. (2002). *A university algebra.* London: Heinemann.