Iteration of complex functions and Newton's method

Jerry Dwyer, Roger Barnard, and David Cook
Texas Tech University, USA
<jerry.dwyer@ttu.edu>
<roger.barnard@ttu.edu>
<dcook2@sandi.net>

Jennifer Corte
University of Tennessee, USA
<jcoorte@math.utk.edu>

This paper discusses some common iterations of complex functions. The presentation is such that similar processes can easily be implemented and understood by undergraduate students. The aim is to illustrate some of the beauty of complex dynamics in an informal setting, while providing a couple of results that are not otherwise readily available in the literature.

Basic ideas in complex variables

Definition: A *complex number* is any number of the form $z = x + iy$ where x and y are real numbers, and the imaginary number $i = \sqrt{-1}$.

The historically minded reader may be interested to note that this representation on the plane was simultaneously developed by Wengel, Gauss and Argand (Needham, 1997) around the end of the 18th century. Indeed, complex numbers had been viewed with skepticism within the scientific community before the geometric interpretation was displayed. This was despite work by Cardano (Markus, 1983) and Bombelli (see www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Bombelli.html) in the late 16th century. Bombelli's work on the solution of cubic equations reinforced the practical value of complex numbers, which previously had only been seen as solutions of quadratic equations. Cauchy and Riemann developed many of the fundamental results of complex analysis in a short time span in the middle of the 19th century. A nice exposition of the historical development of the visual impact of complex numbers is described by Needham and recent work by Devaney (2004) displays the connections between elegant mathematics and striking visual images.

It is often useful to express a complex number in polar coordinates in terms of (r, θ), where r is the distance from the origin, and θ is the angle of rotation. Using trigonometry (see Figure 1), we find $z = r(\cos\theta + i\sin\theta)$ or using Euler's formula $z = re^{i\theta}$.

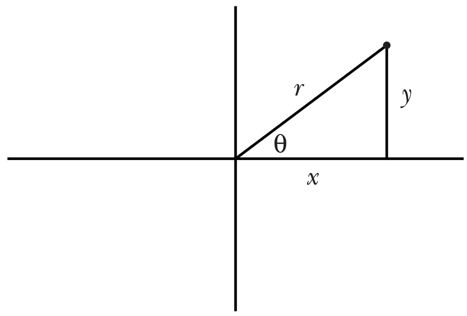

Figure 1. $z = r(\cos\theta + i\sin\theta)$.

A complex function $w = w(z)$ for $z = x + iy$ can be written in the form $w(z) = u(x,y) + iv(x,y)$, where u and v are real-valued functions of the real variables x and y. The roots of a complex function are the solutions to the equation $w(z) = 0$. This equation is true if and only if both $u(x,y) = 0$ and $v(x,y) = 0$.

Definition: A function w is *analytic* at a point z if the derivative of w exists at z and at all points close to z. A point z is a *singular point* of w if w is not analytic at z, but is analytic at some point in every neighbourhood of z (a form of the definition not found in many of the standard complex analysis texts). If w is analytic at all points in some neighbourhood of z (except at z itself), then z is an isolated singular point. Certain isolated singular points, known as *poles*, are important in this investigation.

Iteration

Consider a sequence x_0, $f(x_0)$, $f^2(x_0)$, …, $f^n(x_0)$ where $f^2(x_0)$ means $f(f(x_0))$ and $f^j(x_0)$ represents the jth application of the function f to the x_0 value. We examine the set of iterates

$$\left\{f^n(x_0)\right\}_{n=0}^{\infty}$$

to determine if the terms of this sequence converge, diverge, or oscillate between various values.

Definitions: The initial value x_0 is known as the *seed* and members of the sequence form the *orbit* of x_0. If $f(x) = x$ then x is said to be a *fixed point* of f. If $f^n(x) = x$ for some n, then x is a *periodic point* with period n, defined as the smallest n for which $f^n(x) = x$. If the orbit of x contains some preliminary values before settling at either a fixed point ($f^{n+1}(x) = f^n(x)$ for some $n > 1$) or a periodic orbit ($f^{n+p}(x) = f^n(x)$ for some $n > 1$, where p is the period of the periodic orbit) then x is said to be *eventually fixed* or *eventually periodic*, respectively.

The iterative process is applied in an identical manner when $(w(z))$ is a complex-valued function. The seed value z is a complex number. Fixed points, periodic points, and eventually fixed or periodic points are defined exactly as before.

Definition: The set of all orbits that either converge to a value or diverge (which can also be considered converging to infinity) is called the *Fatou set*. The complement of the Fatou set is the *Julia set*.

A Julia set consists of those orbits that hop around or cycle between various values. The filled Julia set is the Julia set plus all the points whose orbits do not converge to infinity. The colouring of points in the complex plane based on the final destination of their orbit produces a range of beautiful fractal images.

Newton's method

The following is the general formula for Newton's method to find the zeroes of a *real or complex* function g:

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$$

Successive estimates of the zero are iterated until some convergence criteria are satisfied. Newton's method may not converge for every seed value. Specifically, if $g'(x) = 0$ or $g'(x)$ is undefined for any x in the orbit, Newton's method will fail to converge. Also, the original estimate must be close to the desired root or Newton's method may not converge to the desired root.

Pictures from Newton's method

Using MAPLE, a simple algorithm was employed to produce the pictures included in this article. First, a grid was established to determine the seed values to be examined. Next, the grid was sampled to determine the destinations of the orbits of various seed values in the grid. This allowed the various roots to be assigned a colour. Finally, the entire grid was examined, and points coloured according to the destination of orbits. Seed values that did not converge were coloured purple.

Consider $w(z) = z^2 - 1$. The roots of w are $z = \pm 1$. Figure 2 shows the picture generated by this function. Each of the two shaded regions converges to one of the roots of the function. The line dividing the two regions is the y-axis. According to Cayley's Theorem (Cayley, 1879), this line is perpendicular to the segment connecting the two roots. It can be shown that orbits that have seed values on this line remain on the line. To do this we first note that Newton's iteration for the given function is

$$z_{i+1} = \frac{1}{2}\left(z_i + \frac{1}{z_i}\right)$$

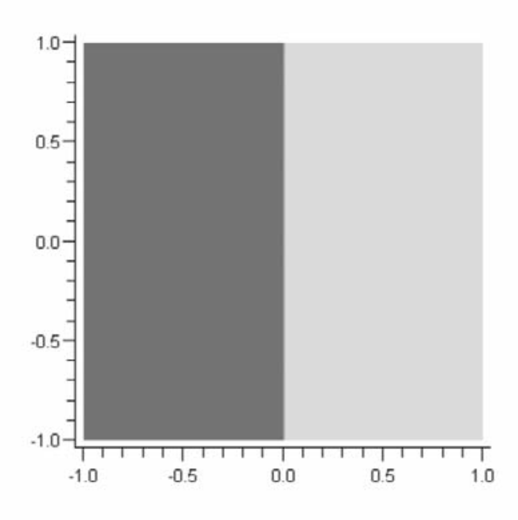

Figure 2. $w(z) = z^2 - 1$.

When the initial value is on the y-axis, then $z_i = iy$ and

$$z_{i+1} = \frac{1}{2}\left(iy + \frac{1}{iy}\right) = \frac{1}{2}\left(iy - \frac{i}{y}\right) = i\left(\frac{1}{2}\left(y - \frac{1}{y}\right)\right)$$

which is clearly on the y-axis also. When the initial value is on the x-axis, then $z_i = x$ and

$$z_{i+1} = \frac{1}{2}\left(x + \frac{1}{x}\right)$$

which can be shown to converge to 1 or –1 depending on whether x is positive or negative.

Figure 3 shows the picture produced by the iteration of Newton's Method for the function, $w(z) = z^3 - 1$. The roots of the function,

$$z = 1,\ \frac{1}{2} \pm \frac{\sqrt{3}}{2} i \text{ (also written as } e^{i\frac{N\Pi}{3}} \text{ for } N = 0, 1 \text{ and } 2)$$

are located in the centre of the three large, shaded regions. These shaded regions are separated by regions of chaotic behaviour that lie along rays that are conjectured to bisect the angles formed by the rays connecting the centre and the roots. To examine this conjecture, one can note that initial seed values may be written in the form $z_0 = Re^{i\theta}$. A line bisecting the angle between the two roots makes an angle of $\frac{\pi}{3}$ with the x-axis. A seed value z_0', which is symmetrical to z_0, with respect to the bisecting line, can then be written in the form

$$z_0' = Re^{i\left(\frac{2\pi}{3} - \theta\right)} = e^{i2\left(\frac{\pi}{3} - \theta\right)} Re^{i\theta} \qquad (*)$$

which is a rotation expressed as multiplication by a factor dependent on the angle measure. Newton's iteration in this case is given by

$$z_{i+1} = \frac{1}{3}\left(2z_i + \frac{1}{z_i^{\ 2}}\right)$$

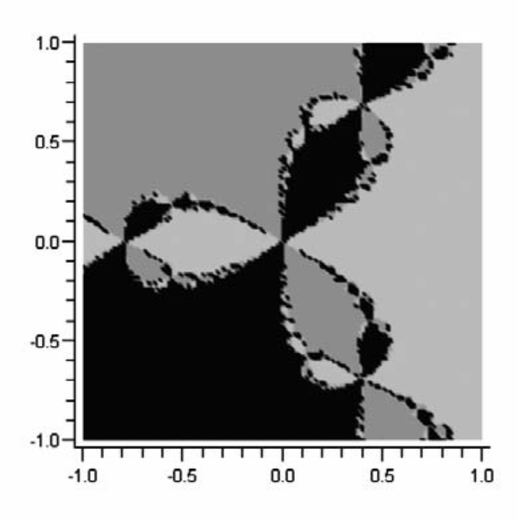

Figure 3. $w(z) = z^3 - 1$.

This involves two transformations. The first, $2z_i$, is simply magnification by a factor of two and does not affect symmetry with respect to the $\frac{\pi}{3}$ ray. The second,

$$\frac{1}{z_i^2}$$

can be written as

$$z_1 = \frac{1}{R^2} e^{-2i\theta}$$

when applied to $z_0 = Re^{i\theta}$. In the same manner, the transformation of

$$z_0' = Re^{i\left(\frac{2\pi}{3} - \theta\right)}$$

can be written as

$$z_1' = \frac{1}{R^2} e^{-2i\left(\frac{2\pi}{3} - \theta\right)}$$

Multiplication of $$z_1 = \frac{1}{R^2} e^{-2i\theta} = \frac{1}{R^2} e^{i(-2\theta)}$$

by the factor $$e^{i2\left(\frac{\pi}{3} - (-2\theta)\right)}$$

as in (*) above, and recalling that

$$e^{i2\frac{\pi}{3}} = e^{-4i\frac{\pi}{3}}$$

shows that z_1' is indeed the image of z_1 with respect to the $\frac{\pi}{3}$ ray. Since both $2z_i$ and

$$\frac{1}{z_i^2}$$

applied to z_i yields a point which is symmetric to z_i, then their sum can easily be shown to yield a symmetric point. This is most clearly seen by using the parallelogram law of vector addition.

Symmetry about the x-axis is also suggested by the diagram. This can be formally shown by noting that the Newton iteration for a cubic function applied to the conjugate $\bar{z} = x - iy$ yields a point which is the conjugate of that obtained when the iteration is applied to z. This is a special case of symmetry with respect to the x axis of any formula which only has real coefficients in all of its terms.

The symmetry issues are discussed more formally by Drexel, Sobey and Bracher (1995) and also by Julia (1918) in his original work. Our purpose here is to present an illustrative approach that is amenable to the reader who may not yet have encountered all of the relevant theoretical background.

Figure 4 shows the function $w(z) = z^7 - 1$. Again note the relationship between the roots and the chaotic regions. The darker region in the centre is a result of the limitations of the computer program. Since $z = 0$ is a zero of w, the region around $z = 0$ represents a neighbourhood of $z = 0$ where the points do not converge in the number of iterations in the program. To see why this is so, we note that the Newton's method iteration includes a z^6th term in the denominator. When the modulus $|z| = \sqrt{x^2 + y^2}$ is small the next iterate is very large in magnitude and requires a large number of iterations to converge back towards the appropriate root. Increasing the number of iterations causes the size of the circle to decrease.

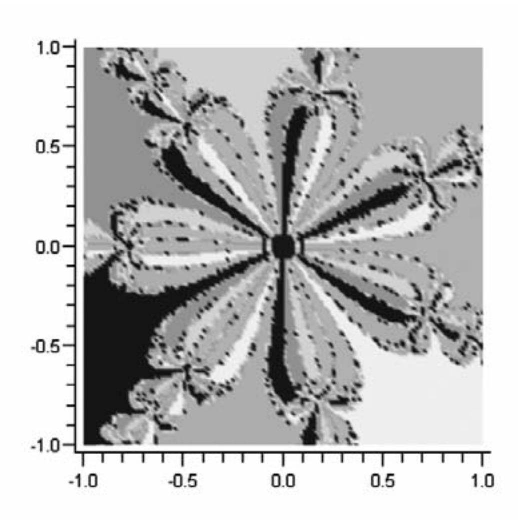

Figure 4. $w(z) = z^7 - 1$.

Figure 5 shows the iterations of the function

$$w(z) = \frac{z^3 - 8}{z^3 + 125}$$

where the zeroes are $z = 2, -1 \pm i\sqrt{3}$

and the poles are $z = 5, -\frac{5}{2} \pm \frac{5\sqrt{3}}{2} i$

We see that the roots lie inside the large circular regions and the poles lie on the edge of the large circular regions. Most importantly, the roots and poles lie along rays extending out from the origin, with each pole lying on the same ray as a root. The poles lie further out from the origin than the roots. These positions are explained when we realise that the poles and roots are simply the cube roots of positive real numbers.

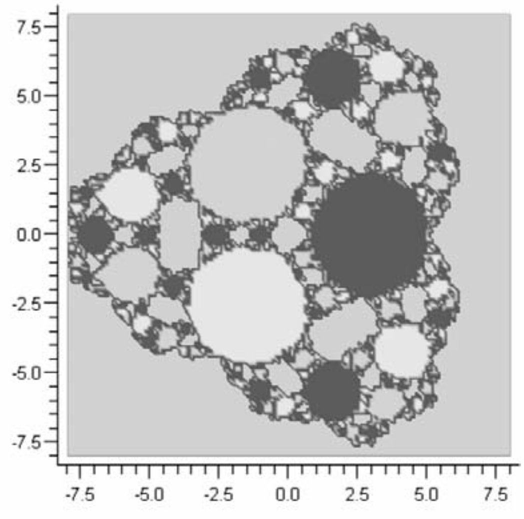

Figure 5. $w(z) = \dfrac{z^3 - 8}{z^3 - 125}$.

In the function

$$w(z) = \frac{z^3 - 8}{z^3 + 125}$$

once again the roots lie in the large regions (although not in the centre), but the poles are located at the end of the chaotic rays extending out from the origin (see Figure 6). This symmetry requires the same analysis as above. These rays can be shown to be the perpendicular bisectors of the line segments connecting the roots. This may be performed as an exercise in elementary geometry. It may also be accomplished by using co-ordinate geometry since the locations of all roots and poles are easily known. In this case the poles are roots of a negative real number.

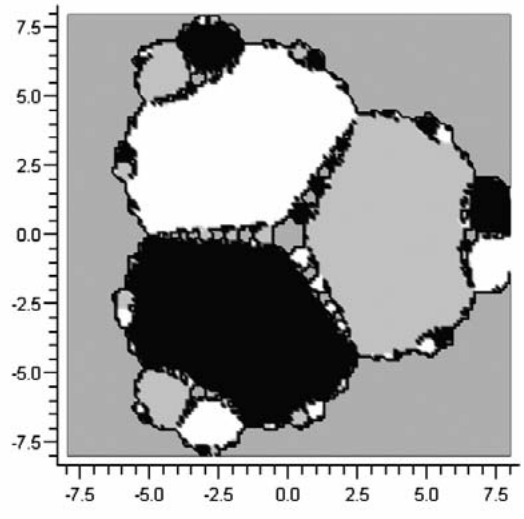

Figure 6. $w(z) = \dfrac{z^3 - 8}{z^3 + 125}$.

Conclusion

In general, one can investigate the function

$$w(z) = \frac{z^3 + a}{z^3 + b}$$

where a and b are arbitrary complex numbers. The reader is encouraged to explore the resulting patterns and those of more complicated functions and the pictures that result from iterations of Newton's method. Numerous topics for teacher workshops or summer research for undergraduates can be based on the material shown. It is the authors' hope that this paper will indeed prove to be a 'seed' value for an exciting 'orbit' through complex iteration.

References

Cayley, A. (1879). The Newton-Fourier imaginary problem. *American Journal of Mathematics, 2*, 97.

Devaney, R. (2004). Cantor and Sierpinski, Julia and Fatou: Complex topology meets complex dynamics. *Notices of the American Mathematical Society, 51*(1), 9–15.

Drexler, M., Sobey, I. J. & Bracher, C. (1995). *On the fractal characteristics of a modified Newton method.* Oxford University Technical Report NA 95/26.

Julia, G. (1918). Mémoire sur l'itération des fonctions rationnelles. *Journal de Mathématiques Pures et Appliquées, 8*, 47–245.

Markus, F. (Ed.) (1983). *Girolamo Cardano (1501–1576): Physician, natural philosopher, mathematician, astrologer and interpreter of dreams.* Boston, MA: Birkhäuser.

Needham, T. (1997). *Visual complex analysis.* Oxford: Oxford University Press.