The zero-at-the-end problem

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A problem given in the Australian Mathematics Competition for the Westpac Awards was stated as follows:

With how many zeros does 2008! end?

In this article, we will solve this problem, and provide further discussion on the related problems.

Getting started

To solve this problem, we rephrase the question as follows:

Given a positive integer k, how many zeros are at the end of the factorial k value?

This is the same as finding the exponent of the biggest power of 10 that can divide $k!$. The next step is the key. Instead of the biggest power of 10, we use the biggest power of 5. So why can we make such a big transition? Those who have seen this know that it is because $10 = 2 \times 5$ and there are more twos dividing $k!$ than fives. Therefore, the power of 5 leads us to the solution. In fact, there exists a nice formula (see Halmos, 1991, pp. 171–173) for calculating the exponent of this power, which is

$$\left[\frac{k}{5}\right] + \left[\frac{k}{5^2}\right] + \left[\frac{k}{5^3}\right] + \ \ldots \qquad (1)$$

where the notation [ ] stands for the greatest integer function. In this formula, the first term $[k/5]$ gives us the number of the multiples of 5 between the integers 1 and k. Then we add $[k/5^2]$ to it because the multiples of 5^2, which are also the multiples of 5, have to be counted one more time. Similarly, the multiples of 5^3 are also the multiples of 5^2 so they have to be counted again, and so on. This is how the entire formula is built up. Hence, to solve our posed problem, we set $k = 2008$ in Formula (1). Therefore, there are

$$\left[\frac{2008}{5}\right] + \left[\frac{2008}{5^2}\right] + \left[\frac{2008}{5^3}\right] + \left[\frac{2008}{5^4}\right] = 401 + 80 + 16 + 3 = 500$$

zeros at the end of 2008!.

An application of the formula

We have solved our original problem, but is there more to learn? One application of the formula (1) is finding the prime factorisation of a "big" factorial. According to the *Fundamental Theorem of Arithmetic* (Rosen, 2000), every positive integer greater than 1 can be written as a product of powers of distinct primes. For example: what is the prime factorisation of 180? Since

$$180 = 18 \times 10 = (3 \times 6) \times (2 \times 5) = 3 \times (2 \times 3) \times 2 \times 5$$

the prime factorisation of 180 is $2^2 \times 3^2 \times 5$ after we collect the common factors. Next, what is the prime factorisation of 7!? By the similar calculation, we have

$$\begin{aligned} 7! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 && \text{by definition} \\ &= 2 \times 3 \times (2 \times 2) \times 5 \times (2 \times 3) \times 7 && \text{since 1 is not prime} \\ &= 2^4 \times 3^2 \times 5 \times 7 \end{aligned}$$

However, how do we find the prime factorisation of a bigger factorial, say 14!? Of course, we can use the above method to obtain the answer, but is there a more effective way to find it? Note that the only primes appearing in 14! are the ones less than 14. So the main question is: *How do we find the exact power of each prime?* The only clue here is Formula (1) which gives us the exponent of the power of 5 in the prime factorisation of k!. So, by similar argument, we can use other primes instead of 5 in this formula. In other words, the sum

$$\left[\frac{k}{p}\right] + \left[\frac{k}{p^2}\right] + \left[\frac{k}{p^3}\right] + \ldots \qquad (2)$$

is the exponent of the power of a prime p in the prime factorisation of k!. For example, $[14/2] + [14/2^2] + [14/2^3] = 11$ means that the power 2^{11} exactly divides 14!. To figure out the powers of other primes, we set $p = 3, 5, 7, 11,$ and 13 in Formula (2), respectively. Therefore, the prime factorisation of 14! is $2^{11} \times 3^5 \times 5^2 \times 7^2 \times 11 \times 13$ and this is how we find the prime factorisation of a "big" factorial.

A generalisation

A further related question is the following:

> Given a positive integer *k*, how many zeros are at the end of *k*! to a different base?

For instance, how do we find the number of zeros at the end of 14! in base 504? According to the fact that $504 = 2^3 \times 3^2 \times 7$ we need one 2^3 one 3^2 and one 7 multiplied together to get one zero in base 504. So how many 2^3s, 3^2s and sevens are there in 14!? Here we need to use the prime factorization of 14!, which is $2^{11} \times 3^5 \times 5^2 \times 7^2 \times 11 \times 13$ shown in the previous section. Since there are eleven twos in 14!, the number of 2^3s in 14! is $[11/3]$. Similarly, the numbers of 3^2s and sevens in 14! are $[5/2]$ and $[2/1]$ respectively. After we obtain the numbers of 2^3s, 3^2s and sevens, we are able to calculate the number of 504s in 14!. With the calculation $\min\{[11/3],[5/2],[2/1]\} = \min\{3,2,2\} = 2$, we have two 504s. Therefore, there are two zeros at the end of 14! in base 504.

These problems form a good model that helps students develop a logical thinking process toward problem solving when they encounter a mathematical problem that they have never seen before. Moreover, the best part is that students can understand these problems and their solution methods without any advanced mathematics.

References

Halmos, P. R. (1991). *Problems for mathematics young and old.* Washington, DC: Mathematical Association of America.

Rosen, K. H. (2000). *Elementary number theory and its application* (4th ed.). New York: Addison-Wesley.