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Geometry and Measurement: A Discussion of Status and Content Options for Elementary School Students with Learning Disabilities

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The purpose of this paper is to present a summary of selected facets of geometry and measurement in elementary school programs and to describe curricula content options designed to demonstrate the feasibility of seeking high level outcomes and meanings for students with learning disabilities. While there are a multitude of published papers relative to arithmetic, the literature specific to geometry and measurement is scant indeed. The illustrations of the area of the circle and the volume of cubes, pyramids, cylinders, and cones are presented to encourage researchers and teachers alike to seek higher-order outcomes for geometry and measurement and avoid the rote drill and practice that often dominates arithmetic instruction at the elementary grade levels.

Key Words: Elementary School, Special Education, Learning Disabilities, Geometry, Measurement, Higher-Level Outcomes, Circles

The focus of this paper is geometry and measurement for elementary school students who, after comprehensive assessments, and consultation with parents and professionals, are identified as having learning disabilities that require special education services. This paper is based on the proposition that programs in mathematics have two primary purposes. First, programs assure students *a wide range of opportunities to know about the many meanings and principles of mathematics* and to *develop proficiency with a variety of ways of doing and representing mathematics*. Second, programs utilize *mathematics activities to enhance students' performance in areas such as language comprehension, social-personal development, cognitive growth, and all other limitations that are often used to describe students with learning disabilities* (Cawley, Hayes & Foley, 2008).

Geometry and measurement are considered essential and meaningful mathematics in the lives of individuals regardless of age. They are interrelated to the extent that one generally includes implications from the other. Yet, one may never realize the importance of this from an examination of the activities in many school mathematics curricula dominated by a "metric" or "Euclidean" focus on shapes and the rules that pertain to shapes. The present framework is consistent with the distinction between an approach to mathematics rooted in an academic perspective versus one rooted in a societal view. The academic perspective views mathematics

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for its own sake, while the societal perspective views mathematics in relation to its effect upon individuals and society.

There are three fundamental reasons for stressing geometry and measurement in school programs for students with learning disabilities. First, geometry and measurement are significant mathematics in everyday life, and they are operational in daily life before topics in number. Second, they offer innumerable opportunities to enhance cognitive performance, communication processes, and language comprehension. Third, geometry and measurement are social mathematics that guide interactions and behavior. Fox (2000) describes three levels of reasoning in geometry. These are hierarchically sequenced as reasoning by (a) resemblance, (b) attributes and (c) properties. Resemblance involves designating something because it looks like the standard; designation by attribute means to use informal language to specify selected characteristics; and, designation by properties involves stating the relationships between or among properties such as knowing that a square is also a parallelogram because it is a quadrilateral with both pairs of opposite sides being parallel. Thus, knowing is knowing beyond facts or definitions; knowing involves comprehension of the relationships between and among items that are factual. Doing is more than the rote completion of a task. Doing, for example, when computing the area of a circle involves knowing that one is working within a region defined by the relationship between the diameter and the circumference. Doing also involves the use of more than one procedure (e.g., different formulas) to complete a task (e.g., to compute the area of a triangle). The key to making *doing* and *knowing* meaningful is to make a distinction between *episodic* and *semantic* learning activities. The former engages the student in real or contrived activities in which there is a combination of teacher guidance and active student participation in problem solving and inquiry-based tasks. This includes activities of a manipulative, pictorial, spoken, or written format. The latter minimizes student participation and places the teacher in a dominate directive mode.

Teacher Education and Elementary School Textbooks and Programs

Teacher education programs comprise one of the many variables that influence what and how topics are presented to students with learning disabilities. Fusco (1993) analyzed 214 course syllabi on the general topic of "methods for teaching students with learning disabilities" from a variety of university based special education programs. The syllabi were analyzed on many variables including the number of class sessions devoted to a topic and the number of class goals and objectives set for the topics in the course. The outcomes of this study showed a degree of inconsistency in the syllabi themselves. For example, a syllabus would state course goals and objectives for a given topic, but not cover the goals and objectives for the topic in class sessions. That aside, the analysis showed about 15 percent of class sessions were scheduled to address mathematics in contrast to over 50 percent of the class sessions scheduled to address language arts. This shows a wide preference for the coverage of language arts in contrast to mathematics in the content presented to special education teachers. In an examination of over 120 syllabi for the course equivalent to "teaching mathematics to elementary school teachers," Paige (2008) found that less than one percent of the syllabi mentioned adaptations for students with special needs or learning disabilities. It appears that the courses in special education cover less mathematics than they do language arts and courses in elementary mathematics education do not include citations relative to students with special needs.

Teacher Education Textbooks

An analysis of the textbooks used in teacher preparation programs provides further insight. Tables 1 through 3 show the total number of pages contained in textbooks, the total number of pages devoted to geometry, and the total number of pages devoted to measurement in professional textbooks used in general education and special education teacher training programs. Table 1 shows the total number of pages in each textbook, the total number and percentage of pages devoted to geometry and measurement respectively in elementary teacher education mathematics textbooks. Tables 2 and 3 include information from textbooks used in special education teacher preparation courses that focus on general methods and mathematics education respectively. The total number of pages, the total number of pages devoted to mathematics, typically a single chapter, and of that, the total number of pages for geometry and measurement are indicated for each text. As one can readily glean from the tables, the emphasis given to mathematics and directed to geometry and measurement represents only a modest stress on these topics. Pre-service students and in-service teachers using these various resources are likely to consider topics such as arithmetic computation to be of more importance than geometry and measurement. As a general view, the texts in mathematics education (i.e., Tables 1 and 3) have a greater, although modest, component of geometry and measurement than the texts of special education (i.e., Table 2). The Buckingham (1953) text in particular has extensive coverage of both geometry and measurement while the Reisman (1977) text is the only source we reviewed that considered topology. Yet awareness of shape, properties of shapes, relationships between and among lines, patterns, and other geometry related concepts play at least as large a role in daily life in society as numbers and computation. Buckingham (1953) provides extensive discussion of the major topics of elementary school mathematics and is a suggested reading for all.

Table 1

General Education Mathematics Education Texts

Author	Pages in Text	Pages Related to Geometry		Pages Related to Measurement	
	N	N	%	N	%
Buckingham (1953)	741	111	15.0	168	22.5
Reisman (1977)	479	38	7.9	16	3.3
Sheffield & Criukshank (2000)	513	47	9.2	39	7.6
Van deWalle (1990)	451	34	7.5	23	5.1

Table 2
Special Education General Methods Texts

Author	Pages in Text	Pages Related to Mathematics		Pages Related to Geometry		Pages Related to Measurement	
	N	N	%	N	%	N	%
Inskeep (1926)	445	50	11.2	0	0.0	0	0.0
Mastropeiri & Scruggs (1987)	406	54	13.3	1	0.2	0	0.0
Mercer (1997)	676	48	7.1	0	0.0	0	0.0

Table 3
Special Education Mathematics Education Texts

Author	Pages in Text	Pages Related to Geometry		Pages Related to Measurement	
	N	N	%	N	%
Bley & Thornton (1989)	513	0	0.0	0	0.0
Cawley, Fitzmaurice-Hayes, & Shaw (1987)	245	27	11.0	19	7.8
Silbert, Carnine, & Stein (1990)	499	8	1.6	12	2.4

These discrepancies result in at least three challenges for teachers and learners: teachers tend to be less comfortable providing instruction in an area for which they have had little preparation; little preparation can result in an attitude that the content is less important and attention is paid to the areas in which greater preparation has been received; learners who might excel in geometry never have the chance to do so. Thus, teachers are given an implied message through their preparation that geometry and measurement is less important and this message is reinforced by the textbooks they rely on for their instruction.

Elementary School Textbooks

Table 4 is derived from the work of Chandler and Brosnan (1994) who identified the number of pages devoted to specific math topics in elementary school textbooks before and after the publication of the *Curriculum and Evaluation Standards for School Mathematics* by the National Council of Teachers of Mathematics (1989). In general, the elementary school textbooks were enlarged approximately 15 percent, yet sections related to geometry and measurement saw little if any increase.

Table 4

Textbook Pages Related to Geometry and Measurement Before and After 1989

Grade	Geometry				Measurement			
	Before		After		Before		After	
	N	%*	N	%*	N	%*	N	%*
1st	12	4	20	5	58	18	75	19
2nd	12	4	20	5	63	19	72	18
3rd	28	8	34	8	42	11	56	12
4th	39	11	48	11	34	9	46	10
5th	47	12	67	15	31	8	35	8
6th	52	13	68	14	30	6	24	6
7th	63	15	97	20	24	6	19	4
8th	80	19	113	23	23	5	14	3

*Percent of total pages in book

Table 5 outlines the content listing of *Project MATH* (Cawley, Fitzmaurice, Goodstein, Lepore, Sedlak, & Althaus, 1972; 1974). *Project MATH* was a comprehensive program designed to address the needs of students with broad developmental lags and/or specific learning disabilities. Overall, there were six content strands that included *sets, numbers*, *fractions*, *patterns*, *geometry,* and *measurement.* A general comparison to programs of elementary school mathematics indicates that the geometry and measurement strands of *Project MATH* represent a greater portion of the total lessons for each level than is customarily found in school curricula. *Project MATH* provided the teacher with a number of curricula formats. For example, the teacher could take a developmental approach and present the content in a spiral format such as a number of lessons in geometry followed by number, followed by sets, and so forth. The teacher might elect an intensified approach and group all geometry lessons together, all fractions together and so forth. Or, the teacher might organize by grouping lessons from different levels (e.g., group levels 3 and 4 together) to address a wider range of student performance levels.

Table 5

Instructional Guides Related to Geometry and Measurement for Project MATH

	Level 1		Level 2		Level 3		Level 4		Total	
Grade Level	K-1.5		1.5-3.0		3.0-4.5		4.5-6.0			
Strand	N	%*	N	%*	N	%*	N	%*	N	%*
Patterns	51	14	20	5	-	-	-	-	71	5
Geometry	61	17	87	22	70	19	40	16	258	19
Measurement	45	13	78	20	101	27	61	24	285	21

*Percent of total pages in the textbook.

Student Performance in Geometry and Measurement

There is a scarcity of data that describe the developmental characteristics of students with learning disabilities in geometry and measurement. A comprehensive assessment of the performance of students with learning disabilities is presented in table 6. These data (Cawley, Fitzmaurice, Shaw, Kahn, & Bates, 1979) are derived from the administration of three out of level subtests of the Mathematics Concept Inventory (MCI) (Cawley, Fitzmaurice, Goodstein, Lepore, Sedlak, & Althaus, 1972; 1974) to a sample of over 800 students with learning disabilities. The inventories were administered to students who were one age level above that for which each measure was designed. Thus, MCI-1, was designed for students at the K-1.5 level and was administered to students in the 1.5-3.0 grade range.

Table 6
Student Performance on Geometry and Measurement Assessments

	MCI-1 (N=89)	MCI-2 (N=354)	MCI-3 (N=340)
Geometry (Cawley et. al., 1972, 1974)	73.9%	73.2%	57.2%
Measurement (Cawley et. al., 1972, 1974)	73.3%	69.6%	71.4%
Ravens Progressive Matrices (Ravens, 1947)	29.8%	48.7%	54.2%
Graham-Kendall* (Graham & Kendall, 1960)	43.7%	17.7%	8.0%
Peabody Picture Vocabulary Test (IQ Score) (Dunn, 1956)	93.82	95.25	98.09
Peabody Picture Vocabulary Test (Mental Age Score) (Dunn, 1956)	7.75	10.44	11.0

*Low score represents higher performance

The table is read as the percent of items answered correctly. The data also include performance on the *Ravens Progressive Matrices Test* (1947) and the *Graham-Kendall Memory for Designs Test* (1960) as each of these involves relationships with geometry as they work with shapes (e.g., memory-for-designs) and patterns of shapes (e.g., matching a pattern to a schemata).

Performance on the MCI approximated seventy-five percent correct for each sample, acknowledging the out-of-level assessment in which students were administered the MCI one step below their grade level. Although these are not recent data, a series of subsequent developmental studies (e.g., Cawley, Parmar, Foley, Salmon & Roy, 2001; Cawley, Parmar, Yan, & Miller, 1996; 1998; Parmar & Cawley, 1994) show similar levels of discrepancy in the mathematics performance of students with learning disabilities.

A cross-age developmental investigation of spatial-geometric understandings by Grobecker and DeLisis (2000) examined the performance of 85 students with learning disabilities and 95 students without learning disabilities. Utilizing a geoboard as a foundation, three problems that involved (a) square enlargement, (b) diamond enlargement, and (c) transformation from a small diamond to a large square were administered to each student. The students worked with the geoboard and made drawings as well. None of the students with learning disabilities performed the geoboard tasks in the most efficient manner in that they moved all 4 pegs instead of just three when only three were necessary. This seems somewhat akin to a study by Cawley and Roy (1997) where a sample of middle school students were given 50 one-inch square tiles and asked to make the largest possible square (i.e., 7 x 7). All made a 7 x 7 square, but none considered the square to be intact and all made extensive efforts to insert the remaining tile. Grobecker and DiLisi (2000) also noted that their students with learning disabilities were less successful in both the imagery and drawing tasks.

Both Strauss and Lehtinen (1947) and Strauss and Kephart (1955) used the geoboard as an integral facet of their studies of the performance of students without disabilities, students with mild developmental disabilities, and students with brain-injury, the latter being somewhat of a precursor to the field of learning disabilities. The initial work included a study of performance wherein three figures (i.e., triangle, oval, and square) were created with tacks on a geoboard. A second set of the same figures was created using solid forms. The board was then hidden from the view of the students and the students were asked to feel the board and then draw a picture of the figure that was felt. On the second set, no difficulties or abnormalities were observed. Differences were noted for the students with brain-injuries and these were attributed to perceptual-motor difficulties existing in the visual and tactual fields. Strauss and Kephart (1955) describe three types of performance on the geoboard. The first as a global type of response in which the responses were constructed of continuous lines of marbles; a second type of response was described as disorganized wherein the students jumped from one side of the board to another; the third type of response was constructive as the students proceeded on the requested task in a normal and organized manner. The test performance of the brain-injured students was incoherent in contrast to the students with developmental disabilities was on the global performance. It would appear from these studies that an interpretation of the performance of the students is related to the theoretical model of the investigators.

Intervention Efforts

Our general approach to intervention has been to initially conduct developmental studies and to then, based on the data in these studies, design curricula and instructional options in geometry and measurement and implement these in a variety of classroom settings. *Developmental* studies provide information on student performance across ages or in comparison to other groups. Intervention efforts are conducted in three stages, namely i*mplementation*, *impact*, and *comparison*. Implementation includes the preparation of the intervention program and an examination of the extent to which it is incorporated into the classroom by teachers and experimenters. For example, assume a program is designed for a 15-week period. A primary implementation concern would be the extent to which the program was adequately covered in 15 weeks and whether it addressed the needs of all students during the period. A second stage is impact, where a determination is made of the effect of the program on the performance of the students. Typically this would involve pretests and posttests, an independent measure and a number of classroom observations. The third stage is comparison where there are both experimental and nonparticpating samples and one or more different treatment options (Cawley, 2002). Our projects have been concentrated mainly at the implementation and impact stages.

An implementation study (Cawley & Sedita, 1997) sought to determine the extent to which 46 middle school students from three home-based classrooms (i.e., severe behavior disorders, learning disabilities, and mild developmental disabilities) would participate and benefit from a 7-week geometry program. The focus of the program was the area of the circle. All students were pretested and post-tested on an author constructed measure of 44 items. The students were provided with an activity-centered program four days per week for forty minutes per day. Table 7 includes an example of pre- and post-test data results collected as part of the 7-week program.

Table 7
Mean Raw Pretest and Posttest Scores of a Sample of Students with Disabilities

Student	Pretest	Posttest
A	15	22
B	28	35
C	18	34
D	32	41

Table 8
Percent of Students Who Responded to Pretest and Posttest Items Correctly

Item	Topic	Pretest % Correct	Posttest % Correct
1	Line	5.9	58.8
2	Shape	100.0	100.0
3	Perimeter/ Circumference	41.2	100.0
4	Area	11.8	41.2
5	Volume	17.6	23.5

Implementation is the extent to which the program is utilized in the classroom and the extent to which students benefit from the program. The data in Tables 7 and 8 illustrate that all participants had a degree of understanding of geometry at pretest. In terms of the standards utilized, the program is considered to be successful when 80 percent of the students attain 100 percent correct at posttest or 100 percent of the students attain 80 percent or higher at posttest. With an overall pretest mean of 26.1 and a posttest mean of 33.2 across the 44 items, the data show an approximation of our standard. However, this is somewhat misleading as illustrated by the item data. What the item data show is that the students did not respond to the more advanced or complex items (i.e., items 3, 4, and 5) as well as those of a more basic quality (i.e., items 1 and 2). For example, item 4 asked students to explain why πr^2 and S^2 x 0.7854 both yield the area of the circle, whereas item 2 asked only that the student identify a rectangle in a multiple choice, four option task. What this indicates is that many of the items and concepts included in the program are known to the students at pretest, and need not be included in a revision. At the same time, work on the more difficult items needs to be examined and the program lengthened to provide for this.

Bierman and Gust (1994) conducted a seven week impact focused instructional program in geometry. The participants were 21 African-American students with a mean chronological age of 11.5 years. Two instructors provided instruction to the students. One taught and the other observed to insure fidelity. All activities were hands-on and covered two- and three-dimensional shapes, lines with open and closed shapes, points and symmetry, and angles and their measurement. Data included the geometry test of the *KeyMath Diagnostic Test* (Connolly, Nachtman, & Pritchett, 1988) and an author developed criterion-referenced test made up of three sections: *vocabulary*, *manipulative*, and *problem solving*. The raw pretest and posttest scores were 40.71 and 49.00 respectively for 60 vocabulary items, 23.48 and 24.24 respectively for 28 manipulation items, and 3.86 and 7.67 respectively for 15 problem solving items. *KeyMath Diagnostic Test* raw scores increased from 14.58 to 17.17 from one year to the next.

Cass and colleagues (2003) conducted a comparison study utilizing a multiple baseline design to examine perimeter and area problem solving among three students determined to have learning disabilities. Criteria were attained when each of the students scored 80 percent or higher on three consecutive days. The students used a geoboard to determine the perimeter or area of different shapes. None of the students solved perimeter or area problems during baseline, but all students reached criteria within a few days. Maintenance evaluations indicated the students maintained their levels of criteria. All students were capable of generalization and utilized their previous experiences on the generalization tasks, an indication that alternative representations of mathematical concepts are an acceptable instructional format.

Geometry and Measurement: The Primary Years (K-4)

Meaningful and higher-order understandings of geometry and measurement are gained by a stress on episodic learning activities in contrast to semantic activities. Relative to the aforementioned, we note this in that all students take up space in the classroom; all students locate points in space whether these points be a desk and chair or the location for a wheel chair; all students traverse paths in moving from one space to another; students generally pass from one room to another in lines which have order-constancy that places one student in front of, in back of, or between other students; and students sit at desks or tables and interact with both horizontal and vertical planes. Most importantly, all students do these things by following essentially the same principles of geometry and measurement. For example, let us examine 'betweenness.' Here the tasks for the teachers are to heighten the students' awareness of (a) the cognition of 'betweenness,' (b) the academic aspects of the topology of 'betweenness,' and the (c) social-personal interactions that take place as individuals assume a relationship of 'betweenness' with one another.

Everyday Geometry and Measurement

Assume it is morning and the household begins to stir. Some members may be awake and conscious of the time-event relationships that are taking place. If one has awoken early, the tendency may be to roll over and catch a few more minutes of sleep. If one has overslept and thinks of being late for work or school, the household may be very active. Suppose for the moment that the household is running behind schedule. Comments that might be heard include, "Hurry up in there. It's my turn! or "Go and eat your breakfast. You need to get dressed or you'll miss the bus!" or "Who left the box in the hallway, I almost fell over it?" or "I can't find my pants!"

A quick review of the above, calls attention to some important concerns with measurement, in this case, time-event relationships. Less evident, but of considerable importance are the geometries of the morning household. We begin with "It's my turn!" as an indication that only one thing can occupy a space at a time. If one is using the space of the bathroom, someone else has to wait until the space is available. The instructions to "Go and eat your breakfast," where breakfast is a time related phenomena, directs the individual to leave a space that is currently occupied, traverse a path to another space (i.e., the kitchen), and to locate a new point in space. Once at the new space, the individual must complete the task of eating, then leave the space, and locate another space to dress. There the individual dresses following a measurement sequence of ordered events such as putting on layers of clothing (i.e., under clothes first then pants, shirt, and sweater), while being aware that the school bus is coming. Some children complete the scenario independently, although with considerable prodding. Others lack independence due to physical or cognitive limitations and still others might not complete the scenario due to behavioral problems. In these instances, the students are assisted and a form of interaction takes place between the child and the helper. With regard to the box in the hallway that someone nearly fell over, an object occupied a space that was not a typical location for the object. The inference is that the object should be located in another point in space. With regard to the individual's inability to locate his or her pants, the pants are missing because they are not in the location expected. All of these instances, along with numerous others, highlight the influence that geometry and measurement play in our daily lives. Generally, neither families nor schools attend to these happenings within the context of measurement and geometry because they are so naturally occurring. It is the fact that they are so naturally occurring that is their beauty.

To ride a swing, the student sits "on" the seat, not "off" the seat. Children sit in "front" of the television set, not "behind" it. When taking a shower they stand "under" the shower, not atop the shower; they stand under the water, not beside it. If taking a bath or being bathed, they sit "in" the water, not "on" the water.

At a very young age, students locate points in space and they are alert to objects changing their position in space. This is evident with infants as they observe that someone is going to feed them and they follow the person or bottle as it moves along a path toward them. This is also evident with mature individuals who follow the stars and their changing positions in the night sky.

What would be the reaction in the teacher's room if a teacher of young children entered and said, "Today I had the kids locating points on a coordinate plane and we followed that with activities that focused on parallel planes and their importance in our everyday lives." Most listeners would likely respond with a look of amazement. What the teacher actually did was arrange the desks in the room by rows and columns to reflect a coordinate plane. The teacher then directed the students to move from one place to another (i.e., "Jason, go from your seat to Scott's seat.") where the student moves up or down the aisle and across the rows. As Jason moved up the aisle, he experienced parallelism in that he traversed a path that was parallel to other aisles. If he remained on this path, he would never reach Scott. Jason encountered perpendicularity when he reached the intersection of the aisle and the row where he had to make the turn to reach Scott. Here he had to make a directional move to go left or right.

Note that the preceding activity did not involve the traditional geometry of shapes or the naming, copying, or reproduction of shapes. Note that none of the measurement activities involved formal measures such as telling time, the length of a string, or the weight of an object. These geometries focus on the relationships that things have in space. These included relationships like inside-outside-on, open-closed, order-constancy, points in space, paths, and traversing paths to locate points and relationships among the horizontal and vertical planes of our environment. We also considered that only one thing can occupy a space at a time and we know from our daily lives (e.g., looking for a parking space, reading that two trains collided on a track, waiting to use the rest room at a ball park, etc.) how well that principle is a part of our way of life. Yet, seldom do we consider these relationships and the lack of awareness that students have of them. It seems that the geometry of school can be only shapes and that measurement can only be the determination of a standard unit. What about informal measures like, "It's hot in here." or "I feel hungry, it must be time to eat." or dealing with the child's question of "How far is it to Grandma's house?" and the fact that we generally respond in units of time (e.g. "It's only fifteen minutes longer.") rather than units of distance (e.g., "It's four more miles.").

How many times do we invoke activities with congruency and similarity when students are asked to retrieve or identify things "just like mine" or to find one "somewhat like mine." The former uses congruency in the principle of same size and same shape whereas the latter employs similarity with objects of the same shape, but not necessarily of the same size. Students may participate in the assembly of objects where the piece or parts that fit a specific space are the same size and same shape. Thus, the principle of congruence is integral to numerous activities of work and play. These tasks could be extended a bit to science where the teacher could stimulate attention to the properties of matter in terms of color, texture, rigidity, and so forth. Students can have experiences with motion and be shown objects from different perspectives as when they slide things from one end of the table to another or when the objects are flipped to another side.

Elementary: Shapes and Measurement

This section describes selected elements of shapes and measurement for the early grades. We begin by listing a few basic and higher-order outcomes for younger students. The list is not all-inclusive.

Basic Outcomes.

1. When given a representation of a circle, square, and triangle, the students can correctly affix a name or term to each.
2. When given a term (e.g., circle) the students can draw or make a representation.

3. The student can identify, illustrate, and label the major components of geometric shapes and regions (e.g., can indicate the perimeter, circumference of different shapes).

High Level Outcomes.

1. When give a set of shapes (e.g., circles, ovals, squares, and rectangles) the students can sort them into subsets of more than one dimension (e.g., shape, color, size) and state a justification for each sort.
2. When given a set of one inch tiles, the student can utilize the tiles to create a square region and count the exterior of each to compute the perimeter. The student can then use the tiles to make regular and irregular shapes of different perimeters and demonstrate the conservation of the original area.
3. The student can count the number of square units in a region to determine area.
4. The student can take tiles used to make a square and change the figure to another shape, but of the same area. This describes the principle of conservation of area.

GEOMETRY AND MEASUREMENT IN THE UPPER GRADES (5-8).

This section will focus on one topic, *area* and primarily the area of a circle. It is for purposes of illustration only and is used to demonstrate the mathematics with which students with learning disabilities can be engaged. The basis of this section is a variety of 7-to-15-week projects involving both general education and special education teachers and students with disabilities in their classrooms, along with a 15-week observation of a single special education teacher during geometry lessons. A few selected basic and higher-order outcomes are listed.

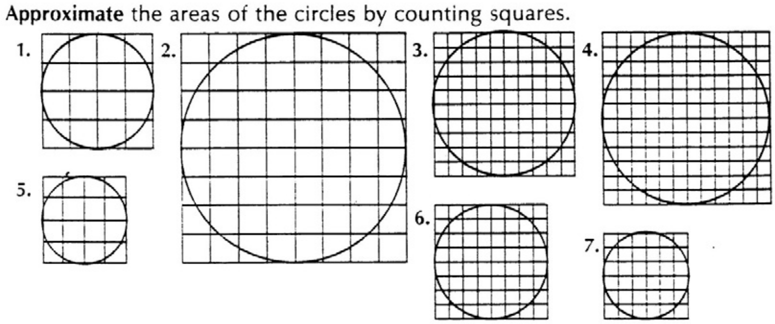
Basic Outcomes

1. The student can identify and name selected elements of a circle (i.e., radius, diameter, circumference, area).
2. The student can determine one-half the diameter on vertical and horizontal planes and separate the circle into regions, each region representing one-fourth of the circle.
3. The student can determine *pi* by measuring the circumference of the circle with a string, measuring the length of the string with a ruler, and then determining the diameter of the circle with a ruler. The student should then divide the circumference by the diameter. The student should do this for six to ten items and note that each time the task is completed the response is approximately 3.14.

Item	Circumference	Diameter	*Pi* (π)
1	25*	8	3.125
2	19*	6	3.17

Approximate value

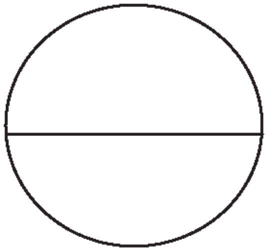
4. The student can place a circle on a square drawn on graph paper with sides equal to the diameter of the circle and note the overlap between the circle and the square as shown below.



5. The student can demonstrate the interrelationships among, diameter, circumference, and *pi* by using the factor x factor = product relationship (e.g., 3.14 x 8 = 25.12, 25.12 ÷ 8 = 3.14, 25.12 ÷ 3.14 = 8 etc.)
6. The students can correctly use the formula πr^2 to compute the area of a circle and the formula *side*^{2} or *length x width* to compute the area of the square.
7. The student can count the number of blocks in any 100 unit square that are outside the circle and note that regardless of the size of the square/circle there are always 22 unit squares outside the circle leaving 78 unit squares inside the circle. Thus, the area of the circle is "78 percent" the area of the square.

Higher Level Outcomes

1. The student can describe the relationship between the area of a circle with a diameter (x) and a square with sides of the same measure (x). That is to say, given a square with sides x and a circle with diameter x the student can show that the area of the circle is approximately 78% of the area of the square. This is done by multiplying the radius squared by 4 to equal the area of the square (Area of Square = $4r^2$) and then by multiplying π by the radius squared to equal the area of the circle (Area of Circle = πr^2). The relationship can be further illustrated by computing $\pi \div 4$ = -.07854 (rounded to ten-thousandths).



D = 8 inches

S = 8 inches

The above illustrates the value of an inductive versus deductive approach to student learning. Supporting formula information for the benefit of the teacher but deemphasized for the students follows:

$$\frac{Area_{circle} = \pi\left(\frac{1}{2}x\right)^2}{Area_{square} = x^2} = \frac{\pi\frac{1}{4}x^2}{x^2} = \frac{\pi}{4}$$

2. The students can review and contrast the formula with the formula S^2 x 0.7854, where "S" is the length of the side of a square that is the same as the diameter of the circle, and explain why both formulas produce approximately the same answer for the area of the circle. For example, in a circle with a diameter of 8 inches and a radius of 4 inches the formula for the area of the circle would be π x 4^2 = 50.2656 square inches. The area of a square with the same dimension as the diameter of the circle (i.e., 8 inches) would be 64 square inches. When the area of the square (i.e., 64 square inches) is multiplied by 0.7854 the resulting area is 50.2656 square inches. This value is equivalent to the ten-thousandths place to the value generated when the formula for the area of a circle (i.e., πr^2) is used (i.e., Cawley, Hayes, & Foley, 2008).

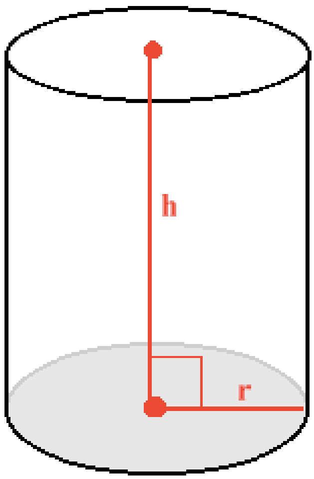
Volume of Various Forms

One way students can compare the volume of a cone with a circular base and the volume of a pyramid that has a square base is to provide the students with a variety of cones and cylinders of different sizes and a variety of cubes (rectangular prisms). Have the students fill a cone with sand or similar substance and then determine the number of cones of sand needed to fill the cylinder; the student can take a pyramid and conduct similar activities by filling the cube. As the students chart the results of these activities, they will determine that it takes three cones to fill a cylinder and three pyramids to fill a cube. Reversing the procedure by beginning with a full cylinder and pouring the contents into cones will show that one cylinder will fill three cones This principle will hold true only when the elements poured from the cone to the cylinder are the same (e.g., they must all be sand as opposed to two different elements (e.g., water and sand)). These activities provide the foundation for student understanding of the different formulas and their interrelationships.

There are a number of formulas that can be utilized in solving problems and determining relationships among units in both geometry and measurement. Often, these formulas are presented in more than one format and have more than one use. For example, the formula for determining the volume of a pyramid and the volume of a cone are the same. Yet, the appearance of the two figures is different. How is it then that the volume of two figures that are so markedly different can be computed from the same formula?

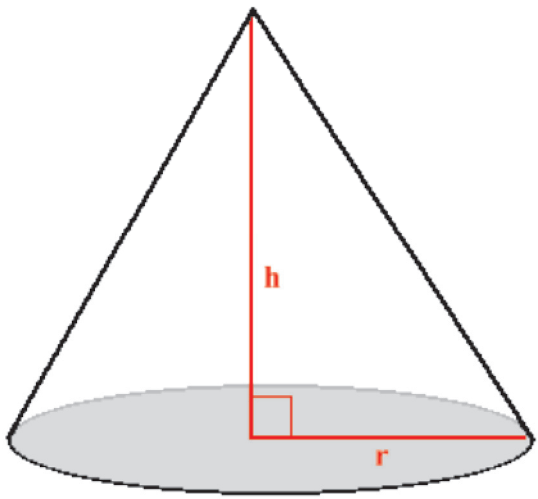
Volume of a Cylinder and Cone

Volume of a Cylinder = area of base x height



(Retrieved October 26, 2008, from http://library.thinkquest.org/20991/geo/solids.html)

Volume of a Cone = 1/3 x area of base x height



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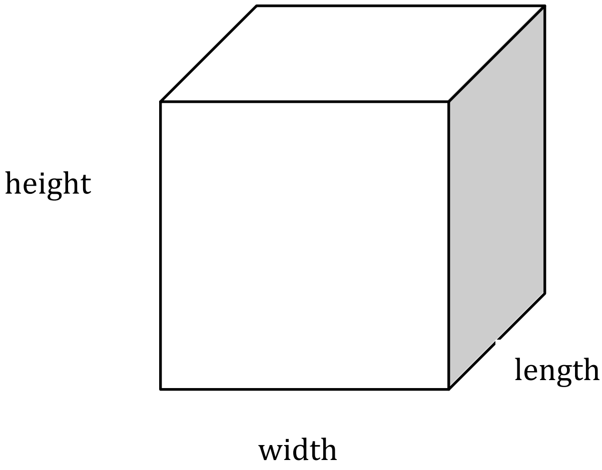
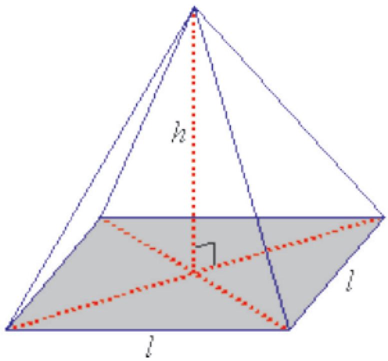
Volume of a Cube and Pyramid

Volume of a Cube = area of base x height



Volume of a Pyramid = 1/3 x area of base x height



(Retrieved October 26, 2008, from http://www.mathsteacher.com.au/year10/ch14_measurement/25_pyramid/21pyramid.htm)

Mastery of the principles presented in the aforementioned can be assessed by having the students:

1. Explain how two figures that are so different from each other can have their volumes determined by the same formula.
2. Explain how you could verify this by analyzing an activity in which they fill a cylinder with sand from cones and fill the cube with sand from pyramids.
3. Explain why two figures that are so different from one another have different volumes even though the same formula is used in each instance.

It is important that the reader be aware that each of the aforementioned is not a single activity or conducted within a single lesson. Our minimum inquiry lasted 7 weeks and extended to as many as 15 weeks. Even with 15 weeks, not all the students attained full competence with all elements.

Cognitive Enhancement

One purpose of programming is to utilize the properties of mathematics to enhance cognitive development, language comprehension, and social-personal development. The following examples illustrate how the properties of mathematics can be used to enhance performance in many of those areas that are commonly cited as reasons why students do not learn mathematics. The mathematics principles are *congruence*, *similarity*, and *numerosity*. The cognitive principles are *visual discrimination, attending*, *analysis*, *hypotheses testing*, and *evaluation* and the behavioral principles are *pondering* and *task commitment.*

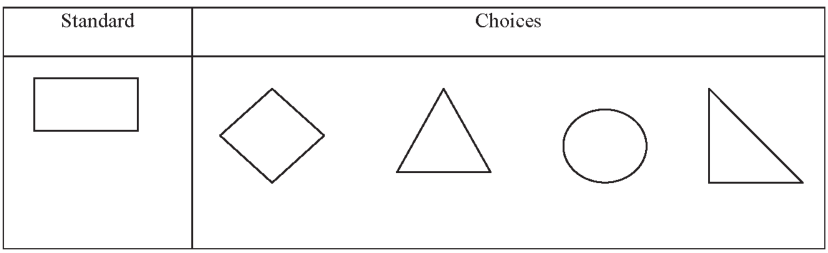
Enhancing Cognitive and Personal Development with Math Activities

Previous components of this paper showed a greater emphasis in teacher education courses for language arts such as reading than for mathematics. We (Cawley, Hayes,

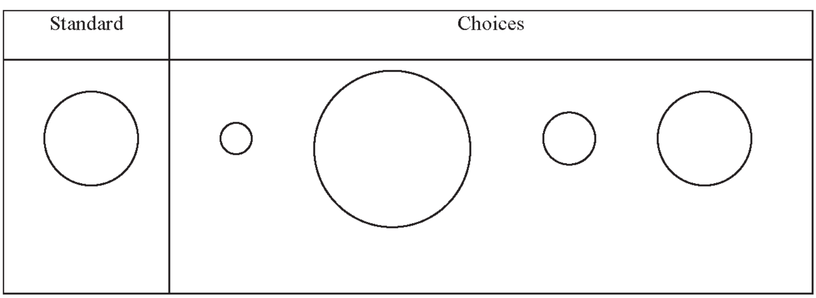
& Foley, 2008) have shown that all components of language arts, including skills such as letter recognition and sound-symbol correspondence and advanced skills such as vocabulary and sentence and passage comprehension can form the basis of mathematics instruction. Further, mathematics learning activities can serve as the basis for the development of cognitive and personal development.

Reisman and Kauffman (1980) cited *attending*, or lack thereof, as a primary element limiting success in mathematics and Myers and Hammill (1969) cited a variety of visual-perceptual weaknesses as characteristics of students with learning difficulties. Bley and Thornton (1989) cite difficulties in figure-ground perception, visual discrimination, spatial organization, and short-term memory problems involving newly presented material. The following section will describe mathematics activities that can be utilized to enhance performance in the areas cited. The tasks used for these illustrations include match-to-sample visual discrimination, two-choice visual discrimination learning, and concept learning activities rooted in the aforementioned.

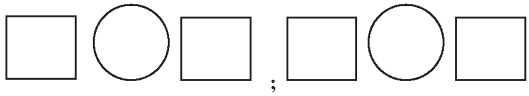
Match-to-Sample visual discrimination tasks present the students with a sample or standard and the students are asked to select from among a number of choices, the choice that matches, or goes best with the sample as shown below. The standard is a four sided figure and the student is to select a four sided figure.



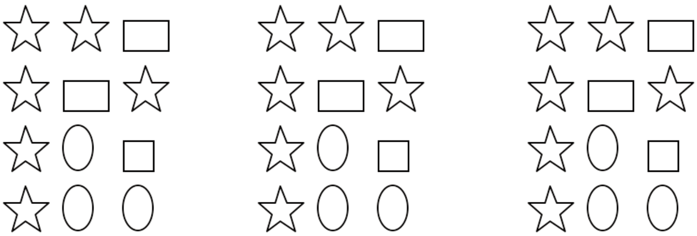
Principle: Congruence



Pattern: The pattern is displayed for the students as shown.



The students are asked, “Which of the examples below is the same as the pattern shown?” The students then mark the option they believe is the best choice.



The correct choice is the second line down as it represents the ABA pattern. (Cawley, Hayes, & Foley, 2008, p. 65).

LANGUAGE DEVELOPMENT AND COMPREHENSION

Vocabulary

The vocabulary of geometry and measurement is both general and specific. It is specific in that meanings for certain terms refer to one thing and one thing only. For example, the term *congruence* refers to figures or objects that are of the same size and shape. Vocabulary is general in that certain terms refer to one or more things that have attributes in common, but are not always the same. For example, the term *square* may refer to a figure with four equal sides and four right angles and these may be of many sizes. The term may also refer to an instrument that is used for measuring right angles or a plaza or one who is a “nerd.” It is important to address both the specific and general (lexical) features of the vocabulary of geometry and measurement.

One way to assess vocabulary knowledge is to use a listening vocabulary test that can be administered to an individual or a group. As a group test, the approach would be to develop a vocabulary worksheet similar to those used to measure vocabulary in the primary grades. The teacher, school, or district could create both a listening version and a reading version and if both were administered to the students, there would be information that would contrast listening and reading.

As a listening test, the test is administered by having the teacher state a term and then the students point to or mark (identify) the picture that best represents the term. The listening format is efficient to administer and is among the more direct means for providing accessibility to the students. Listening is also multi-lingual in that any language can be used to provide the stimulus. One of the basic tenets for assessment is that the developer must recognize that when assessing knowledge, students may not be able to access the procedure required to solve an item they may not be able to respond correctly, not because they do not know the term, but, rather, because they can not access the information during the assessment. This is true when words are presented in print or in written form and students can not read. DeLuca (1996) contrasted the performance of students with learning disabilities on a mathematics listening test and a mathematics reading test. In the former, the students were shown a set of pictures and the examiner stated a term and the student marked a picture that represented the term. In the latter, the student was shown a set of pictures accompanied by a written word. The students were to mark the picture that represented the term. Performance on the listening tests was superior to that of the reading test. Such would also be true in the case of students who are learning English as a second-language and may lack competency with English and when items are administered in English. On the other hand, when assessing a skill it is important to assure that students possess a knowledge base for the skill, otherwise they may default on the skill due to a lack of knowledge rather than a deficiency with the skill.

It is important that vocabulary, terminology, and conceptual representations be presented to students in some form of episodic activity that is guided by real outcomes and accompanied by a variety of alternative illustrations.

The vocabulary of geometry and measurement can be specific and detailed or general and lexical in character. Multiple terms can be linked or associated with one another to provide comparisons or to establish relationships. For example, Buckingham (1953) discusses three types of lines, *straight*, *curved*, and *broken*. He then names three lines according to direction. These are *horizontal*, *vertical*, and *obtuse*.

For teachers who recognize the value of student activity in the learning process the advantages afforded by the content of geometry and measurement are extensive. In turn, the value of the content of geometry and measurement is inestimable with respect to all facets of an individual's life. We have attempted to illustrate ways in which that perfect combination can be promoted in the everyday classroom.

Summary

Our intent in developing this paper is to bring forth attention to geometry and measurement. These topics, while essential to the development and lives of students with LD, suffer from an abominable lack of attention in the literature. Our emphasis on the area of the circle was intended to demonstrate that high levels of conceptual attainment can be presented to students with LD. The fact that the students can use algorithms other than πr^2 to develop the area of a circle reflects a meaningful understanding and is essential to the use of related formulas. It is important that the field of LD direct more attention to topics such as geometry and measurement and broaden the score of mathematics for students in general.

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