Finding the area of a CIRCLE

Didactic explanations in school mathematics

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Learning about the area formulas provides many opportunities for students even at the beginning of junior secondary school to experience mathematical deduction. For example, in easy cases, students can put two triangles together to make a rectangle, and so deduce that the area of a triangle is half the area of a corresponding rectangle. They can dissect a trapezium and rearrange the pieces to make rectangles, parallelograms or rectangles and triangles, and so find the area of a trapezium from the area of other known shapes. The area of a circle, however, provides a new challenge. The curved edge poses a difficult mathematical problem, with an interesting history. In this article, we present several different explanations of the formula for the area of a circle, which have logically different status. Some are "light" versions of a proper mathematical proof, but others are not. However, we believe that they all have a role as didactic explanations in junior secondary mathematics. Explanations in school mathematics must do far more than "prove." The explanations were found in a survey of nine current Australian Year 8 textbooks (see Stacey & Vincent, 2008). We saw a rich and interesting range of possibilities. In the sections below, we show some of the varieties of demonstrations found in our survey.

Approximations

The area of any shape can be approximated by placing it on a grid and counting the squares. The textbook in our sample that demonstrated the counting squares method was careful to explain that this gave only an approximation to the area and was not a valid mathematical method for finding the area of a circle. Counting with a 20 × 20 grid placed over the circle showed that approximately 316 of 400 grid squares fell inside the circle. Hence the area of the circle is approximately 316/400 of the area of the square. Since the area of the square is $4r^2$ the area of the circle is approximately $3.16r^2$ (see Figure 1). If the grid were finer, e.g., dividing the radius into 100 instead of 10 equal parts as in Figure 1, the approximation to π would be more accurate. However, this method does not link the 3.16 with π in any way other than as a numerical coincidence.

The approximation of $3.16r^2$ above comes from an empirical argument: the counting process produces data, which gives the number 3.16 without reasons. However, approximate answers can also be obtained deductively. The same textbook in our sample, which gave the approximation in Figure 1 also presented a simple deductive argument to show that the area of a circle is approximately $3r^2$. By constructing a square inside, and another square outside, a circle of radius r, students can see that the area of the circle must be between $2r^2$ (so is approximately $3r^2$; see Figure 2). This method, of course, underlies the method used by Archimedes (287–212 BCE) to arrive at the area of a circle. He progressively increased the number of sides of the circumscribed (drawn outside) and inscribed (drawn inside) polygons and the circle, using the polygon areas to improve his value for π. Interactive websites (see, for example, HREF1, HREF2) can be used to demonstrate Archimedes' method.

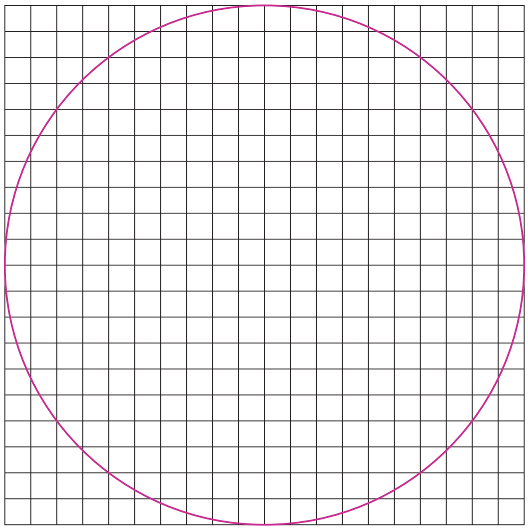

Figure 1

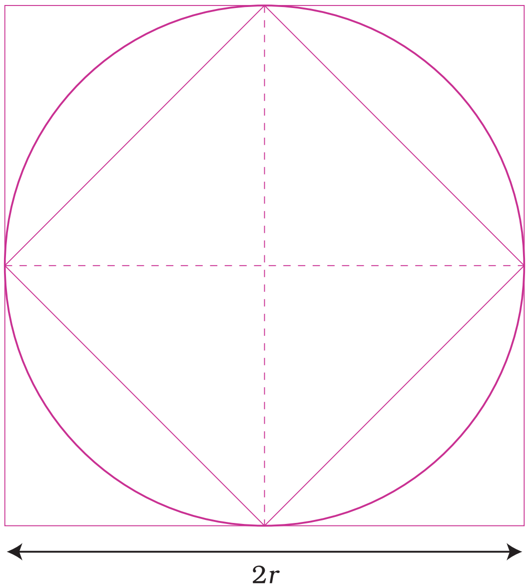

Figure 2

Dissection and rearrangement

Several textbooks demonstrated the method of finding the area by dividing the circle into sectors, then rearranging the "sectors" to form an approximate parallelogram or, by moving a half sector from one end of the "parallelogram" to the other, an approximate rectangle (see Figure 3). The explanation has both general features (e.g., the radius r)

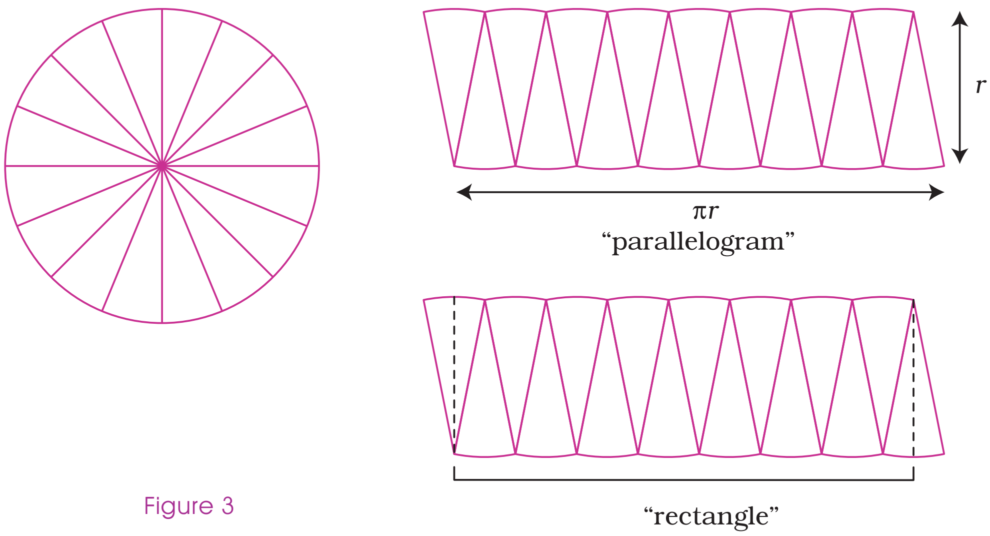

Figure 3

and specific features (e.g., the number of sectors). Even though this "proof" requires very considerable refinement related to the limit processes to become a mathematically acceptable proof, we judge that it functions well as a justification of the circle area formula at around Year 8. One of the textbooks we surveyed prepared students for this explanation by preceding it with a practical version of the dissection in Figure 3, where students cut a photocopy of a circular protractor into sectors, construct the "rectangle" and hence find the area of the protractor. They were asked what would happen if the protractor was cut into more, narrower sectors, thereby acknowledging the limiting processes involved in the mathematical proof.

Another textbook presented a series of diagrams as in Figure 4 to show how the rearrangement of sectors appeared more and more like a parallelogram as the sector angle decreased.

An alternative dissection approach was shown in two textbooks. The circle was dissected into a series of concentric rings, then cut along a radius (see Figure 5). The "opened out" rings were then arranged to form an approximate triangle. The base of the triangle is equal to the circumference of the circle, that is, $2\pi r$, and the perpendicular height of the triangle is r. Using the known rule for the area of a triangle, students can see that the area of the rearranged circle is approximately

$$A = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} r \times 2\pi r$$

that is, $A = \pi r^2$. Like the sector method, this approach involves deductive reasoning, and although refinement is again needed to make it a mathematically acceptable proof, it is a highly appropriate method to justify the circle area rule at this level.

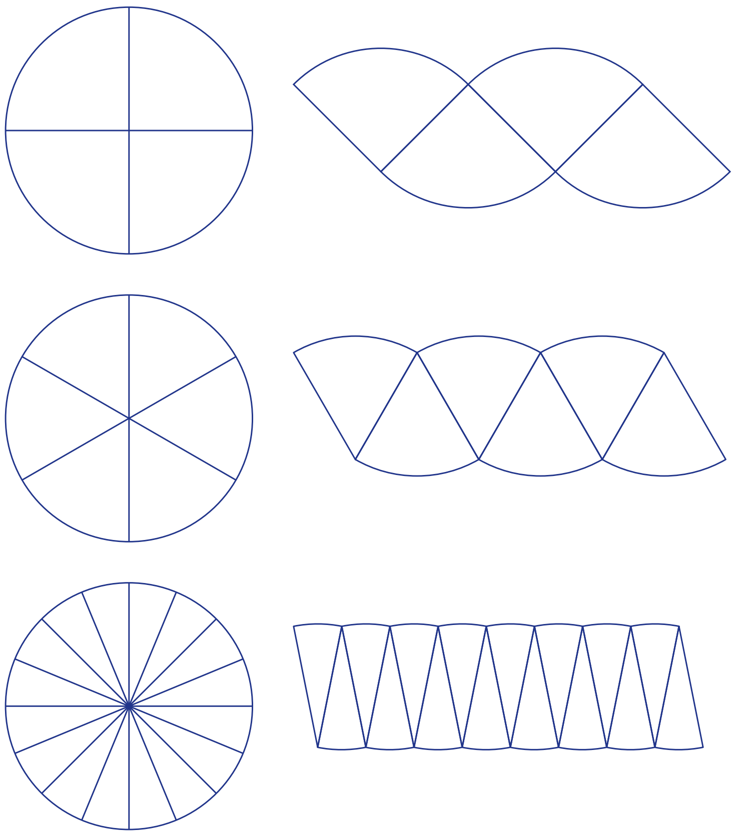

Figure 4

Empirical and deductive reasoning

In the discipline of mathematics, the formula for the area of the circle needs a completely general proof, which is based on deduction from known axioms. As we noted above, students have until now encountered only areas based on figures with straight sides. In the case of the circle, there are some very complicated limiting processes involved in formally proving the formula. However, in school mathematics, explanations of many different types have a role because it is necessary to develop students' conceptual understanding of what area means, help them to be convinced of the

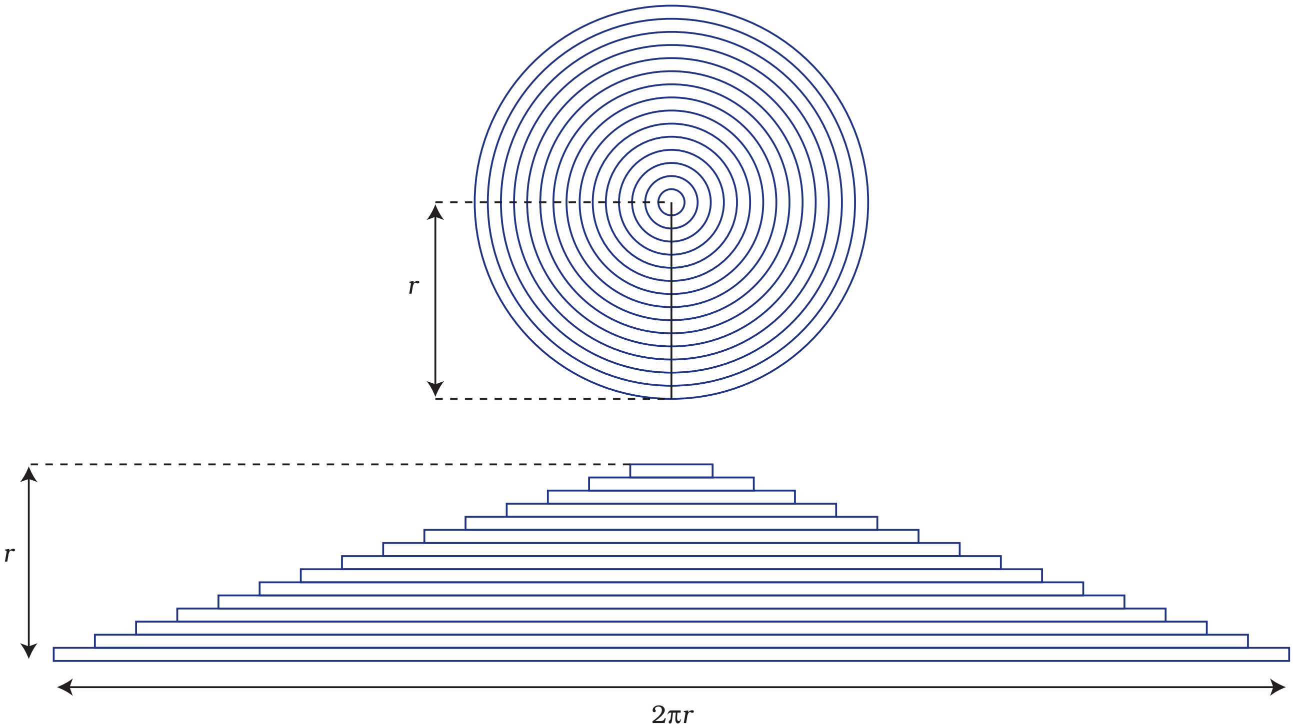

Figure 5

reasonableness of the formula, link it to other ideas, and give them a sense that even though a proper justification is difficult, they can understand some of the reasons why the formula is true. For this reason, all of the methods above have a place in junior secondary mathematics. The empirical counting of squares, for example, can help remind students what area is, as well as to convince them of the approximate answer. The deductive methods can help reinforce the message that there are reasons behind all the formulas in mathematics. Finally, to help students develop their mathematical reasoning, it is important that students are aware that there is a difference in the mathematical quality of the arguments between the empirical "counting squares" method and the approaches that use deductive reasoning.

References

Stacey, K. & Vincent, J. (2008). Modes of reasoning in explanations in Year 8 textbooks. In M. Goos, R. Brown & K. Makar (Eds), *Navigating currents and charting directions, Proceedings of the 31st annual conference of the Mathematics Education Research Group of Australasia* (pp. 475–481). Brisbane: MERGA.

HREF1 (accessed 15 February 2009): http://www.ugrad.math.ubc.ca/coursedoc/math101/notes/integration/archimedes.html.

HREF2 (accessed 15 February 2009): http://www.ugrad.math.ubc.ca/coursedoc/math101/demos/week1/carea.html.