International Education Journal, 2005, 6(2), 175-183.
ISSN 1443-1475 © 2005 *Shannon Research Press*.
http://iej.cjb.net

Differences in problem solving strategies of mathematically gifted and non-gifted elementary students

Astrid Heinze
Westfaelische Wilhelms-University Muenster, Germany *heinzeas@math.uni-muenster.de*

This study presents problem solving strategies and processes of thinking of mathematically gifted elementary children with respect to non-routine word problems. The data stem from a university-based course, especially designed to foster gifted children, ages 6-10 years, through the enrichment of the elementary mathematics curriculum. Videotapes of the children's problem solving processes were transcribed in great detail and provided the basis for the analysis. The presented examples show that mathematically gifted elementary students stand out in the ability to work systematically and quickly, getting an insight into the problem's mathematical structure. Additionally these children stand out in their high ability to verbalise and to explain their solutions. In comparison to non-gifted children these qualities in problem solving show significance. The significance was calculated by non-parametric tests.

Mathematics, elementary, enrichment, problem solving, gifted

INTRODUCTION

Since April 2001 I have worked within a university-based course with mathematically gifted and interested elementary students. The aim of my doctorate study is to gain knowledge of special qualities in mathematically gifted children. The focus is on their problem solving strategies and processes of thought while working on mathematically challenging problems. Dealing with demanding tasks should, moreover, promote the children's interest and talents in mathematics.

SELECTION OF CHILDREN

The children are 6-10 years old and are in the second to fourth grade of elementary school. The selection of the children happened in two steps: The children visiting the course are selected by being nominated by their teachers and parents. Some children had high scores in intelligence tests. After observing the children for weeks or months during the course, the children are selected for the research project. The criteria of mathematical giftedness are tested by these children solving the indicative tasks developed by Käpnick (1998). An intelligence test (WISC III) serves additional information provided that such a test has not yet being taken.

SELECTION OF PROBLEMS

I chose a selection of mathematical problems that were adequate for revealing mathematical giftedness in children, especially in comparison with "normally" talented children. Following Käpnick (1998) and according to what Krutetskii (1976) says about capable primary school children, mathematical giftedness by primary school children can manifest itself in the following criteria:

- recognising patterns and formal structures,
- ability to transfer recognised mathematical structures,
- reversibility of operations and processes, flexibility of mental processes,
- changing the representation of the problem,
- mathematical sensitivity, creativity, and
- mathematical memory.

I studied the children's problem solving strategies while they were working on combinatorial problems (Examples 1 and 2) unsolvable puzzles (Example 3), and sum tasks on series of natural numbers (Example 4). A representative selection of children's solutions to the problems are also presented and explained.

Example 1. Different coloured houses (Hoffmann, 2003, p.94)

Different Coloured Houses

Houses in one city consisting of a basement, a ground and a roof should be painted in different colours. For the basement you have four colours red (r), green (g), yellow (y) und blue (b), but for the ground and roof only the two colours red (r) and green (g).

Find all possibilities to paint the houses in different colours.

This combinatorial problem requires a complex and demanding strategy in order not to forget a possibility. Children solve this problem by putting all possible houses onto a magnetic blackboard. Altogether there are 16 combinations possible, although there is room for 18 combinations on the blackboard. As soon as the children signal that they have found all possible houses they are asked to explain their results.

Lea (10 years old, Fourth Grade) chose the roof and the ground to remain constant. Therefore she combines four houses with green roofs and red ground and then four with a green roof and green ground on the blackboard. For the combinations with red roofs she works analogously. She emphasises this order by leaving one house grey between the houses with green and red roofs as can be seen in Figure 1.

Lea	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	17.	18.
roof	g	g	g	g	g	g	g	g		r	r	r	r	r	r	r	r	
ground	r	r	r	r	g	g	g	g		r	r	r	r	g	g	g	g	
basement	b	g	r	y	b	g	r	y		g	y	r	b	g	r	y	b	

Figure 1. Lea's solution of the "Different coloured houses"

Lea applies the so-called odometer strategy[1] (English, 1997, p.261) by exhausting the constants of roof and ground. She comments on her strategy: "Because here is green and then red and then you can only choose the four colours below; and then once more green and green and all four colours below and that are eight colours and this as well as red roofs and green roofs."[2] The strategy is reflected in Lea's explanation of how she formed the combinations.

In contrast to Lea, Michel (9 years old, Fourth Grade) chose the basement as a first constant. On ground and roof level he forms counterparts, as seen in Figure 2.

[1] English describes the odometer strategy for two- and three-dimensional combinatorial problems. This strategy could be transferred to four-dimensional combinatorial problems as shown in example 2.

[2] Lea's original explanation: „Weil hier ist grün und dann rot und dann kann man ja nur noch nur alle vier unteren Farben nehmen und dann noch grün und grün und dann noch alle vier unteren Farben und das sind dann ja acht Farben und das halt bei roten Dächern und bei grünen Dächern"

Michel	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	17.	18.
roof	g	r	r	g	r	g	r	g	g	r	g	r	g	r	g	r		
ground	g	r	g	r	r	g	g	r	g	r	r	g	r	r	g	g		
basement	y	y	y	y	b	b	b	b	g	g	g	g	r	r	r	r		

Figure 2. Michel's solution of the "Different coloured houses"

Michel explains in detail: “Because of finding all possibilities with red and green for the yellow, for the blue, for the green and for the red basement. And I do not have another colour for the basement. There are four possibilities for each because there are only two colours for ground and roof, so I thought for one basement colour the whole house to be red, the whole house to be green, then once a house with green ground and red roof and once a house with red ground and green roof and this for each basement”.[3] Michel gets additional support for his solution by analysing the problem arithmetically and by justifying why two houses have to be left blank on the blackboard: “It should have been 16 houses but there are 18 and 18 is not a fourth number”.[4] Fourth number means that this number is divisible by four. Michel not only realises the systematic strategy but also the mathematical structure of the problem and explains this in a clear verbal manner.

In her study on problem solving abilities of primary and secondary students, Hoffmann classifies the odometer strategy as an expert strategy into the field of the macro strategies. Macro strategies are higher strategies for creating all possible combinations. In contrast to macro there are micro strategies that merely form part of all possibilities. Among the odometer strategy, Hoffmann distinguishes two further macro strategies as expert strategies. In her study only 12.5 per cent of all primary students (randomly chosen) use one of the macro strategies to solve the “Different coloured houses” problem (Hoffmann 2003, Appendix). In comparison to Hoffmann’s students, the mathematically gifted primary students in my study more frequently use a macro strategy. This difference is statistically significant[5]; it seems that mathematically gifted children recognise higher structures faster and can therefore work more structurally and more systematically on tasks or problems. Not only remarkable however is the systematic and structural procedure but moreover the children’s explanations of why they considered all combinations. Mathematically gifted primary students can explain and verify their systematic procedure significantly[6] more often by comparison with the children in the study of Hoffmann.

The use of the odometer strategy could be observed by a whole string of further combinatorial problems. Some children only use the odometer strategy for obtaining insight into the mathematical structure of the problem for deducing and respectively calculating the total number of combinations. Example 2 demonstrates such a solution.

Example 2. “Combination lock”

Combination Lock
How many different possibilities exist to combine the four figures 0, 1, 2, 3 at the combination lock?

[3] Michel’s original explanation: “Weil ich bei den gelben Kellern alle Möglichkeiten mit rot und grün herausgefunden habe, bei den blauen, bei den grünen, bei den roten und ich hab’ ja keine andere Kellerfarbe. Es gibt bei jedem vier Möglichkeiten, weil es ja nur zwei Farben für Wand und Dach gibt und da hab ich gedacht, bei einer Kellerfarbe das ganze Haus rot, das ganze Haus grün und dann einmal ein Haus mit grüner Wand und rotem Dach und einmal ein Haus mit roter Wand und grünem Dach und das bei jedem Keller so.”

[4] Michel’s original explanation: “Es hätten genau 16 Häuser sein müssen, aber es sind 18 und 18 ist keine 4-stellige, 18 ist keine 4er Zahl”.

[5] Using the Fisher-Tests (Siegel 2000, p.94) with a 0.05 level of significance.

[6] Using a test by Raatz for grouped ordinal data (Lienert 1973, p.235) with a 0.05 level of significance.

As presented in Figure 3, Ingo (7 years old, Second Grade) starts with figure 0 and fixes the second position only when he has formed all possible combinations. He misses one combination for figure 3 in the second position. After finding these first five combinations he deduces immediately that there must be five combinations analogously for the other three figures in the first position (logical thought).

Ingo explains these insights by drawing four lines. One line under the combinations with 0 and three lines held for the further combinations with another figure in the first position. He continues his work with figure 1 in the first position. It is again a complete application of the odometer pattern. This strategy allows Ingo to find all six combinations for figure 1.

Consequently he returns to the combinations with 0 and completes the missing possibility. Ingo transfers the insight that he gained from figure 1 without any difficulties to figure 0. Evidently he knows that for each figure in the first position there must be an equal number of combinations. For figure 2 in the first position Ingo notes analogously six combinations. It is interesting that Ingo does not note these combinations for figure 2 as systematically as for figure 1.

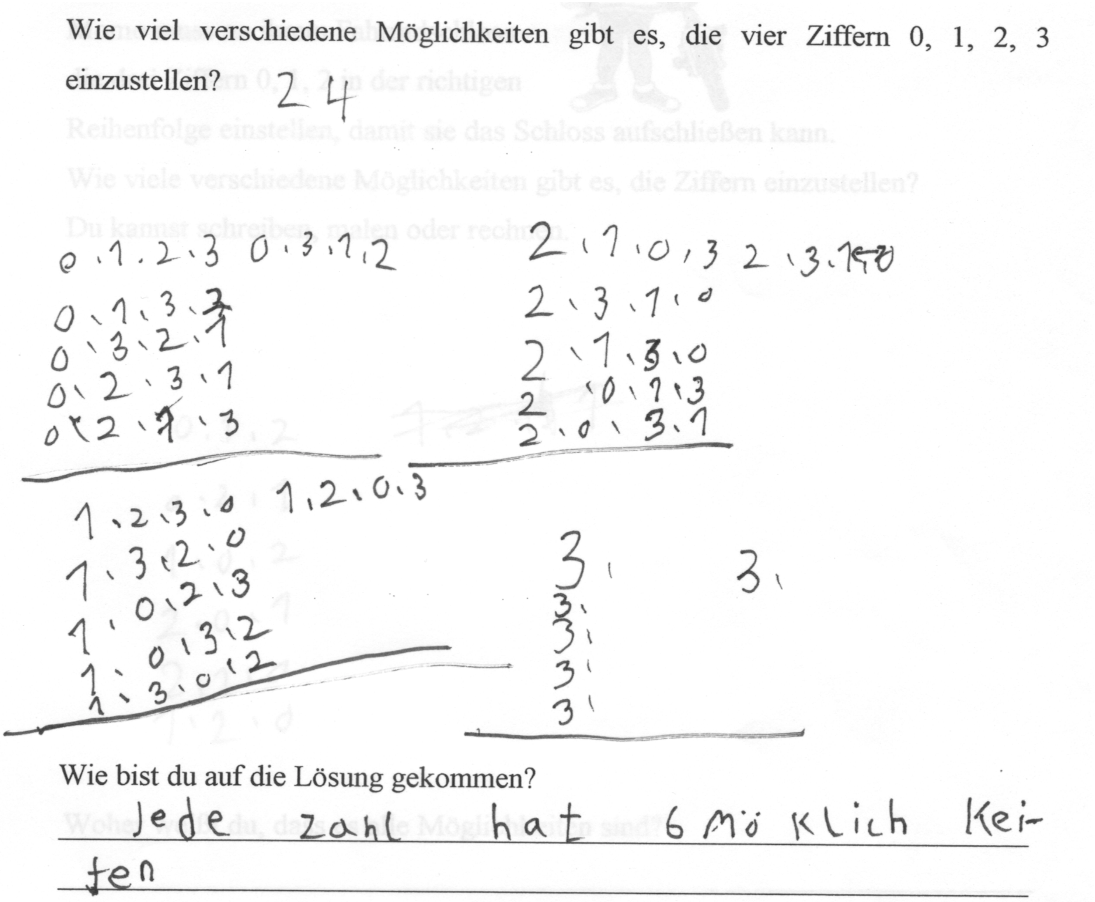


Figure 3. Ingo's solution using the odometer strategy

Perhaps it is not necessary for him because of his knowledge of the existence of six combinations for each figure. For figure 3 Ingo writes only six figures 3 and calculates the total number of combinations by C_4^6. He notes the result, 24, and explains his solution: "Each number has 6 possibilities."[7]. During the problem solving process, Ingo develops a procedure to solve the problem completely using the odometer pattern as a higher strategy. His insight into the mathematical structure of the problem is reflected in his explanation.

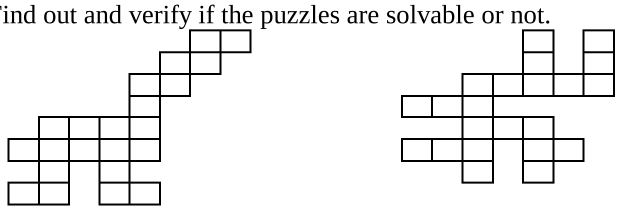
[7] Ingo's original explanation: "Jede Zahl hat 6 Möglichkeiten."

The mathematically gifted primary students' special ability to verbalise appears in another kind of problem similar to the unsolvable puzzle by Burchartz and Stein (2003). As well as Hoffmann, Burchartz and Stein present in their study these problems to primary and secondary students. Therefore a comparison between mathematically gifted primary students and randomly chosen primary students could take place.

Example 3. "Unsolvable Puzzle"[8]

Puzzle

Find out and verify if the puzzles are solvable or not.



Puzzle A Puzzle B

Each puzzle should be covered exactly with the following tiles (each is given once)

1-unit-tile 2-unit-tile 3-unit-tile 4-unit-tile 5-unit-tile

2-unit-angle 3-unit-angle

The unsolvability of puzzle A and B is based on a logical analysis of the possibilities to cover the puzzle. For the 5-unit-tile there is only one possible position in puzzle A and two possible positions in puzzle B, thus logically concluding that there is only one space left for the 4-unit-tile. If the 4-unit-tile and 5-unit-tile are placed in their unique positions, there are no possibilities to place the 3-unit-tile and the 3-unit-angle in puzzle A or the 3-unit-angle in puzzle B. Children have to realise the fixed positions of the big tiles and have to deduce logically the impossibility to cover the whole puzzle with the remaining tiles.

Till and Michel (both 10 years old, Fourth Grade) together solve puzzle A and explain the impossibility of the task as follows: Till: "Firstly there is only one position where the 5-unit-tile fits [he places the 5-unit-tile in its unique position], therefore there is only one where the 4-unit-tile fits [he places the 4-unit-tile in its unique position], and now the remaining tiles could not be used" - Till: "yes you could only place these tiles [he touches the 3-unit-tile and 3-unit-angle] but these two could not be placed anywhere".[9]

Martin (8 years old, Third Grade) comments on the impossibility of puzzle B: "Look at those here [he points at the 4-unit-tile and 5-unit-tile] they always have to be there [he points to the possible positions] or could be placed in this way [he demonstrates the other possible positions for the 4-unit-tile and 5-unit-tile], this is always the same, because they always cover these spaces. And where should this one fit? [He points to the 3-unit-angle]. This one doesn't fit anywhere!"[10]

---

[8] Source: Burchartz and Stein (2003, p.3).

[9] Original explanation of Till and Michel: Till: "Erst mal weil's nur eine Stelle gibt, wo die 5 reinpasst *(legt 5er)* dann gibt es auch nur noch eine wo die 4 reinpasst *(legt 4er)* - Michel: "und den Rest kann man dann nicht mehr verbauen"- Till: "ja und dann kann man nur noch die Teile *(3er und 3er Winkel)* verbauen aber die beiden gehen nicht"

[10] Martin's original explanation: "Guck mal dieses hier *(4er, 5er)* das muss ja hier immer hin oder so *(zeigt die andere mögliche Position)*, das ist ja eigentlich genau das gleiche, weil immer die diese Felder auch bedeckt sind, wo soll der *(3er Winkel)* dann noch hinpassen? Der passt hier nirgendswo dann mehr hin."

The children Till, Michel and Martin discuss the impossibility of the tasks in a logical analysis. First they explain the fixed positions of the two big tiles (5-unit-tile and 4-unit-tile). In puzzle A, there is only one possible position for the 5-unit-tile, explains Till. Martin demonstrates the two possible positions for 5- and 4-unit-tile in puzzle B. These fixed positions in mind they deduce the insolvability of each puzzle by pointing out those tiles that could not be placed. The children's answers consequently form a complete proof for the impossibility of each puzzle. Therefore these explanations belong to the highest category in the classification of Burchartz and Stein (Burchartz and Stein 2003, p.10). A comparison shows the mathematically gifted children to give qualitatively better answers than the normal primary students, displaying a significant difference[11].

Beyond this, mathematically gifted primary school children need fewer attempts in order to recognise the puzzle to be unsolvable. Because of this significantly[12] less time was required to deal with the problem.

The qualitatively better answers and explanations given by mathematically gifted children by solving combinatorial problems and dealing with unsolvable puzzles may be the result of these problems not being typical numeral tasks and of the children being allowed to use material. Example 4 shows a more typical arithmetical problem. Here again the mathematically gifted children's special ability to verbalise and to work more structurally is demonstrated.

Example 4. "**Series of natural numbers of size up to 25**"[13]

Series of Natural Numbers of Size Up to 25

"Find all possible sums consisting of an adjacent sequence of natural numbers. The result may not be larger than 25"

"Normal" primary students, as shown by Schwätzer and Selter, solve this problem by writing different sums rather unstructured. On their way to find a solution they use a large number of different strategies. To explain the reason for completion most of them sort their sums using a superior sort strategy. In comparison to the students in the study of Schwätzer and Selter the mathematically gifted primary students in my study use a macro strategy to find systematically all sums right from the beginning. The difference to the results of Schwätzer and Selter is statistically significant[14]. The following two superior "production strategies" (Schwätzer and Selter, 1997, p.132) were shown by the mathematically gifted children and are presented in Figures 4 and 5.

Firstly the solutions are sorted according to the first addend's size. An "ascending" or "descending" amount of addends can be distinguished. Martin's (9 years old, Fourth Grade) solution, shown in Figure 4, is an example for the strategy "First addend descending".

Another strategy is to sort according to the amount of addends,. Firstly the children write down all sums with two addends, then with three addends, and so on. Second condition in this case is the size of the first addend. Again an "ascending" and "descending" size of the used addends can be distinguished. Sarah's (9 years old, Fourth Grade) solution, presented in Figure 5, is an example for the strategy "Amount of addends ascending".

---

[11] Using a test by Raatz for grouped ordinal data (Lienert 1973, p.235) with a 0.05 level of significance.

[12] Using the U-test by Mann-Whitney (Siegel 2001, p.112) with a 0.05 level of significance.

[13] Source: Schwätzer and Selter (1998, p.125).

[14] Using the Fisher-Tests (Siegel 2001, p.94) with a 0.05 level of significance.

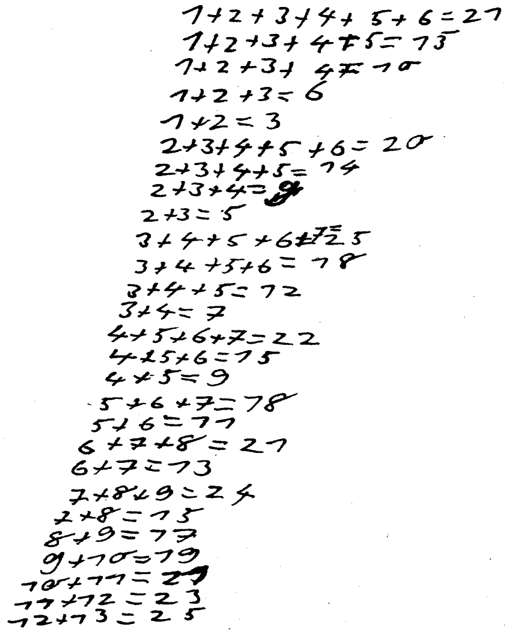


Figure 4. Martin's solution using the "First addend descending" strategy

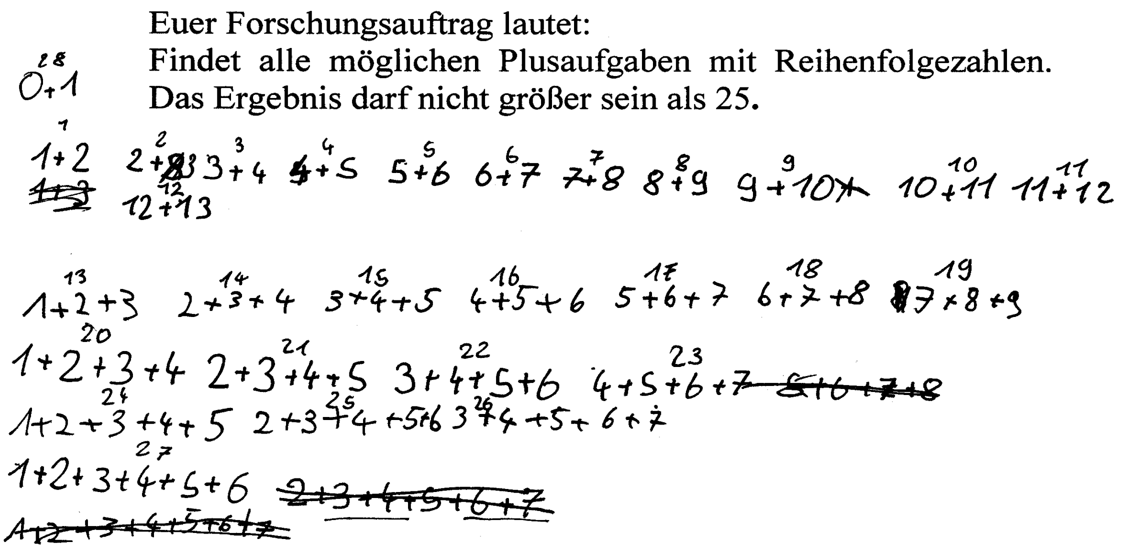


Figure 5. Sarah's solution using the "Amount of addends ascending" strategy

Using a superior production strategy mathematically gifted primary students are able to solve the sum problem faster. They need significantly[15] less time than the children by Schwätzer and Selter (1998). The gifted children do not need to sort their sums subsequent to explain the completeness of the produced sums. Instead of this they explicitly or implicitly explain the completeness with the help of their production strategy.

On one hand, Martin describes his strategy; on the other hand, he also explains the completeness of his solution (see Figure 4) by indicating explicitly when 25 is exceeded. He demonstrates this for the first addend 1 exemplarily: Martin: "At first the 1 then 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12 [he taps the numbers with his pencil]. I add the numbers as long as the next one does not fit anymore. And always when I get this number, for example now 6 here [he taps 1+,..,+6], if I

[15] Using the U-test by Mann-Whitney (Siegel 2001, p.112) with a 0.05 level of significance.

would add the 7, then I would get 28. So I leave it and I always remove the last number, there again one less and there again one less [he taps 1+2+3+4 and 1+2+3] and there again one less until it does not go on anymore."[16]

CONCLUSION

In comparison to "normal" primary students, the mathematically gifted ones need significantly less time to deal with the unsolvable puzzle and the sums. Their procedure is based on a logical analysis (puzzle) and they solve the combinatorial problems and the sum task significantly more systematically by using macro strategies. Some children satisfy the insight in the mathematical structure to solve the combinatorial problems arithmetically. This suggests that there is a special ability of mathematically gifted children to use their insight in mathematical structures to be able to calculate the result.

All examples additionally explain another ability, which has not been considered in criteria lists of mathematically gifted children yet: the high ability to verbalise and to explain/reason. The mathematically gifted children's explanations reflect their understanding of the mathematical structure of the problem and their strategic use. A comparison to "normal" primary students shows that gifted children more often give answers of "better quality". The difference is significant.

On the one hand, my study confirms the typical criteria of mathematical gifted children as formed by Krutetskii and Käpnick, especially the ability to recognise formal structures. On the other hand, it seems necessary to add some further characteristics of mathematically gifted primary students to the list mentioned above. Firstly a criteria which is, as Krutetskii says, typical for mathematically gifted secondary students, is "the ability for logical thought and logical analysis" (Krutetskii, 1976, p.350).

Secondly, two additional criteria have emerged:

- the high ability to verbalise and to explain their solutions, and
- the ability to use the insight in the mathematical structure of a problem in order to solve it by deducing or calculating the solution.

Further solutions and observations in the study confirm these results. However, individual abilities of mathematically gifted children should also be taken into consideration. The different criteria of mathematical giftedness do not have to take effect completely. Rather, they appear individually pronounced and require individual support. But one form of mathematically giftedness can turn out to be an ability to give a precise analysis of problems and reasons for solutions.

The presented problems not only serve to foster mathematically gifted children, but rather supply important knowledge about children's ability to recognise mathematical structures and relationships. They are consequently qualified as part of the identification of mathematically gifted children.

REFERENCES

Burchartz, B. and Stein, M. (in press) The invisible wall project: Reasoning and problem solving processes of primary and lower secondary students. *Mathematical Thinking and Learning.*

[16] Martin's original explanation: "Ich hab immer erst 1 dann 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 und 12 *(tippt mit dem Stift auf die jeweiligen Zahlen)*. So lange da die Zahlen drangehängt, bis es nicht mehr geht, bis dann die nächste nicht mehr dranpasst. Und dann immer wenn ich die Zahl hat z.B. jetzt 6 hier *(tippt auf 1+...+6)* wenn ich da die 7 dranhäng' würde, dann wären es ja schon 28. Dann hab ich die gelassen und dann hab ich immer den letzten weggenommen und da wieder den ein weniger und da wieder ein weniger *(tippt auf 1+...+4 und 1+...+3)* und da wieder ein weniger bis tiefer geht nicht."

English, L.D. (1993). Children's strategies for solving two- and three-dimensional combinatorial problems. *Journal for Research in Mathematics Education*, 24, 255-273.

Hoffmann, A. (2003): *Elementare Bausteine der kombinatorischen Problemlösefähigkeit.* Hildesheim: Franz Becker.

Käpnick, F. (1998). *Mathematisch begabte Kinder. Modelle. Empirische Studien und Förderungsprojekte für das Grundschulalter*. Frankfurt/Main: Lang.

Krutetskii, V.A. (1976). *The Psychology of Mathematical Abilities in School Children*. Chicago: University of Chicago Press.

Lienert, G.A. (1973). *Verteilungsfreie Methoden in der Biostatistik.* Band 1. Meisenheim am Glan: Verlag Anton Hain.

Schwätzer, U. and Selter, C. (1998). Summen von Reihenfolgezahlen – Vorgehensweisen von Viertklässlern bei einer arithmetisch substantiellen Aufgabenstellung. *JMD,* 19(2-3), 123-148.

Siegel, S. (2001). *Nichtparametrische statistische Methoden.* Eschborn: Verlag Dietmar Klotz.